symmetry

Volume 2

Structures of Neutrosophic Triplets, Neutrosophic Duplets, or Neutrosophic Multisets

Edited by
Florentin Smarandache, Xiaohong Zhang and Mumtaz Ali

Printed Edition of the Special Issue Published in *Symmetry*

www.mdpi.com/journal/symmetry

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This is a reprint of articles from the Special Issue published online in the open access journal *Symmetry* (ISSN 2073-8994) from 2018 to 2019 (available at: http://www.mdpi.com/journal/symmetry/special_issues/Algebraic_Structure_Neutrosophic_Triplet_Neutrosophic_Duplet_Neutrosophic_Multiset)

For citation purposes, cite each article independently as indicated on the article page online and as indicated below:

LastName, A.A.; LastName, B.B.; LastName, C.C. Article Title. *Journal Name* **Year**, *Article Number*, Page Range.

Volume 2
ISBN 978-3-03897-475-8 (Pbk)
ISBN 978-3-03897-476-5 (PDF)

Volume 1-2
ISBN 978-3-03897-477-2 (Pbk)
ISBN 978-3-03897-478-9 (PDF)

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About the Special Issue Editors

Florentin Smarandache is a professor of mathematics at the University of New Mexico, USA. He got his M.Sc. in Mathematics and Computer Science from the University of Craiova, Romania, Ph.D. in Mathematics from the State University of Kishinev, and Post-Doctoral in Applied Mathematics from Okayama University of Sciences, Japan. He is the founder of neutrosophic set, logic, probability and statistics since 1995 and has published hundreds of papers on neutrosophic physics, superluminal and instantaneous physics, unmatter, absolute theory of relativity, redshift and blueshift due to the medium gradient and refraction index besides the Doppler effect, paradoxism, outerart, neutrosophy as a new branch of philosophy, Law of Included Multiple-Middle, degree of dependence and independence between the neutrosophic components, refined neutrosophic over-under-off-set, neutrosophic overset, neutrosophic triplet and duplet structures, DSmT and so on to many peer-reviewed international journals and many books and he presented papers and plenary lectures to many international conferences around the world.

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Mumtaz Ali is a Ph.D. research scholar under Principal Supervision of Dr. Ravinesh Deo and also guided by Dr. Nathan Downs. He is originally from Pakistan where he completed his double masters (M.Sc. and M.Phil. in Mathematics) from Quaid-i-Azam University, Islamabad. Mumtaz has been an active researcher in Neutrosophic Set and Logic; proposed the Neutrosophic Triplets. Mumtaz is the author of three books on neutrosophic algebraic structures. Published more than 30 research papers in prestigious journals. He also published two chapters in the edited books. Research Interests: Currently, Mumtaz pursuing his doctoral studies in drought characteristic and atmospheric simulation models using artificial intelligence. He intends to apply probabilistic (copula-based) and machine learning modelling; fuzzy set and logic; neutrosophic set and logic; soft computing; decision support systems; data mining; clustering and medical diagnosis problems.

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Article

Multi-Criteria Decision-Making Method Based on Prioritized Muirhead Mean Aggregation Operator under Neutrosophic Set Environment

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Received: 6 June 2018; Accepted: 9 July 2018; Published: 12 July 2018

Abstract: The aim of this paper is to introduce some new operators for aggregating single-valued neutrosophic (SVN) information and to apply them to solve the multi-criteria decision-making (MCDM) problems. Single-valued neutrosophic set, as an extension and generalization of an intuitionistic fuzzy set, is a powerful tool to describe the fuzziness and uncertainty, and Muirhead mean (MM) is a well-known aggregation operator which can consider interrelationships among any number of arguments assigned by a variable vector. In order to make full use of the advantages of both, we introduce two new prioritized MM aggregation operators, such as the SVN prioritized MM (SVNPMM) and SVN prioritized dual MM (SVNPDMM) under SVN set environment. In addition, some properties of these new aggregation operators are investigated and some special cases are discussed. Furthermore, we propose a new method based on these operators for solving the MCDM problems. Finally, an illustrative example is presented to testify the efficiency and superiority of the proposed method by comparing it with the existing method.

Keywords: neutrosophic set; prioritized operator; Muirhead mean; multicriteria decision-making; aggregation operators; dual aggregation operators

1. Introduction

Multicriteria decision-making (MCDM) is one of the hot topics in the decision-making field to choose the best alternative to the set of the feasible one. In this process, the rating values of each alternative include both precise data and experts' subjective information [1,2]. However, traditionally, it is assumed that the information provided by them are crisp in nature. However, due to the complexity of the system day-by-day, the real-life contains many MCDM problems where the information is either vague, imprecise or uncertain in nature [3]. To deal with it, the theory of fuzzy set (FS) [4] or extended fuzzy sets such as intuitionistic fuzzy set (IFS) [5], interval-valued IFS (IVIFS) [6] are the most successful ones, which characterize the criterion values in terms of membership degrees. Since their existence, numerous researchers were paying more attention to these theories and developed several approaches using different aggregation operators [7–10] and ranking methods [11–13] in the processing of the information values.

It is remarked that neither the FS nor the IFS theory are able to deal with indeterminate and inconsistent data. For instance, consider an expert which gives their opinion about a certain object in such a way that 0.5 being the possibility that the statement is true, 0.7 being the possibility that the statement is false and 0.2 being the possibility that he or she is not sure. Such type of data is not handled with FS, IFS or IVIFS. To resolve this, Smarandache [14] introduced the concept neutrosophic sets (NSs). In NS, each element in the universe of discourse set has degrees of truth membership, indeterminacy-membership and falsity membership, which takes values in the non-standard unit interval $(0^-, 1^+)$. Due to this non-standard unit interval, NS theory is hard to implement on the practical problems. So in order to use NSs in engineering problems more easily, some classes of NSs and their theories were proposed [15,16]. Wang et al. [16] presented the class of NS named as interval NS while in Wang et al. [15], a class of single-valued NS (SVNS) is presented. Due to its importance, several researchers have made their efforts to enrich the concept of NSs in the decision-making process and some theories such as distance measures [17], score functions [18], aggregation operators [19–23] and so on.

Generally, aggregation operators (AOs) play an important role in the process of MCDM problems whose main target is to aggregate a collection of the input to a single number. In that direction, Ye [21] presented the operational laws of SVNSs and proposed the single-valued neutrosophic (SVN) weighted averaging (SVNWA) and SVN weighted geometric average (SVNWGA) operators. Peng et al. [22] defined the improved operations of SVN numbers (SVNNs) and developed their corresponding ordered weighted average/geometric aggregation operator. Nancy and Garg [24] developed the weighted average and geometric average operators by using the Frank norm operations. Liu et al. [25] developed some generalized neutrosophic aggregation operators based on Hamacher operations. Zhang et al. [26] presented the aggregation operators under interval neutrosophic set (INS) environment and Aiwu et al. [27] proposed some of its generalized operators. Garg and Nancy [19] developed a nonlinear optimization model to solve the MCDM problem under the INS environment.

From the above mentioned AOs, it is analyzed that all these studies assume that all the input arguments used during aggregation are independent of each other and hence there is no interrelationship between the argument values. However, in real-world problems, there always occurs a proper relationship between them. For instance, if a person wants to purchase a house then there is a certain relationship between its cost and the locality. Clearly, both the factors are mutually dependent and interacting. In order to consider the interrelationship of the input arguments, Bonferroni mean (BM) [28], Maclaurin symmetric mean (MSM) [29], Heronian mean (HM) [30] etc., are the useful aggregation functions. Yager [31] proposed the concept of BM whose main characteristic is its capability to capture the interrelationship between the input arguments. Garg and Arora [32] presented BM aggregation operators under the intuitionistic fuzzy soft set environment. In these functions, BM can capture the interrelationship between two arguments while others can capture more than two relationships. Taking the advantages of these functions in a neutrosophic domain, Liu and Wang [33] applied the BM to a neutrosophic environment and introduce the SVN normalized weighted Bonferroni mean (SVNNWBM) operator. Wang et al. [34] proposed the MSM aggregation operators to capture the correlation between the aggregated arguments. Li et al. [20] presented HM operators to solve the MCDM problems under SVNS environment. Garg and Nancy [35] presented prioritized AOs under the linguistic SVNS environment to solve the decision-making problems. Wu et al. [36] developed some prioritized weighted averaging and geometric aggregation operators for SVNNs. Ji et al. [37] established the single-valued prioritized BM operator by using the Frank operations. An alternative to these aggregations, the Muirhead mean (MM) [38] is a powerful and useful aggregation technique. The prominent advantage of the MM is that it can consider the interrelationships among all arguments, which makes it more powerful and comprehensive than BM, MSM and HM. In addition, MM has a parameter vector which can make the aggregation process more flexible.

Based on the above analysis, we know the decision-making problems are becoming more and more complex in the real world. In order to select the best alternative(s) for the MCDM problems, it is necessary to express the uncertain information in a more profitable way. In addition, it is important to deal with how to consider the relationship between input arguments. Keeping all these features in mind, and by taking the advantages of the SVNS, we combine the prioritized aggregation and MM and propose prioritized MM (PMM) operator by considering the advantages of both. These considerations have led us to consider the following main objectives for this paper:

1. to handle the impact of the some unduly high or unduly low values provided by the decision makers on to the final ranking;

2. to present some new aggregation operators to aggregate the preferences of experts element;
3. to develop an algorithm to solve the decision-making problems based on proposed operators;
4. to present some example in which relevance of the preferences in SVN decision problems is made explicit.

Since in our real decision-making problems, we always encounter a problem of some attributes' values, provided by the decision makers, whose impact on the decision-making process are unduly high or unduly low; this consequently results in a bad impression on the final results. To handle it, in the first objective we utilize prioritized averaging (PA) as an aggregation function which can handle such a problem very well. To achieve the second objective, we develop two new AOs, named as SVN prioritized MM (SVNPMM) and SVN prioritized dual MM (SVNPDMM) operators, by extending the operations of SVNNs by using MM and PA operators. MM operator is a powerful and useful aggregation technique with the feature that it considers the interrelationships among all arguments which makes it more powerful and comprehensive than BM [28], MSM [29] and HM [30]. Moreover, the MM has a parameter vector which can make the aggregation process more flexible. Several properties and some special cases from the proposed operators are investigated. To achieve the third objective, we establish an MCDM method based on these proposed operators under the SVNS environment where preferences related to each alternative is expressed in terms of SVNNs. An illustrative example is presented to testify the efficiency and superiority of the proposed method by comparative analysis with the other existing methods for fulfilling the fourth objective. Further, apart from these, we verify that the methods proposed in this paper have advantages with respect to existing operators as follows: (1) some of the existing AOs can be taken as a special case of the proposed operators under NSs environment, (2) they consider the interrelationship among all arguments, (3) they are more adaptable and feasible than the existing AOs based on the parameter vector, (4) the presented approach considers the preferences of the decision maker in terms of risk preference as well as risk aversion.

The rest of the manuscript is organized as follows. In Section 2, we briefly review the concepts of SVNS and the aggregation operators. In Section 3, two new AOs based on PA and MM operations are developed under SVNS environment and their desirable properties are investigated. In addition, some special cases of the operators by varying the parametric value are discussed. In Section 4, we explore the applications of SVNN to MCDM problems with the aid of the proposed decision-making method and demonstrate with a numerical example. Finally, Section 5 gives the concluding remarks.

2. Preliminaries

In this section, some basic concepts related to SVNSs have been defined over the universal set X with a generic element $x \in X$.

Definition 1 ([14])**.** *A neutrosophic set (NS) α comprises of three independent degrees in particular truth (μ_α), indeterminacy (ρ_α), and falsity (ν_α) which are characterized as*

$$\alpha = \left\{ \langle x, \mu_\alpha(x), \rho_\alpha(x), \nu_\alpha(x) \mid x \in X \rangle \right\}, \tag{1}$$

where $\mu_\alpha(x)$, $\rho_\alpha(x)$, $\nu_\alpha(x)$ is the subset of the non-standard unit interval $(0^-, 1^+)$ such that $0^- \leq \mu_\alpha(x) + \rho_\alpha(x) + \nu_\alpha(x) \leq 3^+$.

Definition 2 ([16])**.** *A single-valued neutrosophic set (SVNS) α in X is defined as*

$$\alpha = \left\{ \langle x, \mu_\alpha(x), \rho_\alpha(x), \nu_\alpha(x) \mid x \in X \rangle \right\}, \tag{2}$$

where $\mu_\alpha(x)$, $\rho_\alpha(x)$, $\nu_\alpha(x) \in [0, 1]$ such that $0 \leq \mu_\alpha(x) + \rho_\alpha(x) + \nu_\alpha(x) \leq 3$ for all $x \in X$. A SVNS is an instance of an NS.

For convenience, we denote this pair as $\alpha = (\mu_\alpha, \rho_\alpha, \nu_\alpha)$, throughout this article, and called as SVNN with the conditions $\mu_\alpha, \rho_\alpha, \nu_\alpha \in [0,1]$ and $\mu_\alpha + \rho_\alpha + \nu_\alpha \leq 3$.

Definition 3 ([18]). *Let $\alpha = (\mu_\alpha, \rho_\alpha, \nu_\alpha)$ be a SVNN. A score function s of α is defined as*

$$s(\alpha) = \frac{1 + (\mu_\alpha - 2\rho_\alpha - \nu_\alpha)(2 - \mu_\alpha - \nu_\alpha)}{2}. \tag{3}$$

Based on this function, an ordered relation between two SVNNs α and β is stated as, if $s(\alpha) > s(\beta)$ then $\alpha > \beta$.

Definition 4 ([16,22]). *Let $\alpha = (\mu, \rho, \nu)$, $\alpha_1 = (\mu_1, \rho_1, \nu_1)$ and $\alpha_2 = (\mu_2, \rho_2, \nu_2)$ be three SVNNs and $\lambda > 0$ be real number. Then, we have*

1. $\alpha^c = (\nu, \rho, \mu)$;
2. $\alpha_1 \leq \alpha_2$ *if* $\mu_1 \leq \mu_2, \rho_1 \geq \rho_2$ *and* $\nu_1 \geq \nu_2$;
3. $\alpha_1 = \alpha_2$ *if and only if* $\alpha_1 \leq \alpha_2$ *and* $\alpha_2 \leq \alpha_1$;
4. $\alpha_1 \cap \alpha_2 = (\min(\mu_1, \mu_2), \max(\rho_1, \rho_2), \max(\nu_1, \nu_2))$;
5. $\alpha_1 \cup \alpha_2 = (\max(\mu_1, \mu_2), \min(\rho_1, \rho_2), \min(\nu_1, \nu_2))$;
6. $\alpha_1 \oplus \alpha_2 = (\mu_1 + \mu_2 - \mu_1\mu_2, \rho_1\rho_2, \nu_1\nu_2)$;
7. $\alpha_1 \otimes \alpha_2 = (\mu_1\mu_2, \rho_1 + \rho_2 - \rho_1\rho_2, \nu_1 + \nu_2 - \nu_1\nu_2)$;
8. $\lambda\alpha_1 = (1 - (1 - \mu_1)^\lambda, \rho_1^\lambda, \nu_1^\lambda)$;
9. $\alpha_1^\lambda = (\mu_1^\lambda, 1 - (1 - \rho_1)^\lambda, 1 - (1 - \nu_1)^\lambda)$.

Definition 5 ([36]). *For a collection of SVNNs $\alpha_j = (\mu_j, \rho_j, \nu_j)(j = 1, 2, \ldots, n)$, the prioritized weighted aggregation operators are defined as*

1. *SVN prioritized weighted average (SVNPWA) operator*

$$SVNPWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(1 - \prod_{j=1}^{n}(1 - \mu_j)^{\frac{H_j}{\sum_{j=1}^{n} H_j}}, \prod_{j=1}^{n}(\rho_j)^{\frac{H_j}{\sum_{j=1}^{n} H_j}}, \prod_{j=1}^{n}(\nu_j)^{\frac{H_j}{\sum_{j=1}^{n} H_j}}\right), \tag{4}$$

2. *SVN prioritized geometric average (SVNPGA) operator*

$$SVNPGA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\prod_{j=1}^{n}(\mu_j)^{\frac{H_j}{\sum_{j=1}^{n} H_j}}, 1 - \prod_{j=1}^{n}(1 - \rho_j)^{\frac{H_j}{\sum_{j=1}^{n} H_j}}, 1 - \prod_{j=1}^{n}(1 - \nu_j)^{\frac{H_j}{\sum_{j=1}^{n} H_j}}\right), \tag{5}$$

where $H_1 = 1$ and $H_j = \prod_{k=1}^{j-1} s(\alpha_k)$; $(j = 2, \ldots, n)$.

Definition 6 ([38]). *For a non-negative real numbers $h_j(j = 1, 2, \ldots, n)$, (MM) operator over the parameter $P = (p_1, p_2, \ldots, p_n) \in R^n$ is defined as*

$$MM^P(h_1, h_2, \ldots, h_n) = \left(\frac{1}{n!} \sum_{\sigma \in S_n} \prod_{j=1}^{n} h_{\sigma(j)}^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \tag{6}$$

where σ is the permutation of $(1, 2, \ldots, n)$ and S_n is set of all permutations of $(1, 2, \ldots, n)$.

By assigning some special vectors to P, we can obtain some special cases of the MM:

1. If $P = (1, 0, \ldots, 0)$, the MM is reduced to

$$\mathrm{MM}^{(1,0,\ldots,0)}(h_1, h_2, \ldots, h_n) = \frac{1}{n}\sum_{j=1}^{n} h_j, \tag{7}$$

which is the arithmetic averaging operator.

2. If $P = (1/n, 1/n, \ldots, 1/n)$, the MM is reduced to

$$\mathrm{MM}^{(1/n,1/n,\ldots,1/n)}(h_1, h_2, \ldots, h_n) = \prod_{j=1}^{n} h_j^{1/n}, \tag{8}$$

which is the geometric averaging operator.

3. If $P = (1, 1, 0, 0, \ldots, 0)$, then the MM is reduced to

$$\mathrm{MM}^{(1,1,0,0,\ldots,0)}(h_1, h_2, \ldots, h_n) = \left(\frac{1}{n(n+1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} h_i h_j \right)^{1/2}, \tag{9}$$

which is the BM operator [28].

4. If $P = (\overbrace{1, 1, \ldots, 1}^{k}, \overbrace{0, 0, \ldots, 0}^{n-k})$, then the MM is reduced to

$$\mathrm{MM}^{(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,0,\ldots,0}^{n-k})}(h_1, h_2, \ldots, h_n) = \left(\frac{1}{C_k^n} \sum_{\substack{1 \leq i_1 < \\ \ldots < i_k \leq n}} \prod_{j=1}^{k} h_{i_j} \right)^{1/k}, \tag{10}$$

which is the MSM operator [29].

3. Neutrosophic Prioritized Muirhead Mean Operators

In this section, by considering the overall interrelationships among the multiple input arguments, we develop some new prioritized based MM aggregation operators for a collection of SVNNs α_j; $(j = 1, 2, \ldots, n)$, denoted by Ω. Assume that σ is the permutation of $(1, 2, \ldots, n)$ such that $\alpha_{\sigma(j-1)} \leq \alpha_{\sigma(j)}$ for $j = 2, 3, \ldots, n$.

3.1. Single-Valued Neutrosophic Prioritized Muirhead Mean (SVNPMM) Operator

Definition 7. *For a collection of SVNNs $\alpha_j (j = 1, 2, \ldots, n)$, a SVNPMM operator is a mapping SVNPMM : $\Omega \to \Omega$ defined as*

$$SVNPMM(\alpha_1, \alpha_2, \ldots, \alpha_n) \;=\; \left(\frac{1}{n!} \bigoplus_{\sigma \in S_n} \prod_{j=1}^{n} \left(n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j} \alpha_{\sigma(j)} \right)^{p_j} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \tag{11}$$

where $H_1 = 1$, $H_j = \prod_{k=1}^{j-1} s(\alpha_k)$; $(j = 2, \ldots, n)$, S_n is collection of all permutations of $(1, 2, \ldots, n)$ and $P = (p_1, p_1, \ldots, p_n) \in R^n$ be a vector of parameters.

Theorem 1. *For a collection of SVNNs $\alpha_j = (\mu_j, \rho_j, \nu_j)(j = 1, 2, \ldots, n)$, the aggregated value by Equation (11) is again a SVNN and given by*

$$
\begin{aligned}
&SVNPMM(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= \left(\begin{array}{c}
\left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - (1 - \mu_{\sigma(j)})^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\
1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \rho_{\sigma(j)}^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\
1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \nu_{\sigma(j)}^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}
\end{array} \right). \qquad (12)
\end{aligned}
$$

Proof. For SVNN $\alpha_j (j = 1, 2, \ldots, n)$ and by Definition 4, we have

$$
n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}\alpha_{\sigma}(j) = \left(1 - (1 - \mu_{\sigma(j)})^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}, \rho_{\sigma(j)}^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}, \nu_{\sigma(j)}^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)
$$

and

$$
\left(n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}\alpha_{\sigma(j)} \right)^{p_j} = \left(\begin{array}{c}
\left(1 - (1 - \mu_{\sigma(j)})^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j}, 1 - \left(1 - \rho_{\sigma(j)}^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j}, \\
1 - \left(1 - \nu_{\sigma(j)}^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j}
\end{array} \right).
$$

Thus,

$$
\bigoplus_{\sigma \in S_n} \prod_{j=1}^{n} \left(n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}\alpha_{\sigma(j)} \right)^{p_j} = \left(\begin{array}{c}
1 - \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - (1 - \mu_{\sigma(j)})^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j} \right), \\
\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \rho_{\sigma(j)}^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j} \right), \\
\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \nu_{\sigma(j)}^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j} \right)
\end{array} \right).
$$

Now,

$$\mathrm{SVNPMM}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{1}{n!} \bigoplus_{\sigma \in S_n} \prod_{j=1}^{n} \left(n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j} \alpha_{\sigma(j)} \right)^{p_j} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}$$

$$= \left(\begin{array}{l} \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - (1-\mu_{\sigma(j)})^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ 1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \rho_{\sigma(j)}^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ 1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \nu_{\sigma(j)}^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \end{array} \right).$$

Thus Equation (12) holds. Furthermore, $0 \leq \mu_{\sigma(j)}, \rho_{\sigma(j)}, \nu_{\sigma(j)} \leq 1$ so we have

$$1 - \left(1 - \mu_{\sigma(j)}\right)^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \in [0,1]$$

and

$$\prod_{j=1}^{n} \left(1 - (1-\mu_{\sigma(j)})^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j} \in [0,1],$$

which implies that

$$1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - (1-\mu_{\sigma(j)})^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j} \right) \right) \in [0,1].$$

Hence,

$$0 \leq \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - (1-\mu_{\sigma(j)})^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \leq 1.$$

Similarly, we have

$$0 \le 1 - \left(1 - \left(\prod_{\sigma \in S_n}\left(1 - \prod_{j=1}^{n}\left(1 - \rho_{\sigma(j)}^{\,n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \le 1$$

and

$$0 \le 1 - \left(1 - \left(\prod_{\sigma \in S_n}\left(1 - \prod_{j=1}^{n}\left(1 - \nu_{\sigma(j)}^{\,n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \le 1.$$

which complete the proof. □

The working of the proposed operator is demonstrated through a numerical example, which is illustrated as follow.

Example 1. *Let* $\alpha_1 = (0.5,\ 0.2,\ 0.3)$, $\alpha_2 = (0.3,\ 0.5,\ 0.4)$ *and* $\alpha_3 = (0.6,\ 0.5,\ 0.2)$ *be three SVNNs and* $P = (1,\ 0.5,\ 0.3)$ *be the given parameter vector. By utilizing the given information and* $H_j = \prod_{k=1}^{j-1} s(\alpha_k)$; $(j = 2, 3)$, *we get* $H_1 = 1$, $H_2 = 0.74$ *and* $H_3 = 0.2257$. *Therefore,*

$$\begin{aligned}
&\prod_{\sigma \in S_3}\left(1 - \prod_{j=1}^{3}\left(1 - (1 - \mu_{\sigma(j)})^{3\frac{H_{\sigma(j)}}{\sum_{j=1}^{3} H_j}}\right)^{p_j}\right) \\
= &\left\{1 - \left(1 - (1-0.5)^{3\times 0.5087}\right)^{1} \times \left(1 - (1-0.3)^{3\times 0.3765}\right)^{0.5} \times \left(1 - (1-0.6)^{3\times 0.1148}\right)^{0.3}\right\} \\
\times &\left\{1 - \left(1 - (1-0.3)^{3\times 0.3765}\right)^{1} \times \left(1 - (1-0.5)^{3\times 0.5087}\right)^{0.5} \times \left(1 - (1-0.6)^{3\times 0.1148}\right)^{0.3}\right\} \\
\times &\left\{1 - \left(1 - (1-0.6)^{3\times 0.1148}\right)^{1} \times \left(1 - (1-0.3)^{3\times 0.3765}\right)^{0.5} \times \left(1 - (1-0.5)^{3\times 0.5087}\right)^{0.3}\right\} \\
\times &\left\{1 - \left(1 - (1-0.3)^{3\times 0.3765}\right)^{1} \times \left(1 - (1-0.6)^{3\times 0.1148}\right)^{0.5} \times \left(1 - (1-0.5)^{3\times 0.5087}\right)^{0.3}\right\} \\
\times &\left\{1 - \left(1 - (1-0.5)^{3\times 0.5087}\right)^{1} \times \left(1 - (1-0.6)^{3\times 0.1148}\right)^{0.5} \times \left(1 - (1-0.3)^{3\times 0.3765}\right)^{0.3}\right\} \\
\times &\left\{1 - \left(1 - (1-0.6)^{3\times 0.1148}\right)^{1} \times \left(1 - (1-0.5)^{3\times 0.5087}\right)^{0.5} \times \left(1 - (1-0.3)^{3\times 0.3765}\right)^{0.3}\right\} \\
= &\ 0.0052.
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
& \prod_{\sigma \in S_3} \left(1 - \prod_{j=1}^{3} \left(1 - \rho_{\sigma(j)}^{3 \frac{H_{\sigma(j)}}{\sum_{j=1}^{3} H_j}} \right)^{p_j} \right) \\
= & \left\{ 1 - \left(1 - (0.2)^{3 \times 0.5087} \right)^{1} \times \left(1 - (0.5)^{3 \times 0.3765} \right)^{0.5} \times \left(1 - (0.5)^{3 \times 0.1148} \right)^{0.3} \right\} \\
\times & \left\{ 1 - \left(1 - (0.5)^{3 \times 0.3765} \right)^{1} \times \left(1 - (0.2)^{3 \times 0.5087} \right)^{0.5} \times \left(1 - (0.5)^{3 \times 0.1148} \right)^{0.3} \right\} \\
\times & \left\{ 1 - \left(1 - (0.5)^{3 \times 0.1148} \right)^{1} \times \left(1 - (0.5)^{3 \times 0.3765} \right)^{0.5} \times \left(1 - (0.2)^{3 \times 0.5087} \right)^{0.3} \right\} \\
\times & \left\{ 1 - \left(1 - (0.5)^{3 \times 0.3765} \right)^{1} \times \left(1 - (0.5)^{3 \times 0.1148} \right)^{0.5} \times \left(1 - (0.2)^{3 \times 0.5087} \right)^{0.3} \right\} \\
\times & \left\{ 1 - \left(1 - (0.2)^{3 \times 0.5087} \right)^{1} \times \left(1 - (0.5)^{3 \times 0.1148} \right)^{0.5} \times \left(1 - (0.5)^{3 \times 0.3765} \right)^{0.3} \right\} \\
\times & \left\{ 1 - \left(1 - (0.5)^{3 \times 0.1148} \right)^{1} \times \left(1 - (0.2)^{3 \times 0.5087} \right)^{0.5} \times \left(1 - (0.5)^{3 \times 0.3765} \right)^{0.3} \right\} \\
= & \; 0.000093196
\end{aligned}
$$

and

$$
\begin{aligned}
& \prod_{\sigma \in S_3} \left(1 - \prod_{j=1}^{3} \left(1 - \nu_{\sigma(j)}^{3 \frac{H_{\sigma(j)}}{\sum_{j=1}^{3} H_j}} \right)^{p_j} \right) \\
= & \left\{ 1 - \left(1 - (0.3)^{3 \times 0.5087} \right)^{1} \times \left(1 - (0.4)^{3 \times 0.3765} \right)^{0.5} \times \left(1 - (0.2)^{3 \times 0.1148} \right)^{0.3} \right\} \\
\times & \left\{ 1 - \left(1 - (0.4)^{3 \times 0.3765} \right)^{1} \times \left(1 - (0.3)^{3 \times 0.5087} \right)^{0.5} \times \left(1 - (0.2)^{3 \times 0.1148} \right)^{0.3} \right\} \\
\times & \left\{ 1 - \left(1 - (0.2)^{3 \times 0.1148} \right)^{1} \times \left(1 - (0.4)^{3 \times 0.3765} \right)^{0.5} \times \left(1 - (0.3)^{3 \times 0.5087} \right)^{0.3} \right\} \\
\times & \left\{ 1 - \left(1 - (0.4)^{3 \times 0.3765} \right)^{1} \times \left(1 - (0.2)^{3 \times 0.1148} \right)^{0.5} \times \left(1 - (0.3)^{3 \times 0.5087} \right)^{0.3} \right\} \\
\times & \left\{ 1 - \left(1 - (0.3)^{3 \times 0.5087} \right)^{1} \times \left(1 - (0.2)^{3 \times 0.1148} \right)^{0.5} \times \left(1 - (0.4)^{3 \times 0.3765} \right)^{0.3} \right\} \\
\times & \left\{ 1 - \left(1 - (0.2)^{3 \times 0.1148} \right)^{1} \times \left(1 - (0.3)^{3 \times 0.5087} \right)^{0.5} \times \left(1 - (0.4)^{3 \times 0.3765} \right)^{0.3} \right\} \\
= & \; 0.00000093195.
\end{aligned}
$$

Hence, by using Equation (12), we get the aggregated value by SVNPMM is

$$\begin{aligned}
& SVNPMM(\alpha_1, \alpha_2, \alpha_3) \\
= & \left(\begin{array}{c} \left(1-(0.0052)^{1/6}\right)^{1/1.8}, 1-\left(1-(0.000093196)^{1/6}\right)^{1/1.8}, \\ 1-\left(1-(0.00000093195)^{1/6}\right)^{1/1.8} \end{array} \right) \\
= & (0.7415, 0.1246, 0.0562).
\end{aligned}$$

It is observed from the proposed operator that it satisfies the certain properties which are stated as follows.

Theorem 2. *If $\alpha_j = (\mu_j, \rho_j, \nu_j)$ and $\alpha'_j = (\mu'_j, \rho'_j, \nu'_j)$ are two SVNNs such that $\mu_j \leq \mu'_j$, $\rho_j \geq \rho'_j$ and $\nu_j \geq \nu'_j$ for all j, then*

$$SVNPMM(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq SVNPMM(\alpha'_1, \alpha'_2, \ldots, \alpha'_n).$$

This property is called monotonicity.

Proof. For two SVNNs α_j and α'_j, we have $\alpha_{\sigma(j)} \leq \alpha'_{\sigma(j)}$, for all j which implies that $\mu_{\sigma(j)} \leq \mu'_{\sigma(j)}$ and $(1-\mu_{\sigma(j)})^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \geq (1-\mu'_{\sigma(j)})^{n\frac{H'_{\sigma(j)}}{\sum_{j=1}^{n} H'_j}}$, where $H_1 = 1$, $H_j = \prod_{k=1}^{j-1} s(\alpha_k)$ and $H'_1 = 1$, $H'_j = \prod_{k=1}^{j-1} s(\alpha'_k)$ for $(j = 2, 3, \ldots, n)$. Thus,

$$\left(1-(1-\mu_{\sigma(j)})^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j} \leq \left(1-(1-\mu'_{\sigma(j)})^{n\frac{H'_{\sigma(j)}}{\sum_{j=1}^{n} H'_j}}\right)^{p_j}$$

$$\text{and} \quad \prod_{j=1}^{n}\left(1-(1-\mu_{\sigma(j)})^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j} \leq \prod_{j=1}^{n}\left(1-(1-\mu'_{\sigma(j)})^{n\frac{H'_{\sigma(j)}}{\sum_{j=1}^{n} H'_j}}\right)^{p_j}.$$

Further, we have

$$\begin{aligned}
& \prod_{\sigma \in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\mu_{\sigma(j)})^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j}\right) \\
& \quad \geq \prod_{\sigma \in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\mu'_{\sigma(j)})^{n\frac{H'_{\sigma(j)}}{\sum_{j=1}^{n} H'_j}}\right)^{p_j}\right)
\end{aligned}$$

and

$$\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\sigma(j)}\right)^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n}H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}} \geq \left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu'_{\sigma(j)}\right)^{n\frac{H'_{\sigma(j)}}{\sum_{j=1}^{n}H'_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}.$$

Hence, we get

$$\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\sigma(j)}\right)^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n}H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}} \leq \left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu'_{\sigma(j)}\right)^{n\frac{H'_{\sigma(j)}}{\sum_{j=1}^{n}H'_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}.$$

Similarly, we have

$$1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\rho_{\sigma(j)}^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n}H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}} \geq 1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\rho'^{\,n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n}H_j}}_{\sigma(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$

and

$$1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\nu_{\sigma(j)}^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n}H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}} \geq 1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\nu'^{\,n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n}H_j}}_{\sigma(j)}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}.$$

Therefore, by Definition 4, we have

$$SVNPMM(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq SVNPMM(\alpha'_1, \alpha'_2, \ldots, \alpha'_n).$$

□

Theorem 3. *For a collection of SVNNs $\alpha_j = (\mu_j, \rho_j, \nu_j)(j = 1, 2, \ldots, n)$. Let $\alpha^- = (\mu^-, \rho^-, \nu^-)$ and $\alpha^+ = (\mu^+, \rho^+, \nu^+)$ be the lower and upper bound, respectively, of the SVNNs where $\mu^- = \min_j\{\mu_j\}$, $\rho^- = \max_j\{\rho_j\}$, $\nu^- = \max_j\{\nu_j\}$, $\mu^+ = \max_j\{\mu_j\}$, $\rho^+ = \min_j\{\rho_j\}$ and $\nu^+ = \min_j\{\nu_j\}$, then*

$$\alpha^- \leq SVNPMM(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+.$$

This property is called boundedness.

Proof. Since $\min_j\{\mu_j\} \leq \mu_j$, therefore $\min_j\{\mu_j\} \leq \mu_{\sigma(j)}$, which implies

$$\left(1 - \min_j \mu_j\right)^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \geq \left(1 - \mu_{\sigma(j)}\right)^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}$$

and

$$\left(1 - \left(1 - \min_j \mu_j\right)^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j} \leq \left(1 - \left(1 - \mu_{\sigma(j)}\right)^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j}.$$

Then,

$$\prod_{j=1}^{n} \left(1 - \left(1 - \min_j \mu_j\right)^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j} \leq \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{\sigma(j)}\right)^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j}.$$

Further,

$$\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \min_j \mu_j\right)^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j}\right)$$
$$\geq \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{\sigma(j)}\right)^{n \frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j}\right),$$

which implies that

$$\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\min_j \mu_j\right)^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n}H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$

$$\leq \left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\sigma(j)}\right)^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n}H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}},$$

i.e.,

$$\mu^- \leq \left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\sigma(j)}\right)^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n}H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}.$$

In the same manner, we get

$$\rho^- \geq 1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\rho_{\sigma(j)}^{\;n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n}H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}$$

and

$$\nu^- \geq 1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\nu_{\sigma(j)}^{\;n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n}H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}.$$

Hence, $(\mu^-,\rho^-,\nu^-) \leq \text{SVNPMM}(\alpha_1,\alpha_2,\ldots,\alpha_n)$. Similarly, we have

$$\text{SVNPMM}(\alpha_1,\alpha_2,\ldots,\alpha_n) \leq (\mu^+,\rho^+,\nu^+),$$

which completes the proof. □

Theorem 4. *Let $\tilde{\alpha}_j$ be any permutation of α_j then we have*

$$SVNPMM(\alpha_1,\alpha_2,\ldots,\alpha_n) = SVNPMM(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n).$$

This property is called commutativity.

Proof. The proof of this theorem can be easily followed from Equation (12), so we omit it here. □

Theorem 5. *If the priority level of all the SVNNs is taken to be the same then SVNPMM operator reduces to single-valued neutrosophic Muirhead mean (SVNMM) operator. This property is called reducibility.*

Proof. Take $\xi_j = \frac{H_j}{\sum\limits_{j=1}^{n} H_j} = \frac{1}{n}$ for all j denotes the prioritized level. As ξ_j is same for all j, so, we have $(n\xi_j)\alpha_{\sigma(j)} = \alpha_{\sigma(j)}$, which implies

$$\begin{aligned} \text{SVNPMM}(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \left(\frac{1}{n!} \bigoplus_{\sigma \in S_n} \prod_{j=1}^{n} \alpha_{\sigma(j)}^{p_j} \right)^{\frac{1}{\sum\limits_{j=1}^{n} p_j}} \\ &= \text{SVNMM}(\alpha_1, \alpha_2, \ldots, \alpha_n). \end{aligned}$$

□

However, apart from these, the following particular cases are observed from the proposed SVNPMM operator by assigning different values to $P = (p_1, p_2, \ldots, p_n)$.

1. If $P = (1, 0, \ldots, 0)$, then SVNPMM operator becomes the SVN prioritized weighted average (SVNPWA) operator which is given as

$$\begin{aligned} \text{SVNPMM}(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \left(\frac{1}{n!} \bigoplus_{\sigma \in S_n} \left(n \frac{H_{\sigma(1)}}{\sum\limits_{j=1}^{n} H_j} \alpha_{\sigma(1)} \right) \right)^{\frac{1}{\sum\limits_{j=1}^{n} p_j}} \\ &= \bigoplus_{j=1}^{n} \frac{H_j}{\sum\limits_{j=1}^{n} H_j} \alpha_j \\ &= \text{SVNPWA}(\alpha_1, \alpha_2, \ldots, \alpha_n). \end{aligned}$$

2. When $P = (\lambda, 0, \ldots, 0)$, then SVNPMM operator yields to SVN generalized hybrid prioritized weighted average (SVNGHPWA) operator as shown below

$$\begin{aligned} \text{SVNPMM}(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \left(\frac{1}{n!} \bigoplus_{\sigma \in S_n} \left(n \frac{H_{\sigma(1)}}{\sum\limits_{j=1}^{n} H_j} \alpha_{\sigma(1)} \right)^{\lambda} \right)^{\frac{1}{\lambda}} \\ &= \left(\frac{1}{n} \bigoplus_{j=1}^{n} \left(n \frac{H_j}{\sum\limits_{j=1}^{n} H_j} \alpha_j \right)^{\lambda} \right)^{\frac{1}{\lambda}} \\ &= \text{SVNGHPWA}(\alpha_1, \alpha_2, \ldots, \alpha_n). \end{aligned}$$

3. If $P = (1, 1, 0, \ldots, 0)$, then Equation (11) reduces to SVN prioritized bonferroni mean (SVNPBM) operator as below

$$\begin{aligned} \mathrm{SVNPMM}(\alpha_1,\alpha_2,\ldots,\alpha_n) &= \left(\frac{1}{n!} \bigoplus_{\sigma\in S_n} \left(n\frac{H_{\sigma(1)}}{\sum_{j=1}^{n} H_j}\alpha_{\sigma(1)} \right) \left(n\frac{H_{\sigma(2)}}{\sum_{j=1}^{n} H_j}\alpha_{\sigma(2)} \right) \right)^{\frac{1}{2}} \\ &= \left(\frac{n^2}{n!} \bigoplus_{\substack{r,s=1\\ r\neq s}}^{n} \left(\frac{H_r}{\sum_{r=1}^{n} H_r}\alpha_r \right) \left(\frac{H_s}{\sum_{s=1}^{n} H_s}\alpha_s \right) \right)^{\frac{1}{2}} \\ &= \mathrm{SVNPBM}(\alpha_1,\alpha_2,\ldots,\alpha_n). \end{aligned}$$

4. If $P = (\overbrace{1,1\ldots,1}^{t\text{ terms}}, \overbrace{0,0\ldots,0}^{n-t\text{ terms}})$, then SVNPMM operator yields to SVN prioritized Maclaurin symmetric mean (SVNPMSM) operator as follows

$$\mathrm{SVNPMM}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left(\frac{2n^t t}{n!} \bigoplus_{\substack{1<j_1<\cdots\\ <j_t<n}} \bigotimes_{q=1}^{t} \left(\frac{H_{j_q}}{\sum_{r=1}^{n} H_r}\alpha_{j_q} \right) \right)^{\frac{1}{t}}.$$

3.2. Single-Valued Neutrosophic Prioritized Dual Muirhead Mean Operator

In this section, we propose prioritized dual aggregation operator based on the MM under the SVNS environment.

Definition 8. *A SVNPDMM operator is a mapping SVNPDMM* : $\Omega^n \to \Omega$ *given by*

$$SVNPDMM(\alpha_1,\alpha_2,\ldots,\alpha_n) = \frac{1}{\sum_{j=1}^{n} p_j} \left(\prod_{\sigma\in S_n} \bigoplus_{j=1}^{n} \left(p_j\alpha_{\sigma(j)} \right)^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}} \right)^{\frac{1}{n!}}. \tag{13}$$

Theorem 6. *The collective value by using Equation (13) is still a SVNN and is given as*

$$\begin{aligned} &SVNPDMM(\alpha_1,\alpha_2,\ldots,\alpha_n) \\ &= \left(\begin{array}{c} 1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\sigma(j)}^{\,n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ \left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\rho_{\sigma(j)})^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ \left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\sigma(j)})^{n\frac{H_{\sigma(j)}}{\sum_{j=1}^{n} H_j}}\right)^{p_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \end{array} \right). \end{aligned} \tag{14}$$

Proof. The proof follows from Theorem 1. □

In order to illustrate the working of this operator, we demonstrate it through an illustrative example as follows.

Example 2. *If we have taken the data as considered in Example 1 to illustrate the aggregation operator as defined in Theorem 6 then, we have*

$$\begin{aligned}
&\prod_{\sigma\in S_3}\left(1-\prod_{j=1}^{3}\left(1-\mu_{\sigma(j)}^{3\frac{H_{\sigma(j)}}{\sum_{j=1}^{3}H_j}}\right)^{p_j}\right)\\
=\;& \left\{1-\left(1-(0.5)^{3\times0.5087}\right)^{1}\times\left(1-(0.3)^{3\times0.3765}\right)^{0.5}\times\left(1-(0.6)^{3\times0.1148}\right)^{0.3}\right\}\\
\times\;& \left\{1-\left(1-(0.3)^{3\times0.3765}\right)^{1}\times\left(1-(0.5)^{3\times0.5087}\right)^{0.5}\times\left(1-(0.6)^{3\times0.1148}\right)^{0.3}\right\}\\
\times\;& \left\{1-\left(1-(0.6)^{3\times0.1148}\right)^{1}\times\left(1-(0.3)^{3\times0.3765}\right)^{0.5}\times\left(1-(0.5)^{3\times0.5087}\right)^{0.3}\right\}\\
\times\;& \left\{1-\left(1-(0.3)^{3\times0.3765}\right)^{1}\times\left(1-(0.6)^{3\times0.1148}\right)^{0.5}\times\left(1-(0.5)^{3\times0.5087}\right)^{0.3}\right\}\\
\times\;& \left\{1-\left(1-(0.5)^{3\times0.5087}\right)^{1}\times\left(1-(0.6)^{3\times0.1148}\right)^{0.5}\times\left(1-(0.3)^{3\times0.3765}\right)^{0.3}\right\}\\
\times\;& \left\{1-\left(1-(0.6)^{3\times0.1148}\right)^{1}\times\left(1-(0.5)^{3\times0.5087}\right)^{0.5}\times\left(1-(0.3)^{3\times0.3765}\right)^{0.3}\right\}\\
=\;& 0.00042495.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\prod_{\sigma\in S_3}\left(1-\prod_{j=1}^{3}\left(1-(1-\rho_{\sigma(j)})^{3\frac{H_{\sigma(j)}}{\sum_{j=1}^{3}H_j}}\right)^{p_j}\right)\\
=\;& \left\{1-\left(1-(1-0.2)^{3\times0.5087}\right)^{1}\times\left(1-(1-0.5)^{3\times0.3765}\right)^{0.5}\times\left(1-(1-0.5)^{3\times0.1148}\right)^{0.3}\right\}\\
\times\;& \left\{1-\left(1-(1-0.5)^{3\times0.3765}\right)^{1}\times\left(1-(1-0.2)^{3\times0.5087}\right)^{0.5}\times\left(1-(1-0.5)^{3\times0.1148}\right)^{0.3}\right\}\\
\times\;& \left\{1-\left(1-(1-0.5)^{3\times0.1148}\right)^{1}\times\left(1-(1-0.5)^{3\times0.3765}\right)^{0.5}\times\left(1-(1-0.2)^{3\times0.5087}\right)^{0.3}\right\}\\
\times\;& \left\{1-\left(1-(1-0.5)^{3\times0.3765}\right)^{1}\times\left(1-(1-0.5)^{3\times0.1148}\right)^{0.5}\times\left(1-(1-0.2)^{3\times0.5087}\right)^{0.3}\right\}\\
\times\;& \left\{1-\left(1-(1-0.2)^{3\times0.5087}\right)^{1}\times\left(1-(1-0.5)^{3\times0.1148}\right)^{0.5}\times\left(1-(1-0.5)^{3\times0.3765}\right)^{0.3}\right\}\\
\times\;& \left\{1-\left(1-(1-0.5)^{3\times0.1148}\right)^{1}\times\left(1-(1-0.2)^{3\times0.5087}\right)^{0.5}\times\left(1-(1-0.5)^{3\times0.3765}\right)^{0.3}\right\}\\
=\;& 0.0268
\end{aligned}$$

and

$$\begin{aligned}
&\prod_{\sigma\in S_3}\left(1-\prod_{j=1}^{3}\left(1-(1-\nu_{\sigma(j)})^{3\frac{H_{\sigma(j)}}{\sum_{j=1}^{3}H_j}}\right)^{p_j}\right)\\
=&\left\{1-\left(1-(1-0.3)^{3\times0.5087}\right)^{1}\times\left(1-(1-0.4)^{3\times0.3765}\right)^{0.5}\times\left(1-(1-0.2)^{3\times0.1148}\right)^{0.3}\right\}\\
\times&\left\{1-\left(1-(1-0.4)^{3\times0.3765}\right)^{1}\times\left(1-(1-0.3)^{3\times0.5087}\right)^{0.5}\times\left(1-(1-0.2)^{3\times0.1148}\right)^{0.3}\right\}\\
\times&\left\{1-\left(1-(1-0.2)^{3\times0.1148}\right)^{1}\times\left(1-(1-0.4)^{3\times0.3765}\right)^{0.5}\times\left(1-(1-0.3)^{3\times0.5087}\right)^{0.3}\right\}\\
\times&\left\{1-\left(1-(1-0.4)^{3\times0.3765}\right)^{1}\times\left(1-(1-0.2)^{3\times0.1148}\right)^{0.5}\times\left(1-(1-0.3)^{3\times0.5087}\right)^{0.3}\right\}\\
\times&\left\{1-\left(1-(1-0.3)^{3\times0.5087}\right)^{1}\times\left(1-(1-0.2)^{3\times0.1148}\right)^{0.5}\times\left(1-(1-0.4)^{3\times0.3765}\right)^{0.3}\right\}\\
\times&\left\{1-\left(1-(1-0.2)^{3\times0.1148}\right)^{1}\times\left(1-(1-0.3)^{3\times0.5087}\right)^{0.5}\times\left(1-(1-0.4)^{3\times0.3765}\right)^{0.3}\right\}\\
=&\ 0.0791.
\end{aligned}$$

Hence,

$$\begin{aligned}
SVNPDMM(\alpha_1,\alpha_2,\alpha_3) &= \begin{pmatrix} 1-\left(1-(0.00042495)^{\frac{1}{6}}\right)^{\frac{1}{1.8}}, \left(1-(0.0268)^{\frac{1}{6}}\right)^{\frac{1}{1.8}}, \\ \left(1-(0.0791)^{\frac{1}{6}}\right)^{\frac{1}{1.8}} \end{pmatrix}\\
&= (0.1631, 0.6441, 0.5535).
\end{aligned}$$

Similar to SVNPMM operator, it is observed that this SVNPDMM operator also satisfies same properties for a collection of SVNNs $\alpha_j (j = 1, 2, \ldots, n)$ which are stated without proof as below.

(P1) Monotonicity: If $\alpha_j \le \alpha'_j$ for all j, then

$$\text{SVNPDMM}(\alpha_1, \alpha_2, \ldots, \alpha_n) \le \text{SVNPDMM}(\alpha'_1, \alpha'_2, \ldots, \alpha'_n).$$

(P2) Boundedness: If α^-, and α^+ are lower and upper bound of SVNNs then

$$\alpha^- \le \text{SVNPDMM}(\alpha_1, \alpha_2, \ldots, \alpha_n) \le \alpha^+.$$

(P3) Commutativity: For any permutation $(\tilde{\alpha_1}, \tilde{\alpha_2}, \ldots, \tilde{\alpha_n})$ of the $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, we have

$$\text{SVNPDMM}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \text{SVNPDMM}(\tilde{\alpha_1}, \tilde{\alpha_2}, \ldots, \tilde{\alpha_n}).$$

4. Multi-Criteria Decision-Making Approach Based on Proposed Operators

In this section, we present an MCDM approach for solving the decision-making problem under the SVNS environment by using the proposed operators. A practical example from a field of decision-making has been taken to illustrate it.

4.1. Proposed Decision-Making Approach

Consider an MCDM problem which consists of m alternatives $A_1, A_2, \ldots, A_m$ which are evaluated under the n criteria $C_1, C_2, \ldots, C_n$. For this, an expert was invited to evaluate these alternatives under the SVN environment such that their rating values were given in the form of SVNNs. For instance, corresponding to alternative A_i under criterion C_j, when we ask the opinion of an expert about the alternative A_i with respect to the criterion C_j, he or she may observe that the possibility degree in which the statement is good is μ_{ij}, the statement is false is ν_{ij} and the degree in which he or she is unsure is ρ_{ij}. In this case, the evaluation of these alternatives are represented as SVNN $\alpha_{ij} = (\mu_{ij}, \rho_{ij}, \nu_{ij})$ such that $0 \leq \mu_{ij}, \rho_{ij}, \nu_{ij} \leq 1$ and $\mu_{ij} + \rho_{ij} + \nu_{ij} \leq 3$. This collective information is represented in the form of the neutrosophic decision-matrix D which is represented as

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \ldots & C_n \end{array} \\ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mn} \end{pmatrix} \end{array}.$$

Based on this information, the procedure to find the best alternative(s) is summarized as follows:

Step 1: If in the considered decision-making problem, there exist two kinds of criteria, namely the benefit and the cost types, then all the cost type criteria should be normalized into the benefit type by using the following equation

$$r_{ij} = \begin{cases} (\nu_{ij}, \rho_{ij}, \mu_{ij}) & ; \quad \text{for cost type criteria,} \\ (\mu_{ij}, \rho_{ij}, \nu_{ij}) & ; \quad \text{for benefit type criteria.} \end{cases} \tag{15}$$

Step 2: Compute $H_{ij} (i = 1, 2, \ldots, m)$ as

$$H_{ij} = \begin{cases} 1 & ; \quad j = 1, \\ \prod\limits_{k=1}^{j-1} s(r_{ik}) & ; \quad j = 2, \ldots, n. \end{cases} \tag{16}$$

Step 3: For a given parameter $P = (p_1, p_2, \ldots, p_n)$, utilize either SVNPMM or SVNPDMM operator to get the collective values $r_i = (\mu_i, \rho_i, \nu_i) (i = 1, 2, \ldots, m)$ for each alternative as

$$\begin{aligned} r_i &= \text{SVNPMM}(r_{i1}, r_{i2}, \ldots, r_{in}) \\ &= \left(\begin{array}{l} \left(1 - \left(\prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - (1 - \mu_{i\sigma(j)})^{n \frac{H_{i\sigma(j)}}{\sum_{j=1}^{n} H_{ij}}} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ 1 - \left(1 - \left(\prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - \rho_{i\sigma(j)}^{n \frac{H_{i\sigma(j)}}{\sum_{j=1}^{n} H_{ij}}} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ 1 - \left(1 - \left(\prod\limits_{\sigma \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - \nu_{i\sigma(j)}^{n \frac{H_{i\sigma(j)}}{\sum_{j=1}^{n} H_{ij}}} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \end{array} \right) \end{aligned} \tag{17}$$

or

$$r_i = \text{SVNPDMM}(r_{i1}, r_{i2}, \ldots, r_{in})$$

$$= \left(\begin{array}{c} 1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \mu_{i\sigma(j)}^{\,n\frac{H_{i\sigma(j)}}{\sum_{j=1}^{n} H_{ij}}} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - (1 - \rho_{i\sigma(j)})^{\,n\frac{H_{i\sigma(j)}}{\sum_{j=1}^{n} H_{ij}}} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - (1 - \nu_{i\sigma(j)})^{\,n\frac{H_{i\sigma(j)}}{\sum_{j=1}^{n} H_{ij}}} \right)^{p_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \end{array} \right). \quad (18)$$

Step 4: Calculate score values of the overall aggregated values $r_i = (\mu_i, \rho_i, \nu_i)$ $(i = 1, 2, \ldots, m)$ by using equation

$$s(r_i) = \frac{1 + (\mu_i - 2\rho_i - \nu_i)(2 - \mu_i - \nu_i)}{2}. \quad (19)$$

Step 5: Rank all the feasible alternatives $A_i (i = 1, 2, \ldots, m)$ according to Definition 3 and hence select the most desirable alternative(s).

The above mentioned approach has been illustrated with a numerical example discussed in Section 4.2.

4.2. Illustrative Example

A travel agency named Marricot Tripmate has excelled in providing travel related services to domestic and inbound tourists. The agency wants to provide more facilities like detailed information, online booking capabilities, the ability to book and sell airline tickets, and other travel related services to their customers. For this purpose, the agency intends to find an appropriate information technology (IT) software company that delivers affordable solutions through software development. To complete this motive, the agency forms a set of five companies (alternatives), namely, Zensar Tech (A_1), NIIT Tech (A_2), HCL Tech (A_3), Hexaware Tech (A_4), and Tech Mahindra (A_5) and the selection is held on the basis of the different criteria, namely, technology expertise (C_1), service quality (C_2), project management (C_3) and industry experience (C_4). The prioritization relationship for the criterion is $C_1 \succ C_2 \succ C_3 \succ C_4$. In order to access these alternatives, an expert was invited and he gives their preferences toward each alternative in the form of SVNN. Their complete preferences of the expert are summarized in Table 1.

Table 1. Single-valued neutrosophic decision making matrix.

	C_1	C_2	C_3	C_4
A_1	(0.5, 0.3, 0.4)	(0.5, 0.2, 0.3)	(0.2, 0.2, 0.6)	(0.3, 0.2, 0.4)
A_2	(0.7, 0.1, 0.3)	(0.7, 0.2, 0.3)	(0.6, 0.3, 0.2)	(0.6, 0.4, 0.2)
A_3	(0.5, 0.3, 0.4)	(0.6, 0.2, 0.4)	(0.6, 0.1, 0.2)	(0.5, 0.1, 0.3)
A_4	(0.7, 0.3, 0.2)	(0.7, 0.2, 0.2)	(0.4, 0.5, 0.2)	(0.5, 0.2, 0.2)
A_5	(0.4, 0.1, 0.3)	(0.5, 0.1, 0.2)	(0.4, 0.1, 0.5)	(0.4, 0.3, 0.6)

Then, the following steps of the proposed approach have been executed as below

Step 1: As all the criteria values are of the same types, the original decision matrix need not be normalized.

Step 2: Compute $H_{ij}(j = 1,2,3,4)$ by using Equation (16), we get

$$H = \begin{bmatrix} 1 & 0.6650 & 0.4921 & 0.3642 \\ 1 & 0.9000 & 0.7200 & 0.4464 \\ 1 & 0.6650 & 0.5320 & 0.4575 \\ 1 & 0.6650 & 0.5154 & 0.1134 \\ 1 & 0.8250 & 0.6806 & 0.6024 \end{bmatrix}.$$

Step 3: Without loss of generality, we take $P = (0.25, 0.25, 0.25, 0.25)$ and use SVNPMM operator given in Equation (17) to aggregate $r_{ij}(j = 1,2,3,4)$ and hence we get $r_1 = (0.9026, 0.0004, 0.0118)$; $r_2 = (0.9963, 0.0008, 0.0007)$; $r_3 = (0.9858, 0.0001, 0.0029)$; $r_4 = (0.9877, 0.0021, 0.0002)$ and $r_5 = (0.9474, 0.0000, 0.0093)$.

Step 4: By Equation (19), we get $s(r_1) = 0.9959$, $s(r_2) = 0.9992$, $s(r_3) = 0.9998$, $s(r_4) = 0.9978$ and $s(r_5) = 0.9990$.

Step 5: Since $s(r_3) > s(r_2) > s(r_5) > s(r_4) > s(r_1)$ and thus ranking order of their corresponding alternatives is $A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$. Here $\succ$ refers "preferred to". Therefore, A_3 is the best one according to the requirement of the travel agency.

Contrary to this, if we utilize SVNPDMM operator then the following steps are executed as:

Step 1: Similar to above Step 1.

Step 2: Similar to above Step 2.

Step 3: For a parameter $P = (0.25, 0.25, 0.25, 0.25)$, use SVNPDMM operator given in Equation (18) we get $r_1 = (0.0069, 0.7379, 0.9413)$; $r_2 = (0.1034, 0.7423, 0.7782)$; $r_3 = (0.0428, 0.6021, 0.8672)$; $r_4 = (0.0625, 0.8271, 0.6966)$ and $r_5 = (0.0109, 0.5340, 0.9125)$.

Step 4: The evaluated score values by using Equation (19) are $s(r_1) = 0.2226$, $s(r_2) = 0.1628$, $s(r_3) = 0.3396$, $s(r_4) = -0.0554$ and $s(r_5) = 0.4222$.

Step 5: The ranking order of the alternatives, based on the score values, is $A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$ and hence A_5 as the best alternative among the others.

4.3. Comparison Study

If we apply the existing prioritized aggregation operator named as SVN prioritized operator [36] on the considered problem, then the following steps of the Wu et al. [36] approach have been executed as follows:

Step 1: Use SVNPWA operator as given in Equation (4) to calculate the aggregated values $\beta_i(i = 1,2,3,4,5)$ of each alternative A_i are $\beta_1 = (0.4392, 0.2407, 0.3981)$, $\beta_2 = (0.6681, 0.1864, 0.2602)$, $\beta_3 = (0.5461, 0.1929, 0.3414)$, $\beta_4 = (0.6294, 0.2844, 0.2000)$ and $\beta_5 = (0.4291, 0.1141, 0.3232)$.

Step 2: Compute the cross entropy E for each β_i from $A^+ = (1,0,0)$ and $A^- = (0,0,1)$ based on the equation $E(\alpha_1, \alpha_2) = (\sin \mu_1 - \sin \mu_2) \times (\sin(\mu_1 - \mu_2)) + (\sin \rho_1 - \sin \rho_2) \times (\sin(\rho_1 - \rho_2)) + (\sin \nu_1 - \sin \nu_2) \times (\sin(\nu_1 - \nu_2))$ and then evaluate S_{β_i} by using equation $S_{\beta_i} = \frac{E(\beta_i, A^+)}{E(\beta_i, A^+) + E(\beta_i, A^-)}$. The values corresponding to it are: $S_{\beta_1} = 0.4642$, $S_{\beta_2} = 0.1755$, $S_{\beta_3} = 0.3199$, $S_{\beta_4} = 0.1914$ and $S_{\beta_5} = 0.4007$.

Step 3: The final ranking of alternative, according to the values of S_{β_i}, is $A_2 \succ A_4 \succ A_3 \succ A_5 \succ A_1$.

From above, we have concluded that the A_2 is the best alternative and A_1 is the worst one. However, from their approach [36], it has been concluded that they have completely ignored the interrelationships among the multi-input arguments and hence the ranking order are quite different. Thus, from it, we can see the influence of the interrelationships among all the criteria on the decision-making process.

4.4. Influence of Parameter P on the Decision-Making Process

The proposed aggregation operators have two prominent advantages. First, it can reduce the bad effects of the unduly high and low assessments on the final results. Second, it can capture the interrelationship between SVN attributes values. Moreover, both of the two aggregation operators have a parameter vector P, which leads to a more flexibility during the aggregation process. Further, the parameter vector P plays a significant role in the final ranking results. In order to illustrate the influence of the parameter vector $P = (p_1, p_2, \ldots, p_n)$ on the score functions and the ranking results, we set different values to P in the SVNPMM and SVNPDMM operators and their corresponding results are summarized in Table 2. From this table, it is concluded that the score value of each alternative decreases by SVNPMM operator while it increases by SVNPDMM operator. Therefore, based on the decision maker behavior, either A_3 or A_5 are the best alternatives to be chosen for their desired goals. Thus, the parameter vector P can be viewed as decision makers' risk preference.

4.5. Further Discussion

The prominent advantage of the proposed aggregation operators is that the interrelationship among all SVNNs can be taken into consideration. Moreover, it has a parameter vector that leads to flexible aggregation operators. To show the validity and superiorities of the proposed operators, we conduct a comparative analysis whose characteristics are presented in Table 3.

Table 2. Ranking results of alternatives using proposed operators for different values of P.

Parameter Vector P	Operator	Score Values of Alternatives					Ranking Results
		A_1	A_2	A_3	A_4	A_5	
(1, 0, 0, 0)	SVNPMM	0.9975	0.9997	0.9999	0.9989	0.9990	$A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$
	SVNPDMM	0.2184	0.0876	0.2942	-0.1233	0.3632	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
(1, 1, 0, 0)	SVNPMM	0.9844	0.9969	0.9988	0.9920	0.9940	$A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$
	SVNPDMM	0.3638	0.2891	0.4851	0.0162	0.5597	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
(1, 1, 1, 0)	SVNPMM	0.9723	0.9926	0.9968	0.9809	0.9887	$A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$
	SVNPDMM	0.4268	0.3846	0.5529	0.1219	0.6053	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
(1, 1, 1, 1)	SVNPMM	0.9624	0.9868	0.9942	0.9659	0.9851	$A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$
	SVNPDMM	0.4617	0.4507	0.5955	0.2079	0.6341	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
(2, 2, 2, 2)	SVNPMM	0.9443	0.9633	0.9836	0.9189	0.9767	$A_3 \succ A_5 \succ A_2 \succ A_1 \succ A_4$
	SVNPDMM	0.5165	0.5024	0.640	0.3016	0.6698	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
(3, 3, 3, 3)	SVNPMM	0.9322	0.9440	0.9744	0.8896	0.9715	$A_3 \succ A_5 \succ A_2 \succ A_1 \succ A_4$
	SVNPDMM	0.5369	0.5018	0.6490	0.3142	0.6853	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	SVNPMM	0.9824	0.9965	0.9987	0.9903	0.9943	$A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$
	SVNPDMM	0.3652	0.3217	0.4982	0.0490	0.5661	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	SVNPMM	0.9959	0.9992	0.9998	0.9978	0.9990	$A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$
	SVNPDMM	0.2226	0.1628	0.3396	-0.0554	0.4222	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
(2, 0, 0, 0)	SVNPMM	0.9890	0.9984	0.9990	0.9953	0.9931	$A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1$
	SVNPDMM	0.3571	0.1886	0.4228	-0.1009	0.4781	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$
(3, 0, 0, 0)	SVNPMM	0.9814	0.9964	0.9974	0.9898	0.9860	$A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1$
	SVNPDMM	0.4139	0.2426	0.4645	-0.0595	0.5008	$A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$

SVNPMM: single-valued neutrosophic prioritized Muirhead mean, SVNPDMM: single-valued neutrosophic prioritized dual Muirhead mean.

Table 3. Comparison of different approaches and aggregation operators.

Approaches	Whether the Interrelationship of Two Attributes Is Captured	Whether the Interrelationship of Three Attributes Is Captured	Whether the Relationship of Multiple Attributes Is Captured	Whether the Bad Effects of the Unduly High Unduly Low Arguments Can Be Reduced	Whether It Makes the Method Flexible by the Parameter Vector
NWA [21]	×	×	×	×	×
SVNWA [22]	×	×	×	×	×
SVNOWA [22]	×	×	×	×	×
SVNWG [22]	×	×	×	×	×
SVNOWG [22]	×	×	×	×	×
SVNHWA [25]	×	×	×	×	×
SVNHWG [25]	×	×	×	×	×
NWG [21]	×	×	×	×	×
SVNFWG [24]	×	×	×	×	✓
SVNFWA [24]	×	×	×	×	✓
SVNFNPBM [37]	✓	×	×	×	✓
WSVNLMSM [34]	✓	✓	✓	×	✓
SVNNWBM [33]	✓	×	×	×	✓
SVNIGWHM [20]	✓	✓	✓	×	✓
GNNHWA [25]	×	×	×	×	✓
The proposed method	✓	✓	✓	✓	✓

NWA: neutrosophic weighted averaging, SVNWA: single-valued neutrosophic weighted averaging, SVNOWA: single-valued neutrosophic ordered weighted averaging, SVNWG: single-valued neutrosophic weighted geometric, SVNOWG: single-valued neutrosophic ordered weighted geometric, SVNHWA: single-valued neutrosophic hybrid weighted averaging, SVNHWG: single-valued neutrosophic hybrid weighted geometric, NWG: neutrosophic weighted geometric, SVNFWG: single-valued neutrosophic Frank weighted geometric, SVNFWA: single-valued neutrosophic Frank weighted averaging, SVNFNPBM: single-valued neutrosophic Frank normalized prioritized Bonferroni mean, WSVNLMSM: weighted single-valued neutrosophic linguistic Maclaurin symmetric mean, SVNNWBM: single-valued neutrosophic normalized weighted Bonferroni mean, IGWHM: single-valued neutrosophic improved generalized weighted Heronian mean, GNNHWA: generalized neutrosophic number Hamacher weighted averaging.

The approaches in [21,22,25] are based on a simple weighted averaging operator. However, in these approaches, some of the weakness are (1) they assume that all the input arguments are independent, which is somewhat inconsistent with reality; (2) they cannot consider the interrelationship among input arguments. However, on the contrary, the proposed method can capture the interrelationship among input arguments. In addition to that, the proposed operator has an additional parameter P which provide a feasible aggregation process. In addition, some of the existing operators are deduced from the proposed operators. Thus, the proposed method is more powerful and flexible than the methods in [21,22,25].

In [33,37], authors presented an approach based on the BM aggregation operator where they considered the interrelationship between the arguments. However, the main flaws of these approaches are that they consider only two arguments during the interrelationship. On the other hand, in [34] authors have presented an aggregation operator based on MSM by considering two or more arguments during the interrelationship; however, these methods [33,34,37] fail to reflect the interrelationship among all input arguments. Finally, in [20] authors used the Heronian mean AOs without considering any interrelationship between the arguments.

As compared with these existing approaches, the merits of the proposed approach are that it can reflect the interrelationships among all the input arguments. In addition, the proposed operators have an additional parameter P which makes the proposed approach more flexible and feasible.

5. Conclusions

Muirhead mean aggregation operator is more flexible by using a variable and considering the multiple interrelationships between the pairs of the input arguments. On the other hand, SVNS is more of a generalization of the fuzzy set, intuitionistic fuzzy set to describe the uncertainties in the data. In order to combine their advantages, in the present paper, we develop some new MM aggregation operators for the SVNSs including the SVNPMM and the SVNPDMM. The desirable properties of these proposed operators and some special cases are discussed in detail. Moreover, we presented two new methods to solve the MCDM problem based on the proposed operators. The proposed method is more general and flexible, not only by considering the parametric vector P but also by taking into account the multiple interrelationships between the input argument. Apart from this, the remarkable characteristic of the proposed operator is to reflect the correlations of the aggregated arguments by considering the fact that those different criteria having different priority levels. The mentioned approach has been demonstrated through a numerical example and compares their corresponding proposed results with some of the results of existing approaches. From the computed results, it has been observed that the proposed approach can be efficiently utilized to solve decision-making problems where uncertainties and vagueness in the data occur concurrently. Moreover, by changing the values of the parameter P, an analysis has been done which concludes that the proposed operators provide more choices to the decision makers according to their preferences. In addition, it is also regarded as considering the risk preference of decision makers by the parameter P. So, the proposed approach is more suitable and flexible to solve the practical and complex MCDM problems.

In future works, we will apply our proposed method for more practical decision-making problems. In addition, considering the superiority of MM operator, we can extend it to some new fuzzy sets, such as Pythagorean fuzzy sets [39–41], applications to MCDM [42–44], multiplicative sets [45,46] and so on.

Author Contributions: Conceptualization, Methodology, Validation, H.G. and Nancy; Formal Analysis, Investigation, H.G.; Writing-Original Draft Preparation, H.G. and Nancy.; Writing-Review & Editing, H.G.; Investigation and Visualization, H.G.; Funding Acquisition, H.G. and Nancy.

Funding: The second author (Nancy) would like to thank the University Grant Commission, New Delhi, India for providing financial support under Maulana Azad National Fellowship scheme wide File No. F1-17.1/2017-18/MANF-2017-18-PUN-82613/(SA-III/Website) during the preparation of this manuscript.

Acknowledgments: The authors are thankful to the editor and anonymous reviewers for their constructive comments and suggestions that helped us in improving the paper significantly. The authors would like to thank the University Grant Commission, New Delhi, India for providing financial support under Maulana Azad National Fellowship scheme wide File No. F1-17.1/2017-18/MANF-2017-18-PUN-82613/(SA-III/Website) during the preparation of this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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On Neutrosophic Triplet Groups: Basic Properties, NT-Subgroups, and Some Notes

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Received: 20 June 2018; Accepted: 13 July 2018; Published: 17 July 2018

Abstract: As a new generalization of the notion of the standard group, the notion of the neutrosophic triplet group (NTG) is derived from the basic idea of the neutrosophic set and can be regarded as a mathematical structure describing generalized symmetry. In this paper, the properties and structural features of NTG are studied in depth by using theoretical analysis and software calculations (in fact, some important examples in the paper are calculated and verified by mathematics software, but the related programs are omitted). The main results are obtained as follows: (1) by constructing counterexamples, some mistakes in the some literatures are pointed out; (2) some new properties of NTGs are obtained, and it is proved that every element has unique neutral element in any neutrosophic triplet group; (3) the notions of NT-subgroups, strong NT-subgroups, and weak commutative neutrosophic triplet groups (WCNTGs) are introduced, the quotient structures are constructed by strong NT-subgroups, and a homomorphism theorem is proved in weak commutative neutrosophic triplet groups.

Keywords: neutrosophic triplet group (NTG); NT-subgroup; homomorphism theorem; weak commutative neutrosophic triplet group

1. Introduction

The importance of group theory is self-evident. It is widely used in many fields, such as physics, chemistry, engineering, and so on. It is a very good mathematical tool to describe the symmetry of nature [1,2]. As a more general concept, Molaei introduced the new notion of generalized group in 1999 [3,4], and some researchers studied its properties [5,6].

The concept of neutrosophic set is introduced by F. Smarandache, it is a generalization of (intuitionistic) fuzzy sets [7]. The neutrosophic set theory is applied to algebraic structures, multiple attribute decision-making, and so on [8–13]. Recently, F. Smarandache and Mumtaz Ali in [14,15], for the first time, introduced the new notion of neutrosophic triplet group (NTG), which is another generalization of classical group. It is easy to verify that all generalized groups are neutrosophic triplet group. Note that, in this paper, the notion of neutrosophic triplet group, indeed, is the neutrosophic extended triplet group in [14].

Until now, for neutrosophic triplet group, some research articles are published [16–21]. At the same time, there are still some misunderstandings about this new algebraic structure. This paper will clarify some misunderstandings, especially pointing out some erroneous conclusions in [18] and will try to give improved results. In Section 2, we give some examples to illustrate which conclusions are incorrect and some misunderstandings have led to the emergence of these results. In Section 3, we prove some new important properties of neutrosophic triplet groups. In Section 4, we give some new concepts, include NT-subgroups, strong NT-subgroups, and weak commutative neutrosophic triplet groups (WCNTGs), and prove a homomorphism theorem of weak commutative neutrosophic triplet groups.

2. Preliminaries

As we all know, the concept of group is a useful tool to characterize symmetry. In the definition of classical group, unit element has operation invariance for any element, i.e., $e \cdot x = x \cdot e = x$ for all x in a group $(G, \cdot)$, where e in G is the unit element. Moreover, the inverse element x^{-1} of x is also relative to the unit element e, and the inverse element is unique in the classical group. In [14,15], starting from the basic idea of neutrosophic set, a new algebraic structure, neutrosophic triplet group (briefly, NTG), is proposed. In NTG, the unit element is generalized as a neutral element, which is relative and local; that is, each element has its own neutral element; and the original inverse element concept is generalized as an anti (opposite) element, and it is relative to own neutral element, and it cannot be unique. In this way, NTG can express more general symmetry and has important theoretical and applied value.

Definition 1. *Assume that N is an empty set and * is a binary operation on N. Then, N is called a neutrosophic triplet set (NTS) if for any $a \in N$, there exists a neutral of "a" (denoted by neut(a)), and an opposite of "a" (denoted by anti(a)) satisfying ([14,15]):*

$$a * neut(a) = neut(a) * a = a;$$

$$a * anti(a) = anti(a) * a = neut(a).$$

And, the triple (a, neut(a), anti(a)) is called a neutrosophic triplet.

Note that, for a neutrosophic triplet set $(N, *)$, $a \in N$, $neut(a)$ and $anti(a)$ may not be unique. In order not to cause ambiguity, we use the notations $\{neut(a)\}$ and $\{anti(a)\}$; they represent the sets of $neut(a)$ and $anti(a)$, respectively.

Remark 1. *In the original definition in [14,15], the neutral element cannot be a unit element in the usual sense, and then this restriction is removed, using the concept of a neutrosophic extended triplet by F. Smarandache [14]. That is, the classical unit element can be regarded as a special neutral element. Here, the notion of neutrosophic triplet refers to neutrosophic extended triplet.*

Definition 2. *Assume that (N, *) is a neutrosophic triplet set. Then, N is called a neutrosophic triplet group, if it satisfies ([14,15]):*

(1) *The operation * is closed, i.e., $a * b \in N, \forall a, b \in N$;*
(2) *The operation* is associative, i.e., $(a * b) * c = a * (b * c), \forall a, b, c \in N$*

*A neutrosophic triplet group (N, *) is called to be commutative, if $a * b = b * a, \forall a, b \in N$.*

3. Some Counterexamples and Misunderstandings on Neutrosophic Triplet Groups

The research idea of Ref. [18] is very good, but the main results are not true. This section first gives some counterexamples, and then analyzes some of the misunderstandings on neutrosophic triplet groups.

Example 1. *Denote N = {1, 2, 3, 4, 5}; the operation * on N is defined by Table 1. Then, (N, *) is a commutative neutrosophic triplet group, and:*

$$neut(1) = 1, \{anti(1)\} = \{1, 2, 3\}; neut(2) = 3, anti(2) = 2; neut(3) = 3, anti(3) = 3;$$

$$neut(4) = 4, \{anti(4)\} = \{1, 2, 3, 4\}; neut(5) = 4, anti(5) = 5.$$

Table 1. Commutative neutrosophic triplet group.

*	1	2	3	4	5
1	1	1	1	4	5
2	1	3	2	4	5
3	1	2	3	4	5
4	4	4	4	4	5
5	5	5	5	5	4

*Denote H = {1, 2, 3, 4}, then (H, *) is a neutrosophic triplet subgroup (according to Definition 17 in [18]). And,*

$$1H = \{1, 4\}, 2H = \{1, 2, 3, 4\}, 3H = \{1, 2, 3, 4\}, 4H = \{4\}, 5H = \{5\}.$$

This means that Lemma 1 (2), (4), (7), and (9) in [18] are not true:

$$1 \in H, \text{ but } 1H \neq H;$$

$$1H \neq 2H \text{ and } 1H \cap 2H \neq \varnothing;$$

1 ∈ H, but 1H is a neutrosophic triplet subgroup (according to Definition 17 in [18]);

$$|1H| \neq |2H|.$$

Moreover, |H| = 4, |N| = 5, it follows that |H|| |N|; and the number of distinct aH in N (according to Definition 18 in [18]) is no |N| | |H|. This means that Theorem 3 in [18] are not true.

Example 2. *Denote N = {1, 2, 3, 4, 5}, the operation * on N is defined by Table 2. Then, (N, *) is a non-commutative neutrosophic triplet group, and:*

$$neut(1) = 1, anti(1) = 1; neut(2) = 2, anti(2) = 2; neut(3) = 3, anti(3) = 3;$$

$$neut(4) = 4, \{anti(4)\} = \{3, 4\}; neut(5) = 3, anti(5) = 5.$$

Table 2. Non-commutative neutrosophic triplet group.

*	1	2	3	4	5
1	1	1	1	1	1
2	2	2	2	2	2
3	4	4	3	4	5
4	4	4	4	4	4
5	4	4	5	4	3

*Denote H = {1, 2, 3, 4}, then (H, *) is a neutrosophic triplet subgroup (according to Definition 17 in [18]). And:*

$$1H = \{1\}, H1 = \{1, 2, 4\}; 2H = \{2\}, H2 = \{1, 2, 4\}; 3H = \{3, 4\}, H3 = \{1, 2, 3, 4\};$$

$$4H = \{4\}, H4 = \{1, 2, 4\}; 5H = \{4, 5\}, H5 = \{1, 2, 4, 5\}.$$

It follows that Theorem 4 in [18] is not true:

$$anti(1)*(H1) \subseteq H, anti(2)*(H2) \subseteq H, anti(3)*(H3) \subseteq H, anti(4)*(H4) \subseteq H, anti(5)*(H5) \subseteq H;$$

but H is not normal (according to Definition 20 in [18]).

*Moreover, anti(5)*4 = 4 ∈ H, thus 5 =* $_I$ *4(mod H), according to Definition 19 in [18]. But 4* $\neq_I$ *5 (mod H), this means that =* $_I$ *is not an equivalence relation. Therefore, Proposition 2 in [18] is not true.*

4. Some New and Important Properties of Neutrosophic Triplet Groups

As mentioned earlier, from the definition of neutrosophic triplet group, there may be multiple neutral elements *neut*(*a*) of an element *a*. We used more than a dozen personal computers, hoping to find an example to show that neutral elements of an element do not have to be unique. Unfortunately, we spent several months without finding the desired examples. This prompted us to consider another possibility: perhaps because of the associative law, every element in a neutrosophic triplet group has a unique neutral element? Recently, we succeeded to prove that this conjecture is true.

Theorem 1. *Assume that (N, *) is a neutrosophic triplet group. Then:*

(1) *a ∈ N, neut(a) is unique.*
(2) *a ∈ N, neut(a) * neut(a) = neut(a).*

Proof. Assume *s*, *t* ∈ {*neut*(*a*)}. Then $s*a = a*s = a$, $t*a = a*t = a$, and there exists *p*, *q* such that:

$$p*a = a*p = s,\ q*a = a*q = t.$$

Thus:

$$s*t = (p*a)*t = p*(a*t) = p*a = s.$$

On the other hand:

$$s*t = (a*p)*(a*q) = [a*(p*a)]*q = (a*s)*q = a*q = t.$$

Therefore, $s = t = s*t$. This means that *neut(a)* is unique, and *neut(a)* * *neut(a)* = *neut(a)* for any *a* in *N*. □

Remark 2. *For an element a in a neutrosophic triplet group (N, *), although neut(a) is unique, but we can see from Examples 1 and 2 that anti(a) is usually not unique.*

Theorem 2. *Let (N, *) be a neutrosophic triplet group. Then ∀a ∈ N, ∀anti(a) ∈ {anti(a)},*

(1) *neut(a)*p = q*neut(a), for any p, q ∈ {anti(a)};.*
(2) *neut(neut(a)) = neut(a);*
(3) *anti(neut(a))*anti(a) ∈ {anti(a)};*
(4) *neut(a*a)*a = a*neut(a*a) = a; neut(a*a)*neut(a) = neut(a)*neut(a*a) = neut(a);*
(5) *neut(anti(a))*a = a*neut(anti(a)) = a; neut(anti(a))*neut(a) = neut(a)*neut(anti(a)) = neut(a);*
(6) *anti(neut(a))*a = a*anti(neut(a)) = a, for any anti(neut(a)) ∈ {anti(neut(a))};*
(7) *a ∈ {anti(neut(a)*anti(a))};*
(8) *neut(a)*anti(a) ∈ {anti(a)}; anti(a)*neut(a) ∈ {anti(a)};*
(9) *a ∈ {anti(anti(a))}, that is, there exists p ∈ {anti(a)} such that a ∈ {anti(p)};*
(10) *neut(a)*anti(anti(a)) = a.*

Proof.

(1) For any *p*, *q* ∈ {*anti*(*a*)}, according the definition of neutral and opposite element, applying Theorem 1 (1), we have:

$$p*a = a*p = neut(a),\ q*a = a*q = neut(a).$$
$$neut(a)*p = (q*a)*p = q*(a*p) = q*neut(a).$$

(2) For any $anti(a) \in \{anti(a)\}$ and $anti(neut(a)) \in \{anti(neut(a))\}$,

$$[anti(neut(a))*anti(a)]*a = anti(neut(a))*[anti(a)*a] = anti(neut(a))*neut(a) = neut(neut(a)).$$

On the other hand:

$$\{[anti(neut(a))*anti(a)]*a\}*neut(a) = [anti(neut(a))*anti(a)]*[a*neut(a)] =$$
$$[anti(neut(a))*anti(a)]*a = neut(neut(a)).$$

Thus:

$$neut(neut(a))*neut(a) = \{[anti(neut(a))*anti(a)]*a\}*neut(a) = neut(neut(a)).$$

Moreover, the definition of neutral element, $neut(neut(a))*neut(a) = neut(a)$. Therefore, $neut(neut(a)) = neut(a)$.

(3) For any $anti(a) \in \{anti(a)\}$ and $anti(neut(a)) \in \{anti(neut(a))\}$, applying (2), we have:

$$[anti(neut(a))*anti(a)]*a = anti(neut(a))*[anti(a)*a] = anti(neut(a))*neut(a) = neut(neut(a)) = neut(a);$$
$$a*[anti(neut(a))*anti(a)] = [a*neut(a)]*[anti(neut(a))*anti(a)] = a*[neut(a)*anti(neut(a))]*anti(a) =$$
$$a*neut(neut(a))*anti(a) = a*neut(a)*anti(a) = a*anti(a) = neut(a).$$

Thus, $anti(neut(a))*anti(a) \in \{anti(a)\}$.

(4) According to the definition of neutral element, using the associative law, we get:

$$(a*a)*neut(a*a) = (a*a),$$
$$anti(a)*[(a*a)*neut(a*a)] = anti(a)*(a*a),$$
$$[anti(a)*a]*[a*neut(a*a)] = [anti(a)*a]*a,$$
$$neut(a)*[a*neut(a*a)] = neut(a)*a,$$
$$[neut(a)*a]*neut(a*a) = neut(a)*a,$$
$$a*neut(a*a) = a.$$

Similarly, we can get that $neut(a*a)*a = a$. Moreover:

$$neut(a)*neut(a*a) = [anti(a)*a]*neut(a*a) = anti(a)*[a*neut(a*a)] = anti(a)*a = neut(a).$$
$$neut(a*a)*neut(a) = neut(a*a)*[a*anti(a)] = [neut(a*a)*a]*anti(a) = a *anti(a) = neut(a).$$

(5) For any $anti(a) \in \{anti(a)\}$, we have:

$$anti(a)*neut(anti(a)) = anti(a);\ neut(anti(a))*anti(a) = anti(a).$$
$$a*[anti(a)*neut(anti(a))] = a*anti(a);\ [neut(anti(a))*anti(a)]*a = anti(a)*a.$$
$$[a*anti(a)]*neut(anti(a)) = a*anti(a);\ neut(anti(a))*[anti(a)*a] = anti(a)*a.$$
$$neut(a)*neut(anti(a)) = neut(a);\ neut(anti(a))*neut(a) = neut(a).$$
$$a*[neut(a)*neut(anti(a))] = a*neut(a);\ [neut(anti(a))*neut(a)]*a = neut(a)*a.$$
$$[a*neut(a)]*neut(anti(a)) = a*neut(a);\ neut(anti(a))*[neut(a)*a] = neut(a)*a.$$
$$a*neut(anti(a)) = a;\ neut(anti(a))*a = a.$$

Moreover:

$$neut(a)*neut(anti(a)) = [anti(a)*a]*neut(anti(a)) = anti(a)*[a*neut(anti(a))] = anti(a)*a = neut(a).$$
$$neut(anti(a))*neut(a) = neut(anti(a))*[a*anti(a)] = [neut(anti(a))*a]*anti(a) = a *anti(a) = neut(a).$$

(6) For any $anti(neut(a)) \in \{anti(neut(a))\}$, by the definition of opposite element, we have:

$$neut(a)*anti(neut(a)) = anti(neut(a))*neut(a) = neut(neut(a)).$$

Applying (2), $neut(neut(a)) = neut(a)$, we get:

$$neut(a)*anti(neut(a)) = anti(neut(a))*neut(a) = neut(a).$$

Thus:

$$a*[neut(a)*anti(neut(a))] = a*neut(a);\ [anti(neut(a))*neut(a)]*a = neut(a)*a.$$
$$[a*neut(a)]*anti(neut(a)) = a*neut(a);\ anti(neut(a))*[neut(a)*a] = neut(a)*a.$$
$$a*anti(neut(a)) = a;\ anti(neut(a))*a = a.$$

(7) For any $anti(a) \in \{anti(a)\}$, we have:

$$a*anti(a) = anti(a)*a = neut(a).$$
$$[a*neut(a)]*anti(a) = anti(a)*[neut(a)*a] = neut(a).$$
$$a*[neut(a)*anti(a)] = [anti(a)*neut(a)]*a = neut(a).$$

Applying (1), $anti(a)*neut(a) = neut(a)*anti(a)$, thus:

$$a*[neut(a)*anti(a)] = [neut(a)*anti(a)]*a = neut(a).$$

Using (5), $neut(a)*neut(anti(a)) = neut(a)$, it follows that:

$$a*[neut(a)*anti(a)] = [neut(a)*anti(a)]*a = neut(a)*neut(anti(a)).$$

On the other hand, by (1) and Theorem 1 (2):

$$[neut(a)*anti(a)]*[neut(a)*neut(anti(a))] = neut(a)*neut(a)*[anti(a)*neut(anti(a))] = neut(a)*anti(a);$$
$$[neut(a)*neut(anti(a))]*[neut(a)*anti(a)] = neut(a)*[neut(anti(a))*anti(a)]*neut(a) = neut(a)*anti(a).$$

Therefore, $a \in \{anti(neut(a)*anti(a))\}$.

(8) Assume $anti(a) \in \{anti(a)\}$, then $[neut(a)*anti(a)]*a = neut(a)*[anti(a)*a] = neut(a)*neut(a)$. By Theorem 1 (2), $neut(a)*neut(a) = neut(a)$. Thus, $[neut(a)*anti(a)]*a = neut(a)$. On the other hand,

$$a*[neut(a)*anti(a)] = [a*neut(a)]*anti(a) = a*anti(a) = neut(a).$$

Therefore:

$$[neut(a)*anti(a)]*a = a*[neut(a)*anti(a)] = neut(a).$$

This means that $neut(a)*anti(a) \in \{anti(a)\}$. Similarly, we can get $anti(a)*neut(a) \in \{anti(a)\}$.

(9) For any $anti(a) \in \{anti(a)\}$, denote $p = neut(a)*anti(a)$. Using (8) we have $p \in \{anti(a)\}$. Moreover, by Theorem 1 (2):

$$neut(a)*\mathrm{p} = neut(a)*[neut(a)*anti(a)] = [neut(a)*neut(a)]*anti(a) = neut(a)*anti(a) = \mathrm{p}.$$

From this and applying (7), $a \in \{anti(neut(a)*p)\} = \{anti(p)\}$, $p \in \{anti(a)\}$.

(10) Assume $anti(a) \in \{anti(a)\}$ and $anti(anti(a)) \in \{anti(anti(a))\}$, by the definition of opposite element, we have:

$$anti(a)*anti(anti(a)) = neut(anti(a)).$$

Thus:

$$a*[anti(a)*anti(anti(a))] = a*neut(anti(a)).$$
$$[a*anti(a)]*anti(anti(a)) = a*neut(anti(a)).$$
$$neut(a)*anti(anti(a)) = a*neut(anti(a)).$$

Applying (5), $a*neut(anti(a)) = a$, it follows that:

$$neut(a)*anti(anti(a)) = a.$$

□

Example 3. *Let Z_6 = {[0], [1], [2], [3], [4], [5]}, * is classical mod multiplication, then (Z_6, *) is a commutative neutrosophic triplet group, see Example 10 in [16].*

We can show that (they correspond to the conclusions of Theorem 2):

(1) [2]*[4] = [5]*[2], [2]*[5] = [4]*[2], that is, for any $p, q \in \{anti([2])\}$, $neut([2])*p = q*neut([2])$.
(2) $neut(neut([0])) = neut([0]) = [0]$, $neut(neut([1])) = neut([1]) = [1]$, $neut(neut([2])) = neut([2]) = [4]$, $neut(neut([3])) = neut([3]) = [3]$, $neut(neut([4])) = neut([4]) = [4]$, $neut(neut([5])) = neut([5]) = [1]$.
(3) Since $neut([2]) = [4]$, $\{anti([4])\} = \{[1], [4]\}$ and $\{anti([2])\} = \{[2], [5]\}$, so $anti(neut([2])) = anti([4]) = \{[1], [4]\}$, and $[1]*[2] = [2] \in \{anti([2])\}$, $[1]*[5] = [5] \in \{anti([5])\}$, $[4]*[2] = [2] \in \{anti([2])\}$, $[4]*[5] = [2] \in \{anti([2])\}$. This means that $anti(neut([2]))*anti([2]) \in \{anti([2])\}$ for any $anti([2]) \in \{anti([2])\}$ and any $anti(neut([2])) \in \{anti(neut([2]))\}$.
(4) $neut([0]*[0])*[0] = [0]*neut([0]*[0]) = [0]$, $neut([0]*[0])*neut([0]) = neut([0])*neut([0]*[0]) = [0]$; $neut([1]*[1])*[1] = [1]*neut([1]*[1]) = [1]$, $neut([1]*[1])*neut([1]) = neut([1])*neut([1]*[1]) = [1]$; and so on. This means that (4) hold for all $a \in Z_6$.
(5) Since $\{anti([2])\} = \{[2], [5]\}$, so $neut(anti([2])) = [4]$ or [1]. From [4]*[2] = [2]*[4] = [2] and [1]*[2] = [2]*[1] = [2] we know that $neut(anti([2]))*[2] = [2]*neut(anti([2])) = [2]$ for any $anti([2]) \in \{anti([2])\}$ and any $neut(anti([2])) \in \{neut(anti([2]))\}$. Note that, since $\{neut(anti([2]))\} = \{[4], [1]\}$; when $anti([2]) = [5]$, $neut(anti([2])) = [1] \neq neut([2])$, this means that $neut(anti(a)) = neut(a)$ is not true in general.
(6) Since $\{anti(neut([2]))\} = \{[1], [4]\}$, from this and [1]*[2] = [2]*[1] = [2] and [4]*[2] = [2]*[4] = [2] we know that $anti(neut([2]))*[2] = [2]*anti(neut([2])) = [2]$ for any $anti(neut([2])) \in \{anti(neut([2]))\}$. Note that, since $\{anti(neut([2]))\} = \{[1], [4]\}$; when $anti(neut([2])) = [1]$, $anti(neut([2])) \neq neut([2])$, this means that $anti(neut(a)) = neut(a)$ is not true in general.
(7) Since $\{anti(neut([2]))\} = \{[1], [4]\}$ and $\{anti([2])\} = \{[2], [5]\}$, so $\{anti(neut([2]))*anti([2])\} = \{[2], [5]\}$, that is, $[2] \in \{anti(neut([2]))*anti([2])\}$.
(8) Since $neut([2]) = [4]$ and $\{anti([2])\} = \{[2], [5]\}$, from [4]*[2] = [4]*[5] = [2] we know that $neut([2])*anti([2]) \in \{anti([2])\}$.
(9) Since $neut([2]) = [4]$ and $\{anti([2])\} = \{[2], [5]\}$, so $\{anti(anti([2]))\} = \{[2], [5]\}$. Thus, from [4]*[2] = [4]*[5] = [2] we know that $neut([2])*anti(anti([2])) = [2]$ for any $anti([2]) \in \{anti([2])\}$ and $anti(anti([2])) \in \{anti(anti([2]))\}$. Note that, since $\{anti(2)\} = \{[2], [5]\}$; when $anti([2]) = [5]$, $anti(anti([2])) = [5] \neq [2]$, this means that $anti(anti(a)) = a$ is not true in general.

Theorem 3. *Assume that (N, *) is a commutative neutrosophic triplet group. Then $\forall\, a, b \in N$:*

(1) $neut(a) * neut(b) = neut(a*b)$.
(2) $anti(a) * anti(b) \in \{anti(a*b)\}$.

Proof. If $a, b \in N$, then:

$$[neut(a)*neut(b)]*(a*b) = \{[neut(a)*neut(b)]*a\}*b = \{[neut(a)*a]*neut(b)\}*b = [a*neut(b)]*b$$
$$= a*[neut(b)*b] = a*b.$$

Similarly, we have $(a*b)*[neut(a)*neut(b)] = a*b$. That is:

$$(a*b)*[neut(a)*neut(b)] = [neut(a)*neut(b)]*(a*b) = a*b. \tag{1}$$

Moreover, for any $anti(a) \in \{anti(a)\}$ and $anti(b) \in \{anti(b)\}$, we have:

$$[anti(a)*anti(b)]*(a*b) = \{[anti(a)*anti(b)]*a\}*b = \{[anti(a)*a]*anti(b)\}*b = [neut(a)*anti(b)]*b$$
$$= neut(a)*[anti(b)*b] = neut(a)*neut(b).$$

Similarly, we have $(a*b)*[anti(a)*anti(b)] = neut(a)*neut(b)$. That is:

$$(a*b)*[anti(a)*anti(b)] = [anti(a)*anti(b)]*(a*b) = neut(a)*neut(b). \quad (2)$$

Combining (1) and (2), we have $neut(a)*neut(b) \in \{neut(a*b)\}$. From this, by Theorem 1, we get: $neut(a)*neut(b) = neut(a*b)$. Therefore, using (2), we get $anti(a)*anti(b) \in \{anti(a*b)\}$. □

5. NT-subgroups and Weak Commutative Neutrosophic Tripet Groups

The notion of subgroup is an important basic concept for neutrosophic triplet groups, but the definitions in the existing literatures are not consistent (see [14,15,18,20]). In order to avoid ambiguity, this paper gives a new definition and formally named NT-subgroup. Moreover, this section will discuss an important kind of neutrosophic triplet groups, call weak commutative neutrosophic triplet group (WCNTG). We will prove some well-known properties of WCNTG and a homomorphism theorem by special NT-subgroups.

Definition 3. *Assume that (N,*) is a neutrosophic triplet group and H be a nonempty subset of N. Then H is called a NT-subgroup of N if;*

(1) $a*b \in H$ *for all* $a, b \in H$;

(2) *there exists* $anti(a) \in \{anti(a)\}$ *such that* $anti(a) \in H$ *for all* $a \in H$, *where* $\{anti(a)\}$ *is the set of opposite element of a in (N,*).*

Proposition 1. *Assume that (N,*) is a neutrosophic triplet group. If H is a NT- subgroup of N, then* $neut(a) \in H$ *for all* $a \in H$, *where neut(a) is the neutral element of a in (N,*).*

Proof. For any $a \in H$, by Theorem 1 (1) we know that $neut(a)$ is unique. Applying Definition 3, we get that there exists $anti(a) \in H$ and $neut(a) = a*anti(a) \in H$. □

Remark 3. *(1) For a NT-subgroup H of N, where (N, *) is a neutrosophic triplet group,* $a \in H$, *by Definition 3 we know that not all anti(a) is in H; in fact, at least one can be in H. (2) By Proposition 1,* $a \in H$ *implies* $neut(a) \in H$. *But H does not necessarily contain neut(b) for all* $b \in N$. *For example, let* $N = Z_6$ *in Example 3 and H = {[0], [2], [3], [4]}, then H is a NT-subgroup of* $(Z_6, *)$, *and (1)* $[2] \in H$ *but {anti([2])} is not a subset of H; (2)* $\{neut(a) \mid a \in N = Z_6\} = \{[0], [1], [3], [4]\}$ *is not a subset of H.*

Definition 4. *Assume that (N,∗) is a neutrosophic triplet group. N is called a weak commutative neutrosophic triplet group (briefly, WCNTG) if* $a*neut(b) = neut(b)*a$ *for all* $a, b \in N$.

Obviously, every commutative neutrosophic triplet group is weak commutative. The following example shows that there exists non-commutative neutrosophic triplet group which is weak commutative neutrosophic triplet group.

Example 4. *Put N = {1, 2, 3, 4, 5, 6, 7}, and define the operation * on N as Table 3. Then, (N, *) is a non-commutative neutrosophic triplet group, and:*

neut(1) = 1, anti(1) = 1; neut(2) = 1, anti(2) = 2; neut(3) = 1, anti(3) = 3; neut(4) = 1, anti(4) = 4;
neut(5) = 1, anti(5) = 6; neut(6) = 1, anti(6) = 5; neut(7) = 7, {anti(7)} = {1, 2, 3, 4, 5, 6, 7}.

*It is easy to verify that (N, *) is a weak commutative neutrosophic triplet group.*

Table 3. Weak commutative neutrosophic triplet group.

*	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	2	3	4	5	6	7
2	2	1	6	5	4	3	7
3	3	5	1	6	2	4	7
4	4	6	5	1	3	2	7
5	5	3	4	2	6	1	7
6	6	4	2	3	1	5	7
7	7	7	7	7	7	7	7

Proposition 2. *Assume that (N,*) is a neutrosophic triplet group. Then (N,*) is weak commutative if and only if N satisfies the following conditions:*

(1) $neut(a)*neut(b) = neut(b)*neut(a)$ *for all* $a, b \in N$.
(2) $neut(a)*neut(b)*a = a*neut(b)$ *for all* $a, b \in N$.

Proof. If (N, *) is a weak commutative neutrosophic triplet group, then (using Definition 4):

$$neut(a)*neut(b) = neut(b)*neut(a), \forall a, b \in \text{N}.$$

And:

$$neut(a)*neut(b)*a = neut(a)*[neut(b)*a] = neut(a)*[a*neut(b)] = [neut(a)*a]*neut(b) = a*neut(b).$$

Conversely, assume that N satisfies the conditions (1) and (2) above. Then:

$$a*neut(b) = [neut(a)*neut(b)]*a = [neut(b)*neut(a)]*a = neut(b)*[neut(a)*a] = neut(b)*a.$$

From Definition 4 we know that (N, *) is a weak commutative neutrosophic triplet group. □

Proposition 3. *Let (N,*) be a weak commutative neutrosophic triplet group. Then* $\forall\, a, b \in N$:

(1) $neut(a)*neut(b) = neut(b*a)$;
(2) $anti(a)*anti(b) \in \{anti(b*a)\}$.

Proof. If $a, b \in N$, then:

$$[neut(a)*neut(b)]*(b*a) = \{[neut(a)*neut(b)]*b\}*a = \{neut(a)*[b*neut(b)]\}*a = [neut(a)*b]*a$$
$$= [b*neut(a)]*a = b*[neut(a)*a] = b*a.$$

Similarly, we have $(b*a)*[neut(a)*neut(b)] = b*a$. That is:

$$(b*a)*[neut(a)*neut(b)] = [neut(a)*neut(b)]*(b*a) = b*a. \quad (3)$$

Moreover, for any $anti(a) \in \{anti(a)\}$ and $anti(b) \in \{anti(b)\}$, we have:

$$[anti(a)*anti(b)]*(b*a) = \{[anti(a)*anti(b)]*b\}*a = \{anti(a)*[anti(b)*b]\}*a = [anti(a)*neut(b)]*a$$
$$= anti(a)*[neut(b)*a] = anti(a)*[a*neut(b)] = [anti(a)*a]*neut(b) = neut(a)*neut(b).$$

Similarly, we have $(b*a)*[anti(a)*anti(b)] = neut(a)*neut(b)$. That is:

$$(b*a)*[anti(a)*anti(b)] = [anti(a)*anti(b)]*(b*a) = neut(a)*neut(b). \quad (4)$$

Combining (3) and (4), we have $neut(a)*neut(b) \in \{neut(b*a)\}$. From this, by Theorem 1, we get $neut(a)*neut(b) = neut(b*a)$. Therefore, using (4), we get $anti(a)*anti(b) \in \{anti(b*a)\}$. □

Definition 5. *Let (N,*) be a neutrosophic triplet group and H be a NT-subgroup of N. Then H is called a strong NT- subgroup of N if:*

(1) *$neut(a) \in H$ for all $a \in N$.*
(2) *if there exists $anti(a) \in \{anti(a)\}$ and $p \in N$ such that $anti(a)*b*neut(p) \in H$, then there exists $anti(b) \in \{anti(b)\}$ and $q \in N$ such that $a*anti(b)*neut(q) \in H$; and the inverse is true.*

Example 5. *Let (N, *) be the neutrosophic triplet group in Example 4 and $H_1 = \{1, 7\}$, $H_2 = \{1, 5, 6, 7\}$. Then H_1 and H_2 are two strong NT-subgroups of N.*

Proposition 4. *Let ((N, *) be a group (as a special neutrosophic triplet group) and H be a normal subgroup of N. Then ((N, *) is a weak commutative neutrosophic triplet group and H is a strong NT-subgroup of N.*

Proof. For group (*N*, *) with identity *e*, $neut(a) = e$ and $anti(a) = a^{-1}$ for any $a \in N$.

It is easy to verify that $a* neut(b) = neut(b)* a$ for all $a, b \in N$. From this, by Definition 4 we know that *(N, *)* is a weak commutative neutrosophic triplet group.

For normal subgroup *H*, by Definition 3, *H* is a NT-subgroup of *N*. Moreover, *H* satisfies the condition in Definition 5 (1).

Now, assume that there exists $anti(a) \in \{anti(a)\}$ and $p \in N$ such that $anti(a)*b*neut(p) \in H$, this means that $a^{-1}*b \in H$. Denote $h = a^{-1}*b \in H$. Then $a = b*h^{-1}$. Since *H* is a normal subgroup of *N*, $h^{-1} \in H$ and there exists $h_1 \in H$ such that $b*h^{-1} = h_1*b$. Thus, $a = h_1*b$, $a*b^{-1} = h_1 \in H$. That is, there exists $b^{-1} = anti(b) \in \{anti(b)\}$ and $a \in N$ such that $a*anti(b)*neut(a) = a*b^{-1}*e = a*b^{-1} = h_1 \in H$. Similarly, we can prove the inverse is true.

Therefore, *H* satisfies the condition in Definition 5 (2), and *H* is a strong NT-subgroup of *N*. □

Theorem 4. *Let (N, *) be a weak commutative neutrosophic triplet group and H be a strong NT-subgroup of N. Define binary relation $\approx_H$ on N as follows: $\forall a, b \in N$:*

$$a \approx_H b \text{ if and only if there exists } anti(a) \in \{anti(a)\} \text{ and } p \in N \text{ such that } anti(a)*b*neut(p) \in H.$$

Then:

(1) *the binary relation $\approx_H$ is an equivalent relation on N;*
(2) *$a \approx_H b$ implies $c*a \approx_H c*b$ for all $c \in N$;*
(3) *$a \approx_H b$ implies $a*c \approx_H b*c$ and $c*a \approx_H c*b$ for all $c \in N$;*
(4) *denote the equivalent class contained a by $[a]_H$, and denote $N/H = \{[a]_H \mid a \in N\}$, define binary operation * on N/H as follows: $[a]_H *[b]_H = [a*b]_H, \forall a, b \in N$. We can obtain a homomorphism from (N, *) to (N/H, *), that is, $f: N \to N/H$; $f(a) = [a]_H$ for all $a \in N$.*

Proof.

(1) For any $a \in N$, applying Theorem 1 we have:

$$anti(a)*a*neut(a) = [anti(a)*a]*neut(a) = neut(a)*neut(a) = neut(a) \in \mathrm{H}.$$

Thus $a \approx_H a$.

- If $a \approx_H b$, then there exists $anti(a) \in \{anti(a)\}$ and $p \in N$ such that $anti(a)*b*neut(p) \in H$. Denote $h = anti(a)*b*neut(p)$, then $h \in H$ and:

$$a*h = a*[anti(a)*b*neut(p)],$$
$$a*h = neut(a)*b*neut(p),$$
$$a*h = b*neut(a)*neut(p), \text{ (by Definition 4)}$$
$$anti(b)*(a*h) = anti(b)*[b*neut(a)*neut(p)],$$
$$[anti(b)*a]*h = neut(b)*neut(a)*neut(p),$$
$$\{[anti(b)*a]*h\}*anti(h) = [neut(b)*neut(a)*neut(p)]*anti(h),$$
$$anti(b)*a*neut(h) = [neut(b)*neut(a)*neut(p)]*anti(h).$$

  Applying Definition 3 we have $[neut(b)*neut(a)*neut(p)]*anti(h) \in H$, thus $anti(b)*a*neut(h) \in H$, this means that $b \approx_H a$.
- If $a \approx_H b$ and $b \approx_H c$, then there exists $anti(a) \in \{anti(a)\}$, $anti(a) \in \{anti(a)\}$, $p \in N$ and $q \in N$ such that $anti(a)*b*neut(p) \in H$, $anti(b)*c*neut(q) \in H$. Denote $h_1 = anti(a)*b*neut(p)$, $h_2 = anti(b)*c*neut(q)$, then $h_1 \in H$, $h_2 \in H$ and:

$$b*h_2 = b*[anti(b)*c*neut(q)] = [b*anti(b)]*[c*neut(q)] = neut(b)*c*neut(q).$$

  From this and $h_1 = anti(a)*b*neut(p)$, using Definition 4 we get:

$$\begin{aligned} &h_1*h_2 \\ &= [anti(a)*b*neut(p)]*h_2 \\ &= [anti(a)*b]*[neut(p)*h_2] \\ &= [anti(a)*b]*[h_2*neut(p)] \\ &= anti(a)*(b*h_2)*neut(p) \\ &= anti(a)*[neut(b)*c*neut(q)]*neut(p) \\ &= anti(a)*[neut(b)*c]*[neut(q)*neut(p)] \\ &= anti(a)*[c*neut(b)]*[neut(q)*neut(p)] \\ &= [anti(a)*c]*[neut(b)*neut(q)*neut(p)] \end{aligned}$$

  By Definition 3 we have $h_1*h_2 \in H$; using Proposition 3 (1), $neut(b)*neut(q)*neut(p) = neut(p*q*b)$. Hence:

$$anti(a)*c*neut(p*q*b) = h_1*h_2 \in H.$$

  This means that $a \approx_H c$. Therefore, $\approx_H$ is an equivalent relation on N.

(2) Assume $a \approx_H b$. Then there exists $anti(a) \in \{anti(a)\}$ and $p \in N$ such that $anti(a)*b*neut(p) \in H$. Denote:$h = anti(a)*b*neut(p)$, then $h \in H$ and:

$$\begin{aligned} &h*neut(c) \\ &= [anti(a)*b*neut(p)]*neut(c) \\ &= [anti(a)*b]*[neut(p)*neut(c)] \\ &= [anti(a)*b]*[neut(c)*neut(p)] \\ &= anti(a)*[b*neut(c)]*neut(p) \\ &= anti(a)*[neut(c)*b]*neut(p) \\ &= [anti(a)*neut(c)]*[b*neut(p)] \\ &= [anti(a)*anti(c)*c]*[b*neut(p)] \\ &= [anti(a)*anti(c)]*(c*b)*neut(p). \end{aligned}$$

Using Proposition 3 (2), $anti(a)*anti(c) \in \{anti(c*a)\}$. Thus, there exists $anti(c*a) \in \{anti(c*a)\}$ such that:

$$anti(c*a)*(c*b)*neut(p) = h*neut(c) \in H.$$

This means that $(c*a) \approx_H (c*b)$.

(1) Assume $a \approx_H b$. Then there exists $anti(a) \in \{anti(a)\}$ and $p \in N$ such that $anti(a)*b*neut(p) \in H$. Applying Definition 5 (2), there exists $anti(b) \in \{anti(b)\}$ and $q \in N$ such that $a*anti(b)*neut(q) \in H$. Denote $h = a*anti(b)*neut(q)$, then $h \in H$ and:

$$\begin{aligned} & neut(c)*h \\ & = neut(c)*[a*anti(b)*neut(q)] \\ & = [neut(c)*a]*[anti(b)*neut(q)] \\ & = [a*neut(c)]*[anti(b)*neut(q)] \\ & = \{a*[c*anti(c)]\}*[anti(b)*neut(q)] \\ & = (a*c)*[anti(c)*anti(b)]*neut(q). \end{aligned}$$

Using Proposition 3 (2), $anti(c)*anti(b) \in \{anti(b*c)\}$. Thus, there exists $anti(b*c) \in \{anti(b*c)\}$ such that:

$$(a*c)*anti(b*c)*neut(q) = neut(c)*h \in H.$$

Applying Definition 5 (2), there exists $anti(a*c) \in \{anti(a*c)\}$ and $r \in N$ such that:

$$anti(a*c)*(b*c)*neut(r) \in H.$$

This means that $(a*c) \approx_H (b*c)$.

(2) Using (1)–(3) we can obtain (4). □

Example 6. *Let N = {1, 2, 3, 4, 5, 6, 7}. The operation * on N is defined as Table 4. Then, (N, *) is a non- commutative neutrosophic triplet group, and:*

$$neut(1) = 1, anti(1) = 1; neut(2) = 1, anti(2) = 2;$$
$$neut(3) = 1, anti(3) = 3; neut(4) = 1, anti(4) = 4;$$
$$neut(5) = 1, anti(5) = 6; neut(6) = 1, anti(6) = 5; neut(7) = 7, anti(7) = 7.$$

Table 4. Weak commutative neutrosophic triplet group and its strong neutrosophic triplet (NT)-subgroup.

*	1	2	3	4	5	6	7
1	1	2	3	4	5	6	1
2	2	1	6	5	4	3	2
3	3	5	1	6	2	4	3
4	4	6	5	1	3	2	4
5	5	3	4	2	6	1	5
6	6	4	2	3	1	5	6
7	1	2	3	4	5	6	7

It is easy to verify that (N, *) is a weak commutative neutrosophic triplet group. Denote H = {1, 5, 6, 7}. Then H is a strong NT-subgroups of N.

Thus we can get that (they correspond to the conclusions of Theorem 4):

(1) The relation $\approx_H$ is an equivalent relation on N and $N/H = \{\{1, 5, 6, 7\}, \{2, 3, 4\}\}$.
(2) $1 \approx_H 5$ implies $2*1 = 2 \approx_H 4 = 2*5$, and so on.
(3) $1 \approx_H 5$ implies $1*2 = 2 \approx_H 3 = 5*2$, and so on.
(4) $(N/H, *) = \{[1]_H, [2]_H\}$, $(N, *)\overset{f}{\simeq}(N/H, *)$, where $f(1) = f(5) = f(6) = f(7) = [1]_H$, and $f(2) = f(3) = f(4) = [2]_H$.

Remark 4. *Applying Proposition 4 we know that Theorem 4 is a generalization of homomorphism basic theorem in classical group theory. Moreover, Theorem 4 is also a generalization of related results in [17].*

6. Conclusions

This paper studied furtherly neutrosophic triplet group (NTG) and obtained some important results. First, some examples are given to show that some results in [18] are not true. Second, some new properties of neutrosophic triplet groups are presented, in particular, the fact of unique neutral element in every neutrosophic triplet group is proved. Third, the notions of NT-subgroup and strong NT-subgroup are proposed, a special kind of NTG (called weak commutative neutrosophic triplet group) is studied, and a homomorphism theorem is presented. All these results are interesting for exploring the structure characterizations of NTG. As the next research topics, we will explore the structures of some special NTG and their relationships with related logic algebras (such as BE-algebras and pseudo-BCI algebras [21–23]).

Author Contributions: X.Z. and Q.H. proposed the original idea of the research and wrote the first draft. F.S. gave related guidance, and X.A. participated in the study.

Acknowledgments: The research was supported by National Natural Science Foundation of China (Grant No. 61573240, 61473239).

Conflicts of Interest: The authors declare no conflict of interest.

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Multi-Granulation Neutrosophic Rough Sets on a Single Domain and Dual Domains with Applications

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Received: 28 June 2018; Accepted: 18 July 2018; Published: 23 July 2018

Abstract: It is an interesting direction to study rough sets from a multi-granularity perspective. In rough set theory, the multi-particle structure was represented by a binary relation. This paper considers a new neutrosophic rough set model, multi-granulation neutrosophic rough set (MGNRS). First, the concept of MGNRS on a single domain and dual domains was proposed. Then, their properties and operators were considered. We obtained that MGNRS on dual domains will degenerate into MGNRS on a single domain when the two domains are the same. Finally, a kind of special multi-criteria group decision making (MCGDM) problem was solved based on MGNRS on dual domains, and an example was given to show its feasibility.

Keywords: neutrosophic rough set; MGNRS; dual domains; inclusion relation; decision-making

1. Introduction

As we all know, Pawlak first proposed a rough set in 1982, which was a useful tool of granular computing. The relation is an equivalent in Pawlak's rough set. After that, many researchers proposed other types of rough set theory (see the work by the following authors [1–8]).

In 1965, Zadeh presented a new concept of the fuzzy set. After that, a lot of scholars studied it and made extensions. For example, Atanassov introduced an intuitionistic fuzzy set, which gives two degrees of membership of an element; it is a generalization of the fuzzy set. Smarandache introduced a neutrosophic set in 1998 [9,10], which was an extension of the intuitionistic fuzzy set. It gives three degrees of membership of an element (T.I.F). Smarandache and Wang [11] proposed the definition of a single valued neutrosophic set and studied its operators. Ye [12] proposed the definition of simplified neutrosophic sets and studied their operators. Zhang et al. [13] introduced a new inclusion relation of the neutrosophic set and told us when it was used by an example, and its lattice structure was studied. Garg and Nancy proposed neutrosophic operators and applied them to decision-making problems [14–16]. Now, some researchers have combined the fuzzy set and rough set and have achieved many running results, such as the fuzzy rough set [17] and rough fuzzy set. Broumi and Smarandache [18] proposed the definition of a rough neutrosophic set and studied their operators and properties. In 2016, Yang et al. [19] proposed the definition of single valued neutrosophic rough sets and studied their operators and properties.

Under the perspective of granular computing [20], the concept of a rough set is shown by the upper and lower approximations of granularity. In other words, the concept is represented by the known knowledge, which is defined by a single relationship. In fact, to meet the user's needs or achieve the goal of solving the problem, it is sometimes necessary to use multiple relational representation concepts on the domain, such as illustrated by the authors of [21]. In a grain calculation, an equivalence relation in the domain is a granularity, and a partition is considered as a granularity space [22]. The approximation that is defined by multiple equivalence relationships is a multi-granularity approximation and multiple partitions are considered as multi-granularity spaces; the resulting rough set is named a multi-granularity rough set, which has been proposed by Qian and Liang [23]. Recently, many scholars [24,25] have studied it and made extensions. Huang et al. [26] proposed the notion of intuitionistic fuzzy multi-granulation rough sets and studied their operators. Zhang et al. [27] introduced two new multi-granulation rough set models and investigated their operators. Yao et al. [28] made a summary about the rough set models on the multi-granulation spaces.

Although there have been many studies regarding multi-granulation rough set theory, there have been fewer studies about the multi-granulation rough set model on dual domains. Moreover, a multi-granulation rough set on dual domains is more convenient, for example, medical diagnosis in clinics [22,29]. The symptoms are the uncertainty index sets and the diseases are the decision sets. They are associated with each other, but they belong to two different domains. Therefore, it is necessary to use two different domains when solving the MCGDM problems. Sun et al. [29] discussed the multi-granulation rough set models based on dual domains; their properties were also obtained.

Although neutrosophic sets and multi-granulation rough sets are both useful tools to solve uncertainty problems, there are few research regarding their combination. In this paper, we proposed the definition of MGNRS as a rough set generated by multi-neutrosophic relations. It is useful to solve a kind of special group decision-making problem. We studied their properties and operations and then built a way to solve MCGDM problems based on the MGNRS theory on dual domains.

The structure of the article is as follows. In Section 2, some basic notions and operations are introduced. In Section 3, the notion of MGNRS is proposed and their properties are studied. In Section 4, the model of MGNRS on dual domains is proposed and their properties are obtained. Also, we obtained that MGNRS on dual domains will degenerate into MGNRS on a single domain when the two domains are same. In Section 5, an application of the MGNRS to solve a MCGDM problem was proposed. Finally, Section 6 concludes this paper and provides an outlook.

2. Preliminary

In this section, we review several basic concepts and operations of the neutrosophic set and multi-granulation rough set.

Definition 1 ([11])**.** *A single valued neutrosophic set B is denoted by* $\forall\, y \in Y$*, as follows:*

$$B(y) = (T_B(y), I_B(y), F_B(y))$$

$T_B(y)$, $I_B(y)$, $F_B(y) \in [0,1]$ *and satisfies* $0 \le T_B(y) + I_B(y) + F_B(y) \le 3$.

As a matter of convenience, 'single valued neutrosophic set' is abbreviated to 'neutrosophic set' later. In this paper, $NS(Y)$ denotes the set of all single valued neutrosophic sets in Y, and $NR(Y \times Z)$ denotes the set of all of the neutrosophic relations in $Y \times Z$.

Definition 2 ([11])**.** *If A and C are two neutrosophic sets, then the inclusion relation, union, intersection, and complement operations are defined as follows:*

(1) $A \subseteq C$ *iff* $\forall\, y \in Y$, $TA(y) \le TC(y)$, $IA(y) \ge IC(y)$ *and* $FA(y) \ge FC(y)$
(2) $A^c = \{(y, F_A(y), 1 - I_A(y), T_A(y)) \mid y \in Y\}$
(3) $A \cap C = \{(y, T_A(y) \wedge T_C(y), I_A(y) \vee I_C(y), F_A(y) \vee F_C(y)) \mid y \in Y\}$
(4) $A \cup C = \{(y, T_A(y) \vee T_C(y), I_A(y) \wedge I_C(y), F_A(y) \wedge F_C(y)) \mid y \in Y\}$

Definition 3 ([19]). *If (U, R) is a single valued neutrosophic approximation space. Then* ∀ *B* ϵ *SVNS(U), the lower approximation* $\underline{N}(B)$ *and upper approximation* $\overline{N}(B)$ of B are defined as follows:

$$T_{\underline{N}(B)}(y) = \min_{z \in U}[\max(F_R(y,z), T_B(z))],\ I_{\underline{N}(B)}(y) = \max_{z \in U}[\min((1 - I_R(y,z)), I_B(z))],$$
$$F_{\underline{N}(B)}(y) = \max_{z \in U}[\min(T_R(y,z), F_B(z))]$$

$$T_{\overline{N}(B)}(y) = \max_{z \in U}[\min(T_R(y,z), T_B(z))],\ I_{\overline{N}(B)}(y) = \min_{z \in U}[\max(I_R(y,z), I_B(z))],$$
$$F_{\overline{N}(B)}(y) = \min_{z \in U}[\max(F_R(y,z), F_B(z))]$$

The pair $\left(\underline{N}(B), \overline{N}(B)\right)$ is called the single valued neutrosophic rough set of B, with respect to (U, R).

According to the operation of neutrosophic number in [16], the sum of two neutrosophic sets in *U* is defined as follows.

Definition 4. *If C and D are two neutrosophic sets in U, then the sum of C and D is defined as follows:*

$$C + D = \{\langle y, C(y) \oplus D(y)\rangle \mid y \in U\}.$$

Definition 5 ([30]). *If* $b = (T_b, I_b, F_b)$ *is a neutrosophic number,* $n^* = (T_{b^*}, I_{b^*}, F_{b^*}) = (1, 0, 0)$ *is an ideal neutrosophic number. Then, the cosine similarity measure is defined as follows:*

$$S(b, b^*) = \frac{T_b \cdot T_{b^*} + I_b \cdot I_{b^*} + F_b \cdot F_{b^*}}{\sqrt{{T_b}^2 + {I_b}^2 + {F_b}^2} \cdot \sqrt{(T_{b^*})^2 + (I_{b^*})^2 + (F_{b^*})^2}}$$

3. Multi-Granulation Neutrosophic Rough Sets

In this part, we propose the concept of MGNRS and study their characterizations. MGNRS is a rough set generated by multi-neutrosophic relations, and when all neutrosophic relations are same, MGNRS will degenerated to neutrosophic rough set.

Definition 6. *Assume U is a non-empty finite domain, and* R_i $(1 \le i \le n)$ *is the binary neutrosophic relation on U. Then, (U,* R_i*) is called the multi-granulation neutrosophic approximation space (MGNAS).*

Next, we present the multi-granulation rough approximation of a neutrosophic concept in an approximation space.

Definition 7. *Let the tuple ordered set (U,* R_i*)* $(1 \le i \le n)$ *be a MGNAS. For any B* ϵ *NS (U), the three memberships of the optimistic lower approximation* $\underline{M}^o(B)$*and optimistic upper approximation*$\overline{M}^o(B)$ *in (U,* R_i*) are defined, respectively, as follows:*

$$T_{\underline{M}^o(B)}(y) = \max_{i=1}^{n} \min_{z \in U}\left(\max\left(F_{R_i}(y,z), T_B(z)\right)\right)\ I_{\underline{M}^o(B)}(y) = \min_{i=1}^{n} \max_{z \in U}\left(\min\left((1 - I_{R_i}(y,z)), I_B(z)\right)\right),$$
$$F_{\underline{M}^o(B)}(y) = \min_{i=1}^{n} \max_{z \in U}\left(\min\left(T_{R_i}(y,z), F_B(z)\right)\right),\ T_{\overline{M}^o(B)}(y) = \min_{i=1}^{n} \max_{z \in U}\left(\min\left(T_{R_i}(y,z), T_B(z)\right)\right),$$
$$I_{\overline{M}^o(B)}(y) = \max_{i=1}^{n} \min_{z \in U}\left(\max\left(I_{R_i}(y,z), I_B(z)\right)\right),\ F_{\overline{M}^o(B)}(y) = \max_{i=1}^{n} \min_{z \in U}\left(\max\left(F_{R_i}(y,z), F_B(z)\right)\right).$$

Then, $\underline{M}^o(B), \overline{M}^o(B)$ ϵ NS(U). In addition, B is called a definable neutrosophic set on (U, R_i) when $\underline{M}^o(B) = \overline{M}^o(B)$. Otherwise, the pair $\left(\underline{M}^o(B), \overline{M}^o(B)\right)$ is called an optimistic MGNRS.

Definition 8. *Let the tuple ordered set (U, R_i) ($1 \le i \le n$) be a MGNAS. For any B ϵ NS(U), the three memberships of pessimistic lower approximation $\underline{M}^p(B)$ and pessimistic upper approximation $\overline{M}^p(B)$ in (U, R_i) are defined, respectively, as follows:*

$$T_{\underline{M}^p(B)}(y) = \min_{i=1}^{n} \min_{z \in U}\big(\max\big(F_{R_i}(y,z), T_B(z)\big)\big),\ I_{\underline{M}^p(B)}(y) = \max_{i=1}^{n} \max_{z \in U}\big(\min\big((1 - I_{R_i}(y,z)), I_B(z)\big)\big),$$
$$F_{\underline{M}^p(B)}(y) = \max_{i=1}^{n} \max_{z \in U}\big(\min\big(T_{R_i}(y,z), F_B(z)\big)\big),\ T_{\overline{M}^p(B)}(y) = \max_{i=1}^{n} \max_{z \in U}\big(\min\big(T_{R_i}(y,z), T_B(z)\big)\big),$$
$$I_{\overline{M}^p(B)}(y) = \min_{i=1}^{n} \min_{z \in U}\big(\max\big(I_{R_i}(y,z), I_B(z)\big)\big),\ F_{\overline{M}^p(B)}(y) = \min_{i=1}^{n} \min_{z \in U}\big(\max\big(F_{R_i}(y,z), F_B(z)\big)\big).$$

Similarly, B is called a definable neutrosophic set on (U, R_i) when $\underline{M}^p(B) = \overline{M}^p(B)$. Otherwise, the pair $\left(\underline{M}^p(B), \overline{M}^p(B)\right)$ is called a pessimistic MGNRS.

Example 1. *Define MGNAS (U, R_i), where U = {z_1, z_2, z_3} and R_i ($1 \le i \le 3$) are given in Tables 1–3*

Table 1. Neutrosophic relation R_1.

R_1	z_1	z_2	z_3
z_1	(0.4, 0.5, 0.4)	(0.5, 0.7, 0.1)	(1, 0.8, 0.8)
z_2	(0.5, 0.6, 1)	(0.2, 0.6, 0.4)	(0.9, 0.2, 0.4)
z_3	(1, 0.2, 0)	(0.8, 0.9, 1)	(0.6, 1, 0)

Table 2. Neutrosophic relation R_2.

R_2	z_1	z_2	z_3
z_1	(0.9, 0.2, 0.4)	(0.3, 0.9, 0.1)	(0.1, 0.7, 0)
z_2	(0.4, 0.5, 0.1)	(0, 0.1, 0.7)	(1, 0.8, 0.8)
z_3	(1, 0.5, 0)	(0.4, 0.4, 0.2)	(0.1, 0.5, 0.4)

Table 3. Neutrosophic relation R_3.

R_3	z_1	z_2	z_3
z_1	(0.7, 0.7, 0)	(0.4, 0.8, 0.9)	(1, 0.4, 0.5)
z_2	(0.8, 0.2, 0.1)	(1, 0.1, 0.8)	(0.1, 0.3, 0.5)
z_3	(0, 0.8, 1)	(1, 0, 1)	(1, 1, 0)

Suppose a neutrosophic set on U is as follows: $C(z_1)$ = (0.2, 0.6, 0.4), $C(z_2)$ = (0.5, 0.4, 1), $C(z_3)$ = (0.7, 0.1, 0.5); by Definitions 7 and 8, we can get the following:

$$\underline{M}^o(C)(z_1) = (0.4, 0.3, 0.4), \underline{M}^o(C)(z_2) = (0.5, 0.4, 0.5), \underline{M}^o(C)(z_3) = (0.7, 0.4, 0.4)$$
$$\overline{M}^o(C)(z_1) = (0.3, 0.6, 0.4), \overline{M}^o(C)(z_2) = (0.5, 0.4, 0.5), \overline{M}^o(C)(z_3) = (0.4, 0.6, 0.5)$$
$$\underline{M}^p(C)(z_1) = (0.2, 0.6, 0.5), \underline{M}^p(C)(z_2) = (0.2, 0.6, 0.1), \underline{M}^p(C)(z_3) = (0.2, 0.6, 0.1)$$
$$\overline{M}^p(C)(z_1) = (0.7, 0.4, 0.4), \overline{M}^p(C)(z_2) = (0.7, 0.2, 0.4), \overline{M}^p(C)(z_3) = (0.7, 0.4, 0.4)$$

Proposition 1. *Assume (U, R_i) is MGNAS, R_i ($1 \le i \le n$) is the neutrosophic relations. $\forall$ C ϵ NS(U), $\underline{M}^o(C)$ and $\overline{M}^o(C)$ are the optimistic lower and upper approximation of C. Then,*

$$\underline{M}^o(C) = \bigcup_{i=1}^{n} \underline{N}(C)\overline{M}^o(C) = \bigcap_{i=1}^{n} \overline{N}(C)$$

where

$$\underline{N}(C)(y) = \bigcap_{z \in U}({R_i}^c(y,z) \cup C(z)), \overline{N}(C)(y) = \bigcup_{z \in U}(R_i(y,z) \cap C(z))$$

Proof. They can be proved by Definitions 7.

Proposition 2. *Assume (U, R_i) be MGNAS, R_i ($1 \le i \le n$) be neutrosophic relations. $\forall$ C ϵ NS(U), $\underline{M}^p(C)$ and $\overline{M}^p(C)$ are the pessimistic lower and upper approximation of C. Then*

$$\underline{M}^p(C) = \bigcap_{i=1}^{n} \underline{N}(C)\overline{M}^p(C) = \bigcup_{i=1}^{n} \overline{N}(C)$$

where

$$\underline{N}(C)(y) = \bigcap_{z \in U}(R_i{}^c(y,z) \cup C(z)), \overline{N}(C)(y) = \bigcup_{z \in U}(R_i(y,z) \cap C(z))$$

Proof. Proposition 2 can be proven by Definition 8.

Proposition 3. *Assume (U, R_i) is MGNAS, R_i ($1 \le i \le n$) is the neutrosophic relations. $\forall$ C, D ϵ NS(U), we have the following:*

(1) $\underline{M}^o(C) = \sim \overline{M}^o(\sim C), \underline{M}^p(C) = \sim \overline{M}^p(\sim C)$;
(2) $\overline{M}^o(C) = \sim \underline{M}^o(\sim C), \overline{M}^p(C) = \sim \underline{M}^p(\sim C)$;
(3) $\underline{M}^o(C \cap D) = \underline{M}^o(C) \cap \underline{M}^o(D), \underline{M}^p(C \cap D) = \underline{M}^p(C) \cap \underline{M}^p(D)$;
(4) $\overline{M}^o(C \cup D) = \overline{M}^o(C) \cup \overline{M}^o(D), \overline{M}^p(C \cup D) = \overline{M}^p(C) \cup \overline{M}^p(D)$;
(5) $C \subseteq D \Rightarrow \underline{M}^o(C) \subseteq \underline{M}^o(D), \underline{M}^p(C) \subseteq \underline{M}^p(D)$;
(6) $C \subseteq D \Rightarrow \overline{M}^o(C) \subseteq \overline{M}^o(D), \overline{M}^p(C) \subseteq \overline{M}^p(D)$;
(7) $\underline{M}^o(C \cup D) \supseteq \underline{M}^o(C) \cup \underline{M}^o(D), \underline{M}^p(C \cup D) \supseteq \underline{M}^p(C) \cup \underline{M}^p(D)$;
(8) $\overline{M}^o(C \cap D) \subseteq \overline{M}^o(C) \cap \overline{M}^o(D), \overline{M}^p(C \cap D) \subseteq \overline{M}^p(C) \cap \overline{M}^p(D)$.

Proof. (1), (2), (5), and (6) can be taken directly from Definitions 7 and 8. We only show (3), (4), (7), and (8).

(3) From Proposition 1, we have the following:

$$\begin{aligned} \underline{M}^o(C \cap D)(y) = & \bigcup_{i=1}^{n}\left(\bigcap_{z \in U}(R_i{}^c(y,z) \cup (C \cap D)(z))\right) \\ & = \bigcup_{i=1}^{n}\left(\bigcap_{z \in U}((R_i{}^c(y,z) \cup C(z)) \cap (R_i{}^c(y,z) \cup D(z)))\right) \\ & = \left(\bigcup_{i=1}^{n}\left(\bigcap_{z \in U}(R_i{}^c(y,z) \cup C(z))\right)\right) \cap \left(\bigcup_{i=1}^{n}\left(\bigcap_{z \in U}(R_i{}^c(y,z) \cup D(z))\right)\right) \\ & = \underline{M}^o C(y) \cap \underline{M}^o D(y). \end{aligned}$$

Similarly, from Proposition 2, we can get the following:

$$\underline{M}^p(C \cap D)(y) = \underline{M}^p C(y) \cap \underline{M}^p D(y).$$

(4) According to Propositions 1 and 2, in the same way as (3), we can get the proof.

(7) From Definition 7, we have the following:

$$\begin{aligned} T_{\underline{M}^o(C \cup D)}(y) & = \max_{i=1}^{n} \min_{z \in U}\{\max[F_{R_i}(y,z), (\max(T_C(z), T_D(z)))]\} \\ & = \max_{i=1}^{n} \min_{z \in U}\{\max[(\max(F_{R_i}(y,z), T_C(z))), (\max(F_{R_i}(y,z), T_D(z)))]\} \\ & \ge \max\left\{\left[\max_{i=1}^{n} \min_{z \in U}(\max(F_{R_i}(y,z), T_C(z)))\right], \left[\max_{i=1}^{n} \min_{z \in U}(\max(F_{R_i}(y,z), T_D(z)))\right]\right\} \\ & = \max\left(T_{\underline{M}^o(C)}(y), T_{\underline{M}^o(D)}(y)\right). \end{aligned}$$

$$
\begin{aligned}
I_{\underline{M}^{o}(C\cup D)}(y) &= \min_{i=1}^{n}\max_{z\in U}\{\min[(1-I_{R_i}(y,z)),(\min(I_C(z),I_D(z)))]\}\\
&= \min_{i=1}^{n}\max_{z\in U}\{\min[(\min((1-I_{R_i}(y,z)),I_C(z))),(\min((1-I_{R_i}(y,z)),I_D(z)))]\}\\
&\le \min\left\{\left[\min_{i=1}^{n}\max_{z\in U}(\min((1-I_{R_i}(y,z)),I_C(z)))\right],\left[\min_{i=1}^{n}\max_{z\in U}(\min((1-I_{R_i}(y,z)),I_D(z)))\right]\right\}\\
&= \min\left(I_{\underline{M}^{o}(C)}(y),I_{\underline{M}^{o}(D)}(y)\right).\\
F_{\underline{M}^{o}(C\cup D)}(y) &= \min_{i=1}^{n}\max_{z\in U}\{\min[T_{R_i}(y,z),(\min(F_C(z),F_D(z)))]\}\\
&= \min_{i=1}^{n}\max_{z\in U}\{\min[\min(T_{R_i}(y,z),F_C(z))],[\min(T_{R_i}(y,z),F_D(z))]\}\\
&\le \min\left\{\left[\min_{i=1}^{n}\max_{z\in U}(\min(T_{R_i}(y,z),F_C(z)))\right],\left[\min_{i=1}^{n}\max_{z\in U}(\min(T_{R_i}(y,z),F_D(z)))\right]\right\}\\
&= \min\left(F_{\underline{M}^{o}(C)}(y),F_{\underline{M}^{o}(D)}(y)\right).
\end{aligned}
$$

Hence, $\underline{M}^{o}(C\cup D)\supseteq\underline{M}^{o}(C)\cup\underline{M}^{o}(D)$.

Also, according to Definition 8, we can get $\underline{M}^{p}(C\cup D)\supseteq\underline{M}^{p}(C)\cup\underline{M}^{p}(D)$.

(8) From Definition 7, we have the following:

$$
\begin{aligned}
T_{\overline{M}^{o}(C\cap D)}(y) &= \min_{i=1}^{n}\max_{z\in U}\{\min[T_{R_i}(y,z),(\min(T_C(z),T_D(z)))]\}\\
&= \min_{i=1}^{n}\max_{z\in U}\{\min[(\min(T_{R_i}(y,z),T_C(z))),(\min(T_{R_i}(y,z),T_D(z)))]\}\\
&\le \min\left\{\left[\min_{i=1}^{n}\max_{z\in U}(\min(T_{R_i}(y,z),T_C(z)))\right],\left[\min_{i=1}^{n}\max_{z\in U}(\min(T_{R_i}(y,z),T_D(z)))\right]\right\}\\
&= \min\left(T_{\overline{M}^{o}(C)}(y),T_{\overline{M}^{o}(D)}(y)\right).
\end{aligned}
$$

$$
\begin{aligned}
I_{\overline{M}^{o}(C\cap D)}(y) &= \max_{i=1}^{n}\min_{z\in U}\{\max[I_{R_i}(y,z),(\max(I_C(z),I_D(z)))]\}\\
&= \max_{i=1}^{n}\min_{z\in U}\{\max[(\max(I_{R_i}(y,z),I_C(z))),(\max(I_{R_i}(y,z),I_D(z)))]\}\\
&\le \min\left\{\left[\max_{i=1}^{n}\min_{z\in U}(\max(I_{R_i}(y,z),I_C(z)))\right],\left[\max_{i=1}^{n}\min_{z\in U}(\max(I_{R_i}(y,z),I_D(z)))\right]\right\}\\
&= \min\left(I_{\overline{M}^{o}(C)}(y),I_{\overline{M}^{o}(D)}(y)\right).\\
F_{\overline{M}^{o}(C\cap D)}(y) &= \max_{i=1}^{n}\min_{z\in U}[F_{R_i}(y,z)\vee(F_C(z)\vee F_D(z))]\\
&= \max_{i=1}^{n}\min_{z\in U}[(F_{R_i}(y,z)\vee F_C(z))\vee(F_{R_i}(y,z)\vee F_D(z))]\\
&\ge \left[\max_{i=1}^{n}\min_{z\in U}(F_{R_i}(y,z)\vee F_C(z))\right]\vee\left[\max_{i=1}^{n}\min_{z\in U}(F_{R_i}(y,z)\vee F_D(z))\right]\\
&= \max\left(F_{\overline{M}^{o}(C)}(y),F_{\overline{M}^{o}(D)}(y)\right).
\end{aligned}
$$

Hence, $\overline{M}^{o}(C\cap D)\subseteq\overline{M}^{o}(C)\cap\overline{M}^{o}(D)$.

Similarly, according Definition 8, we can get $\overline{M}^{p}(C\cap D)\subseteq\overline{M}^{p}(C)\cap\overline{M}^{p}(D)$.

Next, we will give an example to show that maybe $\underline{M}^{o}(C\cup D)\neq\underline{M}^{o}(C)\cup\underline{M}^{o}(D)$.

Example 2. *Define MGNAS (U, R_i), where U = {z_1, z_2, z_3} and R_i ($1\le i\le 3$) are given in Example 1.*

Suppose there are two neutrosophic sets on universe U, as follows: C(z_1) = (0.5, 0.1, 0.2), C(z_2) = (0.5, 0.3, 0.2), C(z_3) = (0.6, 0.2, 0.1), D(z_1) = (0.7, 0.2, 0.1), D(z_2) = (0.4, 0.2, 0.1), D(z_3) = (0.2, 0.2, 0.5), we have ($C\cup D$)(z_1) = (0.7, 0.1, 0.1), ($C\cup D$)(z_2) = (0.5, 0.2, 0.1), ($C\cup D$)(z_3) = (0.6, 0.2, 0.1), ($C\cap D$)(z_1) = (0.5, 0.1, *0.2), $(C \cap D)(z_2)$ = (0.4, 0.2, 0.2), $(C \cap D)(z_3)$ = (0.2, 0.2, 0.5). Then, from Definitions 7 and 8, we can get the following:*

$$\underline{M}^o(C)(z_1) = (0.5, 0, 0.2), \underline{M}^o(C)(z_2) = (0.5, 0.1, 0.2), \underline{M}^o(C)(z_3) = (0.5, 0.1, 0.2);$$
$$\underline{M}^o(D)(z_1) = (0.4, 0, 0.1), \underline{M}^o(D)(z_2) = (0.2, 0.1, 0.2), \underline{M}^o(D)(z_3) = (0.4, 0.1, 0.2);$$
$$\underline{M}^o(C \cup D)(z_1) = (0.5, 0, 0.1), \underline{M}^o(C \cup D)(z_2) = (0.5, 0.1, 0.1), \underline{M}^o(C \cup D)(z_3) = (0.5, 0.1, 0.1)$$
$$(\underline{M}^o(C) \cup \underline{M}^o(D))(z_1) = (0.5, 0, 0.1), (\underline{M}^o(C) \cup \underline{M}^o(D))(z_2) = (0.5, 0.1, 0.2),$$
$$(\underline{M}^o(C) \cup \underline{M}^o(D))(z_3) = (0.5, 0.1, 0.2)$$

So, $\underline{M}^o(C \cup D) \neq \underline{M}^o(C) \cup \underline{M}^o(D)$.
Also, there are examples to show that maybe $\underline{M}^p(C \cup D) \neq \underline{M}^p(C) \cup \underline{M}^p(D)$, $\overline{M}^o(C \cap D) \neq \overline{M}^o(C) \cap \overline{M}^o(D)$, $\overline{M}^p(C \cap D) \neq \overline{M}^p(C) \cap \overline{M}^p(D)$. We do not say anymore here.

4. Multi-Granulation Neutrosophic Rough Sets on Dual Domains

In this section, we propose the concept of MGNRS on dual domains and study their characterizations. Also, we obtain that the MGNRS on dual domains will degenerate into MGNRS, defined in Section 3, when the two domains are same.

Definition 9. *Assume that U and V are two domains, and $R_i \in NS(U \times V)$ $(1 \leq i \leq n)$ is the binary neutrosophic relations. The triple ordered set (U, V, R_i) is called the (two-domain) MGNAS.*

Next, we present the multi-granulation rough approximation of a neutrosophic concept in an approximation space on dual domains.

Definition 10. *Let (U, V, R_i) $(1 \leq i \leq n)$ be (two-domain) MGNAS. $\forall$ B $\in$ NS(V) and y $\in$ U, the three memberships of the optimistic lower and upper approximation $\underline{M}^o(B)$, $\overline{M}^o(B)$ in (U, V, R_i) are defined, respectively, as follows:*

$$T_{\underline{M}^o(B)}(y) = \max_{i=1}^{n} \min_{z \in V}\left[\max\left(F_{R_i}(y,z), T_B(z)\right)\right] I_{\underline{M}^o(B)}(y) = \min_{i=1}^{n} \max_{z \in V}\left[\min\left(\left(1 - I_{R_i}(y,z)\right), I_B(z)\right)\right]$$

$$F_{\underline{M}^o(B)}(y) = \min_{i=1}^{n} \max_{z \in V}\left[\min\left(T_{R_i}(y,z), F_B(z)\right)\right] T_{\overline{M}^o(B)}(y) = \min_{i=1}^{n} \max_{z \in V}\left[\min\left(T_{R_i}(y,z), T_B(z)\right)\right]$$
$$I_{\overline{\sum_{i=1}^{n} R_i}^{o}(B)}(y) = \max_{i=1}^{n} \min_{z \in V}\left[\max\left(I_{R_i}(y,z), I_B(z)\right)\right] F_{\overline{M}^o(B)}(y) = \max_{i=1}^{n} \min_{z \in V}\left[\max\left(F_{R_i}(y,z), F_B(z)\right)\right]$$

Then $\underline{M}^o(B)$, $\overline{M}^o(B) \in NS(U)$. In addition, B is called a definable neutrosophic set on (U, V, R_i) on dual domains when $\underline{M}^o(B) = \overline{M}^o(B)$. Otherwise, the pair $\left(\underline{M}^o(B), \overline{M}^o(B)\right)$ is called an optimistic MGNRS on dual domains.

Definition 11. *Assume (U, V, R_i) $(1 \leq i \leq n)$ is (two-domain) MGNAS. $\forall$ B $\in$ NS(V) and y $\in$ U, the three memberships of the pessimistic lower and upper approximation $\underline{M}^p(B)$, $\overline{M}^p(B)$ in (U, V, R_i) are defined, respectively, as follows:*

$$T_{\underline{M}^p(B)}(y) = \min_{i=1}^{n} \min_{z \in V}\left[\max\left(F_{R_i}(y,z), T_B(z)\right)\right], I_{\underline{M}^p(B)}(y) = \max_{i=1}^{n} \max_{z \in V}\left[\min\left(\left(1 - I_{R_i}(y,z)\right), I_B(z)\right)\right],$$

$$F_{\underline{M}^p(B)}(y) = \max_{i=1}^{n} \max_{z \in V}\left[\min\left(T_{R_i}(y,z), F_B(z)\right)\right], T_{\overline{M}^p(B)}(y) = \max_{i=1}^{n} \max_{z \in V}\left[\min\left(T_{R_i}(y,z), T_B(z)\right)\right],$$
$$I_{\overline{M}^p(B)}(y) = \min_{i=1}^{n} \min_{z \in V}\left[\max\left(I_{R_i}(y,z), I_B(z)\right)\right], F_{\overline{M}^p(B)}(y) = \min_{i=1}^{n} \min_{z \in V}\left[\max\left(F_{R_i}(y,z), F_B(z)\right)\right].$$

Then, B is called a definable neutrosophic set on (U, V, R_i) when $\underline{M}^p(B) = \overline{M}^p(B)$. Otherwise, the pair $\left(\underline{M}^p(B), \overline{M}^p(B)\right)$ is called a pessimistic MGNRS on dual domains.

Remark 1. *Note that if U = V, then the optimistic and pessimistic MGNRS on the dual domains will be the same with the optimistic and pessimistic MGNRS on a single domain, which is defined in Section 3*

Proposition 4. *Assume (U, V, R_i) ($1 \le i \le n$) is (two-domain) MGNAS, R_i ($1 \le i \le n$) is the neutrosophic relations. ∀ C, D ϵ NS(U), we have the following:*

(1) $\underline{M}^o(C) = \sim \overline{M}^o(\sim C), \underline{M}^p(C) = \sim \overline{M}^p(\sim C)$;
(2) $\overline{M}^o(C) = \sim \underline{M}^o(\sim C), \overline{M}^p(C) = \sim \underline{M}^p(\sim C)$;
(3) $\underline{M}^o(C \cap D) = \underline{M}^o(C) \cap \underline{M}^o(D), \underline{M}^p(C \cap D) = \underline{M}^p(C) \cap \underline{M}^p(D)$;
(4) $\overline{M}^o(C \cup D) = \overline{M}^o(C) \cup \overline{M}^o(D), \overline{M}^p(C \cup D) = \overline{M}^p(C) \cup \overline{M}^p(D)$;
(5) $C \subseteq D \Rightarrow \underline{M}^o(C) \subseteq \underline{M}^o(D), \underline{M}^p(C) \subseteq \underline{M}^p(D)$;
(6) $C \subseteq D \Rightarrow \overline{M}^o(C) \subseteq \overline{M}^o(D), \overline{M}^p(C) \subseteq \overline{M}^p(D)$;
(7) $\underline{M}^o(C \cup D) \supseteq \underline{M}^o(C) \cup \underline{M}^o(D), \underline{M}^p(C \cup D) \supseteq \underline{M}^p(C) \cup \underline{M}^p(D)$;
(8) $\overline{M}^o(C \cap D) \subseteq \overline{M}^o(C) \cap \overline{M}^o(D), \overline{M}^p(C \cap D) \subseteq \overline{M}^p(C) \cap \overline{M}^p(D)$.

Proof. These propositions can be directly proven from Definitions 10 and 11.

5. An Application of Multi-Granulation Neutrosophic Rough Set on Dual Domains

Group decision making [31] is a useful way to solve uncertainty problems. It has developed rapidly since it was first proposed. Its essence is that in the decision-making process, multiple decision makers (experts) are required to participate and negotiate in order to settle the corresponding decision-making problems. However, with the complexity of the group decision-making problems, what we need to deal with is the multi-criteria problems, that is, multi-criteria group decision making (MCGDM). The MCGDM problem is to select or rank all of the feasible alternatives in multiple, interactive, and conflicting standards.

In this section, we build a neo-way to solve a kind of special MCGDM problem using the MGNRS theory. We generated the rough set according the multi-neutrosophic relations and then used it to solve the decision-making problems. We show the course and methodology of it.

5.1. Problem Description

Firstly, we describe the considered problem and we show it using a MCGDM example of houses selecting.

Let $U = \{x_1, x_2, \ldots, x_m\}$ be the decision set, where x_1 represents very good, x_2 represents good, x_3 represents less good, ... , and x_m represents not good. Let $V = \{y_1, y_2, \ldots, y_n\}$ be the criteria set to describe the given house, where y_1 represents texture, y_2 represents geographic location, y_3 represents price, ... , and y_n represents solidity. Suppose there are k evaluation experts and all of the experts give their own evaluation for criteria set y_j (y_j ϵ V) ($j = 1, 2, \ldots, n$), regarding the decision set elements x_i (x_i ϵ U) ($i = 1, 2, \ldots, m$). In this paper, let the evaluation relation $R_1, R_2, \ldots, R_k$ between V and U given by the experts, be the neutrosophic relation, $R_1, R_2, \ldots, R_k$ ϵ SNS ($U \times V$). That is, R_l (x_i, y_j) ($l = 1, 2, \ldots, k$) represents the relation of the criteria set y_j and the decision set element x_i, which is given by expert l, based on their own specialized knowledge and experience. For a given customer, the criterion of the customer is shown using a neutrosophic set, C, in V, according to an expert's opinion. Then, the result of this problem is to get the opinion of the given house for the customer.

Then, we show the method to solve the above problem according to the theory of optimistic and pessimistic MGNRS on dual domains.

5.2. New Method

In the first step, we propose the multi-granulation neutrosophic decision information system based on dual domains for the above problem.

According to Section 5.1′s description, we can get the evaluation of each expert as a neutrosophic relation. Then, all of the binary neutrosophic relations R_l given by all of the experts construct a relation set $\mathcal{R}$ (i.e., $R_l \in \mathcal{R}$). Then, we get the multi-granulation neutrosophic decision information systems based on dual domains, denoted by $(U, V, \mathcal{R})$.

Secondly, we compute $\underline{M}^o(C)$, $\overline{M}^o(C)$, $\underline{M}^p(C)$, $\overline{M}^p(C)$ for the given customer, regarding $(U, V, \mathcal{R})$.

Thirdly, according to Definition 4, we computed the sum of the optimistic and pessimistic multi-granulation neutrosophic lower and upper approximation.

Next, according Definition 5, we computed the cosine similarity measure. Define the choice $x*$ with the idea characteristics value $\alpha * = (1, 0, 0)$ as the ideal choice. The bigger the value of $S(\alpha_{x_i}, \alpha^*)$ is, the closer the choice x_i with the ideal alternative $x*$, so the better choice x_i is.

Finally, we compared $S(\alpha_{x_i}, \alpha^*)$ and ranked all of the choices that the given customer can choose from and we obtained the optimal choice.

5.3. Algorithm and Pseudo-Code

In this section, we provide the algorithm and pseudo-code given in table Algorithm 1.

Algorithm 1. Multi-granulation neutrosophic decision algorithm.

Input Multi-granulation neutrosophic decision information systems $(U, V, \mathcal{R})$.
Output The optimal choice for the client.
Step 1 Computing $\underline{M}^o(C)$, $\overline{M}^o(C)$, $\underline{M}^p(C)$, $\overline{M}^p(C)$ of neutrosophic set C about $(U, V, \mathcal{R})$;
Step 2 From Definition 4., we get $\underline{M}^o(C) + \overline{M}^o(C)$ and $\underline{M}^p(A) + \overline{M}^p(A)$;
Step 3 From Definition 5., we computer $S^o(\alpha_{x_i}, \alpha^*)$ and $S^p(\alpha_{x_i}, \alpha^*)$ $(i = 1, 2, \ldots, m)$;
Step 4 The optimal decision-making is to choose x_h if
$S(\alpha_{x_h}, \alpha^*) = \max_{i \in \{1,2,\cdots,m\}}(S(\alpha_{x_i}, \alpha^*))$.
pseudo-code
Begin
Input $(U, V, \mathcal{R})$, where U is the decision set, V is the criteria set, and $\mathcal{R}$ denotes the binary neutrosophic relation between criteria set and decision set.
Calculate $\underline{M}^o(C)$, $\overline{M}^o(C)$, $\underline{M}^p(C)$, $\overline{M}^p(C)$. Where $\underline{M}^o(C)$, $\overline{M}^o(C)$, $\underline{M}^p(C)$, $\overline{M}^p(C)$, which represents the optimistic and pessimistic multi-granulation lower and upper approximation of C, which is defined in Section 4.
Calculate $\underline{M}^o(C) + \overline{M}^o(C)$ and $\underline{M}^p(C) + \overline{M}^p(C)$, which is defined in Definition 4.
Calculate $S^o\left(\underline{M}^o(C) + \overline{M}^o(C), \alpha^*\right)$ and $S^p\left(\underline{M}^p(C) + \overline{M}^p(C), \alpha^*\right)$, which is defined in Definition 5.
For $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $l = 1, 2, \ldots, k$;
If $S^o(\alpha_{x_i}, \alpha^*) < S^o\left(\alpha_{x_j}, \alpha^*\right)$, then $S^o\left(\alpha_{x_j}, \alpha^*\right) \to$ Max,
else $S^o(\alpha_{x_i}, \alpha^*) \to$ Max,
If $S^o(\alpha_{x_l}, \alpha^*) > Max$, then $S^o(\alpha_{x_l}, \alpha^*) \to$ Max;
Print Max;
End

5.4. An Example

In this section, we used Section 5.2’s way of solving a MCGDM problem, using the example of buying houses.

Let $V = \{y_1, y_2, y_3, y_4\}$ be the criteria set, where y_1 represents the texture, y_2 represents the geographic location, y_3 represents the price, and y_4 represents the solidity. Let $U = \{z_1, z_2, z_3, z_4\}$ be a decision set, where z_1 represents very good, z_2 represents good, z_3 represents less good, and z_4 represents not good.

Assume that there are three experts. They provide their opinions about all of the criteria sets y_j ($y_j \in V$) ($j = 1, 2, 3, 4$) regarding the decision set elements z_i ($x_i \in U$) ($i = 1, 2, 3, 4$). Like the discussion in Section 5.1, the experts give three evaluation relations, R_1, R_2, and R_3, which are neutrosophic relations between V and U, that is, $R_1, R_2, R_3 \in NR(U \times V)$. $T_{Rk}(z_i, y_j)$ shows the expert, k, give the truth membership of y_j to z_i; $I_{Rk}(z_i, y_j)$ shows the expert, k, give the indeterminacy membership of y_j to z_i; $F_{Rk}(z_i, y_j)$ shows the expert, k, give the falsity membership of y_j to z_i. For example, the first value (0.2, 0.3, 0.4) in Table 4, of 0.2 shows that the truth membership of the texture for the given house is very good, 0.3 shows that the indeterminacy membership of the texture for the given house is very good, and 0.4 shows that the falsity membership of the texture for the given house is very good.

Table 4. Neutrosophic relation R_1.

R_1	y_1	y_2	y_3	y_4
z_1	(0.2, 0.3, 0.4)	(0.3, 0.5, 0.4)	(0.4, 0.6, 0.2)	(0.1, 0.3, 0.5)
z_2	(0.8, 0.7, 0.1)	(0.2, 0.5, 0.6)	(0.6, 0.6, 0.7)	(0.4, 0.6, 0.3)
z_3	(0.5, 0.7, 0.2)	(0.6, 0.2, 0.1)	(1, 0.9, 0.4)	(0.5, 0.4, 0.3)
z_4	(0.4, 0.6, 0.3)	(0.5, 0.5, 0.4)	(0.3, 0.8, 0.4)	(0.2, 0.9, 0.8)

So, we build the multi-granulation neutrosophic decision information system (U, V, $\mathcal{R}$) for the example.

Assume that the three experts give three evaluation relations, the results are given in Tables 4–6.

Table 5. Neutrosophic relation R_2.

R_2	y_1	y_2	y_3	y_4
z_1	(0.3, 0.4, 0.5)	(0.6, 0.7, 0.2)	(0.1, 0.8, 0.3)	(0.5, 0.3, 0.4)
z_2	(0.5, 0.5, 0.4)	(1, 0, 1)	(0.8, 0.1, 0.8)	(0.7, 0.8, 0.5)
z_3	(0.7, 0.2, 0.1)	(0.3, 0.5, 0.4)	(0.6, 0.1, 0.4)	(1, 0, 0)
z_4	(1, 0.2, 0)	(0.8, 0.1, 0.5)	(0.1, 0.2, 0.7)	(0.2, 0.2, 0.8)

Table 6. Neutrosophic relation R_3.

R_3	y_1	y_2	y_3	y_4
z_1	(0.6, 0.2, 0.2)	(0.3, 0.1, 0.7)	(0, 0.2, 0.9)	(0.8, 0.3, 0.2)
z_2	(0.1, 0.1, 0.7)	(0.2, 0.3, 0.8)	(0.7, 0.1, 0.2)	(0, 0, 1)
z_3	(0.8, 0.4, 0.1)	(0.9, 0.5, 0.3)	(0.2, 0.1, 0.6)	(0.7, 0.2, 0.3)
z_4	(0.6, 0.2, 0.2)	(0.2, 0.2, 0.8)	(1, 1, 0)	(0.5, 0.3, 0.1)

Assume C is the customer's evaluation for each criterion in V, and is given by the following:

$$C(y_1) = (0.6, 0.5, 0.5),\ C(y_2) = (0.7, 0.3, 0.2),\ C(y_3) = (0.4, 0.5, 0.9),\ C(y_4) = (0.3, 0.2, 0.6).$$

From Definitions 10 and 11, we can compute the following:

$$\underline{M}^o(C)(z_1) = (0.4, 0.5, 0.4),\ \underline{M}^o(C)(z_2) = (0.5, 0.4, 0.6),\ \underline{M}^o(C)(z_3) = (0.3, 0.3, 0.6),$$
$$\underline{M}^o(C)(z_4) = (0.6, 0.4, 0.4)$$

$$\overline{M}^o(C)(z_1) = (0.4, 0.3, 0.5),\ \overline{M}^o(C)(z_2) = (0.4, 0.5, 0.7),\ \overline{M}^o(C)(z_3) = (0.6, 0.3, 0.4),$$
$$\overline{M}^o(C)(z_4) = (0.5, 0.5, 0.5)$$
$$\underline{M}^p(C)(z_1) = (0.3, 0.5, 0.6),\ \underline{M}^p(C)(z_2) = (0.3, 0.5, 0.8),\ \underline{M}^p(C)(z_3) = (0.3, 0.5, 0.9),$$
$$\underline{M}^p(C)(z_4) = (0.3, 0.5, 0.9)$$
$$\overline{M}^o(C)(z_1) = (0.6, 0.3, 0.2),\ \overline{M}^o(C)(z_2) = (0.7, 0.2, 0.5),\ \overline{M}^o(C)(z_3) = (0.7, 0.2, 0.2),$$
$$\overline{M}^o(C)(z_4) = (0.7, 0.2, 0.4)$$

According Definition 4, we have the following:

$$\left(\underline{M}^o(C)+\overline{M}^o(C)\right)(z_1)=(0.64,0.15,0.2),\ \left(\underline{M}^o(C)+\overline{M}^o(C)\right)(z_2)=(0.7,0.2,0.42),$$
$$\left(\underline{M}^o(C)+\overline{M}^o(C)\right)(z_3)=(0.72,0.09,0.24),\ \left(\underline{M}^o(C)+\overline{M}^o(C)\right)(z_4)=(0.8,0.2,0.2)$$

$$\left(\underline{M}^p(C)+\overline{M}^p(C)\right)(z_1)=(0.72,0.15,0.12),\ \left(\underline{M}^p(C)+\overline{M}^p(C)\right)(z_2)=(0.79,0.1,0.4),$$
$$\left(\underline{M}^p(C)+\overline{M}^p(C)\right)(z_3)=(0.79,0.1,0.18),\ \left(\underline{M}^p(C)+\overline{M}^p(C)\right)(z_4)=(0.79,0.1,0.36)$$

Then, according Definition 5, we have the following:

$$S^o(\alpha_{z_1},\alpha^*)=0.9315, S^o(\alpha_{z_2},\alpha^*)=0.8329, S^o(\alpha_{z_3},\alpha^*)=0.8588, S^o(\alpha_{z_4},\alpha^*)=0.9428. \quad (1)$$

$$S^p(\alpha_{z_1},\alpha^*)=0.9662, S^p(\alpha_{z_2},\alpha^*)=0.8865, S^p(\alpha_{z_3},\alpha^*)=9677, S^p(\alpha_{z_4},\alpha^*)=0.9040. \quad (2)$$

Then, we have the following:

$$S^o(\alpha_{z_4},\alpha^*)>S^o(\alpha_{z_1},\alpha^*)>S^o(\alpha_{z_3},\alpha^*)>S^o(\alpha_{z_2},\alpha^*). \quad (3)$$

$$S^p(\alpha_{z_3},\alpha^*)>S^p(\alpha_{z_1},\alpha^*)>S^p(\alpha_{z_4},\alpha^*)=S^p(\alpha_{z_2},\alpha^*). \quad (4)$$

So, the optimistic optimal choice is to choose x_4, that is, this given house is "not good" for the customer; the pessimistic optimal choice is to choose x_3, that is, this given house is "less good" for the customer.

6. Conclusions

In this paper, we propose the concept of MGNRS on a single domain and dual domains, and obtain their properties. I addition, we obtain that MGNRS on dual domains will be the same as the MGNRS on a single domain when the two domains are same. Then, we solve a kind of special group decision-making problem (based on neutrosophic relation) using MGNRS on dual domains, and we show the algorithm and give an example to show its feasibility.

In terms of the future direction, we will study other types of combinations of multi-granulation rough sets and neutrosophic sets and obtain their properties. At the same time, exploring the application of MGNRS in totally dependent-neutrosophic sets (see [32]) and related algebraic systems (see [33–35]), and a new aggregation operator, similarity measure, and distance measure (see [36–39]), are also meaningful research directions for the future.

Author Contributions: X.Z. and C.B. initiated the research and wrote the paper, S.S. participated in some of the research work, and F.S. supervised the research work and provided helpful suggestions.

Funding: This research received no external funding.

Acknowledgments: This work was supported by the National Natural Science Foundation of China (Grant No. 61573240, 61473239) and the Graduate Student Innovation Project of Shanghai Maritime University 2017ycx082.

Conflicts of Interest: The authors declare no conflicts of interest.

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Fundamental Homomorphism Theorems for Neutrosophic Extended Triplet Groups

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Received: 17 July 2018; Accepted: 31 July 2018; Published: 3 August 2018

Abstract: In classical group theory, homomorphism and isomorphism are significant to study the relation between two algebraic systems. Through this article, we propose neutro-homomorphism and neutro-isomorphism for the neutrosophic extended triplet group (NETG) which plays a significant role in the theory of neutrosophic triplet algebraic structures. Then, we define neutro-monomorphism, neutro-epimorphism, and neutro-automorphism. We give and prove some theorems related to these structures. Furthermore, the Fundamental homomorphism theorem for the NETG is given and some special cases are discussed. First and second neutro-isomorphism theorems are stated. Finally, by applying homomorphism theorems to neutrosophic extended triplet algebraic structures, we have examined how closely different systems are related.

Keywords: neutro-monomorphism; neutro-epimorphism; neutro-automorphism; fundamental neutro-homomorphism theorem; first neutro-isomorphism theorem; and second neutro-isomorphism theorem

1. Introduction

Groups are finite or infinite set of elements which are vital to modern algebra equipped with an operation (such as multiplication, addition, or composition) that satisfies the four basic axioms of closure, associativity, the identity property, and the inverse property. Groups can be found in geometry studied by "Felix klein in 1872" [1], characterizing phenomenality like symmetry and certain types of transformations. Group theory, firstly introduced by "Galois" [2], with the study of polynomials has applications in physics, chemistry, and computer science, and also puzzles like the Rubik's cube as it may be expressed utilizing group theory. Homomorphism is both a monomorphism and an epimorphism maintaining a map between two algebraic structures of the same type (such as two groups, two rings, two fields, two vector spaces) and isomorphism is a bijective homomorphism defined as a morphism, which has an inverse that is also morphism. Accordingly, homomorphisms are effective in analyzing and calculating algebraic systems as they enable one to recognize how intently distinct systems are associated. Similar to the classical one, neuro-homomorphism is the transform between two neutrosophic triplet algebraic objects N and H. That is, if elements in N satisfy some algebraic equation involving binary operation "*", their images in H satisfy the same algebraic equation. A neutro-isomorphism identifies two algebraic objects with one another. The most common use of neutro-homomorphisms and neutro-isomorphisms in this study is to deal with homomorphism theorems which allow for the identification of some neutrosophic triplet quotient objects with certain other neutrosophic triplet subgroups, and so on.

The neutrosophic logic and a neutrosophic set, firstly made known by Florentin Smarandache [3] in 1995, has been widely applied to several scientific fields. This study leads to a new direction, exploration, path of thinking to mathematicians, engineers, computer scientists, and many other researchers, so the area of study grew extremely and applications were found in many areas of neutrosophic logic and sets such as computational modelling [4], artificial intelligence [5], data mining [6], decision making problems [7], practical achievements [8], and so forth. Florentin Smarandache and Mumtazi Ali investigated the neutrosophic triplet group and neutrosophic triplet as expansion of matter plasma, nonmatter plasma, and antimatter plasma [9,10]. By using the concept of neutrosophic theory Vasantha and Smarandache introduced neutrosophic algebraic systems and N-algebraic structures [11] and this was the first neutrosofication of algerbraic structures. The characterization of cancellable weak neutrosophic duplet semi-groups and cancellable NTG are investigated [12] in 2017. Florentin Smarandache and Mumtaz Ali examined the applications of the neutrosophic triplet field and neutrosophic triplet ring [13,14] in 2017. Şahin Mehmet and Abdullah Kargın developed the neutrosophic triplet normed space and neutrosophic triplet inner product [15,16]. The neutrosophic triplet G-module and fixed point theorem for NT partial metric space are given in literature [17,18]. Similarity measures of bipolar neutrosophic sets and single valued triangular neutrosophic numbers and their appliance to multi-attribute group decision making investigated in [19,20]. By utilizing distance-based similarity measures, refined neutrosophic hierchical clustering methods are achieved in [21]. Single valued neutrosophic sets to deal with pattern recognition problems are given with their application in [22]. Neutrosophic soft lattices and neutrosophic soft expert sets are analyzed in [23,24]. Centroid single valued neutrosophic numbers and their applications in MCDM is considered in [25]. Bal Mikail, Moges Mekonnen Shalla, and Necati Olgun reviewed neutrosophic triplet cosets and quotient groups [26] by using the concept of NET in 2018. The concepts concerning neutrosophic sets and neutrosophic modules are described in [27,28], respectively. A method to handle MCDM problems under the SVNSs are introduced in [29]. Bipolar neutrosophic soft expert set theory and its basic operations are defined in [30].

The other parts of a paper is coordinated thusly. Subsequently, through the literature analysis in the first section and preliminaries in the second section, we investigated neutro-monomorphism, neutro-epimorphism, neutro-isomorphism, and neutro-automorphism in Section 3 and a fundamental homomorphism theorem for NETG in Section 4. We give and prove the first neutro-isomorphism theorem for NETG in Section 5, and then the second neutro-isomorphism theorem for NETG is given in Section 6. Finally, results are given in Section 7.

2. Preliminaries

In this section, we provide basic definitions, notations and facts which are significant to develop the paper.

2.1. Neutrosophic Extended Triplet

Let U be a universe of discourse, and (N, $*$) a set included in it, endowed with a well-defined binary law $*$.

Definition 1 ([3]). *The set N is called a neutrosophic extended triplet set if for any $x \in N$ there exist $e^{neut(x)} \in N$ and $e^{anti(x)} \in N$. Thus, a neutrosophic extended triplet is an object of the form $(x, e^{neut(x)}, e^{anti(x)})$ where $e^{neut(x)}$ is extended neutral of x, which can be equal or different from the classical algebraic unitary element if any, such that*

$$x * e^{neut(x)} = e^{neut(x)} * x = x$$

and $e^{anti(x)} \in N$ is the extended opposite of x such that

$$x * e^{anti(x)} = e^{anti(x)} * x = e^{neut(x)}$$

In general, for each $x \in N$ there are many existing $e^{neut(x)'s}$ and $e^{anti(x)'s}$.

Theorem 1 ([11]). *Let (N, $*$) be a commutative NET with respect to $*$ and $a, b \in N$;*

(i) $neut(a) * neut(b) = neut(a * b)$;
(ii) $anti(a) * anti(b) = anti(a * b)$;

Theorem 2 ([11]). *Let $(N, *)$ be a commutative NET with respect to $*$ and $a \in N$;*

(i) $neut(a) * neut(a) = neut(a)$;
(ii) $anti(a) * neut(a) = neut(a) * anti(a) = anti(a)$

2.2. NETG

Definition 2 ([3]). *Let $(N, *)$ be a neutrosophic extended triplet set. Then $(N, *)$ is called a NETG, if the following classical axioms are satisfied.*

(a) *$(N, *)$ is well defined, i.e., for any $x, y \in N$ one has $x * y \in N$.*
(b) *$(N, *)$ is associative, i.e., for any $x, y, z \in N$ one has $x * (y * z) = (x * y) * z$.*

We consider, that the extended neutral elements replace the classical unitary element as well the extended opposite elements replace the inverse element of classical group. Therefore, NETGs are not a group in classical way. In the case when NETG enriches the structure of a classical group, since there may be elements with more extended opposites.

2.3. Neutrosophic Extended Triplet Subgroup

Definition 3 ([26]). *Given a NETG $(N, *)$, a neutrosophic triplet subset H is called a neutrosophic extended triplet subgroup of N if it itself forms a neutrosophic extended triplet group under $*$. Explicity this means*

(1) *The extended neutral element $e^{neut(x)}$ lies in H.*
(2) *For any $x, y \in H, x * y \in H$.*
(3) *If $x \in H$ then $e^{anti(x)} \in H$.*

In general, we can show $H \leq N$ as $x \in H$ and then $e^{anti(x)} \in H$, i.e $x * e^{anti(x)} = e^{neut(x)} \in H$.

Definition 4. *Suppose that N is NETG and $H_1, H_2 \leq N$.H_1 and H_2 are called neutrosophic triplet conjugates of N if $n \in N$ thereby $H_1 = nH_2(anti(n))$.*

2.4. Neutro-Homomorphism

Definition 5 ([26]). *Let $(N_1, *)$ and $(N_2, \circ)$ be two NETGs. A mapping $f : N_1 \to N_2$ is called a neutro-homomorphism if*

(a) *For any $x, y \in N$, we have*

$$f(x * y) = f(x) \circ f(y)$$

(b) *If $(x, neut(x), anti(x)$ is a neutrosophic extended triplet from N_1, then*

$$f(neut(x)) = neut(f(x))$$

and

$$f(anti(x)) = anti(f(x)).$$

Definition 6 ([26]). *Let f: $N_1 \rightarrow N_2$ be a neutro-homomorphism from a NETG (N_1, $*$) to a NETG (N_2, $\circ$). The neutrosophic triplet image of f is*

$$Im(f) = \{f(g) : g \in N_1, *\}.$$

Definition 7 ([26]). *Let f: $N_1 \rightarrow N_2$ be a neutro-homomorphism from a NETG (N_1, $*$) to a NETG (N_2, $\circ$) and B $\subseteq N_2$. Then*

$$f^{-1}(B) = \{x \in N_1 : f(x) \in B\}$$

is the neutrosophic triplet inverse image of B under f.

Definition 8 ([26]). *Let $f : N_1 \rightarrow N_2$ be a neutro-homomorphism from a NETG (N_1, $*$) to a NETG (N_2, $\circ$). The neutrosophic triplet kernel of f is a subset*

$$Ker(f) = \{x \in N_1 : f(x) = neut(x)\} \; of \; N_1,$$

where neut(x) denotes the neutral element of N_2.

Definition 9. *The neutrosophic triplet kernel of ϕ is called the neutrosophic triplet center of NETG N and it is denoted by Z(N). Explicitly,*

$$\begin{aligned} Z(N) &= \{a \in N : \varphi_a = neut_N\} \\ &= \{a \in N : ab(anti(a)) = b, \forall b \in N\} \\ &= \{a \in N : ab = ba, \forall b \in N\}. \end{aligned}$$

Hence Z(N) is the neutrosophic triplet set of elements in N that commute with all elements in N. Note that obviously Z(N) is a neutrosophic triplet. We have $Z(N) = N$ in the case that N is abelian.

Definition 10 ([26]). *Let N be a NETG and $H \subseteq N. \forall x \in N$, the set $xh/h \in H$ is called neutrosophic triplet coset denoted by xH. Analogously,*

$$Hx = hx/h \in H$$

and

$$(xH)anti(x) = (xh)anti(x)/h \in H.$$

When $h \leq N, xH$ is called the left neutrosophic triplet coset of H in N containing x, and Hx is called the right neutrosophic triplet coset of H in N containing x. $\mid xH \mid$ and $\mid Hx \mid$ are used to denote the number of elements in xH and Hx, respectively.

2.5. Neutrosophic Triplet Normal Subgroup and Quotient Group

Definition 11 ([26]). *A neutrosophic extended triplet subgroup H of a NETG of N is called a neutrosophic triplet normal subgroup of N if $aH(anti(a)) \subseteq H, \forall x \in N$ and we denote it as $H \trianglelefteq N$ and $H \triangleleft N$ if $H \neq N$.*

Example 1. *Let N be NETG. {neut} $\triangleleft$ N and N $\trianglelefteq$ N.*

Definition 12 ([26]). *If N is a NETG and $H \trianglelefteq N$ is a neutrosophic triplet normal subgroup, then the neutrosophic triplet quotient group N/H has elements $xH : x \in N$, the neutrosophic triplet cosets of H in N, and operation $(xH)(yH) = (xy)H$.*

3. Neutro-Monomorphism, Neutro-Epimorphism, Neutro-Isomorphism, Neutro-Automorphism

In this section, we define neutro-monomorphism, neutro-epimorphism, neutro-isomorphism, and neutro-automorphism. Then, we give and some important theorems related to them.

3.1. Neutro-Monomorphism

Definition 13. *Assume that* (N_1, $*$) *and* (N_2, $\circ$) *be two NETG's. If a mapping* $f : N_1 \to N_2$ *of NETG is only one to one (injective) f is called neutro-monomorphism.*

Theorem 3. *Let* (N_1, $*$) *and* (N_2, $\circ$) *be two NETG's.* $\varphi : N_1 \to N_2$ *is a neutro-monomorphism of NETG if and only if* $ker\varphi = \{neut_{N1}\}$.

Proof. Assume φ is injective. If $a \in \ker\varphi$, then

$$\varphi(a) = neut_{N2} = \varphi(neut_{N1}), \forall a \in N_1$$

and hence by injectivity $a = neut_{N1}$. Conversely, assume $\ker\varphi = \varphi(neut_{N1})$. Let $a,b \in N_1$ such that $\varphi(a) = \varphi(b)$. We need to show that $a = b$.

$$\begin{aligned} neut_H &= \varphi(b)anti(\varphi(a)) \\ &= \varphi(b)\varphi(anti(a)) \\ &= \varphi(b(anti(a))). \end{aligned}$$

Thus $b(anti(a))) \in ker\varphi$, and hence, by assumption $ker\varphi = \varphi(neut_{N1})$. We conclude that $b(anti(a))) = neut_{N1}, i.e., a = b$. □

Definition 14. *Let* (N_1, $*$) *and* (N_2, $\circ$) *be two NETG's. If a mapping* $f : N_1 \to N$ *is only onto (surjective) f is called neutro-epimorphism.*

Theorem 4. *Let N and H be two NETG's. If* $\varphi : N \to H$ *is a neutro-homomorphism of NETG, then so is* φ^{-1}: $H \to N$.

Proof. Let $x = \varphi(a), y = \varphi(b), \forall a,b \in N$ and $\forall x,y \in H$. So $a = anti(\varphi(x)), b = anti(\varphi(y))$. Now

$$\begin{aligned} anti(xy) &= \varphi(\varphi(a)\varphi(b)) \\ &= anti(\varphi(ab) = ab \\ &= anti(\varphi(x))anti(\varphi(y)). \end{aligned}$$

□

Theorem 5. *Let N be NETG and* $a,b \in N$. *The map* $\phi : N \to AutN$. *Then,* $a \to \varphi_a$, *is a neutro-homomorphism.*

Proof. For any fixed $n \in N$, we have

$$\begin{aligned} \varphi_{ab}(N) = abn(anti(ab)) &= abn(anti(a))anti(b) \\ &= \varphi_a(bn(anti(b)) = \varphi_a\varphi_b(n), \\ \text{So } \varphi_{ab} &= \varphi_a\varphi_b, i.e., \phi(ab) = \phi(a)\phi(b). \end{aligned}$$

It is in fact has anti-neutral element i.e., $\varphi(anti(n)) = anti(\varphi_n)$. Since $\varphi_n anti(\varphi_n(a)) = n(anti(n)an)anti(n) = a$, and so φ_n is injective. □

Theorem 6. *Let $f : N \to H$ be a neutro-homomorphism of NETG N and H. For $h \in H$ and $x \in f^{-1}(h), f^{-1}(h) = x \in kerf$.*

Proof. (1) Let's show that $f^{-1}(h) \subseteq x\ kerf$. If $x \in f^{-1}(h)$, then $f(x) = h$ and $b \in f^{-1}(h)$, then $f(b) = h$. If $f(x) = f(y)$, then:

$$anti(f(x))f(x) = anti(f(x))f(b)(\text{by theorem 1})$$

$$neut_H = f(anti(x))f(b)\ (\text{by definition 1})$$

$$\Rightarrow anti(x)b \in kerf.$$

For at least $k \in kerf, anti(x)b = k$. If $b = xk$, then,

$$b \in xkerf \Rightarrow f^{-1}(h) \subseteq xkerf \tag{1}$$

(2) Let's show that $xkerf \subseteq f^{-1}(h)$. Let $b \in xkerf$. For at least $k \in kerf, b = xk$

$$\Rightarrow f(b) = f(xk) = f(x)f(k) = h\ neut_H = h$$

If $f^{-1}(h) = b$ and $b \in f^{-1}(h)$, then

$$xkerf \subseteq f^{-1}(h) \tag{2}$$

by (1) and (2), we obtain $xkerf = f^{-1}(h)$.

□

Theorem 7. *Let $\varphi : N_1 \to N_2$ be a neutro-homomorphism of NETG N_1 and N_2.*

(1) $If\ H_2 \trianglelefteq N_2,\ then\ \varphi^{-1}(H_2) \trianglelefteq N_1$.
(2) $If H_1 \trianglelefteq N_1\ and\ \varphi\ is\ a\ neutro-epimorhism\ then\ \varphi(H_1) \trianglelefteq N_2$.

Proof. (1) If $x \in \varphi^{-1}(H_2)$ and $a \in N_1$, then $\varphi(x) \in H_2$ and so $\varphi((ax)(anti(a)) = \varphi(a)\varphi(x)anti(\varphi(a)) \in H_2$. Since H_2 is neutrosophic triplet normal subgroup. We conclude $ax(anti(a)) \in \varphi^{-1}(H_2)$.

(2) Since H_1 is neutrosophic triplet normal subgroup, we have $\varphi(a)\varphi(H_1)anti(\varphi(a)) \subseteq \varphi(H_1)$. Since we assume φ is surjective, every $b \in N_2$ can be written as $b = \varphi(a), a \in N_1$. Therefore, $b\varphi(H_1)anti(b) \in \varphi(H_1)$.

□

Theorem 8 ([26]). *Let $f : N \to H$ be a neutro-homomorphism from a NETG N to a NETG H. $Kerf \lhd N$.*

Theorem 9. *Let N be NETG and $H \trianglelefteq N$. The map $\varphi : N \to N/H,\ n \to nH,$ is a neutro-homomorphism with neutrosophic triplet kernel $ker\varphi = H$.*

Proof. We have $\varphi(ab) = (ab)H = (aH)(bH) = \varphi(a)\varphi(a)$, so ϕ is a neutro-homomorphism. As to the neutrosophic triplet kernel, $a \in ker\varphi \Leftrightarrow \varphi(a) = H$ (since H is neutral in N/H) $\Leftrightarrow aH = H$ (by definition of ϕ) $\Leftrightarrow a \in H$. □

Theorem 10. *Let N be NETG and $H \subseteq N$ be a non-empty neutrosophic extended triplet subset. Then $H \trianglelefteq N$, if and only if there exists a neutro-homomorphism $\varphi : N_1 \to N_2$ with $H = ker\varphi$.*

Proof. Its straight forward. □

3.2. Neutro-Isomorphism

Definition 15. *Let $(N_1, *)$ and $(N_2, \circ)$ be two NETGs. If a mapping $f : N_1 \to N_2$ neutro-homomorphism is one to one and onto f is called neutro-isomorphism. Here, N_1 and N_2 are called neutro-isomorphic and denoted as $N_1 \cong N_2$.*

Theorem 11. *Let $(N_1, *)$ and $(N_2, \circ)$ be two NETG's. If $f : N_1 \to N_2$ is a neutro-isomorphism of NETG's, then so is $f^{-1} : N_2 \to N_1$.*

Proof. It is obvious to show that f is one to one and onto. Now let's show that f is neutro-homomorphism. Let $x = \varphi(a), y = \varphi(b), \forall a, b \in N_1, \forall x, y \in N_2$ and so, $a = anti(\varphi(x)), b = anti(\varphi(y))$. Now $anti(xy) = anti(\varphi(\varphi(a)\varphi(b))) = anti(\varphi(\varphi(ab))) = ab = anti(\varphi(x))anti(\varphi(y))$. □

3.3. Neutro-Automorphism.

Definition 16. *Let $(N_1, *)$ and $(N_2, \circ)$ be two NETG'S. If a mapping $f : N_1 \to N_2$ is one to one and onto f is called neutro-automorphism.*

Definition 17. *Let N be NETG. $\varphi \in AutN$ is called a neutro-inner automorphism if there is a $n \in N$ such that $\varphi = \varphi_n$.*

Proposition 1. *Let N be a NETG. For $a \in N, f_a : N \to N$ such that $x \to ax(anti(a)$ is a neutro-automorphism $(AutN)$.*

Proof. (1) $\forall x, y \in N$, we have to show that

$$f(x) = f(y) \Rightarrow x = y.ax(anti(a)) = ay(anti(a)) \Rightarrow ax(anti(a))a = ay(anti(a))a \Rightarrow ax(neut(a)) = ay(neut(a)) \Rightarrow$$

Therefore, f is one to one.

(2) $\forall x, y \in N$, we have to show that

$$f(x) = ax(anti(a)) = y.ax(anti(a))a = ya \Rightarrow ax(neut(a)) = ya \Rightarrow ax = ya \Rightarrow anti(a)ax = anti(a)ya \Rightarrow neut(a)x =$$

So, f is onto. Therefore, f_a is a neutro-automorphism.
□

Lemma 1. *Let a be an element of NETG N such that* $a^2 = a$. *Then* $a = neut(a)$.

Proof. We have

$$= (anti(a) * a) * a \text{ for } anti(a) \in N \text{ (anti axiom)}$$
$$= anti(a) * a^2 \text{ (associativity axiom)}$$
$$= anti(a) * a \text{ (by assumption)}$$
$$= neut(a) \text{ (by definition of anti)}$$

□

Theorem 12. *Let* N *be NETG and* $H_1, H_2 \leq N$. *Then the neutrosophic extended triplet set* $H_1H_2 = \{ab : a \in H_1, b \in H_2\}$ *is a neutrosophic extended triplet subgroup in the case that* $H_1H_2 = H_2H_1$.

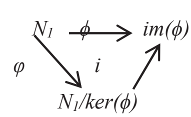
Proof. Suppose H_1H_2 is a neutrosophic extended triplet subgroup. Then, for all $a \in H_1, b \in H_2$, we have $anti(a)anti(b) \in H_1H_2$, *i.e.*, $H_2H_1 \subseteq H_1H_2$. But also for $h \in H_1H_2$ we find $a \in H_1, b \in H_2$ thereby $anti(h) = ab$, and then $h = anti(b)anti(a) \in H_2H_1$. So $H_1H_2 \subseteq H_2H_1$, that's, $H_1H_2 = H_2H_1$. On the other hand, assume that $H_1H_2 = H_2H_1$. Then $\forall a, a' \in H_1, b, b' \in H_2$ we have $aba'b' \in aH_2H_1b' = aH_1H_2b' = H_1H_2$. Furthermore, $\forall a \in H_1, b \in H_2$ we have $anti(ab) = anti(b)anti(a) \in H_2H_1 = H_1H_2$. □

4. Fundamental Theorem of Neutro-Homomorphism

The fundamental theorem of neutro-homomorphism relates the structure of two objects between which a neutrosophic kernel and image of the neutro-homomorphism is given. It is also significant to prove neutro-isomorphism theorems. In this section, we give and prove the fundamental theorem of neutro-homomorphism. Then, we discuss a few special cases. Finally, we give examples by using NETG.

Theorem 13. *Let* N_1, N_2 *be NETG's and* $\phi : N_1 \to N_2$ *be a neutro-homomorphism. Then,* $N_1/ker(\phi) \cong im(\phi)$. *Furthermore if* φ *is neutro-epimorphism, then*

$$N_1/ker\phi \cong N_2.$$



Proof. We will construct an explicit map $i : N_1/ker(\phi) \to im(\phi)$ and prove that it is a neutro-isomorphism and well defined. Since $ker(\varphi)$ is neutrosophic triplet normal subgroup of N_1. Let $K = ker(\phi)$, and recall that $N_1/K = \{aK : a \in N_1\}$. Define $i : N_1/K \to im(\phi), i : nK \to \phi(n), n \in N_1$. Thus, we need to check the following conditions.

(1) i is well defined
(2) i is injective
(3) i is surjective
(4) i is a neutro-homomorphism

(1) We must show that if $aK = bK$, then $i(aK) = (bK)$. Suppose $aK = bK$. We have $aK = bK \Rightarrow anti(b)aK = K \Rightarrow anti(b)a \in K$. Here, $neut_{(n2)} = \phi(anti(b)a) = \phi(anti(b)\phi(a) = anti(\phi(b))\phi(a) \Rightarrow \phi(a) = \phi(b)$. Hence, $i(aK) = \phi(a) = \phi(b) = i(bK)$. Therefore, it is well defined.

(2) We must show that $i(aK) = i(bK) \Rightarrow aK = bK$. Suppose that $i(aK) = i(bK)$. Then

$$i(aK) = i(bK) \Rightarrow aK = bK.$$

$$\Rightarrow \phi(anti(b))\ \phi(a) = neut_{(n2)} \Rightarrow \phi(anti(b)a) = neut_{(n2)} \Rightarrow anti(b)a \in K$$

$$\Rightarrow anti(b)aK = K\ (aN_2 = N_2 \Leftrightarrow a \in N_2).$$

Thus, i is injective.

(3) We must show that for any element in the domain (N_1/K) gets mapped to it by i. let's pick any element $\phi(a) \in im(\phi)$. By definition, $i(aK) = \phi(a)$, hence i is surjective.

(4) We must show that $i(aK\,bK) = i(aK)i(bK).i(aK\,bK) = i(abK)(aK\,bK = abK) = \phi(ab) = \phi(a)\phi(b) = i(aK\,bK) = i(aK)i(bK)$. Thus, i is a neutro-homomorphism.

In summary, since $i : N_1/K \to im(\phi)$ is a well-defined neutro-homomorphism that is injective and surjective. Therefore, it is a neutro-isomorphism. *i.e.*, $N_1/K \cong im(\phi)$, and the fundamental theorem of neutro-homomorphism is proven. □

Corollary 1 (A Few Special Cases of Fundamental Theorem of Neutro-homomorphism).

- *Let N = (1, 1, 1) be a trivial neutrosophic extended triplet. If φ: $N_1 \to N_2$ is an embedding, then neutrosophic ker(φ) = {neut(1) = $1N_1$}. The Theorem 12 says that im(φ) $\cong$ {$N_1/1N_1$} $\cong$ N_1.*
- *If φ: $N_1 \to N_2$ is a map $\varphi(n)$ = neut(1) = $1N_2$ for all $n_2 \in N_1$, then neutrosophic ker(φ) = N_1, so Theorem 13 says that $1N_2$ = im(φ) $\cong$ N_1/N_1.*

Example 2. *The neutrosophic extended triplet alternating group A_n (the neutrosophic extended triplet subgroup of even permutation in NETG S_n) has index 2 in S_n.*

Solution. *To prove that [S_n:A_n] = 2. We will construct a surjective neutro-homomorphism ϕ: $S_n \to Z_2$ with neutrosophic triplet $ker\phi = A_n$. Here the neutrosophic extended triplets of Z_2 are (0, 0, 0) and (1, 1, 1). If this is achieved, it would follow that $S_n / A_n \cong Z_2$, so $\mid S_n/A_n \mid = \mid Z_2 \mid = 2$, and therefore [$S_n$:$A_n$] = $\mid S_n/A_n \mid$ = 2, as desired. Define ϕ: $S_n \to Z_2$ by* $\phi(f) = \begin{cases} [0]\ if\ f\ is\ even \\ [1]\ if\ f\ is\ odd \end{cases}$

By construction ϕ is surjective. To prove that ϕ is a neutro-homomorphism we need to show that $\phi(x) + \phi(y) = \phi(xy)$, $\forall x, y \in S_n$. Here if x and y are both even or both odd, then xy is even. If x is even and y is odd, or if x is odd and y is even, then xy is odd. Let us see these four different cases as follows:

(1) x and y are both even. Then xy is also even. So, ϕ(x) = ϕ(y) = ϕ(xy) = [0]. Since [0] + [0] = [0] holds.

(2) x is even, and y is odd. Then xy is odd. So, ϕ(x) + ϕ(y) = [0] + [1] = [1] = ϕ(xy).

(3) x is odd, and y is even. This case is analogous to case 2.

(4) x and y are both odd. Then xy is even, so ϕ(x) + ϕ(y) = [1] + [1] = [0] = ϕ(xy). Thus, we verified that ϕ is a neutro-homomorphism. Finally, neutrosophic trplet kerϕ = $\{x \in S_n: \phi(x) = [0]_2\}$ is the neutrosophic extended triplet set of all even permutations, so neutrosophic triet $ker\varphi = A_n$.

5. First Neutro-Isomorphism Theorem

The first neutro-isomorphism theorem relates two neutrosophic triplet quotient groups involving products and intersections of neutrosophic extended triplet subgroups. In this section, we give and prove the first neutro-isomorphism theorem. Finally, we give an example by using NETG.

Theorem 14. *Let N be NETG and H, K be two neutrosophic extended triplet subgroup of N and H is a neutrosophic triplet normal in K. Then*

(a) *HK is neutrosophic triplet subgroup of N.*

(b) *$H \bigcap K$ is neutrosophic triplet normal subgroup in K.*

(c) $\frac{HK}{H} \cong \frac{\mathrm{K}}{H \bigcap K}$

Proof. (a) Let $xy \in HK$. If $x = h_1k_1$ and $y = h_2k_2, h_1h_2 \in H$ and $k_1, k_2 \in K$. Consider

$$\begin{aligned} x(anti(y)) = (h_1k_1) \quad & anti(h_2k_2) \\ &= (h_1k_1)anti(k_2)anti(h_2) \\ &= h_1(k_1(anti(k_2)))anti(h_2), (k_3 = k_1(anti(k2)) : k_3 \in K \\ &= h_1k_3(anti(h_2)) \\ &= h_1k_3(anti(h_2))anti(k_3)k_3 \\ &= h_1k_3(anti(h_2))anti(k_3)k_3 \\ &= h_1h_2k_3 \; because \; H \lhd kso \; h_3 = k_3(anti(h_2))anti(k_3) \in H \\ &\Rightarrow x(anti(y) = h_4k_3 \in HK, (h_4 = h_1h_2) \\ &\Rightarrow HK \text{ is NETG of } N. \end{aligned}$$

(b) We have to prove $H \cap K$ is neutrosophic triplet normal subgroup in k or $H \cap K \lhd k$. Let $x \in H \cap K$ and $x \in K$. If $x \in H$ and $x \in K$, then $kx(anti(k)) \in H$ because $H \lhd k$ and $kx(anti(k)) \in K$ because $xk \in K$. Thus, $kx(anti(k)) \in H \cap K$. Since $H \cap K \lhd k$.

(c) $\frac{HK}{H} \cong \frac{K}{H \bigcap K}$. Let $H \bigcap K = D$, so $\frac{K}{D} = \frac{K}{H \bigcap K}$. Now let's define a mapping φ: $HK \rightarrow \frac{K}{D}$ by $\phi(hk) = KD$.

1. φ is well defined

$$h_1k_1 = h_2k_2, h_1h_2 \in H \text{ and } k_1k_2 \in K$$

$$\begin{aligned} & k_1h_1' = k_2h_2' \\ & \Rightarrow anti(k_2)k_1h_1' = h_2' \\ & \Rightarrow anti(k_2)k_1 = h_2'(anti(h_1)), h_2'(anti(h_1)) \in H \\ & \Rightarrow anti(k_2)k_1 \in H, \text{ but } anti(k_2)k_1 \in K \\ & \Rightarrow anti(k_2)k_1 \in H \cap K = D \\ & \Rightarrow anti(k_2)k_1 \in D \\ & \Rightarrow anti(k_2)k_1D = D \\ & \Rightarrow k_1D = k_2D \\ & \Rightarrow \phi(h_1k_1) = \phi(h_2k_2). \end{aligned}$$

2. φ is neutro-homomorphism.

$$\begin{aligned} \Phi(h_1k_1.h_2k_2) &= \phi(h_1(k_1h_2)k_2 \\ &= \phi(h_1h2'k_1k_2) \\ &= K_1k_2D \\ &= k_1Dk_2D \\ &= \phi(h_1k_1).\phi(h_2k_2) \end{aligned}$$

3. φ is onto.

Since for every $KD \in K/D, \ni neut.k \in HK$ under φ such that $\phi(neut.k) = KD$. Hence, φ is onto. Now by Theorem 13,

$$HK/Ker\phi \cong K/D$$

Now it is enough to prove that $ker\phi = H$. Let $h \in H, h(neut) \in HK$. Thus

$$\begin{aligned} \phi(h) &= \phi(h.neut) = neut.D = D \\ &\Rightarrow \phi(h) = D \\ &\Rightarrow h \in ker\phi.i.eH \subseteq ker\phi \end{aligned}$$

Conversly, $hk \in ker\phi$, where $h \in H$ and $k \in K$. If $\phi(hk) = D$, then

$$\begin{aligned} KD = D \quad &\Rightarrow k \in D = H \cap K \\ &\Rightarrow h \in H \text{ and } k \in K \\ &\Rightarrow hk \subseteq H \\ &\Rightarrow ker\phi \subseteq H. \text{ Thus } H = ker\phi \end{aligned}$$

by (1) $\frac{HK}{H} \cong \frac{K}{H \bigcap K}$. □

Example 3. *Let N be NETG. Neutro-isomorphism theorems are for instance useful in the calculation of NETG orders, since neutro-isomorphic groups have the same order. If $H \leq N$ and $K \trianglelefteq N$ so that HK is finite, then Lagrange's theorem [26] in neutrosophic triplet with theorem 13 yield*

$$\begin{aligned} |HK|/|K| &= |HK:K| \\ &= |HK/K| \\ &= |H/H \bigcap K| \\ &= |H:H \bigcap K| \\ &= |H|/|H \bigcap K|, \text{ that is} \\ |HK| &= |H|\,|K|/|H \bigcap K| \end{aligned}$$

6. Second Neutro-Isomorphism Theorem

The second neutro- isomorphism theorem is extremely useful in analyzing the neutrosophic extended normal subgroups of a neutrosophic triplet quotient group. In this section, we give and prove the second neutro-homomorphism theorem for NETG.

Theorem 15. *Let N be a NETG. Let H and K be neutrosophic triplet normal subgroup of N with $K \subseteq H$. Then $H/K \lhd N/K$ and $N/KH/K \cong N/H$*

Proof. Consider the natural map *Ψ:N→N/H*. The neutrosophic triplet kernel, *H* contains *K*. Thus, by the universal property of N/K, it follows that there is a neutro-homomorphism $N/K \to N/H$. This map is clearly surjective. In fact, it sends the neutrosophic triplet left coset nK to the neutrosophic triplet left coset nH. Now suppose that nK is in the neutrosophic triplet kernel. Then the neutrosophic triplet left coset nH is the neutral neutrosophic triplet coset, that is, $nH = H$, so that $n \in H$. Thus the neutrosophic triplet kernel consists of those neutrosophic triplet left cosets of the form nK, for $n \in H$, that is, H/K.

1. Ψ is well defined. Let $ak = bk$.

$$anti(b)ak = k$$
$$anti(b)a \in k$$
$$\Rightarrow K \lhd H$$
$$anti(b)a \in H$$
$$aH = bH(anti(b)aH = H)$$
$$\Psi(ak) = \Psi(bk)$$

2. Ψ is neutro-homomorphism

$$a_k, b_k \in N/K$$
$$\Psi(a_k b_k) = \Psi(abk) = abH = aHbH = \Psi(ak)\Psi(bk).$$

3. Ψ is onto
   For all $y = aH \in N/H, x = ak \in N/K \Rightarrow \Psi(x) = y$.
4. kerΨ = H/K
   The neutral element of N/H is H. Therefore

$$ker\ \Psi\colon \{xk \in N/K : \Psi(xk) = H\}$$
$$= \{xk \in N/K : \Psi(xk) = xH = H\}$$
$$= \{xk \in N/K : x \in H\}$$
$$= \{xk \in H/K\}$$
$$= H/K.$$

By Theorem 13 $N/KH/K \cong N/H$.

□

7. Conclusions

This paper is mainly focused on fundamental homomorphism theorems for neutrosophic extended triplet groups. We gave and proved the fundamental theorem of neutro-homomorphism, as well as first and second neutro-isomorphism theorems explained for NETG. Furthermore, we define neutro-monomorphism, neutro-epimorphism, neutro-automorphism, inner neutro-automorphism, and center for neutrosophic extended triplets. Finally, by applying them to neutrosophic algebraic structures, we have examined how closely different systems are related. By using the concept of a fundamental theorem of neutro-homomorphism and neutro-isomorphism theorems, the relation between neutrosophic algebraic structures (neutrosophic triplet ring, neutrosophic triplet field, neutrosophic triplet vector space, neutrosophic triplet normed space, neutrosophic modules, etc.) can be studied and the field of study in neutrosophic algebraic structures will be extended.

Author Contributions: M.Ç. defined and studied preliminaries and the fundamental theorem of neutro-homomorphism theorem with examples and wrote part of the paper. M.M.S. defined and studied first and second neutro-isomorphism theorems with example. The revision and submission of the paper was completed by N.O.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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Different Forms of Triangular Neutrosophic Numbers, De-Neutrosophication Techniques, and their Applications

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Received: 10 June 2018; Accepted: 3 July 2018; Published: 7 August 2018

Abstract: In this paper, we introduce the concept of neutrosophic number from different viewpoints. We define different types of linear and non-linear generalized triangular neutrosophic numbers which are very important for uncertainty theory. We introduced the de-neutrosophication concept for neutrosophic number for triangular neutrosophic numbers. This concept helps us to convert a neutrosophic number into a crisp number. The concepts are followed by two application, namely in imprecise project evaluation review technique and route selection problem.

Keywords: linear and non-linear neutrosophic number; de-neutrosophication methods

1. Introduction

1.1. Theory of Uncertainty and Uncertainty Quantification

Uncertainty theory playsanimportant role in modeling sciences and engineering problems. However, there is a basic question regarding how we can define or use the uncertainty concept in our mathematical modeling. Researchers around the globe defined many approaches to defining them, and give their various recommendations to using uncertainty theory. There are several literaturestudiesthatclassify some basic uncertain parameters. It should be noted that there is no unique reorientation of the uncertain parameter. For the problem's purpose or decision makers' choice, it can be varied and presented as a different application. We now, here, give some info about uncertain parameters, and show how they differ from eachother using the concept of uncertainty using some definition, flowcharts, and diagrams. In this paper, we recommend the researcher to take the uncertain parameter as a parametric interval valued neutrosophic number.

Some basic differences between some uncertain parameters:

If we take Interval number [1] then we can see,

1. The information belongs to a certain interval
2. There is no concept of membership function

If we take Fuzzy number [2,3], then we can see,

1. The concept of belongingness of the elements comes
2. The use of membership function is present

If we take Intuitionistic fuzzy number [4], then we can see,

1. The concept of belongingness and non-belongingness of the elementscomes
2. The use of membership and non-membership function is present

If we take Neutrosophic fuzzy number [5], then we can see,

1. The concept of truthiness, falsity, and indeterminacy of the elements comes
2. The use of membership function for truthiness, falsity, and indeterminacy is present

Please follow the idea given in the flowchart below, as shown in Figure 1:

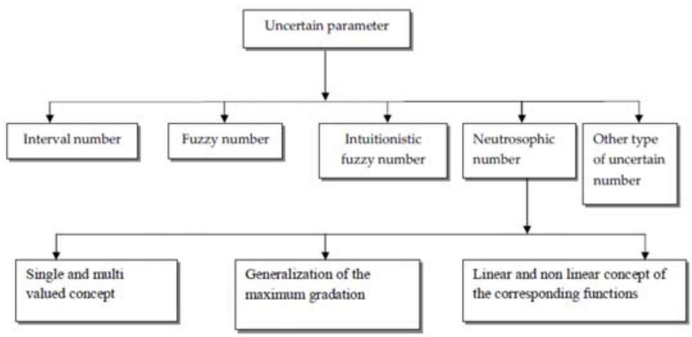


Figure 1. Flowchart for different uncertain parameter.

1.2. Neutrosophic Number

Fuzzy systems (FSs) and Intuitionistic fuzzy systems (IFSs) cannot successfully deal with a situation where the conclusion is adequate, unacceptable, and decision-maker declaration is uncertain. Therefore, some novel theories are mandatory for solving the problem with uncertainty. The neutrosophic sets (NSs) [5] reflect on the truth membership, indeterminacy membership, and falsity membership concurrently, which is more practical and adequate than FSs and IFSs in commerce, which areuncertain, incomplete, and inconsistent in sequence. Single-valued neutrosophic sets are an extension of NSs which were introduced by Wang et al. [6]. Ye [7] introduced simplify neutrosophic sets, and Peng et al. [8,9] definite their novel operations and aggregation operators. Finally, there are different extensions of NSs, such as interval neutrosophic set [10], bipolar neutrosophic sets [11], and multi-valued neutrosophic sets [12,13]. The decision-making problem [14–38] is very important in study, when it is with uncertainty.

Although many researchers and scientists have worked in the recently developed neutrosophic method, and applied it in the field of decision making, there is, however, still some viewpoints regarding defining neutrosophic numbers in different forms, and their corresponding de-impreciseness is very important.

1.3. Ranking and De-Impreciseness

The ranking and de-impreciseness of the imprecise numbers are not a new concept.However, what is the basic concept of the above-said important results and what is the relation. Figure 2 shows the flowchart for de-impreciseness and ranking.

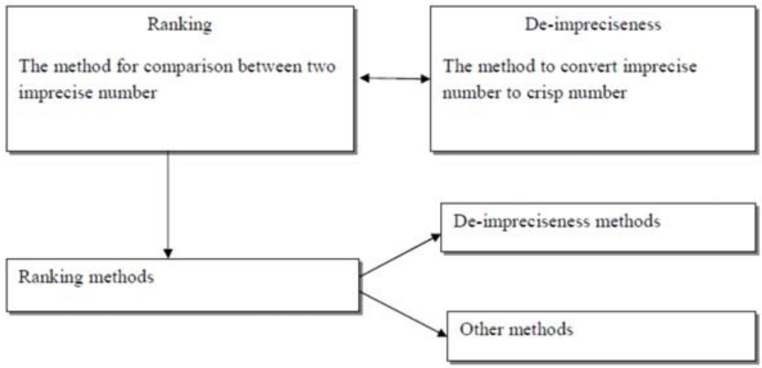


Figure 2. Flowchart for de-impreciseness and ranking.

Ranking is a concept where we can compare two imprecise numbers, and de-impreciseness is a technique where the imprecise number is converted to a crisp number. Somewhere, the decision maker takes the two concepts as the same. In this case, they convert the imprecise number into crisp number, and compares them on the basis of crisp value.

1.4. Structure of the Paper

The paper is organized as follows. In Section 1, the basic concept on imprecise set theory and neutrosophic set theory are discussed. Section 2 contains the preliminaries section. Section 3 goes for the known definition of neutrosophic sets and numbers. Single valued linear neutrosophic number and its variation are showing in Section 3. In Section 4, we address the basic concept of neutrosophic non-linear number and generalized neutrosophic number. In Section 5, the de-neutrosophication of linear neutrosophic triangular fuzzy number is performed. The PERT problem is considered in Section 6. The application in assignment problem, considering aproblem, is taken in Section 7. The conclusions are written in Section 8.

2. Neutrosophic Number

Definition 1. *(Neutrosophic Set) A set $\widetilde{S_{neu}}$ in the universal discourse X, which is denoted generically by x, is said to be a neutrosophic set if* $\widetilde{S_{neu}} = \{\langle x; [\pi_{\widetilde{S_{neu}}}(x), \mu_{\widetilde{S_{neu}}}(x), \vartheta_{\widetilde{S_{neu}}}(x)]\rangle \vdots x \in X\}$, *where* $\pi_{\widetilde{S_{neu}}}(x) : X \to [0,1]$ *is called the truth membership function which represents the degree of confidence,* $\mu_{\widetilde{S_{neu}}}(x) : X \to [0,1]$ *is called the indeterminacy membership function which represents the degree of uncertainty, and* $\vartheta_{\widetilde{S_{neu}}}(x) : X \to [0,1]$ *is called the falsity membership function which represents the degree of scepticism on the decision given the decision maker.*

$\pi_{\widetilde{S_{neu}}}(x), \mu_{\widetilde{S_{neu}}}(x) \& \vartheta_{\widetilde{S_{neu}}}(x)$ exhibits the following relation:

$$0 \leq \pi_{\widetilde{S_{neu}}}(x) + \mu_{\widetilde{S_{neu}}}(x) + \vartheta_{\widetilde{S_{neu}}}(x) \leq 3$$

Definition 2. *(Single Valued Neutrosophic Set) Neutrosophic set $\widetilde{S_{neu}}$ in the definition 2.3, is called a Single Valued Neutrosophic Set $(\widetilde{S_{neu}})$ if x is a single valued independent variable. Thus* $\widetilde{S_{neu}} = \{\langle x; [\pi_{\widetilde{S_{neu}}}(x), \mu_{\widetilde{S_{neu}}}(x), \vartheta_{\widetilde{S_{neu}}}(x)]\rangle \vdots x \in X\}$, *where* $\pi_{\widetilde{S_{neu}}}(x), \mu_{\widetilde{S_{neu}}}(x) \& \vartheta_{\widetilde{S_{neu}}}(x)$ *represents the truth, indeterminacy, and falsity membership function, respectively, as stated in definition 2.3, and also exhibits the same relationship as stated earlier.*

If there exists three points, $a_0, b_0 \& c_0$, for which $\pi_{\widetilde{S_{neu}}}(a_0) = 1, \mu_{\widetilde{S_{neu}}}(b_0) = 1 \ \& \vartheta_{\widetilde{S_{neu}}}(c_0) = 1$, then the $\widetilde{S_{neu}}$ is called neut-normal.

A $\widetilde{S_{neu}}$ is said to be neut-convex, which implies that it is a subset of a real line, by satisfying the following conditions:

1. $\pi_{\widetilde{S_{neu}}}\langle \rho a_1 + (1-\rho)a_2 \rangle \geq min\langle \pi_{\widetilde{S_{neu}}}(a_1), \pi_{\widetilde{S_{neu}}}(a_2) \rangle$
2. $\mu_{\widetilde{S_{neu}}}\langle \rho a_1 + (1-\rho)a_2 \rangle \leq max\langle \mu_{\widetilde{S_{neu}}}(a_1), \mu_{\widetilde{S_{neu}}}(a_2) \rangle$
3. $\vartheta_{\widetilde{S_{neu}}}\langle \rho a_1 + (1-\rho)a_2 \rangle \leq max\langle \vartheta_{\widetilde{S_{neu}}}(a_1), \vartheta_{\widetilde{S_{neu}}}(a_2) \rangle$

where, $a_1 \& a_2 \in \mathbb{R}$ and $\rho \in [0,1]$.

Definition 3. *(Single Valued Neutrosophic Number)Single Valued Neutrosophic Number* $(\widetilde{z})$ *is defined as* $\widetilde{z} = \langle [(p^1, q^1, r^1, s^1); \alpha], [(p^2, q^2, r^2, s^2); \beta], [(p^3, q^3, r^3, s^3); \gamma] \rangle$ *where* $\alpha, \beta, \gamma \in [0,1]$*, the truth membership function* $(\pi_{\widetilde{z}}) : \mathbb{R} \to [0, \alpha]$*, the indeterminacy membership function* $(\mu_{\widetilde{z}}) : \mathbb{R} \to [\beta, 1]$*, and the falsity membership function* $(\vartheta_{\widetilde{z}}) : \mathbb{R} \to [\gamma, 1]$ *is given as:*

$$\pi_{\widetilde{z}}(x) = \begin{cases} \pi_{\widetilde{zl}}(x) & p^1 \leq x \leq q^1 \\ \alpha & q^1 \leq x \leq r^1 \\ \pi_{\widetilde{zu}}(x) & r^1 \leq x \leq s^1 \\ 0 & \text{otherwise} \end{cases},$$

$$\mu_{\widetilde{z}}(x) = \begin{cases} \mu_{\widetilde{zl}}(x) & p^2 \leq x \leq q^2 \\ \beta & q^2 \leq x \leq r^2 \\ \mu_{\widetilde{zu}}(x) & r^2 \leq x \leq s^2 \\ 1 & \text{otherwise} \end{cases}$$

$$\vartheta_{\widetilde{z}}(x) = \begin{cases} \vartheta_{\widetilde{zl}}(x) & p^3 \leq x \leq q^3 \\ \gamma & q^3 \leq x \leq r^3 \\ \vartheta_{\widetilde{zu}}(x) & r^3 \leq x \leq s^3 \\ 1 & \text{otherwise} \end{cases}$$

3. Single Valued Linear Neutrosophic Number

1. Triangular Single Valued Neutrosophic number of Type 1: The quantity of the truth, indeterminacy and falsity are not dependent: A Triangular Single Valued Neutrosophic number of Type 1 is defined as $\widetilde{\mathrm{A}}_{\mathrm{Neu}} = (\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3; \mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3; \mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3)$ whose truth membership, indeterminacy and falsity membership is defined as follows:

$$T_{\widetilde{A}_{Neu}}(x) = \begin{cases} \frac{x-p_1}{p_2-p_1} & \text{when } p_1 \leq x < p_2 \\ 1 & \text{when } x = p_2 \\ \frac{p_3-x}{p_3-p_2} & \text{when } p_2 < x \leq p_3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_{\widetilde{A}_{Neu}}(x) = \begin{cases} \frac{x-p_1}{p_2-p_1} & \text{when } p_1 \leq x < p_2 \\ 1 & \text{when } x = p_2 \\ \frac{p_3-x}{p_3-p_2} & \text{when } p_2 < x \leq p_3 \\ 0 & \text{otherwise} \end{cases}$$

$$I_{\widetilde{A}_{Neu}}(x) = \begin{cases} \frac{q_2-x}{q_2-q_1} & \text{when } q_1 \leq x < q_2 \\ 0 & \text{when } x = q_2 \\ \frac{x-q_2}{q_3-q_2} & \text{when } q_2 < x \leq q_3 \\ 1 & \text{otherwise} \end{cases}$$

and

$$T_{\widetilde{A}_{Neu}}(x) = \begin{cases} \frac{x-p_1}{p_2-p_1} & \text{when } p_1 \le x < p_2 \\ 1 & \text{when } x = p_2 \\ \frac{p_3-x}{p_3-p_2} & \text{when } p_2 < x \le p_3 \\ 0 & \text{otherwise} \end{cases}$$

where, $0 \le T_{\widetilde{A}_{Neu}}(x) + I_{\widetilde{A}_{Neu}}(x) + F_{\widetilde{A}_{Neu}}(x) \le 3, x \in \widetilde{A}_{Neu}$.

The parametric form of the above type number is $(\widetilde{A}_{Neu})_{\alpha,\beta,\gamma} = [T_{Neu1}(\alpha), T_{Neu2}(\alpha); I_{Neu1}(\beta), I_{Neu2}(\beta); F_{Neu1}(\gamma), F_{Neu2}(\gamma)]$,
where,

$$T_{Neu1}(\alpha) = p_1 + \alpha(p_2 - p_1)$$
$$T_{Neu2}(\alpha) = p_3 - \alpha(p_3 - p_2)$$
$$I_{Neu1}(\beta) = q_2 - \beta(q_2 - q_1)$$
$$I_{Neu2}(\beta) = q_2 + \beta(q_3 - q_2)$$
$$F_{Neu1}(\gamma) = r_2 - \gamma(r_2 - r_1)$$
$$F_{Neu2}(\gamma) = r_2 + \gamma(r_3 - r_2)$$

here, $0 < \alpha \le 1, 0 < \beta \le 1, 0 < \gamma \le 1$ and $0 < \alpha + \beta + \gamma \le 3$

Example 1. *Take* $\widetilde{A}_{Ne} = (10, 15, 20; 14, 16, 22; 12, 15, 19)$.

The parametric representation is

$$T_{Ne1}(\alpha) = 10 + 5\alpha$$
$$T_{Ne2}(\alpha) = 20 - 5\alpha$$
$$I_{Ne1}(\beta) = 16 - 2\beta$$
$$I_{Ne1}(\beta) = 16 + 6\beta$$
$$F_{Ne1}(\gamma) = 15 - 3\gamma$$
$$F_{Ne2}(\gamma) = 15 + 4\gamma$$

Table 1 and Figure 3 show the value of $T_{Ne1}(\alpha)$, $T_{Ne2}(\alpha)$, $I_{Ne1}(\beta)$, $I_{Ne1}(\beta)$, $F_{Ne1}(\gamma)$, and $F_{Ne2}(\gamma)$ and graphical representation of triangular single valued neutrosophic numbers (TrSVNNs) respectively.

Table 1. Value of $T_{Ne1}(\alpha)$, $T_{Ne2}(\alpha)$, $I_{Ne1}(\beta)$, $I_{Ne1}(\beta)$, $F_{Ne1}(\gamma)$, and $F_{Ne2}(\gamma)$.

α,β,γ	$T_{Ne1}(\alpha)$	$T_{Ne2}(\alpha)$	$I_{Ne1}(\beta)$	$I_{Ne1}(\beta)$	$F_{Ne1}(\gamma)$	$F_{Ne2}(\gamma)$
0	10	20	16	16	15	15
0.1	10	19.5	15.8	16.6	14.7	15.4
0.2	11	19	15.6	17.2	14.4	15.8
0.3	11.5	18.5	15.4	17.8	14.1	16.2
0.4	12	18	15.2	18.4	13.8	16.6
0.5	12.5	17.5	15	19	13.5	17
0.6	13	17	14.8	19.6	13.2	17.4
0.7	13.5	16.5	14.6	20.2	12.9	17.8
0.8	14	16	14.4	20.8	12.6	18.2
0.9	14.5	15.5	14.2	21.4	12.3	18.6
1	15	15	14	22	12	19

2. Triangular Single Valued Neutrosophic Number of Type 2: The quantity of indeterminacy and falsity are dependent: A triangular single valued neutrosophic number (TrSVNN) of Type 2 is defined as $\widetilde{A}_{Neu} = (p_1, p_2, p_3; q_1, q_2, q_3; u_{Neu}, y_{Neu})$ whose truth membership, indeterminacy, and falsity membership are defined as follows:

$$T_{\widetilde{A}_{Neu}}(x) = \begin{cases} \frac{x-p_1}{p_2-p_1} & \text{when } p_1 \le x < p_2 \\ 1 & \text{when } x = p_2 \\ \frac{p_3-x}{p_3-p_2} & \text{when } p_2 < x \le p_3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$I_{\widetilde{A}_{Neu}}(x) = \begin{cases} \frac{q_2-x+u_{Neu}(x-q_1)}{q_2-q_1} & \text{when } q_1 \le x < q_2 \\ u_{Neu} & \text{when } x = q_2 \\ \frac{x-q_2+u_{Neu}(q_3-x)}{q_3-q_2} & \text{when } q_2 < x \le q_3 \\ 1 & \text{otherwise} \end{cases}$$

and

$$F_{\widetilde{A}_{Neu}}(x) = \begin{cases} \frac{q_2-x+y_{Neu}(x-q_1)}{q_2-q_1} & \text{when } q_1 \le x < q_2 \\ y_{Neu} & \text{when } x = q_2 \\ \frac{x-q_2+y_{Neu}(q_3-x)}{q_3-q_2} & \text{when } q_2 < x \le q_3 \\ 1 & \text{otherwise} \end{cases}$$

where, $0 \le T_{\widetilde{A}_{Neu}}(x) + I_{\widetilde{A}_{Neu}}(x) + F_{\widetilde{A}_{Neu}}(x) \le 2$, $x \in \widetilde{A}_{Neu}$.

The parametric form of the above type number is $(\widetilde{A}_{Neu})_{\alpha,\beta,\gamma} = [T_{Neu1}(\alpha), T_{Neu2}(\alpha); I_{Neu1}(\beta), I_{Neu2}(\beta); F_{Neu1}(\gamma), F_{Neu2}(\gamma)]$, where

$$\begin{aligned} T_{Neu1}(\alpha) &= p_1 + \alpha(p_2 - p_1) \\ T_{Neu2}(\alpha) &= p_3 - \alpha(p_3 - p_2) \\ I_{Neu1}(\beta) &= \frac{q_2 - u_{Neu}q_1 - \beta(q_2-q_1)}{1-u_{Neu}} \\ I_{Neu2}(\beta) &= \frac{q_2 - u_{Neu}q_3 + \beta(q_3-q_2)}{1-u_{Neu}} \\ F_{Neu1}(\gamma) &= \frac{q_2 - y_{Neu}q_1 - \gamma(q_2-q_1)}{1-y_{Neu}} \\ F_{Neu2}(\gamma) &= \frac{q_2 - y_{Neu}q_3 + \gamma(q_3-q_2)}{1-y_{Neu}}. \end{aligned}$$

Here, $0 < \alpha \le 1$, $u_{Neu} < \beta \le 1$, $y_{Neu} < \gamma \le 1$ and $0 < \beta + \gamma \le 1$ and $0 < \alpha + \beta + \gamma \le 2$.

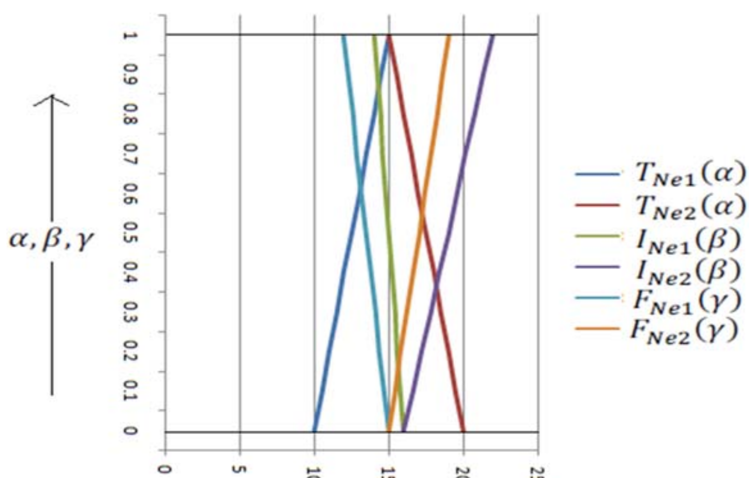


Figure 3. Graphical representation of TrSVNNs.

Example 2. *Take* $\widetilde{A}_{Ne} = (10, 15, 20; 14, 16, 22; 0.4, 0.5)$

The parametric representation is,

$$T_{Ne1}(\alpha) = 10 + 5\alpha$$
$$T_{Ne2}(\alpha) = 20 - 5\alpha$$
$$I_{Ne1}(\beta) = \frac{1}{3}(52 - 10\beta)$$
$$I_{Ne1}(\beta) = 12 + 10\beta$$
$$F_{Ne1}(\gamma) = 18 - 4\gamma$$
$$F_{Ne2}(\gamma) = 10 + 12\gamma.$$

Table 2 and Figure 4 show the value of $T_{Ne1}(\alpha)$, $T_{Ne2}(\alpha)$, $I_{Ne1}(\beta)$, $I_{Ne1}(\beta)$, $F_{Ne1}(\gamma)$, and $F_{Ne2}(\gamma)$ and graphical representation of type-2 TrSVNNs.

Table 2. Value of $T_{Ne1}(\alpha)$, $T_{Ne2}(\alpha)$, $I_{Ne1}(\beta)$, $I_{Ne1}(\beta)$, $F_{Ne1}(\gamma)$, and $F_{Ne2}(\gamma)$.

α,β,γ	$T_{Ne1}(\alpha)$	$T_{Ne2}(\alpha)$	$I_{Ne1}(\beta)$	$I_{Ne1}(\beta)$	$F_{Ne1}(\gamma)$	$F_{Ne2}(\gamma)$
0	10	20	–	–	–	–
0.1	10.5	19.5	–	–	–	–
0.2	11	19	–	–	–	–
0.3	11.5	18.5	–	–	–	–
0.4	12	18	16	16	–	–
0.5	12.5	17.5	15.6667	17	16	16
0.6	13	17	15.3333	18	15.6	17.2
0.7	13.5	16.5	15.	19	15.2	18.4
0.8	14	16	14.6667	20	14.8	19.6
0.9	14.5	15.5	14.3333	21	14.4	20.8
1	15	15	14	22	14	22

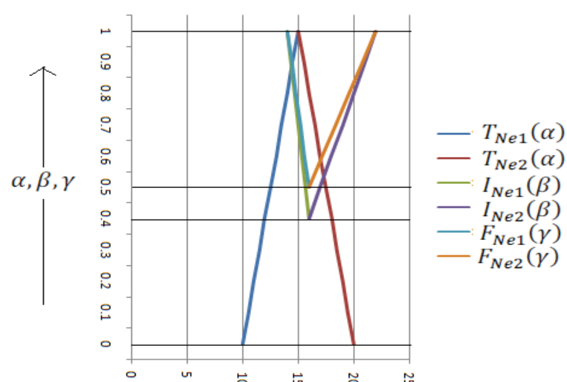


Figure 4. Graphical representation of type-2 TrSVNNs.

3. Triangular Single Valued Neutrosophic number of Type 3: The quantity of the truth, indeterminacy, and falsity are dependent: A TrSVNN of Type 3 is defined as $\widetilde{A}_{Neu} = (p_1, p_2, p_3; w_{Ne}, u_{Neu}, y_{Neu})$, whose truth membership, indeterminacy, and falsity membership are defined as follows:

$$T_{\widetilde{A}_{Neu}}(x) = \begin{cases} w_{Neu}\frac{x-p_1}{p_2-p_1} \text{ when } p_1 \le x < p_2 \\ w_{Neu} \text{ when } x = p_2 \\ w_{Neu}\frac{p_3-x}{p_3-p_2} \quad \text{when } p_2 < x \le p_3 \\ 0 \text{ otherwise} \end{cases}$$

and

$$I_{\widetilde{A}_{Neu}}(x) = \begin{cases} \frac{p_2-x+u_{Neu}(x-p_1)}{p_2-p_1} \text{ when } p_1 \le x < p_2 \\ u_{Neu} \text{ when } x = p_2 \\ \frac{x-p_2+u_{Neu}(p_3-x)}{p_3-p_2} \quad \text{when } p_2 < x \le p_3 \\ 1 \text{ otherwise} \end{cases}$$

and

$$I_{\widetilde{A}_{Neu}}(x) = \begin{cases} \frac{p_2-x+u_{Neu}(x-p_1)}{p_2-p_1} \text{when } p_1 \le x < p_2 \\ u_{Neu} \text{ when } x = p_2 \\ \frac{x-p_2+u_{Neu}(p_3-x)}{p_3-p_2} \quad \text{when } p_2 < x \le p_3 \\ 1 \text{ otherwise} \end{cases}$$

where, $0 \le T_{\widetilde{A}_{Neu}}(x) + I_{\widetilde{A}_{Neu}}(x) + F_{\widetilde{A}_{Neu}}(x) \le 1$, $x \in \widetilde{A}_{Neu}$.

The parametric form of the above type number is $(\widetilde{A}_{Neu})_{\alpha,\beta,\gamma} = [T_{Neu1}(\alpha), T_{Neu2}(\alpha); I_{Neu1}(\beta), I_{Neu2}(\beta); F_{Neu1}(\gamma), F_{Neu2}(\gamma)]$, where

$$\begin{aligned} T_{Neu1}(\alpha) &= p_1 + \tfrac{\alpha}{w_{Neu}}(p_2 - p_1) \\ T_{Neu2}(\alpha) &= p_3 - \tfrac{\alpha}{w_{Neu}}(p_3 - p_2) \\ I_{Neu1}(\beta) &= \tfrac{p_2 - u_{Neu}p_1 - \beta(p_2-p_1)}{1-u_{Neu}} \\ I_{Neu2}(\beta) &= \tfrac{p_2 - u_{Neu}p_3 + \beta(p_3-p_2)}{1-u_{Neu}} \\ F_{Neu1}(\gamma) &= \tfrac{p_2 - y_{Neu}p_1 - \gamma(p_2-p_1)}{1-y_{Neu}} \\ F_{Neu2}(\gamma) &= \tfrac{p_2 - y_{Neu}p_3 + \gamma(p_3-p_2)}{1-y_{Neu}}. \end{aligned}$$

Here, $0 < \alpha \le w_{Neu}$, $u_{Neu} < \beta \le 1$, $y_{Neu} < \gamma \le 1$, and $0 < \alpha + \beta + \gamma \le 1$.

Example 3. *Take* $\widetilde{A}_{Ne} = (14, 16, 22; 0.5, 0.8, 0.7)$

The parametric representation is,

$$\begin{aligned} T_{Ne1}(\alpha) &= 14 + 4\alpha \\ T_{Ne2}(\alpha) &= 22 - 12\alpha \\ I_{Ne1}(\beta) &= 16 - \tfrac{5}{2}\beta \\ I_{Ne1}(\beta) &= 16 + \tfrac{15}{2}\beta \\ F_{Ne1}(\gamma) &= 16 - \tfrac{20}{7}\gamma \\ F_{Ne2}(\gamma) &= 16 + \tfrac{60}{7}\gamma. \end{aligned}$$

Table 3 and Figure 5 show the value of $T_{Ne1}(\alpha)$, $T_{Ne2}(\alpha)$, $I_{Ne1}(\beta)$, $I_{Ne1}(\beta)$, $F_{Ne1}(\gamma)$ and $F_{Ne2}(\gamma)$. and Graphical representation of type-3 TrSVNNs

Table 3. Value of $T_{Ne1}(\alpha)$, $T_{Ne2}(\alpha)$, $I_{Ne1}(\beta)$, $I_{Ne1}(\beta)$, $F_{Ne1}(\gamma)$ and $F_{Ne2}(\gamma)$.

α,β,γ	$T_{Ne1}(\alpha)$	$T_{Ne2}(\alpha)$	$I_{Ne1}(\beta)$	$I_{Ne1}(\beta)$	$F_{Ne1}(\gamma)$	$F_{Ne2}(\gamma)$
0	14	22				
0.1	14.4	20.8				
0.2	14.8	19.6				
0.3	15.2	18.4				
0.4	15.6	17.2				
0.5	16	16				
0.6						
0.7					16	16
0.8			16	16	16.2857	15.1429
0.9			15.75	16.75	16.5714	14.2857
1			15.5	17.5	16.8571	13.4286

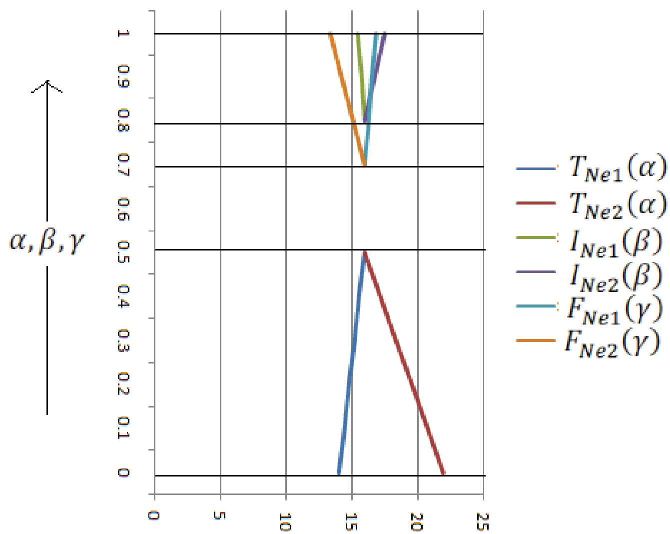


Figure 5. Graphical representation of type-3 TrSVNNs.

Different Operational Laws of Two Triangular Neutrosophic Numbers: If $\widetilde{A}_{Neu}$ and $\widetilde{B}_{Neu}$ are two single valued neutrosophic numbers with nine components having truthmembership $T_{\widetilde{A}_{Neu}}\&T_{\widetilde{B}_{Neu}}$, indeterminacymembership $I_{\widetilde{A}_{Neu}}\&I_{\widetilde{B}_{Neu}}$, and falsitymembership $F_{\widetilde{A}_{Neu}}\&F_{\widetilde{B}_{Neu}}$, respectively, such as:

$$\widetilde{A}_{Neu} = \langle a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3\rangle \text{ and } \widetilde{B}_{Neu} = \langle a_4, a_5, a_6; b_4, b_5, b_6; c_4, c_5, c_6\rangle$$

where a, band c are the scores given by the decision maker in the scale, ranging from lower limit L_l to upper limit U_l.

- Addition

$$\widetilde{C}_{Neu} = \widetilde{A}_{Neu} + \widetilde{B}_{Neu}$$
$$= \left\langle \begin{array}{c} \{min(a_1+a_4, U_l), min(a_2+a_5, U_l), min(a_3 + raphical\ representation\ of\ type\ 3\ TrSVNNsa_6, U_l)\}; \\ \{min(b_1+b_4, U_l), min(b_2+b_5, U_l), min(b_3+b_6, U_l)\}; \{min(c_1+c_4, U_l), min(c_2+c_5, U_l), min(c_3+c_6, U_l)\} \end{array} \right\rangle$$

- Negative of SVNNs

$$\begin{aligned}
&\widetilde{S}_{Neu} = -\widetilde{A}_{Neu} \\
&= \langle -a_3, -a_2, -a_1; -b_3, -b_2, -b_1; -c_3, -c_2, -c_1 \rangle
\end{aligned}$$

- Subtraction

$$\begin{aligned}
&\widetilde{D}_{Neu} = \widetilde{A}_{Neu} - \widetilde{B}_{Neu} \\
&= \widetilde{A}_{Neu} + (-\widetilde{B}_{Neu}) \\
&\qquad = \left\langle \begin{array}{c} \{max(a_1 - a_6, L_l), max(a_2 - a_5, L_l), max(a_3 - a_4, L_l)\}; \\ \{max(b_1 - b_6, L_l), max(b_2 - b_5, L_l), max(b_3 - b_4, L_l)\}; \\ \{max(c_1 - c_6, L_l), max(c_2 - c_5, L_l), max(c_3 - c_4, L_l)\} \end{array} \right\rangle
\end{aligned}$$

- Multiplications

$$\begin{aligned}
\widetilde{D}_{Neu} &= \widetilde{A}_{Neu} - \widetilde{B}_{Neu} \\
&= \widetilde{A}_{Neu} + (-\widetilde{B}_{Neu}) \\
= &\left\langle \begin{array}{c} \{max(a_1 - a_6, L_l), max(a_2 - a_5, L_l), max(a_3 - a_4, L_l)\}; \\ \{max(b_1 - b_6, L_l), max(b_2 - b_5, L_l), max(b_3 - b_4, L_l)\}; \\ \{max(c_1 - c_6, L_l), max(c_2 - c_5, L_l), max(c_3 - c_4, L_l)\} \end{array} \right\rangle
\end{aligned}$$

- Multiplication by a constant

$$\begin{aligned}
\widetilde{E}_{Neu} &= k\left[\widetilde{A}_{Neu}\right] \\
&= k \times \langle a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3 \rangle \\
&= \langle ka_1, ka_2, ka_3; kb_1, kb_2, kb_3; kc_1, kc_2, kc_3 \rangle
\end{aligned}$$

- Inverse of SVNNs

$$\begin{aligned}
\widetilde{F}_{Neu} &= \widetilde{A}_{Neu}^{-1} = \frac{1}{\langle a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3 \rangle} \\
&= \langle \tfrac{1}{a_3}, \tfrac{1}{a_2}, \tfrac{1}{a_1}; \tfrac{1}{b_3}, \tfrac{1}{b_2}, \tfrac{1}{b_1}; \tfrac{1}{c_3}, \tfrac{1}{c_2}, \tfrac{1}{c_1} \rangle \; for \; (a, b, c) > 0 \\
&= \langle \tfrac{1}{a_1}, \tfrac{1}{a_2}, \tfrac{1}{a_3}; \tfrac{1}{b_1}, \tfrac{1}{b_2}, \tfrac{1}{b_3}; \tfrac{1}{c_1}, \tfrac{1}{c_2}, \tfrac{1}{c_3} \rangle \; for \; (a, b, c) < 0
\end{aligned}$$

- Divisions

$$\begin{aligned}
&\widetilde{G}_{Neu} \widetilde{A}_{Neu} \div \widetilde{B}_{Neu} \\
&= \widetilde{A}_{Neu} \div \widetilde{B}_{Neu} \\
&= \langle a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3 \rangle \times \langle \tfrac{1}{a_6}, \tfrac{1}{a_5}, \tfrac{1}{a_4}; \tfrac{1}{b_6}, \tfrac{1}{b_5}, \tfrac{1}{b_4}; \tfrac{1}{c_6}, \tfrac{1}{c_5}, \tfrac{1}{c_4} \rangle \\
&= \left\langle \begin{array}{c}
\left\{ \begin{array}{c} min(\frac{a_1}{a_4}, \frac{a_1}{a_5}, \frac{a_1}{a_6}, \frac{a_2}{a_4}, \frac{a_2}{a_5}, \frac{a_2}{a_6} = vision\ of\ SVNNs, \frac{a_3}{a_4}, \frac{a_3}{a_5}, \frac{a_3}{a_6}), \\ mean(\frac{a_1}{a_4}, \frac{a_1}{a_5}, \frac{a_1}{a_6}, \frac{a_2}{a_4}, \frac{a_2}{a_5}, \frac{a_2}{a_6} = vision\ of\ SVNNs, \frac{a_3}{a_4}, \frac{a_3}{a_5}, \frac{a_3}{a_6}), \\ \max(\frac{a_1}{a_4}, \frac{a_1}{a_5}, \frac{a_1}{a_6}, \frac{a_2}{a_4}, \frac{a_2}{a_5}, \frac{a_2}{a_6} = vision\ of\ SVNNs, \frac{a_3}{a_4}, \frac{a_3}{a_5}, \frac{a_3}{a_6}) \end{array} \right\}; \\
\left\{ \begin{array}{c} min(\frac{b_1}{b_4}, \frac{b_1}{b_5}, \frac{b_1}{b_6}, \frac{b_2}{b_4}, \frac{b_2}{b_5}, \frac{b_2}{b_6}, \frac{b_3}{b_4}, \frac{b_3}{b_5}, \frac{b_3}{b_6}), \\ mean(\frac{b_1}{b_4}, \frac{b_1}{b_5}, \frac{b_1}{b_6}, \frac{b_2}{b_4}, \frac{b_2}{b_5}, \frac{b_2}{b_6}, \frac{b_3}{b_4}, \frac{b_3}{b_5}, \frac{b_3}{b_6}), \\ \max(\frac{b_1}{b_4}, \frac{b_1}{b_5}, \frac{b_1}{b_6}, \frac{b_2}{b_4}, \frac{b_2}{b_5}, \frac{b_2}{b_6}, \frac{b_3}{b_4}, \frac{b_3}{b_5}, \frac{b_3}{b_6}), \end{array} \right\}; \\
\left\{ \begin{array}{c} min(\frac{c_1}{c_4}, \frac{c_1}{c_5}, \frac{c_1}{c_6}, \frac{c_2}{c_4}, \frac{c_2}{c_5}, \frac{c_2}{c_6}, \frac{c_3}{c_4}, \frac{c_3}{c_5}, \frac{c_3}{c_6}), \\ mean(\frac{c_1}{c_4}, \frac{c_1}{c_5}, \frac{c_1}{c_6}, \frac{c_2}{c_4}, \frac{c_2}{c_5}, \frac{c_2}{c_6}, \frac{c_3}{c_4}, \frac{c_3}{c_5}, \frac{c_3}{c_6}), \\ max(\frac{c_1}{c_4}, \frac{c_1}{c_5}, \frac{c_1}{c_6}, \frac{c_2}{c_4}, \frac{c_2}{c_5}, \frac{c_2}{c_6}, \frac{c_3}{c_4}, \frac{c_3}{c_5}, \frac{c_3}{c_6}), \end{array} \right\}
\end{array} \right\rangle
\end{aligned}$$

Example 4. *If* $\widetilde{A}_{Neu} = \langle 5, 10, 15; 2.5, 5, 7.5; 10, 17.5, 25\rangle$ *and* $\widetilde{B}_{Neu} = \langle 4, 6, 8; 3, 6, 9; 1, 1.75, 2.5\rangle$ *are two single valued neutrosophic numbers with independent truth, indeterminate, and false values in the scale of 0 to 25, then find the* $\widetilde{A}_{Neu} + \widetilde{B}_{Neu}$, $\widetilde{A}_{Neu} - \widetilde{B}_{Neu}$, $\widetilde{A}_{Neu} \times \widetilde{B}_{Neu}$, $\frac{\widetilde{A}_{Neu}}{\widetilde{B}_{Neu}}$ *and* $k\widetilde{B}_{Neu}$ *where k = 3.*

- Addition

$$\widetilde{A}_{Neu} + \widetilde{B}_{Neu} = \langle 9, 16, 23; 5.5, 11, 16.5; 11, 19.25, 25\rangle,$$

- Subtraction

$$\widetilde{A}_{Neu} - \widetilde{B}_{Neu} = \langle 0, 4, 11; 0, 0, 4.5; 7.5, 15.75, 24\rangle$$

- Multiplication

$$\widetilde{A}_{Neu} \times \widetilde{B}_{Neu} = \langle 20, 60, 120; 7.5, 30, 67.5; 10, 30.625, 62.5\rangle$$

- Division

$$\frac{\widetilde{A}_{Neu}}{\widetilde{B}_{Neu}} = \langle 0.625, 1.806, 3.75; 0.278, 1.0185, 2.5; 4, 11.5, 25\rangle,$$

- Multiplication by a constant

$$k\widetilde{B}_{Neu} = \langle 12, 18, 24; 9, 18, 27; 3, 5.25, 7.5\rangle$$

4. Neutrosophic Non-Linear Number and Generalized Neutrosophic Number

4.1. Single Valued Non-Linear Triangular Neutrosophic Number with Nine Components

A single valued non-linear triangular neutrosophic number with nine components is defined as $\widetilde{A}_{Neu} = (p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3)$, whose truth membership, indeterminacy, and falsity membership is defined as:

$$T_{\widetilde{A}_{Neu}}(x) = \begin{cases} \left(\frac{x-p_1}{p_2-p_1}\right)^{a_1} & \text{when } p_1 \leq x < p_2 \\ 1 & \text{when } x = p_2 \\ \left(\frac{p_3-x}{p_3-p_2}\right)^{a_2} & \text{when } p_2 < x \leq p_3 \\ 0 & otherwise \end{cases}$$

and

$$I_{\widetilde{A}_{Neu}}(x) = \begin{cases} \left(\frac{x-q_1}{q_2-q_1}\right)^{b_1} & when\ q_1 \leq x < q_2 \\ 0 & when\ x = q_2 \\ \left(\frac{x-q_3}{q_3-q_2}\right)^{b_2} & when\ q_2 < x \leq q_3 \\ 1 & otherwise \end{cases}$$

and

$$F_{\widetilde{A}_{Neu}}(x) = \begin{cases} \left(\frac{x-r_1}{r_2-r_1}\right)^{c_1} & when\ r_1 \leq x < r_2 \\ 0 & when\ x = r_2 \\ \left(\frac{x-r_3}{r_3-r_2}\right)^{c_2} & when\ r_2 < x \leq r_3 \\ 1 & otherwise \end{cases}$$

where, $0 \leq T_{\widetilde{A}_{Neu}}(x) + I_{\widetilde{A}_{Neu}}(x) + F_{\widetilde{A}_{Neu}}(x) \leq 3$, $x \in \widetilde{A}_{Neu}$.

Note. If $a_1, a_2, b_1, b_2, c_{1,}c_2 = 1$, then single valued non-linear triangular neutrosophic number with nine components will be converted into single valued linear triangular neutrosophic number with nine components.

4.2. Single Valued Generalized Triangular Neutrosophic Number with Nine Components

A single valued triangular neutrosophic number with nine components is defined as $\widetilde{A}_{Ne} = (p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3)$, whose truth membership, indeterminacy, and falsity membership is defined as:

$$T_{\widetilde{A}_{Neu}}(x) = \begin{cases} \omega\frac{x-p_1}{p_2-p_1} \text{ when } p_1 \leq x < p_2 \\ \omega \quad \text{when } x = p_2 \\ \omega\frac{p_3-x}{p_3-p_2} \quad \text{when } p_2 < x \leq p_3 \\ 0 \quad otherwise \end{cases}$$

and

$$I_{\widetilde{A}_{Neu}}(x) = \begin{cases} \rho\frac{x-q_1}{q_2-q_1} \text{ when } q_1 \leq x < q_2 \\ 0 \quad \text{when } x = q_2 \\ \rho\frac{x-q_3}{q_3-q_2} \quad \text{when } q_2 < x \leq q_3 \\ \rho \quad otherwise \end{cases}$$

and

$$F_{\widetilde{A}_{Neu}}(x) = \begin{cases} \lambda\frac{x-r_1}{r_2-r_1} \text{ when } r_1 \leq x < r_2 \\ 0 \quad \text{when } x = r_2 \\ \lambda\frac{x-r_3}{r_3-r_2} \quad \text{when } r_2 < x \leq r_3 \\ \lambda \quad otherwise \end{cases}$$

where, $0 \leq T_{\widetilde{A}_{Neu}}(x) + I_{\widetilde{A}_{Neu}}(x) + F_{\widetilde{A}_{Neu}}(x) \leq 3, x \in \widetilde{A}_{Neu}$.

4.3. Single Valued Generalized Non-Linear Triangular Neutrosophic Number with Nine Components

A single valued non-linear triangular neutrosophic number with nine components is defined as $\widetilde{A}_{Neu} = (p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3)$, whose truth membership, indeterminacy, and falsity membership is defined as:

$$T_{\widetilde{A}_{Neu}}(x) = \begin{cases} \omega\left(\frac{x-p_1}{p_2-p_1}\right)^{a_1} \text{ when } p_1 \leq x < p_2 \\ \omega \quad \text{when } x = p_2 \\ \omega\left(\frac{p_3-x}{p_3-p_2}\right)^{a_2} \quad \text{when } p_2 < x \leq p_3 \\ 0 \quad otherwise \end{cases}$$

and

$$I_{\widetilde{A}_{Neu}}(x) = \begin{cases} \rho\left(\frac{x-q_1}{q_2-q_1}\right)^{b_1} \text{ when } q_1 \leq x < q_2 \\ 0 \quad \text{when } x = q_2 \\ \rho\left(\frac{x-q_3}{q_3-q_2}\right)^{b_2} \quad \text{when } q_2 < x \leq q_3 \\ \rho \quad otherwise \end{cases}$$

and

$$F_{\widetilde{A}_{Neu}}(x) = \begin{cases} \lambda\left(\frac{x-r_1}{r_2-r_1}\right)^{c_1} when\ r_1 \leq x < r_2 \\ 0\ when\ x = r_2 \\ \lambda\left(\frac{x-r_3}{r_3-r_2}\right)^{c_2} \quad when\ r_2 < x \leq r_3 \\ \lambda \quad otherwise \end{cases}$$

where, $0 \leq T_{\widetilde{A}_{Neu}}(x) + I_{\widetilde{A}_{Neu}}(x) + F_{\widetilde{A}_{Neu}}(x) \leq 3, x \in \widetilde{A}_{Neu}$.

Note. if $a_1, a_2, b_1, b_2, c_{1,}c_2 = 1$, then single valued generalized non-linear triangular neutrosophic number with nine components will be converted into single valued generalized linear triangular neutrosophic number with nine components.

5. De-Neutrosophication of Linear Neutrosophic Triangular Fuzzy Number

De-Neutrosophication Using Removal Area Method

Let us consider a linear neutrosophic triangular fuzzy number as follows:

$$\widetilde{A}_{Ne} = (a, b, c; d, e, f; g, h, k)$$

whose pictorial representation is as follows.

Firstly, we consider the graphical representation of linear neutrosophic triangular fuzzy number in Figure 6.

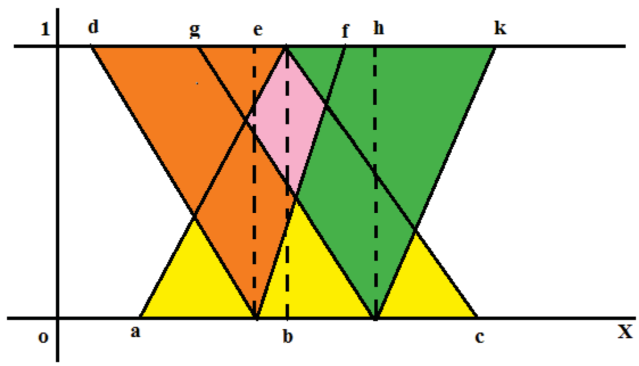


Figure 6. Linear neutrosophic number.

We consider an ordinary number $k \in R$ and a fuzzy number $\check{A}$ for the lower triangle, then left side removal of $\check{A}$ with respect to k is $R_l(\check{A}, k)$, defined as the area bounded by k and the left side of the fuzzy number $\check{A}$. Similarly, the right side removal of $\check{A}$ with respect to k is $R_r(\check{A}, k)$. Also consider an ordinary number $k \in R$ and a fuzzy number $\check{B}$ for the left most upper triangle(Δdef), then the left side removal of $\check{B}$ with respect to k is $R_l(\check{B}, k)$, defined as the area bounded by k and the left side of the fuzzy number $\check{B}$. Similarly, the right side removal of $\check{B}$ with respect to k is $R_r(\check{B}, k)$. A fuzzy number $\check{C}$ for the right most upper triangle(Δghk), then left side removal of $\check{C}$ with respect to k is $R_l(\check{C}, k)$, defined as the area bounded by k and the left side of the fuzzy number $\check{C}$. Similarly, the right side removal of $\check{C}$ with respect to k is $R_r(\check{C}, k)$.

Mean is defined as $(\check{A}, k) = \frac{R_l(\check{A},k)+R_r(\check{A},k)}{2}$, $R(\check{B}, k) = \frac{R_l(\check{B},k)+R_r(\check{B},k)}{2}$, $R(\check{C}, k) = \frac{R_l(\check{C},k)+R_r(\check{C},k)}{2}$.

Then, we defined the defuzzification of a linear neutrosophic triangular fuzzy as $R(\check{D}, k) = \frac{R(\check{A},k)+R(\check{B},k)+R(\check{C},k)}{3}$.

For $k = 0$,

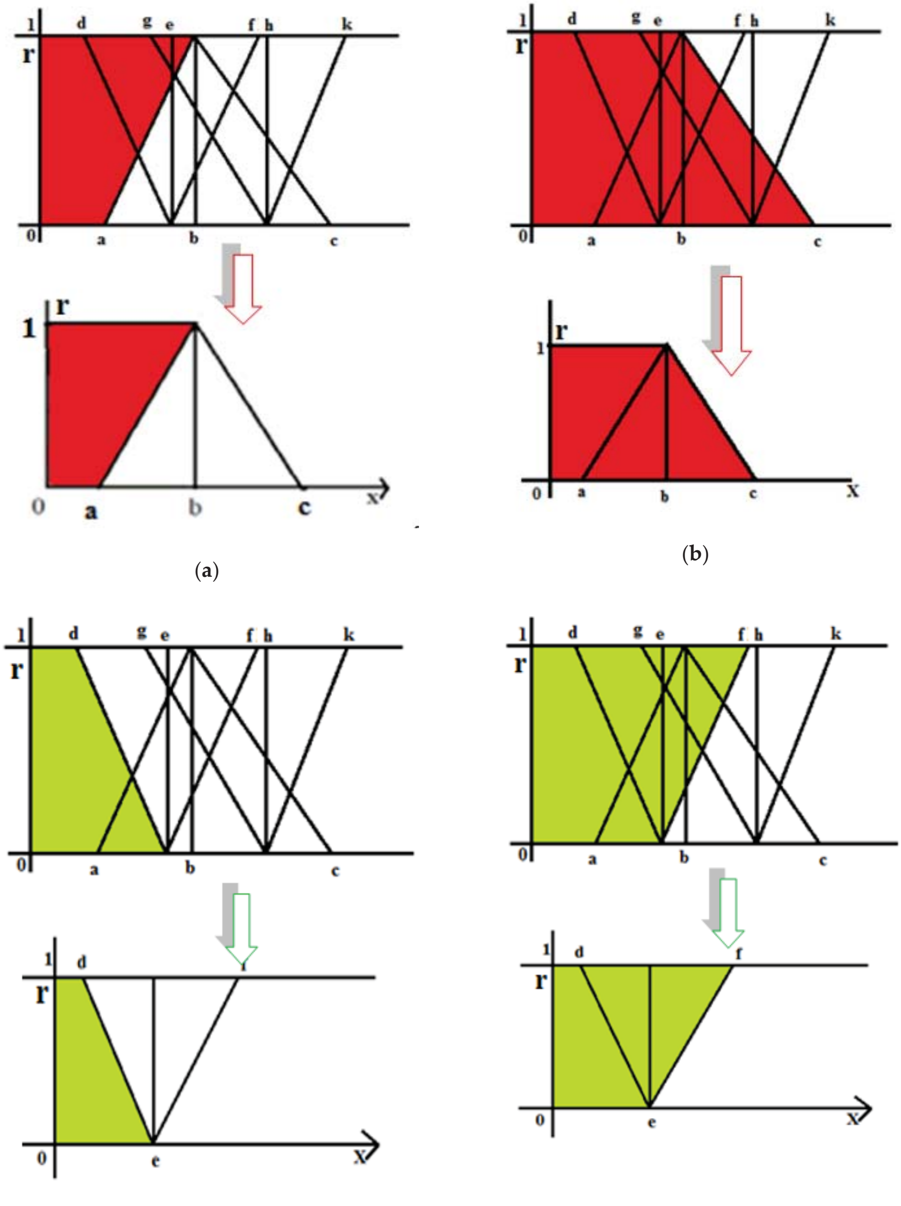
$$R(\check{A}, 0) = \frac{R_l(\check{A},0)+R_r(\check{A},0)}{2}$$
$$R(\check{B}, 0) = \frac{R_l(\check{B},0)+R_r(\check{B},0)}{2}$$
$$R(\check{C}, 0) = \frac{R_l(\check{C},0)+R_r(\check{C},0)}{2}$$

Then,

$$R(\check{D}, 0) = \frac{R(\check{A}, 0) + R(\check{B}, 0) + R(\check{C}, 0)}{3}$$

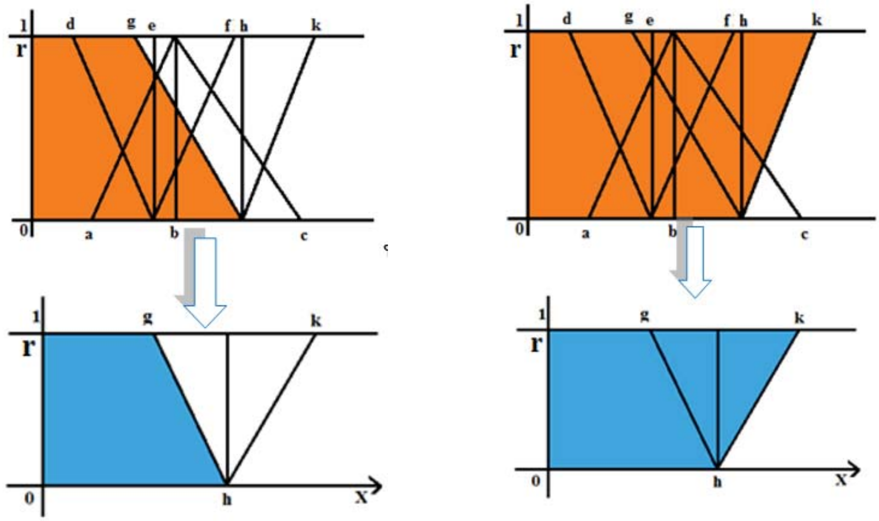
We take $\check{A} = (a, b, c)$, $\check{B} = (d, e, f)$, $\check{C} = (g, h, k)$.

Figure 7 shows the pictorial representation of de-neutrosophication.



(**a**) (**b**) (**c**) (**d**)

Figure 7. *Cont.*



(**e**) (**f**)

Figure 7. Pictorial representation of de-neutrosophication. (**a**) Area of trapezium OABR; (**b**) Area of trapezium OABR; (**c**) Area of trapezium OEDR;(**d**) Area of trapezium OEFR; (**e**) Area of trapezium OHGR: (**f**) Area of trapezium OHKR.

Then,

$$R_l(\check{A},0) = \text{Area of trapezium OABR} = \frac{(a+b)}{2}.1$$
$$R_l(\check{A},0) = \text{Area of trapezium OABR} = \frac{(a+b)}{2}.1$$
$$R_l(\check{B},0) = \text{Area of trapezium OABR} = \frac{(d+e)}{2}.1$$
$$R_r(\check{B},0) = \text{Area of trapezium OABR} = \frac{(e+f)}{2}.1$$
$$R_l(\check{C},0) = \text{Area of trapezium OABR} = \frac{(g+h)}{2}.1$$
$$R_r(C,0) = \text{Area of trapezium OABR} = \frac{(k+h)}{2}.1.$$

Hence, $(\check{A},0) = \frac{(a+2b+c)}{4}, R(\check{B},0) = \frac{(d+2e+f)}{4}, R(\check{C},0) = \frac{(g+2h+k)}{4}$.
So, $R(\check{D},0) = \frac{(a+2b+c+d+2e+f+g+2h+k)}{12}$.

Example 5. *Finding De-neutrosophication value of Neutrosophic number.*

Table 4 shows the de-neutrosophication value of Neutrosophic number.

Table 4. De-neutrosophication value of Neutrosophic number.

Experiment No.	Neutrosophic Number	De-Neutrosophication Value
Set 1	$\check{A} = (\mathbf{1}, \mathbf{2}, \mathbf{3}; \mathbf{0.5}, \mathbf{1.5}, \mathbf{2.5}; \mathbf{1.2}, \mathbf{2.7}, \mathbf{3.5})$	2.0083
Set 2	$\check{\boldsymbol{B}} = (\mathbf{0.5}, \mathbf{1.5}, \mathbf{2.5}; \mathbf{0.3}, \mathbf{1.3}, \mathbf{2.2}; \mathbf{0.7}, \mathbf{1.7}, \mathbf{2.2})$	1.45
Set 3	$\check{C} = (\mathbf{0.3}, \mathbf{1.2}, \mathbf{2.8}; \mathbf{0.5}, \mathbf{1.5}, \mathbf{2.5}; \mathbf{0.8}, \mathbf{1.7}, \mathbf{2.7})$	1.533
Set 4	$\check{D} = (\mathbf{1}, \mathbf{3}, \mathbf{5}; \mathbf{0.5}, \mathbf{1.5}, \mathbf{2.5}; \mathbf{1.2}, \mathbf{2.7}, \mathbf{4.5})$	2.425

6. PERT in Triangular Neutrosophic Environment and the Proposed Model

The full form of PERT method is project evaluation and review technique, which is a project management tool used to schedule, organize, and coordinate tasks within a project. It is basically a method to analyze the tasks involved in completing a given project, especially the time needed to complete each task, and to identify the minimum time needed to complete the total project.

PERT planning involves the following steps:

1. Identify the specific activities and milestones.
2. Determine the proper sequence of the activities.
3. Construct a network diagram.
4. Estimate the time required for each activity.
5. Determine the critical path.
6. Update the PERT chart as the project progresses.

The main objective of PERT is to facilitate decision making and to reduce both the time and cost required to complete a project. PERT is intended for very large-scale, one-time, non-routine, complex projects with a high degree of dependency, projects which require a series of activities, some of which must be performed sequentially, and others that can be performed in parallel with other activities. PERT has been mainly used in new projects which have large uncertainty with respect to design of a structure, technology, and networking system. To take care of associated uncertainties, we introduced triangular neutrosophic environment for PERT activity duration.

The three time estimates for activity duration are as follows:

Optimistic time ($\check{o}$): Generally, the shortest time in which the activity can be completed. It is common practice to specify optimistic time to be three standards deviations from the mean so that there is approximately a 1% chance that the activity will be completed within the optimistic time.

Pessimistic time ($\check{p}$): Generally, the longest time that an activity might require. Three standard deviations from the mean are commonly used for the pessimistic time.

Most likely time ($\check{m}$): Generally, it is the completion time, in normal circumstances, having the highest probability. Note that this time is different from the expected time.

Note 2. In Ref. [22], the authors introduced the concept of score and accuracy function to compute the crisp value of a trapezoidal neutrosophic number. In our proposed model, we choose all the three different times (optimistic, pessimistic, most likely) as triangular neutrosophic number.

To obtain the crisp value, we introduced the de-neutrosophication value $R(\check{D},0) = \frac{(a+2b+c+d+2e+f+g+2h+k)}{12}$ of triangular neutrosophic number $(a,b,c;d,e,f;g,h,k)$.

Now, the expected time and standard deviation can be calculated by the formula $E_{jk} = \frac{o+4m+p}{6}$ and $\sigma_{jk} = \frac{p-o}{6}$, where o, p, and m are all crisp value of optimistic, pessimistic, and most likely time estimations, respectively.

Now, we use CPM method for further calculation of earliest/latest time, critical path, and float.

In forward pass, starting with a time of zero for the first event, the computation proceeds from left to right, up to the final event. For any activity (i,j), let ES_i denote the earliest time of event i, then $ES_j = ES_i + t_{ij}$. If more than one activity enters an event, the earliest start time for that event is computed as $ES_j = \max\{ES_i + t_{ij}\}$ for all activities emanating from node i entering into j.

In case of backward pass, starting with the final node, the computation proceeds from right to left, up to the initial event. For any activity (i,j), let LF_i denote the latest finished time of event i, then $LF_i = LF_j - t_{ij}$. If more than one activity enters an event, the latest finish time for that event is computed as $LF_i = \min\{LF_j - t_{ij}\}$ for all activities emanating from node j entering into i.

After calculating the critical path, compute project length variance, which is the sum of the variances of all the critical activities. Next, calculate the standard normal variable $Z = \frac{T_s - T_e}{\sigma}$, where T_s is the scheduled time to complete the project, and T_e is the normal expected project length duration.

Using a normal curve, we can estimate the probability of completing the project within a specified time. The steps of the said method are shown in Figure 8. We also set the numerical value for the said problem to show the importance of our method in Table 5.

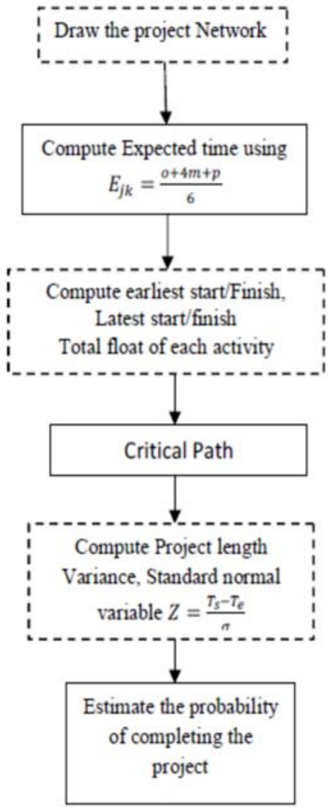


Figure 8. Flowchart for the solution procedure.

Table 5. Set of Neutrosophic value of the above problem.

	Description	Predecessors	Optimistic Time	Pessimistic Time	Most Likely Time
A	Selection of Officer and Force Member	–	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,5,9;1.5,4.5,6.5;4,7,10)	(1.5,3.5,5.5;1,2,3;3,4.5,6)
B	Selection of Site and do Site Survey	–	(**1, 5, 8; 1, 3, 6; 4, 7, 9**)	(1,2,3;0.5,1.5,2.5;1.5,2.5,3.5)	(1,5,8;1.5,3,6.5;4,7,9)
C	Selection of Arms	A	(1,4,7;1,3,5;3.5,6,7.5)	(1,1.5,4;0.5,1,2.5;1.25,3,4.25)	(1,5,9;1.5,4.5,6.5;4,7,10)
D	Final Plan and Blueprint	B	(1,3,5;0.5,2.5,3.5;2.5,4,6)	(1.5,3.5,5.5;1,2,3;3,4.5,6)	(0.5,2.5,4.5;1,2,3;1.5,3.5,5.5)
E	Bring Utilities to the Site	B	(0.5,2.5,4.5;0.5,1.5,3.5;2,4,6)	(1,5,9;1.5,4.5,6.5;4,7.5,10.5)	(1.5,2.5,3.5;1,1.5,3;2,3,4)
F	Interview	A	(2,4,6;1.5,2.5,3.5;3,5,7)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)	(1,4,7;1,3,5;3.5,6,7.5)
G	Acquisition and take delivery of arms	C	(0.5,2.5,4.5;1,2,3;2,4,6)	(1,5,8;1.5,3,6.5;4,7,9)	(1,5,9;1.5,4.5,6.5;4,7,10)
H	Construct the battlefield	D	(1.5,3.5,5.5;1,2,3;3,4.5,6)	(0.5,3.5,6.5;0.5,2.5,4.5;3,5,7)	(1,2,3;0.5,1.5,2.5;1.5,2.5,3.5)
I	Developed networking system	A	(1,5,9;1.5,4.5,6.5;4,7,10)	(0.5,2.5,4.5;1,2,3;1.5,3.5,5.5)	(**1, 5, 8; 1, 3, 6; 4, 7, 9**)
J	Run the system	E,G,H	(0.5,3.5,6.5;0.5,2.5,4.5;3,5,7)	(1.5,2.5,3.5;1,1.5,3;2,3,4)	(1,2,3;0.5,1.5,2.5;1.2,2.7,3.5)
K	Training for all	F,I,J	(1,5,8;1.5,3.5,6.5;4,6,8.5)	(1,4,7;1,3,5;3.5,6,7.5)	(1,1.5,4;0.5,1,2.5;1.25,3,4.25)

Draw the project network and find the probability that the project is completed in 16 days.

Solution. Now, we solve the problem by the following steps, as shown in Table 6, Figures 9 and 10.

Step-1.

Table 6. The value of E_{jk} and $\sigma_{jk}{}^2$ for the above problem.

Optimistic Time (o)	Pessimistic Time (p)	Most Likely Time (m)	$E_{jk} = \frac{o+4m+p}{6}$	$\sigma_{jk}{}^2 = \left(\frac{p-o}{6}\right)^2$
2.26	5.42	3.33	3.50	0.277
4.92	2.00	4.92	4.43	0.244
4.67	1.71	5.42	4.68	0.243
2.96	3.33	2.67	2.83	0.004
2.75	5.54	2.42	3.00	0.216
3.83	2.26	4.67	4.13	0.068
2.83	4.92	2.00	2.63	0.121
3.33	3.50	2.26	2.65	0.001
5.42	2.67	4.92	4.63	0.210
3.50	2.42	2.00	2.32	0.032
4.88	4.67	1.71	2.73	0.001

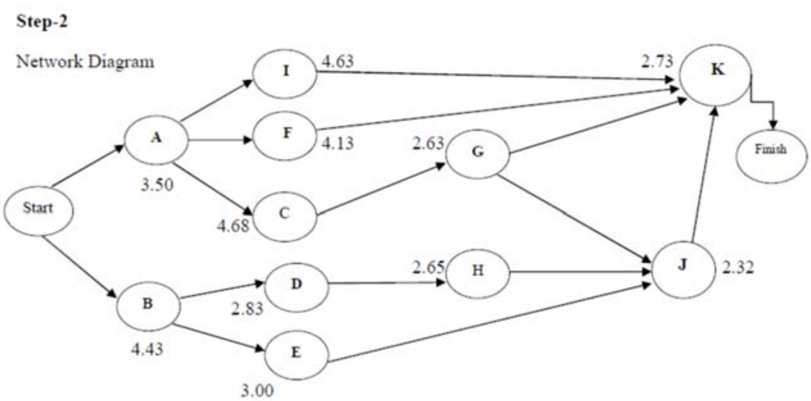


Figure 9. The network diagram for the problem.

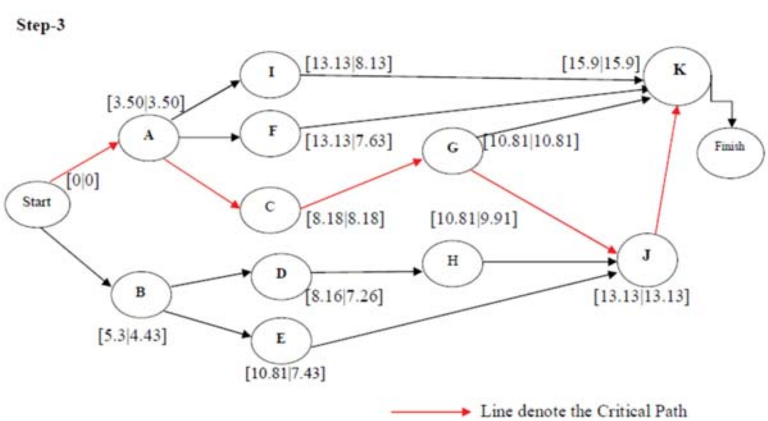


Figure 10. Critical path analysis for the problem.

Therefore, the expected project duration is 15.9 days.
Critical path A→C→G→J→K.
Project length variance $\sigma^2 = 0.962$, standard deviation0.98.
Probability that the project will be finished within 16 days is $P(z \leq \frac{16-15.9}{0.98}) = P(z \leq 0.1)$
Area under the normal curve $P(z \leq 0.1) = 0.5 + \varnothing(0.1)$ = 0.5398
The related normal curve is drawn in Figure 11.

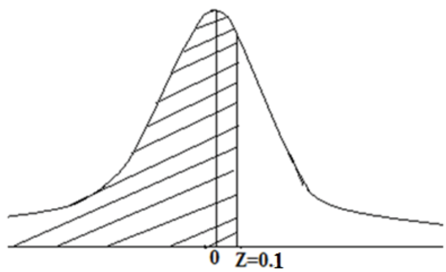


Figure 11. The normal curve for the above problem.

7. Application of Triangular Neutrosophic Fuzzy Number in Assignment Problem Using De-Neutrosophic Value

The assignment problem is very important for transferring goods from one place to another place. In the assignment problem, if uncertainty occurs, then it is more complicated to solve. By the concept of impreciseness and its corresponding crispified value, we can easily handle the assignment problem. In this section, we take a route selection problem with neutrosophic cost data and solve the problem.

We consider a problem of assigning three different trucks to three different destinations. The assigning costs that are the travelling costs in rupees are given here. How should the trucks be dispatched so as to minimize the total travelling cost? Note, that all the costs are triangular neutrosophic numbers.

Let us consider that the transportation cost for the three trucks are neutrosophic in nature. For that viewpoint, we take that the cost of the three trucks are as follows in Table 1, in units of dollar. Each component represents the moneys in units of dollars.

Here, red car denotes Truck 1, yellow car denotes Truck 2, and green car denotes Truck 3 as shown in the Figure 12.

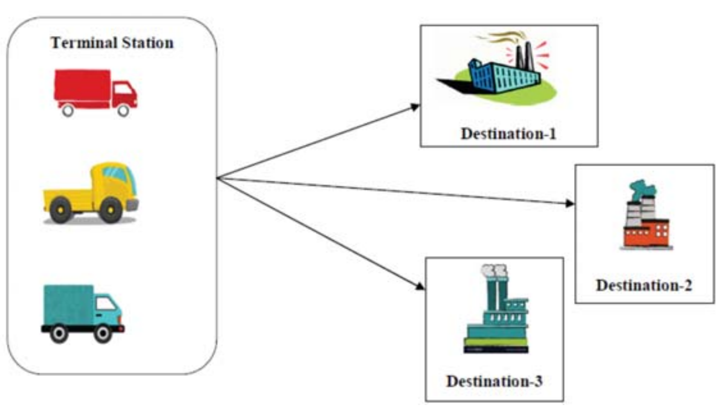


Figure 12. Pictorial representation of the problem.

We apply the defuzzification result of triangular neutrosophic number from Table 7.

Table 7. Neutrosophic value for the transportation costs.

	Destination-1	Destination-2	Destination-3
Truck 1	(1,4,7;1,3,5;3.5,6,7.5)	(0.5,2.5,4.5;1,2,3;1.5,3.5,5.5)	(1,3,5;0.5,1.5,3.5;2,4,6)
Truck 2	(1,2,3;0.5,1.5,2.5;1.5,2.5,3.5)	(1,1.5,4;0.5,1,2.5;1.25,3,4.25)	(1.5,2.5,3.5;1,1.5,3;2,3,4)
Truck 3	(2,4,6;1.5,2.5,4.5;3,5,7)	(1,5,8;1.5,4.5,7.5;4,6.5,9)	(1,5,8;1.5,3,6.5;4,7,9)

$R(\check{D},0) = \frac{(a+2b+c+d+2e+f+g+2h+k)}{12}$ to convert the numbers into a crisp number. Then, we have the following Table 8.

Table 8. De-neutrosophication value for the transportation costs.

	Destination-1	Destination-2	Destination-3
Truck 1	4.25	2.67	2.92
Truck 2	2.00	1.71	2.75
Truck 3	3.92	5.25	5.08

Now, we consider row minimum from each row, and subtract it from the other element (row-wise). Thus, we get Table 9.

Table 9. Row minimum from each row, and subtract it from the other element (row-wise).

	Destination-1	Destination-2	Destination-3
Truck 1	1.58	0	0.25
Truck 2	0.29	0	1.04
Truck 3	0	1.33	1.16

Now, we consider column minimum from each column and subtract it from the other element (column-wise). Thus, we get Table 10.

Table 10. Column minimum from each column and subtract it from the other element (column-wise).

	Destination-1	Destination-2	Destination-3
Truck 1	1.58	0	0
Truck 2	0.29	0	0.79
Truck 3	0	1.33	0.91

Here, the minimum number of straight lines to cover all the zeros is 3 (which is also equal to the order of the matrix), as shown in Table 11.

Table 11. Minimum number of straight lines to cover all the zeros.

	Destination-1	Destination-2	Destination-3
Truck 1	1.58	0	0
Truck 2	0.29	0	0.79
Truck 3	0	1.33	0.91

From the Table 12, we see that if the Truck1 goes to Destination-3, Truck2 goes to Destination-2, and Truck3 goes to Destination-1, then the carrying is minimum.

Table 12. Transformed table.

	Destination-1	Destination-2	Destination-3
Truck 1	1.58	0	[**0**]
Truck 2	0.29	[**0**]	0.79
Truck 3	[**0**]	1.33	0.91

That means from the Figure 13 Truck-1→Destination-3, Truck-2→Destination-2, Truck-3→Destination-1.

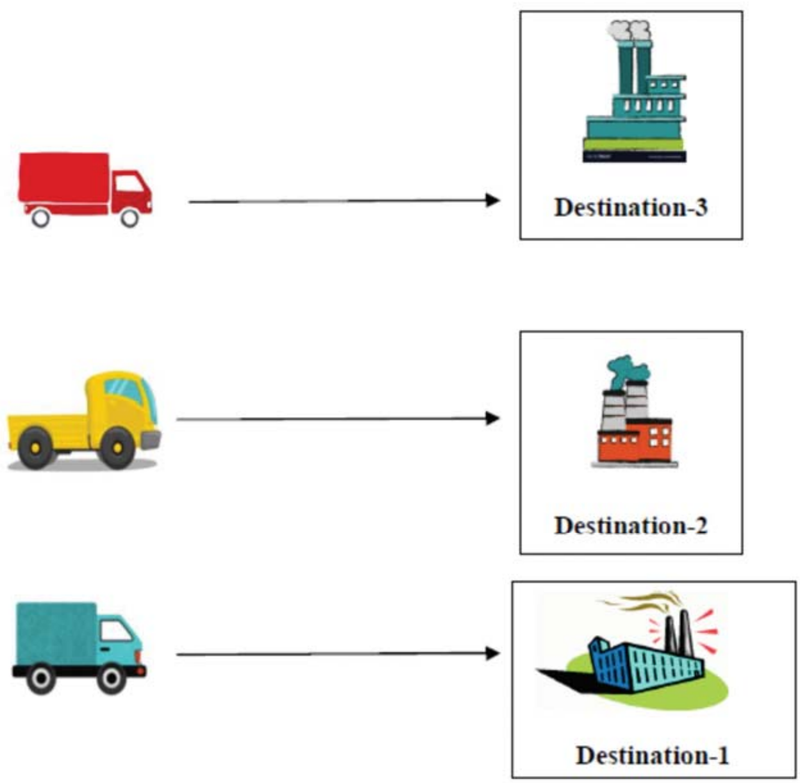


Figure 13. Pictorial representation of the solution.

The corresponding Min cost = (3.92 + 1.71 + 2.92) = 8.55 units of dollar.
Then, we get Table 13.

Table 13. Neutrosophic value of destinations.

	Destination-1	Destination-2	Destination-3
Truck 1	(1,4,7;1,3,5;3.5,6,7.5)	(0.5,2.5,4.5;1,2,3;1.5,3.5,5.5)	(1,3,5;0.5,1.5,3.5;2,4,6)
Truck 2	(1,2,3;0.5,1.5,2.5;1.5,2.5,3.5)	(1,1.5,4;0.5,1,2.5;1.25,3,4.25)	(1.5,2.5,3.5;1,1.5,3;2,3,4)
Truck 3	(2,4,6;1.5,2.5,4.5;3,5,7)	(1,5,8;1.5,4.5,7.5;4,6.5,9)	(1,5,8;1.5,3,6.5;4,7,9)

Ye [21] built up the concept of score function and accuracy function. The score function S and the accuracy function H are applied to compare the grades of triangular fuzzy numbers (TFNS). These functions show that greater is the value, the greater is the TFNS, and by using these, concept paths can be ranked.

We apply the result of triangular neutrosophic number.

Let, $\check{A} = (a, b, c; d, e, f; g, h, k)$ be a triangular neutrosophic fuzzy number, then the score function is defined as $S(\check{A}) = \frac{\{8+(a+2b+c)-(d+2e+f)-(g+2h+k)\}}{12}$, and accuracy function is defined as $H(\check{A}) = \frac{\{(a+2b+c)-(g+2h+k)\}}{4}$.

In order to make comparisons between two triangular neutrosophic values, Ye [21] presented the order relations between two triangular neutrosophic values.

Let $\check{A}_1 = (a_1, b_1, c_1; d_1, e_1, f_1; g_1, h_1, k_1)$ and $\check{A}_2 = (a_2, b_2, c_2; d_2, e_2, f_2; g_2, h_2, k_2)$ be two triangular neutrosophic values, then the ranking method is defined as follows.

(i) if $S(\check{A}_1) > S(\check{A}_2)$, then $\check{A}_1 > \check{A}_2$

(ii) if $S(\check{A}_1) = S(\check{A}_2)$ and $H(\check{A}_1) > H(\check{A}_2)$, then $\check{A}_1 > \check{A}_2$

We apply the score function result of triangular neutrosophic number $S(\check{A}) = \frac{\{8+(a+2b+c)-(d+2e+f)-(g+2h+k)\}}{12}$ to convert the numbers into a crisp number.

Then we have the following table, as shown in Table 14.

Table 14. Converted the numbers into a crisp number.

	Destination-1	Destination-2	Destination-3
Truck 1	−0.92	−0.33	−0.25
Truck 2	0.00	−0.04	−0.08
Truck 3	−0.58	−1.42	−1.17

Take the most negative cost (−1.42), add it with all the elements of the matrix we get Table 15.

Table 15. Corrosponing positive value table.

	Destination-1	Destination-2	Destination-3
Truck 1	0.50	1.09	1.17
Truck 2	1.42	1.38	1.34
Truck 3	0.84	0.00	0.25

Now, we consider row minimum from each row and subtract it from the other elements (row-wise). Thus, we get Table 16.

Table 16. Row minimum from each row and subtract it from the other elements (row-wise).

	Destination-1	Destination-2	Destination-3
Truck 1	0	0.59	0.67
Truck 2	0.08	0.04	0
Truck 3	0.84	0	0.25

Now, we consider column minimum from each column, and subtract it from the other elements (column-wise). Thus, we get Table 17.

Table 17. Column minimum from each column, and subtract it from the other elements (column-wise).

	Destination-1	Destination-2	Destination-3
Truck 1	0	0.59	0.67
Truck 2	0.08	0.04	0
Truck 3	0.84	0	0.25

Here, the minimum number of straight lines to cover all the zeros is 3(which is also equal to the order of the matrix), as shown in Table 18.

Table 18. Minimum number of straight lines to cover all the zeros is 3.

	Destination-1	Destination-2	Destination-3
Truck 1	0	0.59	0.67
Truck 2	0.08	0.04	0
Truck 3	0.84	0	0.25

From the Table 19, we see that if the Truck1 goes to Destination-1, Truck2 goes to Destination-3, and Truck3 goes to Destination-2, then the carrying is minimum.

Table 19. Decision table.

	Destination-1	Destination-2	Destination-3
Truck 1	[0]	0.59	0.67
Truck 2	0.08	0.04	[0]
Truck 3	0.84	[0]	0.25

That means from the Figure 14 the destination is as follows Truck1→Destination-1, Truck2→Destination-3, Truck3→Destination-2.

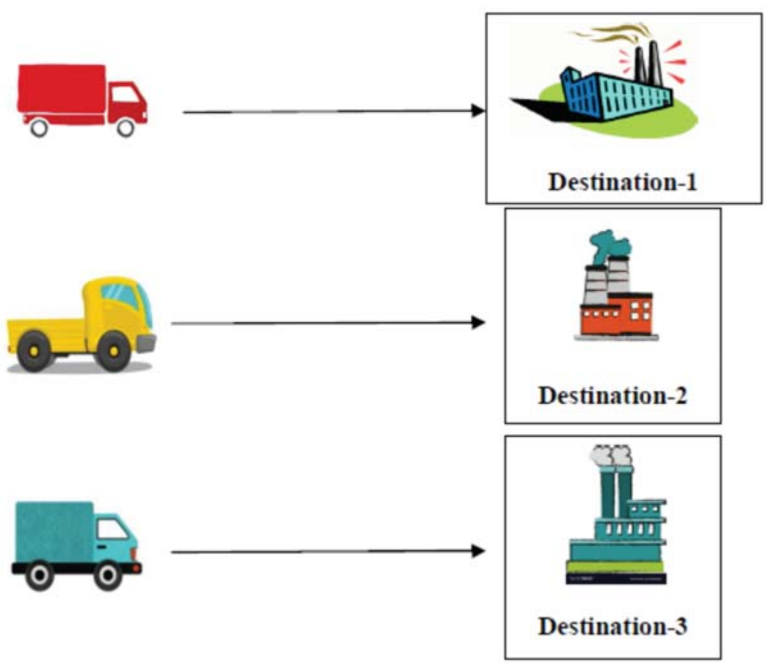


Figure 14. Pictorial representation of the solution.

The corresponding Min cost = (−0.92 − 1.42 − 0.08) = −2.42 units of dollar.

Note: Since, using de-neutrosophic value, we observe that min cost is 8.55 units of dollar, whereas using score function, we get min cost in negative quantity that is loss, hence de-neutrosophication gives us a better result than the score function.

8. Conclusions

The theory of uncertainty plays a key role in applied mathematical modeling. The concept of neutrosophic number is very popular nowadays. The formation and de-neutrosophication of the corresponding number can be very important for the researcher who deals with uncertainty and decision-making problems. In this paper, we construct the concept triangular neutrosophic number from different viewpoints, which is not defined earlier. We use the concept of linear and non-linear form with generalization of the pick value of truth, falsity, and indeterminacy functions by considering triangular neutrosophic numbers, which are very important for uncertainty theory. We introduced the de-neutrosophication concept for triangular neutrosophic numbers. This concept helps us to convert a neutrosophic number into a crisp number, which is surely helpful for decision-making problems. In future, we can extend the concept into different types of neutrosophic numbers, which can be more applicable in modeling with uncertainty.

Author Contributions: Conceptualization, A.C., S.P.M. and A.A.; Methodology, N.S. and S.S.; Vdation, S.A. and S.S.; Formal Analysis, A.C., S.P.M. and A.A.; Investigation, N.S. and S.A.

Acknowledgments: The authors acknowledge the financial support from Universiti Putra Malaysia under Putra-IPB grant: GP-IPB/2017/9542402.

Conflicts of Interest: The authors declare no conflict of interest.

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Applications of Neutrosophic Bipolar Fuzzy Sets in HOPE Foundation for Planning to Build a Children Hospital with Different Types of Similarity Measures

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Received: 8 July 2018; Accepted: 3 August 2018; Published: 9 August 2018

Abstract: In this paper we provide an application of neutrosophic bipolar fuzzy sets in daily life's problem related with HOPE foundation that is planning to build a children hospital, which is the main theme of this paper. For it we first develop the theory of neutrosophic bipolar fuzzy sets which is a generalization of bipolar fuzzy sets. After giving the definition we introduce some basic operation of neutrosophic bipolar fuzzy sets and focus on weighted aggregation operators in terms of neutrosophic bipolar fuzzy sets. We define neutrosophic bipolar fuzzy weighted averaging ($\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$) and neutrosophic bipolar fuzzy ordered weighted averaging ($\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$) operators. Next we introduce different kinds of similarity measures of neutrosophic bipolar fuzzy sets. Finally as an application we give an algorithm for the multiple attribute decision making problems under the neutrosophic bipolar fuzzy environment by using the different kinds of neutrosophic bipolar fuzzy weighted/fuzzy ordered weighted aggregation operators with a numerical example related with HOPE foundation.

Keywords: neutrosophic set; bipolar fuzzy set; neutrosophic bipolar fuzzy set; neutrosophic bipolar fuzzy weighted averaging operator; similarity measure; algorithm; multiple attribute decision making problem

MSC: (2010 Mathematics Subject Classifications) 62C05; 62C86; 03B52; 03E72; 90B50; 91B06; 91B10; 46S40; 47H99

1. Introduction

Zadeh [1] started the theory of fuzzy set and since then it has been a significant tool in learning logical subjects. It is applied in many fields, see [2]. There are numbers of over simplifications/generalization of Zadeh's fuzzy set idea to interval-valued fuzzy notion [3], intuitionistic fuzzy set [4], L-fuzzy notion [5], probabilistic fuzzy notion [6] and many others. Zhang [7,8], provided the generality of fuzzy sets as bipolar fuzzy sets. The extensions of fuzzy sets with membership grades from $[-1, 1]$, are the bipolar fuzzy sets. The membership grade $[-1, 0)$ of a section directs in bipolar fuzzy set that the section fairly fulfils the couched stand-property, the membership grade $]0, 1]$ of a section shows that the section fairly fulfils the matter and the membership grade 0 of a section resources that the section is unrelated to the parallel property. While bipolar fuzzy sets and intuitionistic fuzzy sets aspect parallel to one another, they are really distinct sets (see [3]). When we calculate the place of an objective in a universe, positive material conveyed for a collection of thinkable spaces and negative material conveyed for a collection of difficult spaces [9]. Naveed et al. [10–12], discussed theoretical aspects of bipolar fuzzy sets in detail. Smarandache [13], gave the notion of neutrosophic sets as a generalization of intutionistic fuzzy sets. The applications of Neutrosophic set theory are found in many fields (see http://fs.gallup.unm.edu/neutrosophy.htm). Recently Zhang et al. [14], Majumdar et al. [15], Liu et al. [16,17], Peng et al. [18] and Sahin et al. [19] have discussed various uses of neutrosophic set theory in deciding problems. Now a days, neutrosophic sets are very actively used in applications and MCGM problems. Bausys and Juodagalviene [20], Qun et al. [21], Zavadskas et al. [22], Chan and Tan [23], Hong and Choi [24], Zhan et al. [25] studied the applications of neutrosophic cubic sets in multi-criteria decision making in different directions. Anyhow, these approaches use the maximum, minimum operations to workout the aggregation procedure. This leads to subsequent loss of data and, therefore, inaccurate last results. How ever this restriction can be dealt by using famous weighted averaging (WA) operator [26] and the ordered weighted averaging (OWA) operator [27]. Medina and Ojeda-Aciego [28], gave t-notion lattice as a set of triples related to graded tabular information explained in a non-commutative fuzzy logic. Medina et al. [28] introduces a new frame work for the symbolic representation of informations which is called to as signatures and given a very useful technique in fuzzy modelling. In [29], Nowaková et al., studied a novel technique for fuzzy medical image retrieval (FMIR) by vector quantization (VQ) with fuzzy signatures in conjunction with fuzzy S-trees. In [30] Kumar et al., discussed data clustering technique, Fuzzy C-Mean algorithem and moreover Artificial Bee Colony (ABC) algorithm. In [31] Scellato et al.,discuss the rush of vehicles in urban street networks. Recently Gulistan et al. [32], combined neutrosophic cubic sets and graphs and gave the concept of neutrosophic cubic graphs with practical life applications in different areas. For more application of neutrosophic sets, we refer the reader to [33–37]. Since, the models presented in literature have different limitations in different situations. We mainly concern with the following tools:

(1) Neutrosophic sets are the more summed up class by which one can deal with uncertain informations in a more successful way when contrasted with fuzzy sets and all other versions of fuzzy sets. Neutrosophic sets have the greater adaptability, accuracy and similarity to the framework when contrasted with past existing fuzzy models.
(2) And bipolar fuzzy sets are proved to very affective in uncertain problems which can characterized not only the positive characteristics but also the negative characteristics of a certain problem.

We try to blend these two concepts together and try to develop a more powerful tool in the form of neutrosophic bipolar fuzzy sets. In this work we initiate the study of neutrosophic bipolar fuzzy sets which are the generalization of bipolar fuzzy sets and neutrosophic sets. After introducing the definition we give some basic operations, properties and applications of neutrosophic bipolar fuzzy sets. And the rest of the paper is structured as follows; Section 2 provides basic material from the existing literature to understand our proposal. Section 3 consists of the basic notion and properties of neutrosophic bipolar fuzzy set. Section 4 gives the role of weighted aggregation operator in terms of neutrosophic bipolar fuzzy sets. We define neutrosophic bipolar fuzzy weighted averaging operator $(\mathcal{N}^{\mathcal{B}}\mathcal{FWA})$ and neutrosophic bipolar fuzzy ordered weighted averaging $(\mathcal{N}^{\mathcal{B}}\mathcal{FOWA})$ operators. Section 5 includes different kinds of similarity measures. In Section 6, an algorithm for the multiple attribute decision making problems under the neutrosophic bipolar fuzzy environment by using the different kinds of similarity measures of neutrosophic bipolar fuzzy sets and neutrosophic bipolar fuzzy weighted/fuzzy ordered weighted aggregation operators is proposed. In Section 7, we provide a daily life example related with HOPE foundation, which shows the applicability of the algorithm provided in Section 6. In Section 8, we provide a comparison with the previous existing methods. In Section 9, we discuss conclusion and some future research directions.

2. Preliminaries

Here we provide some basic material from the literature for subsequent use.

Definition 1. *Let $\mathcal{Y}$ be any nonempty set. Then a bipolar fuzzy set [7,8], is an object of the form*

$$B = \langle u, \langle \mu^+(u),\ \mu^-(u) \rangle : u \in \mathcal{Y} \rangle,$$

and $\mu^+(u) : \mathcal{Y} \to [0,1]$ and $\mu^-(u) : \mathcal{Y} \to [-1,0]$, $\mu^+(u)$ is a positive material and $\mu^-(u)$ is a negative material of $u \in \mathcal{Y}$. For simplicity, we donate the bipolar fuzzy set as $B = \langle \mu^+, \mu^- \rangle$ in its place of $B = \langle u, \langle \mu^+(u), \mu^-(u) \rangle : u \in \mathcal{Y} \rangle$.

Definition 2. *Let $B_1 = \langle \mu_1^+, \mu_1^- \rangle$ and $B_2 = \langle \mu_2^+, \mu_2^- \rangle$ be two bipolar fuzzy sets [7,8], on $\mathcal{Y}$. Then we define the following operations.*

(1) $B_1' = \{\langle 1 - \mu_1^+(u), -1 - \mu_1^-(u) \rangle\}$;
(2) $B_1 \cup B_2 = \langle \max(\mu_1^+(u), \mu_2^+(u)), \min(\mu_1^-(u), \mu_2^-(u)) \rangle$;
(3) $B_1 \cap B_2 = \langle \min(\mu_1^+(u), \mu_1^+(u)), \max(\mu_1^-(u), \mu_2^-(u)) \rangle$.

Definition 3. *A neutrosophic set [13], is define as:*

$$L = \{\langle x, \boldsymbol{Tru}_L(x), \boldsymbol{Ind}_L(x), \boldsymbol{Fal}_L(x) \rangle : x \in X\},$$

where X is a universe of discoveries and L is characterized by a truth-membership function $\boldsymbol{Tru}_L : X \to]0^-, 1^+[$, an indtermency-membership function $\boldsymbol{Ind}_L : X \to]0^-, 1^+[$ and a falsity-membership function $\boldsymbol{Fal}_L : X \to]0^-, 1^+[$ such that $0 \le \boldsymbol{Tru}_L(x) + \boldsymbol{Ind}_L(x) + \boldsymbol{Fal}_L(x) \le 3$.

Definition 4. *A single valued neutrosophic set [16], is define as:*

$$L = \{\langle x, \boldsymbol{Tru}_L(x), \boldsymbol{Ind}_L(x), \boldsymbol{Fal}_L(x) \rangle : x \in X\},$$

where X is a universe of discoveries and L is characterized by a truth-membership function $\boldsymbol{Tru}_L : X \to [0,1]$, an indtermency-membership function $\boldsymbol{Ind}_L : X \to [0,1]$ and a falsity-membership function $\boldsymbol{Fal}_L : X \to [0,1]$ such that $0 \le \boldsymbol{Tru}_L(x) + \boldsymbol{Ind}_L(x) + \boldsymbol{Fal}_L(x) \le 3$.

Definition 5. *Let [16]*

$$L = \{\langle x, \boldsymbol{Tru}_L(x), \boldsymbol{Ind}_L(x), \boldsymbol{Fal}_L(x) \rangle : x \in X\},$$

and

$$B = \{\langle x, \boldsymbol{Tru}_B(x), \boldsymbol{Ind}_B(x), \boldsymbol{Fal}_B(x) \rangle : x \in X\},$$

be two single valued neutrosophic sets. Then

(1) *$L \subset B$ if and only if $\boldsymbol{Tru}_L(x) \le \boldsymbol{Tru}_B(x), \boldsymbol{Ind}_L(x) \le \boldsymbol{Ind}_B(x), \boldsymbol{Fal}_L(x) \ge \boldsymbol{Fal}_B(x)$.*
(2) *$L = B$ if and only if $\boldsymbol{Tru}_L(x) = \boldsymbol{Tru}_B(x), \boldsymbol{Ind}_L(x) = \boldsymbol{Ind}_B(x), \boldsymbol{Fal}_L(x) = \boldsymbol{Fal}_B(x)$, for any $x \in X$.*
(3) *The complement of L is denoted by L^c and is defined by*

$$L^c = \{\langle x, \boldsymbol{Fal}_L(x), 1 - \boldsymbol{Ind}_L(x), \boldsymbol{Tru}_L(x) \rangle / x \in X\}.$$

(4) *The intersection*

$$L \cap B = \{\langle x, \min\{\boldsymbol{Tru}_L(x), \boldsymbol{Tru}_B(x)\},\ \max\{\boldsymbol{Ind}_L(x), \boldsymbol{Ind}_B(x)\}, \max\{\boldsymbol{Fal}_L(x), \boldsymbol{Fal}_B(x)\} \rangle : x \in X\}.$$

(5) *The Union*

$$L \cup B = \{\langle x, \max\{\boldsymbol{Tru}_L(x), \boldsymbol{Tru}_B(x)\}, \min\{\boldsymbol{Ind}_L(x), \boldsymbol{Ind}_B(x)\}, \min\{\boldsymbol{Fal}_L(x), \boldsymbol{Fal}_B(x)\}\rangle : x \in X\}.$$

Definition 6. *Let $\tilde{A}_1 = \langle \boldsymbol{Tru}_1, \boldsymbol{Ind}_1, \boldsymbol{Fal}_1\rangle$ and $\tilde{A}_2 = \langle \boldsymbol{Tru}_2, \boldsymbol{Ind}_2, \boldsymbol{Fal}_2\rangle$ be two single valued neutrosophic number [16]. Then, the operations for NNs are defined as below:*

(1) $\lambda\tilde{A} = \left\langle 1-(1-\boldsymbol{Tru}_1)^{\lambda}, \boldsymbol{Ind}_1^{\lambda}, \boldsymbol{Fal}_1^{\lambda}\right\rangle$;

(2) $\tilde{A}_1^{\lambda} = \left\langle \boldsymbol{Tru}_1^{\lambda}, 1-(1-\boldsymbol{Ind}_1)^{\lambda}, 1-(1-\boldsymbol{Fal}_1)^{\lambda}\right\rangle$;

(3) $\tilde{A}_1 + \tilde{A}_2 = \langle \boldsymbol{Tru}_1 + \boldsymbol{Tru}_2 - \boldsymbol{Tru}_1\boldsymbol{Tru}_2, \boldsymbol{Ind}_1\boldsymbol{Ind}_2, \boldsymbol{Fal}_1\boldsymbol{Fal}_2\rangle$;

(4) $\tilde{A}_1\tilde{A}_2 = \langle \boldsymbol{Tru}_1\boldsymbol{Tru}_2, \boldsymbol{Ind}_1 + \boldsymbol{Ind}_2\_\boldsymbol{Ind}_1\boldsymbol{Ind}_2, \boldsymbol{Fal}_1 + \boldsymbol{Fal}_2 - \boldsymbol{Fal}_1\boldsymbol{Fal}_2\rangle$ *where* $\lambda > 0$.

Definition 7. *Let $\tilde{A}_1 = \langle \boldsymbol{Tru}_1, \boldsymbol{Ind}_1, \boldsymbol{Fal}_1\rangle$ be a single valued neutrosophic number [16]. Then, the score function $s(\tilde{A}_1)$,accuracy function $L(\tilde{A}_1)$, and certainty function $c(\tilde{A}_1)$, of an NNs are define as under:*

(1) $s(\tilde{A}_1) = \frac{(\boldsymbol{Tru}_1 + 1 - \boldsymbol{Ind}_1 + 1 - \boldsymbol{Fal}_1)}{3}$;

(2) $L(\tilde{A}_1) = \boldsymbol{Tru}_1 - \boldsymbol{Fal}_1$;

(3) $c(\tilde{A}_1) = \boldsymbol{Tru}_1$.

3. Neutrosophic Bipolar Fuzzy Sets and Operations

In this section we apply bipolarity on neutrosophic sets and initiate the notion of neutrosophic bipolar fuzzy set with the help of Section 2, which is the generalization of bipolar fuzzy set. We also study some basic operation on neutrosophic bipolar fuzzy sets.

Definition 8. *A neutrosophic bipolar fuzzy set is an object of the form $\mathcal{N}^{\mathcal{B}} = (\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$ where*

$$\begin{aligned}
\mathcal{N}^{\mathcal{B}+} &= \langle u, \langle \boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}}\rangle : u \in \mathcal{Y}\rangle, \\
\mathcal{N}^{\mathcal{B}-} &= \langle u, \langle \boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}}\rangle : u \in \mathcal{Y}\rangle,
\end{aligned}$$

where $\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}} : \mathcal{Y} \to [0,1]$ and $\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}} : \mathcal{Y} \to [-1,0]$.

Note: In the Definition 8, we see that a neutrosophic bipolar fuzzy sets $\mathcal{N}^{\mathcal{B}} = (\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$, consists of two parts, positive membership functions $\mathcal{N}^{\mathcal{B}+}$ and negative membership functions $\mathcal{N}^{\mathcal{B}-}$. Where positive membership function $\mathcal{N}^{\mathcal{B}+}$ denotes what is desirable and negative membership function $\mathcal{N}^{\mathcal{B}-}$ denotes what is unacceptable. Desirable characteristics are further characterize as: $\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}}$ denotes what is desirable in past, $\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}}$ denotes what is desirable in future and $\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}}$ denotes what is desirable in present time. Similarly $\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}}$ denotes what is unacceptable in past, $\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}}$ denotes what is unacceptable in future and $\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}}$ denotes what is unacceptable in present time.

Definition 9. *Let $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$ and $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ be two neutrosophic bipolar fuzzy sets. Then we define the following operations:*

(1) $\mathcal{N}_1^{\mathcal{B}^c} = \left\{\left\langle 1-\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}, 1-\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}, -1-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}} \text{ and } 1-\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}, 1-\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}, -1-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}\right\rangle\right\}$;

(2)

$$\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}} = \left\langle \begin{array}{c} \max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}), \max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}), \min(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}), \\ \max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}), \max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}), \min(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}) \end{array} \right\rangle;$$

(3)

$$\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}} = \left\langle \begin{array}{c} \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}), \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}), \max(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}), \\ \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}), \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}), \max(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}). \end{array} \right\rangle.$$

Definition 10. *Let $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$ and $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ be two neutrosophic bipolar fuzzy sets. Then we define the following operations:*

(1)

$$\mathcal{N}_1^{\mathcal{B}+} \oplus \mathcal{N}_2^{\mathcal{B}+} = \left\langle \begin{array}{c} \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}} + \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}} - \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}} \cdot \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}} + \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}} - \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}} \cdot \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}, \\ -(\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}\right| \cdot \left|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}\right|) \end{array} \right\rangle,$$

and

$$\mathcal{N}_1^{\mathcal{B}-} \oplus \mathcal{N}_2^{\mathcal{B}-} = \left\langle \begin{array}{c} \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}} + \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}} - \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}} \cdot \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}} + \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}} - \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}} \cdot \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}, \\ -(\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}\right| \cdot \left|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}\right|) \end{array} \right\rangle;$$

(2)

$$\mathcal{N}_1^{\mathcal{B}+} \otimes \mathcal{N}_2^{\mathcal{B}+} = \left\langle \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}} \cdot \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}} \cdot \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}} + \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}} - (\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}\right| \cdot \left|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}\right|) \right\rangle,$$

and

$$\mathcal{N}_1^{\mathcal{B}-} \otimes \mathcal{N}_2^{\mathcal{B}-} = \left\langle \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}} \cdot \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}} \cdot \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}} + \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}} - (\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}\right| \cdot \left|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}\right|) \right\rangle;$$

(3)

$$\mathcal{N}_1^{\mathcal{B}+} - \mathcal{N}_2^{\mathcal{B}+} = \left\langle \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}), \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}),\ \max(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}) \right\rangle,$$

and

$$\mathcal{N}_1^{\mathcal{B}-} - \mathcal{N}_2^{\mathcal{B}-} = \left\langle \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}), \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}),\ \max(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}) \right\rangle.$$

Definition 11. *Let $\mathcal{N}^{\mathcal{B}} = (\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$ be a neutrosophic bipolar fuzzy set and $\lambda > 0$. Then,*

(1)

$$\begin{aligned} \lambda\mathcal{N}^{\mathcal{B}+} &= \langle 1-(1-\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}})^{\lambda}, 1-(1-\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}})^{\lambda}, -\left|\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}}\right|^{\lambda}\rangle, \\ \lambda\mathcal{N}^{\mathcal{B}-} &= \langle 1-(1-\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}})^{\lambda}, 1-(1-\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}})^{\lambda}, -\left|\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}}\right|^{\lambda}\rangle. \end{aligned}$$

(2)

$$\begin{aligned} \mathcal{N}^{\mathcal{B}+^{\lambda}} &= \langle (\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}+}})^{\lambda}, (\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}+}})^{\lambda}, -1+\left|-1+\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}+}}\right|^{\lambda}\rangle, \\ \mathcal{N}^{\mathcal{B}-\lambda} &= \langle (\boldsymbol{Tru}_{\mathcal{N}^{\mathcal{B}-}})^{\lambda}, (\boldsymbol{Ind}_{\mathcal{N}^{\mathcal{B}-}})^{\lambda}, -1+\left|-1+\boldsymbol{Fal}_{\mathcal{N}^{\mathcal{B}-}}(u)\right|^{\lambda}\rangle. \end{aligned}$$

Theorem 1. *Let $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$, $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ and $\mathcal{N}_3^{\mathcal{B}} = (\mathcal{N}_3^{\mathcal{B}+}, \mathcal{N}_3^{\mathcal{B}-})$ be neutrosophic bipolar fuzzy sets. Then, the following properties hold:*

(1) *Complementary law: $(\mathcal{N}_1^{\mathcal{B}c})^c = \mathcal{N}_1^{\mathcal{B}}$.*

(2) *Idempotent law:*

$$
\begin{aligned}
(i)\ \mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_1^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}},\\
(ii)\ \mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_1^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}}.
\end{aligned}
$$

(3) *Commutative law:*

$$
\begin{aligned}
(i)\ \mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}} &= \mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_1^{\mathcal{B}},\\
(ii)\ \mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}} &= \mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_1^{\mathcal{B}},\\
(iii)\ \mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}} &= \mathcal{N}_2^{\mathcal{B}} \oplus \mathcal{N}_1^{\mathcal{B}},\\
(iv)\ \mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}} &= \mathcal{N}_2^{\mathcal{B}} \otimes \mathcal{N}_1^{\mathcal{B}}.
\end{aligned}
$$

(4) *Associative law:*

$$
\begin{aligned}
(i)\ (\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}}) \cup \mathcal{N}_3^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}} \cup (\mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}),\\
(ii)\ (\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}}) \cap \mathcal{N}_3^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}} \cap \left(\mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}\right),\\
(iii)\ (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}) \oplus \mathcal{N}_3^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}} \oplus (\mathcal{N}_2^{\mathcal{B}} \oplus \mathcal{N}_3^{\mathcal{B}}),\\
(iv)\ (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}}) \otimes \mathcal{N}_3^{\mathcal{B}} &= \mathcal{N}_1^{\mathcal{B}} \otimes (\mathcal{N}_2^{\mathcal{B}} \otimes \mathcal{N}_3^{\mathcal{B}}).
\end{aligned}
$$

(5) *Distributive law:*

$$
\begin{aligned}
(i)\ \mathcal{N}_1^{\mathcal{B}} \cup (\mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}}) \cap (\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}),\\
(ii)\ \mathcal{N}_1^{\mathcal{B}} \cap (\mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}}) \cup (\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}),\\
(iii)\ \mathcal{N}_1^{\mathcal{B}} \oplus (\mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}) \cup (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_3^{\mathcal{B}}),\\
(iv)\ \mathcal{N}_1^{\mathcal{B}} \oplus (\mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}) \cap (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_3^{\mathcal{B}}),\\
(v)\ \mathcal{N}_1^{\mathcal{B}} \otimes (\mathcal{N}_2^{\mathcal{B}} \cup \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}}) \cup (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_3^{\mathcal{B}}),\\
(vi)\ \mathcal{N}_1^{\mathcal{B}} \otimes (\mathcal{N}_2^{\mathcal{B}} \cap \mathcal{N}_3^{\mathcal{B}}) &= (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}}) \cap (\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_3^{\mathcal{B}}).
\end{aligned}
$$

(6) *De Morgan$'$s laws:*

$$
\begin{aligned}
(i)\ (\mathcal{N}_1^{\mathcal{B}} \cup \mathcal{N}_2^{\mathcal{B}})^c &= \mathcal{N}_1^{\mathcal{B}c} \cap \mathcal{N}_2^{\mathcal{B}c},\\
(ii)\ \left(\mathcal{N}_1^{\mathcal{B}} \cap \mathcal{N}_2^{\mathcal{B}}\right)^c &= \mathcal{N}_1^{\mathcal{B}c} \cup \mathcal{N}_2^{\mathcal{B}c},\\
(iii)\ (\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}})^c &\neq \mathcal{N}_1^{\mathcal{B}c} \otimes \mathcal{N}_2^{\mathcal{B}c},\\
(iv)\ \left(\mathcal{N}_1^{\mathcal{B}} \otimes \mathcal{N}_2^{\mathcal{B}}\right)^c &\neq \mathcal{N}_1^{\mathcal{B}c} \oplus \mathcal{N}_2^{\mathcal{B}c}.
\end{aligned}
$$

Proof. Straightforward. □

Theorem 2. *Let $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$ and $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ be two neutrosophic bipolar fuzzy sets and let $\mathcal{N}_3^{\mathcal{B}} = \mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}$ and $\mathcal{N}_4^{\mathcal{B}} = \lambda\mathcal{N}_1^{\mathcal{B}}\ (\lambda > 0)$. Then both $\mathcal{N}_3^{\mathcal{B}}$ and $\mathcal{N}_4^{\mathcal{B}}$ are also neutrosophic bipolar fuzzy sets.*

Proof. Straightforward. □

Theorem 3. *Let $\mathcal{N}_1^{\mathcal{B}} = (\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_1^{\mathcal{B}-})$ and $\mathcal{N}_2^{\mathcal{B}} = (\mathcal{N}_2^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}-})$ be two neutrosophic bipolar fuzzy sets, $\lambda, \lambda_1, \lambda_2 > 0$. Then, we have:*

$$(i)\ \lambda(\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}}) = \lambda\mathcal{N}_1^{\mathcal{B}} \oplus \lambda\mathcal{N}_2^{\mathcal{B}},$$
$$(ii)\ \lambda_1\mathcal{N}_1^{\mathcal{B}} \oplus \lambda_2\mathcal{N}_2^{\mathcal{B}} = (\lambda_1 \oplus \lambda_2)\mathcal{N}_1^{\mathcal{B}}.$$

Proof. Straightforward. □

4. Neutrosophic Bipolar Fuzzy Weighted/Fuzzy Ordered Weighted Aggregation Operators

After defining neutrosophic bipolar fuzzy sets and some basic operations in Section 3. We in this section as applications point of view we focus on weighted aggregation operator in terms of neutrosophic bipolar fuzzy sets. We define $(\mathcal{N}^{\mathcal{B}}\mathcal{FWA})$ and $(\mathcal{N}^{\mathcal{B}}\mathcal{FOWA})$ operators.

Definition 12. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be the collection of neutrosophic bipolar fuzzy values. Then we define $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$ as a mapping $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k : \Omega^n \to \Omega$ by*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) = k_1\mathcal{N}_1^{\mathcal{B}} \oplus k_2\mathcal{N}_2^{\mathcal{B}} \oplus, ..., \oplus k_n\mathcal{N}_n^{\mathcal{B}}.$$

If $k = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$ then the $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$ operator is reduced to

$$\mathcal{N}^{\mathcal{B}}\mathcal{FA}\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) = \frac{1}{n}\left(\mathcal{N}_1^{\mathcal{B}} \oplus \mathcal{N}_2^{\mathcal{B}} \oplus, ..., \oplus \mathcal{N}_n^{\mathcal{B}}\right).$$

Theorem 4. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be the collection of neutrosophic bipolar fuzzy values. Then*

$$\left.\begin{array}{l} \mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_j^{\mathcal{B}+}\right) = \begin{bmatrix} 1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\ 1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\ -\Pi_{j=1}^n \left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}\right| \end{bmatrix} \\ \mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}, ..., \mathcal{N}_j^{\mathcal{B}-}\right) = \begin{bmatrix} 1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\ 1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\ -\Pi_{j=1}^n \left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}\right| \end{bmatrix} \end{array}\right\}. \quad (1)$$

Proof. Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be a collection of neutrosophic bipolar fuzzy values. We first prove the result for $n = 2$. Since

$$\begin{aligned} k_1\mathcal{N}_L^{\mathcal{B}+} &= \left[1 - \left(1 - \boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}, 1 - \left(1 - \boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}, -(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}+}}\right|)^{k_1}\right], \\ k_1\mathcal{N}_L^{\mathcal{B}-} &= \left[1 - \left(1 - \boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}, 1 - \left(1 - \boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}, -(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}-}}\right|)^{k_1}\right], \\ k_1\mathcal{N}_b^{\mathcal{B}+} &= \left[1 - \left(1 - \boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}, 1 - \left(1 - \boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}, -(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}+}}\right|)^{k_2}\right], \\ k_1\mathcal{N}_b^{\mathcal{B}+} &= \left[1 - \left(1 - \boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}, 1 - \left(1 - \boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}, -(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}-}}\right|)^{k_2}\right], \end{aligned}$$

then

$$\begin{aligned}
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}},\mathcal{N}_b^{\mathcal{B}}\right) &= k_1\mathcal{N}_1^{\mathcal{B}}\oplus k_2\mathcal{N}_2^{\mathcal{B}},\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}+},\mathcal{N}_b^{\mathcal{B}+}\right) &= k_1\mathcal{N}_1^{\mathcal{B}+}\oplus k_2\mathcal{N}_2^{\mathcal{B}+},\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}-},\mathcal{N}_b^{\mathcal{B}-}\right) &= k_1\mathcal{N}_1^{\mathcal{B}-}\oplus k_2\mathcal{N}_2^{\mathcal{B}-},\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}+},\mathcal{N}_b^{\mathcal{B}+}\right) &= \left[\begin{array}{c}
2-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}-\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}-\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}\right)\\
\times\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}\right),\\
2-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}-\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}-\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}\right)\\
\times\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2}\right),\\
-(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}+}}\right|)^{k_1}(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}+}}\right|)^{k_2}
\end{array}\right],\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}+},\mathcal{N}_b^{\mathcal{B}+}\right) &= \left[\begin{array}{c}
1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2},1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}+}}\right)^{k_1}\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}+}}\right)^{k_2},\\
-(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}+}}\right|)^{k_1}(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}+}}\right|)^{k_2}
\end{array}\right],
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}-},\mathcal{N}_b^{\mathcal{B}-}\right) &= k_1\mathcal{N}_1^{\mathcal{B}-}\oplus k_2\mathcal{N}_2^{\mathcal{B}-},\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}-},\mathcal{N}_b^{\mathcal{B}-}\right) &= \left[\begin{array}{c}
2-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}-\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}-\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}\right)\\
\times\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}\right),\\
2-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}-\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}-\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}\right)\\
\times\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2}\right),\\
-(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}-}}\right|)^{k_1}(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}-}}\right|)^{k_2}
\end{array}\right],\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}-},\mathcal{N}_b^{\mathcal{B}-}\right) &= \left[\begin{array}{c}
1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}\left(1-\boldsymbol{Tru}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2},1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_L^{\mathcal{B}-}}\right)^{k_1}\left(1-\boldsymbol{Ind}_{\mathcal{N}_b^{\mathcal{B}-}}\right)^{k_2},\\
-(\left|\boldsymbol{Fal}_{\mathcal{N}_L^{\mathcal{B}-}}\right|)^{k_1}(\left|\boldsymbol{Fal}_{\mathcal{N}_b^{\mathcal{B}-}}\right|)^{k_2}
\end{array}\right].
\end{aligned}$$

So $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_L^{\mathcal{B}},\mathcal{N}_b^{\mathcal{B}}\right)=k_1\mathcal{N}_1^{\mathcal{B}}\oplus k_2\mathcal{N}_2^{\mathcal{B}}$. If result is true for $n=k$, that is

$$\begin{aligned}
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+},...,\mathcal{N}_j^{\mathcal{B}+}\right) &= \left[\begin{array}{c}
1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j},\\
1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j},\\
-\Pi_{j=1}^{k}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}\right|
\end{array}\right],\\
\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-},...,\mathcal{N}_j^{\mathcal{B}-}\right) &= \left[\begin{array}{c}
1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j},\\
1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j},\\
-\Pi_{j=1}^{k}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}\right|
\end{array}\right],
\end{aligned}$$

then, when $k+1$, we have

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+},...,\mathcal{N}_j^{\mathcal{B}+}\right) = \begin{bmatrix} 1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}+\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_{k+1}^{\mathcal{B}+}}\right)^{k_{k+1}}\right) \\ -(1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j})\times\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_{k+1}^{\mathcal{B}+}}\right)^{k_{k+1}}\right), \\ 1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}+\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_{k+1}^{\mathcal{B}+}}\right)^{k_{k+1}}\right) \\ -(1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j})\times\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_{k+1}^{\mathcal{B}+}}\right)^{k_{k+1}}\right), \\ -\Pi_{j=1}^{k+1}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}\right| \end{bmatrix}$$

$$= \begin{bmatrix} 1-\Pi_{j=1}^{k+1}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\ 1-\Pi_{j=1}^{k+1}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}, \\ -\Pi_{j=1}^{k+1}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j}\right| \end{bmatrix},$$

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-},...,\mathcal{N}_j^{\mathcal{B}-}\right) = \begin{bmatrix} 1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}+\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_{k+1}^{\mathcal{B}-}}\right)^{k_{k+1}}\right) \\ -(1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j})\times\left(1-\left(1-\boldsymbol{Tru}_{\mathcal{N}_{k+1}^{\mathcal{B}-}}\right)^{k_{k+1}}\right), \\ 1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}+\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_{k+1}^{\mathcal{B}-}}\right)^{k_{k+1}}\right) \\ -(1-\Pi_{j=1}^{k}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right)^{k_j})\times\left(1-\left(1-\boldsymbol{Ind}_{\mathcal{N}_{k+1}^{\mathcal{B}-}}\right)^{k_{k+1}}\right), \\ -\Pi_{j=1}^{k+1}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}\right| \end{bmatrix}$$

$$= \begin{bmatrix} 1-\Pi_{j=1}^{k+1}\left(1-\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\ 1-\Pi_{j=1}^{k+1}\left(1-\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}, \\ -\Pi_{j=1}^{k+1}\left|\left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}}\right)^{k_j}\right| \end{bmatrix}.$$

So result holds for $n=k+1$. □

Theorem 5. *Let* $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+},\mathcal{N}_j^{\mathcal{B}-})$ *be the collection of neutrosophic bipolar fuzzy values and* $k = (k_1,k_2,...,k_n)^T$ *is the weight vector of* $\mathcal{N}_j^{\mathcal{B}}$ $(j=1,2,...,n)$*, with* $k_j \in [0,1]$ *and* $\Sigma_{j=1}^{n}k_j = 1$*. Then we have the following:*

(1) *(Idempotency): If all* $\mathcal{N}_j^{\mathcal{B}\sim}$ $(j=1,2,...,n)$ *are equal, i.e.,*$\mathcal{N}_j^{\mathcal{B}} = \mathcal{N}_j^{\mathcal{B}}$*, for all j, then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}},\mathcal{N}_2^{\mathcal{B}},...,\mathcal{N}_n^{\mathcal{B}}\right) = \mathcal{N}^{\mathcal{B}}.$$

(2) *(Boundary):*

$$\mathcal{N}^{\mathcal{B}^-} \leq \mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}},\mathcal{N}_2^{\mathcal{B}},...,\mathcal{N}_n^{\mathcal{B}}\right) \leq \mathcal{N}^{\mathcal{B}^+}, \text{ for every } k.$$

(3) *(Monotonicity) If* $\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}} \leq \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+*}}$, $\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}} \leq \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+*}}$ *and* $\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}} \geq \boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-*}}$*, for all j, then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_1^{\mathcal{B}},\mathcal{N}_2^{\mathcal{B}},...,\mathcal{N}_n^{\mathcal{B}}\right) \leq \mathcal{N}^{\mathcal{B}}\mathcal{FWA}_k\left(\mathcal{N}_{1^*}^{\mathcal{B}},\mathcal{N}_{2^*}^{\mathcal{B}},...,\mathcal{N}_{n^*}^{\mathcal{B}}\right), \text{ for every } k.$$

Definition 13. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be the $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$ be a collection of neutrosophic bipolar fuzzy values. An neutrosophic bipolar fuzzy OWA($\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$) operator of dimension is a mapping $\mathcal{N}^{\mathcal{B}}\mathcal{FOWA} : \Omega^n \to \Omega$ defined by*

$$\begin{aligned}
\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_n^{\mathcal{B}+}\right) &= k_1\mathcal{N}_{\sigma(1)}^{\mathcal{B}+} \oplus k_2\mathcal{N}_{\sigma(2)}^{\mathcal{B}+} \oplus, ..., \oplus k_n\mathcal{N}_{\sigma(n)}^{\mathcal{B}+}, \\
\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}, ..., \mathcal{N}_n^{\mathcal{B}-}\right) &= k_1\mathcal{N}_{\sigma(1)}^{\mathcal{B}-} \oplus k_2\mathcal{N}_{\sigma(2)}^{\mathcal{B}-} \oplus, ..., \oplus k_n\mathcal{N}_{\sigma(n)}^{\mathcal{B}-},
\end{aligned}$$

where $(\sigma(1), \sigma(2), ..., \sigma(n))$ is a permutation of $(1, 2, ..., n)$ such that $\mathcal{N}_{\sigma(j-1)}^{\mathcal{B}} \geq \mathcal{N}_{\sigma(j)}^{\mathcal{B}}$ for all j. If $k = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$ then BFOWA operator is reduced to BFA operator having dimension n.

Theorem 6. *Let $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ be the collection of neutrosophic bipolar fuzzy values. Then*

$$\left.\begin{array}{l}
\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_n^{\mathcal{B}+}\right) = \left[\begin{array}{c}
1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}+}}\right)^{k_j}, \\
1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Ind}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}+}}\right)^{k_j}, \\
-\Pi_{j=1}^n \left|\left(\boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}+}}\right)^{k_j}\right|
\end{array}\right], \\
\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_n^{\mathcal{B}+}\right) = \left[\begin{array}{c}
1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}-}}\right)^{k_j}, \\
1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Ind}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}-}}\right)^{k_j}, \\
-\Pi_{j=1}^n \left|\left(\boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}-}}\right)^{k_j}\right|
\end{array}\right],
\end{array}\right\}, \tag{2}$$

where

$$k = (k_1, k_2, ..., k_n)^T,$$

is the weight vector of $\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$ operator with $k_j \in [0, 1]$ and $\Sigma_{j=1}^n k_j = 1$, for all $j = 1, 2, ..., n$, i.e., all $\mathcal{N}_j^{\mathcal{B}\sim}$ $(j = 1, 2, ..., n)$, are reduced to the following form:

$$\begin{aligned}
\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}, ..., \mathcal{N}_n^{\mathcal{B}+}\right) &= 1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}+}}\right)^{k_j}, \\
\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}, ..., \mathcal{N}_n^{\mathcal{B}-}\right) &= 1 - \Pi_{j=1}^n \left(1 - \boldsymbol{Tru}_{\mathcal{N}_{(\sigma(j))}^{\mathcal{B}-}}\right)^{k_j}.
\end{aligned}$$

Theorem 7. *Let $\mathcal{N}_j^{\mathcal{B}\sim} = \langle \mathcal{N}_{\mathcal{N}_j^{\mathcal{B}\sim}}^{\mathcal{B}+}, \mathcal{N}_{\mathcal{N}_j^{\mathcal{B}\sim}}^{\mathcal{B}-} \rangle$ $(j = 1, 2, ..., n)$ be a collection of neutrosophic bipolar fuzzy values and*

$$k = (k_1, k_2, ..., k_n)^T,$$

is the weighting vector of $\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$ operator with $k_j \in [0, 1]$ and $\Sigma_{j=1}^n k_j = 1$; then we have the following.

(1) *Idempotency: If all $\mathcal{N}_j^{\mathcal{B}}$ $(j = 1, 2, ..., n)$ are equal, i.e., $\mathcal{N}_j^{\mathcal{B}} = \mathcal{N}^{\mathcal{B}}$, for all j, then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) = \mathcal{N}^{\mathcal{B}}.$$

(2) *Boundary:*

$$\mathcal{N}^{\mathcal{B}^-} \leq \mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) \leq \mathcal{N}^{\mathcal{B}^+},$$

for where k, where $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ *be the* $\mathcal{N}^{\mathcal{B}}\mathcal{FWA}$ $\mathcal{N}_j^{\mathcal{B}+} = \langle \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}, \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}, \boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}} \rangle$ $(j = 1, 2, ..., n)$ *and* $\mathcal{N}_j^{\mathcal{B}-} = \langle \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}, \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}, \boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}} \rangle$ $(j = 1, 2, ..., n)$ *be a collection of neutrosophic bipolar fuzzy values*

$$\begin{aligned} \mathcal{N}^{\mathcal{B}^-} &= \left[\min_j \left(\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}-}}\right), \min_j \left(\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}-}}\right), -\max_j \left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}}\right)\right], \\ \mathcal{N}^{\mathcal{B}^+} &= \left[\max_j \left(\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}}\right), \max_j \left(\boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}}\right), -\min_j \left(\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}+}}\right)\right]. \end{aligned}$$

(3) *Monotonicity: Let* $\mathcal{N}_j^{\mathcal{B}+^*}$ *and* $\mathcal{N}_j^{\mathcal{B}-^*}$ $(j = 1, 2, ..., n)$ *be a collection of neutrosophic bipolar fuzzy values. If* $\boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+}} \leq \boldsymbol{Tru}_{\mathcal{N}_j^{\mathcal{B}+*}}, \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+}} \leq \boldsymbol{Ind}_{\mathcal{N}_j^{\mathcal{B}+*}}$ *and* $\boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-}} \geq \boldsymbol{Fal}_{\mathcal{N}_j^{\mathcal{B}-*}}$*, for all j, then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) \leq BFWL_k\left(\mathcal{N}_{1^*}^{\mathcal{B}}, \mathcal{N}_{2^*}^{\mathcal{B}}, ..., \mathcal{N}_{n^*}^{\mathcal{B}}\right), \text{ for every } k.$$

(4) *Commutativity: Let* $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ *be a collection of neutrosophic bipolar fuzzy values. Then*

$$BFOWL_k\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) = BFOWL_k\left(\mathcal{N}_1^{\mathcal{B}'}, \mathcal{N}_2^{\mathcal{B}'}, ..., \mathcal{N}_n^{\mathcal{B}'}\right),$$

for every w, where $\left(\mathcal{N}_1^{\mathcal{B}'}, \mathcal{N}_2^{\mathcal{B}'}, ..., \mathcal{N}_n^{\mathcal{B}'}\right)$ *is any permutation of* $\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right)$.

Theorem 8. *Let* $\mathcal{N}_j^{\mathcal{B}} = (\mathcal{N}_j^{\mathcal{B}+}, \mathcal{N}_j^{\mathcal{B}-})$ *be a collection of neutrosophic bipolar fuzzy values*

$$k = (k_1, k_2, ..., k_n)^T,$$

is the weighting vector of $\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}$ *operator with*

$$k_j \in [0, 1] \text{ and } \Sigma_{j=1}^n k_j = 1;$$

then we have the following:

(1) *If* $k = (1, 0, ..., 0)^T$*, then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) = \max_j\left(\mathcal{N}_j^{\mathcal{B}}\right).$$

(2) *If* $k = (0, 0, ..., 1)^T$*, then*

$$\mathcal{N}^{\mathcal{B}}\mathcal{FOWA}_k\left(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}, ..., \mathcal{N}_n^{\mathcal{B}}\right) = \min_j\left(\mathcal{N}_j^{\mathcal{B}}\right).$$

(3) *If* $k_j = 1, k_i = 0$*, and* $i \neq j$*, then*

$$\mathcal{BFOWA}_k\left(\mathcal{N}_1^{\mathcal{B}\sim}, \mathcal{N}_2^{\mathcal{B}\sim}, ..., \mathcal{N}_n^{\mathcal{B}\sim}\right) = \mathcal{N}_{\sigma(j)}^{\mathcal{B}\sim},$$

where $\mathcal{N}_{\sigma(j)}^{\mathcal{B}}$ *is the largest of* $\mathcal{N}_i^{\mathcal{B}}$ $(i = 1, 2, ..., n)$.

5. Similarity Measures of Neutrosophic Bipolar Fuzzy Sets

In Section 4 we define different aggregation operators with the help of operations defined in Section 3. Next in this section we are aiming to define some similarity measures which will be used in the next Section 6. A comparisons of several different fuzzy similarity measures as well as their aggregations have been studied by Beg and Ashraf [38,39]. Theoretical and computational properties of the measures was further investigated with the relationships between them [15,40–42]. A review, or even a listing of all these similarity measures is impossible. Here in this section we define different kinds of similarity measures of neutrosophic bipolar fuzzy sets.

5.1. Neutrosophic Bipolar Fuzzy Distance Measures

Definition 14. *A function* $E : \mathcal{N}^{\mathcal{B}}FSs\,(X) \rightarrow [0,1]$ *is called an entropy for* $\mathcal{N}^{\mathcal{B}}FSs\,(X)$*,*

(1) $E\left(\mathcal{N}^{\mathcal{B}}\right) = 1 \Leftrightarrow \mathcal{N}^{\mathcal{B}}$ *is a crisp set.*

(2) $E\left(\mathcal{N}^{\mathcal{B}}\right) = 0 \Leftrightarrow$

$$\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x) = -\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x), \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x) = -\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x), \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x) = -\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x)\ \forall\ x \in X.$$

(3) $E\left(\mathcal{N}^{\mathcal{B}}\right) = E\left(\mathcal{N}^{\mathcal{B}c}\right)$ *for each* $\forall \mathcal{N}^{\mathcal{B}} \in BFSs\,(X)$.

(4) $E\left(\mathcal{N}_1^{\mathcal{B}}\right) \leq E\left(\mathcal{N}_2^{\mathcal{B}}\right)$ *if* $\mathcal{N}_1^{\mathcal{B}}$ *is less than* $\mathcal{N}_2^{\mathcal{B}}$*, that is,*

$$\begin{aligned}
\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x) &\leq \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x)\ ,\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \leq \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x),\ \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \geq \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x), \\
\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x) &\leq \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x)\ ,\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x) \leq \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x),\ \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x) \geq \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x),
\end{aligned}$$

for $\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x) \leq \left|\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x)\right|$

$$or\ \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \geq \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x),\ \boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \geq \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x),$$

and

$$\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x) \leq \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x) \leq \mathcal{N}_{B_2}^{\mathcal{B}-}(x)\ for\ \boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x) \geq \boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x).$$

Definition 15. *Let* $X = \{x_1, x_2, ..., x_n\}$ *and* $\mathcal{N}^{\mathcal{B}} = (\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$ *be an* $\mathcal{N}^{\mathcal{B}}FS$*. The entropy of* $\mathcal{N}^{\mathcal{B}}FS$ *is denoted by* $E(\mathcal{N}^{\mathcal{B}+}, \mathcal{N}^{\mathcal{B}-})$ *and given by*

$$\left.\begin{aligned}
E(\mathcal{N}^{\mathcal{B}+}) &= \frac{1}{n}\sum_{i=1}^{n} \frac{\min((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}\acute{}(\mathbf{x})),\min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}\acute{}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}\acute{}(\mathbf{x})\right|)}{\max((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}\acute{}(\mathbf{x})),\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}\acute{}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(\mathbf{x})\right|)} \\
E(\mathcal{N}^{\mathcal{B}-}) &= \frac{1}{n}\sum_{i=1}^{n} \frac{\min((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}\acute{}(\mathbf{x})),\min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}\acute{}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}\acute{}(\mathbf{x})\right|)}{\max((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}\acute{}(\mathbf{x})),\max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}\acute{}(\mathbf{x})),\left|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(\mathbf{x})\right|)}
\end{aligned}\right\}, \tag{3}$$

and for a neutrosophic bipolar fuzzy number $\mathcal{N}^{\mathcal{B}} = \langle \mathcal{N}_L^{\mathcal{B}+}, \mathcal{N}_L^{\mathcal{B}-} \rangle$*, the bipolar fuzzy entropy is given by*

$$\left.\begin{aligned}
E(\mathcal{N}_L^{\mathcal{B}+}) &= \frac{\min((\boldsymbol{Tru}_{L_1^+}\acute{}(\mathbf{x}),\min(\boldsymbol{Ind}_{\mathbf{L}_1^+}(\mathbf{x})\acute{}),\left|\boldsymbol{Fal}_{L_1^+}(\mathbf{x})\acute{}\right|)}{\max(\boldsymbol{Tru}_{L_1^+}(\mathbf{x})\acute{},\max(\boldsymbol{Ind}_{\mathbf{L}_1^+}(\mathbf{x})\acute{}),\left|\boldsymbol{Fal}_{\mathbf{L}_1^+}\acute{}(\mathbf{x})\right|)} \\
E(\mathcal{N}_L^{\mathcal{B}-}) &= \frac{\min((\boldsymbol{Tru}_{L_1^-}\acute{}(\mathbf{x}),\min(\boldsymbol{Ind}_{\mathbf{L}_1^-}(\mathbf{x})\acute{}),\left|\boldsymbol{Fal}_{L_1^-}(\mathbf{x})\acute{}\right|)}{\max(\boldsymbol{Tru}_{L_1^-}(\mathbf{x})\acute{},\max(\boldsymbol{Ind}_{\mathbf{L}_1^-}(\mathbf{x})\acute{}),\left|\boldsymbol{Fal}_{\mathbf{L}_1^-}\acute{}(\mathbf{x})\right|)}
\end{aligned}\right\}. \tag{4}$$

Definition 16. *Let* $X = \{x_1, x_2, ..., x_n\}$*. We define the Hamming distance between* $\mathcal{N}_1^{\mathcal{B}}$ *and* $\mathcal{N}_2^{\mathcal{B}}$ *belonging to* $\mathcal{N}^{\mathcal{B}}FSs(X)$ *defined as follows:*

(1) *The Hamming distance:*

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = \frac{1}{2}\sum_{j=1}^{n}(|\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +|\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +||\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)||) \\ & \textit{Hamming distance for positive neutrosophic bipolar sets} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = \frac{1}{2}\sum_{j=1}^{n}(|\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)| \\ & +|\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)| \\ & +||\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)||) \\ & \textit{Hamming distance for negative neutrosophic bipolar sets} \end{array}\right\}. \tag{5}$$

(2) *The normalized Hamming distance:*

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = \frac{1}{2n}\sum_{j=1}^{n}(|\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +|\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +||\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)||) \\ & \textit{normalized Hamming distance for positive neutrosophic bipolar sets} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = \frac{1}{2n}\sum_{j=1}^{n}(|\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +|\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)| \\ & +||\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)||) \\ & \textit{normalized Hamming distance for negative neutrosophic bipolar sets} \end{array}\right\}. \tag{6}$$

(3) *The Euclidean distance:*

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) = & \sqrt{\begin{array}{l} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^2 \end{array}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) = & \sqrt{\begin{array}{l} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^2 \end{array}} \end{array}\right\}. \tag{7}$$

(4) *The normalized Euclidean distance:*

$$\left.\begin{array}{ll} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) = & \sqrt{\begin{array}{l} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^2 \end{array}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) = & \sqrt{\begin{array}{l} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^2 \end{array}} \end{array}\right\}. \tag{8}$$

(5) *Based on the geometric distance formula, we have*

$$\left.\begin{array}{rcl} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = & \left[\begin{array}{c} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = & \left[\begin{array}{c} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \end{array}\right\}. \quad (9)$$

(6) *Normalized geometric distance formula:*

$$\left.\begin{array}{rcl} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = & \left[\begin{array}{c} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = & \left[\begin{array}{c} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \end{array}\right\}, \quad (10)$$

where $\alpha > 0$.

(i) *If* $\alpha = 1$, *then Equations (9) and (10), reduce to Equations (5) and (6).*
(ii) *If* $\alpha = 2$, *then Equations (9) and (10), reduce to Equations (7) and (8).*
(iii) *We define a weighted distance as follows:*

$$\left.\begin{array}{rcl} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = & \left[\frac{1}{2}\sum_{j=1}^{n}k_j\left(\begin{array}{c} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^L \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^L \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right|^L \end{array}\right)\right]^{\frac{1}{\alpha}} \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = & \left[\frac{1}{2}\sum_{j=1}^{n}k_j\left(\begin{array}{c} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^L \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^L \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))\right|^L \end{array}\right)\right]^{\frac{1}{\alpha}} \end{array}\right\}, \quad (11)$$

where $k = (k_1, k_2, \ldots, k_n)^T$ *is the weight vector of* $x_j (j = 1, 2, \ldots, n)$, *and* $\alpha > 0$.

(i) *Especially, if* $\alpha = 1$, *then Equation (11) is reduced as*

$$\left.\begin{array}{rcl} d(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = & \left[\frac{1}{2}\sum_{j=1}^{n}k_j\left(\begin{array}{c} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right| \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right| \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right| \end{array}\right)\right] \\ d(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = & \left[\frac{1}{2}\sum_{j=1}^{n}k_j\left(\begin{array}{c} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right| \\ +\left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right| \\ +\left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right| \end{array}\right)\right] \end{array}\right\}. \quad (12)$$

If $k = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, *then Equation (11) goes to Equation (10), and Equation (12) goes to Equation (6).*

(ii) *If* $\alpha = 2$, *then Equation (11) is reduced to the as:*

$$\left.\begin{aligned} d(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}) &= \sqrt{\begin{array}{c} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^2 \end{array}} \\ d(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}) &= \sqrt{\begin{array}{c} \frac{1}{2}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^2 \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^2 \end{array}} \end{aligned}\right\}. \tag{13}$$

If $k = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, *then Equation (13) is reduced to Equation (8).*

5.2. Similarity Measures of Neutrosophic Bipolar Fuzzy Set

Definition 17. *Let* $\hat{s}$ *be a mapping* $\hat{s} : \Omega(X)^2 \to [0,1]$, *then the degree of similarity between* $\mathcal{N}_1^{\mathcal{B}} \in \Omega(X)$ *and* $\mathcal{N}_2^{\mathcal{B}} \in \Omega(X)$ *is defined as* $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}})$, *which satisfies the following properties: [43,44].*

(1) $0 \le \hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}) \le 1$;
(2) $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}) = 1$ *if* $\mathcal{N}_1^{\mathcal{B}} = \mathcal{N}_2^{\mathcal{B}}$;
(3) $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}) = \hat{s}(\mathcal{N}_2^{\mathcal{B}}, \mathcal{N}_1^{\mathcal{B}})$;
(4) *If* $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}}) = 0$ *and* $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_3^{\mathcal{B}}) = 0$, $\mathcal{N}_3^{\mathcal{B}} \in \Omega(X)$, *then* $\hat{s}(\mathcal{N}_2^{\mathcal{B}}, \mathcal{N}_3^{\mathcal{B}}) = 0$. *We define a similarity measure of* $\mathcal{N}_1^{\mathcal{B}}$ *and* $\mathcal{N}_2^{\mathcal{B}}$ *as:*

$$\left.\begin{aligned} \hat{s}(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}) &= 1 - \left[\begin{array}{c} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \\ \hat{s}(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}) &= 1 - \left[\begin{array}{c} \frac{1}{2n}\sum_{j=1}^{n}(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))^L \\ +(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))^L \end{array}\right]^{\frac{1}{\alpha}} \end{aligned}\right\}, \tag{14}$$

where $\alpha > 0$, *and* $\hat{s}(\mathcal{N}_1^{\mathcal{B}}, \mathcal{N}_2^{\mathcal{B}})$ *is the degree of similarity of* $\mathcal{N}_1^{\mathcal{B}}$ *and* $\mathcal{N}_2^{\mathcal{B}}$. *Now by considering the weight of every element we have,*

$$\left.\begin{aligned} \hat{s}(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}) &= 1 - \left[\frac{1}{2}\sum_{j=1}^{n} k_j \left(\begin{array}{c} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^L \\ + \left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))\right|^L \\ + \left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j))\right|^L \end{array}\right)\right]^{\frac{1}{\alpha}} \\ d(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}) &= 1 - \left[\frac{1}{2}\sum_{j=1}^{n} k_j \left(\begin{array}{c} \left|(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^L \\ + \left|(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))\right|^L \\ + \left|(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j) - \boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j))\right|^L \end{array}\right)\right]^{\frac{1}{\alpha}} \end{aligned}\right\}. \tag{15}$$

If we give equal importance to every member then Equation (15) is reduced to Equation (14). Similarly we may use

$$\left.\begin{aligned}
\hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) &= 1-\left[\frac{\begin{array}{r}\sum_{j=1}^{n}\Big(\Big|\big(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\big)\Big|^{\alpha}\\ +\Big|\big(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\big)\Big|^{L}\\ +\Big|\big(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)\big)\Big|^{L}\end{array}}{\begin{array}{r}\sum_{j=1}^{n}\Big(\Big|\big(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\big)\Big|^{\alpha}\\ +\Big|\big(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\big)\Big|^{L}\\ +\Big|\big(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)\big)\Big|^{L}\Big)\end{array}}\right]^{\frac{1}{\alpha}}\\
\hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) &= 1-\left[\frac{\begin{array}{r}\sum_{j=1}^{n}\Big(\Big|\big(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\big)\Big|^{\alpha}\\ +\Big|\big(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\big)\Big|^{L}\\ +\Big|\big(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)\big)\Big|^{L}\end{array}}{\begin{array}{r}\sum_{j=1}^{n}\Big(\Big|\big(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\big)\Big|^{\alpha}\\ +\Big|\big(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\big)\Big|^{L}\\ +\Big|\big(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)\big)\Big|^{L}\Big)\end{array}}\right]^{\frac{1}{\alpha}}
\end{aligned}\right\}. \tag{16}$$

Now by considering the weight of every element we have

$$\left.\begin{aligned}
\hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) &= 1-\left[\frac{\begin{array}{r}\sum_{j=1}^{n}k_j\Big(\Big|\big(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\big)\Big|^{\alpha}\\ +\Big|\big(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\big)\Big|^{L}\\ +\Big|\big(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\big)\Big|^{L}\end{array}}{\begin{array}{r}\sum_{j=1}^{n}k_j\Big(\Big|\big(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\big)\Big|^{\alpha}\\ +\Big|\big(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\big)\Big|^{L}\\ +\Big|\big(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)\big)\Big|^{L}\Big)\end{array}}\right]^{\frac{1}{\alpha}}\\
\hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) &= 1-\left[\frac{\begin{array}{r}\sum_{j=1}^{n}k_j\Big(\Big|\big(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\big)\Big|^{\alpha}\\ +\Big|\big(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\big)\Big|^{L}\\ +\Big|\big(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\big)\Big|^{L}\end{array}}{\begin{array}{r}\sum_{j=1}^{n}k_j\Big(\Big|\big(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\big)\Big|^{\alpha}\\ +\Big|\big(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\big)\Big|^{L}\\ +\Big|\big(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)-\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)\big)\Big|^{L}\Big)\end{array}}\right]^{\frac{1}{\alpha}}
\end{aligned}\right\}. \tag{17}$$

If we give equal importance to every member, then Equation (17) is reduced to Equation (16).

5.3. Similarity Measures Based on the Set-Theoretic Approach

Definition 18. *Let $\mathcal{N}_1^{\mathcal{B}} \in \Omega(X)$ and $\mathcal{N}_2^{\mathcal{B}} \in \Omega(X)$. Then, we define a similarity measure $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$ from the point of set-theoretic view as:*

$$\left.\begin{aligned}
\hat{s}(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}) &= \frac{\begin{gathered}\textstyle\sum_{j=1}^{n} \langle \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j), \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) \\ + \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j), \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) \\ + \min(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|, |\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|)\end{gathered}}{\begin{gathered}\textstyle\sum_{j=1}^{n} \langle \max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j), \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) \\ + \max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j), \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) \\ + \max(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|, |\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|))\end{gathered}} \\
\hat{s}(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}) &= \frac{\begin{gathered}\textstyle\sum_{j=1}^{n} \langle \min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j), \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)) \\ + \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j), \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)) \\ + \min(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|, |\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))|))\end{gathered}}{\begin{gathered}\textstyle\sum_{j=1}^{n} \langle \max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j), \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)) \\ + \max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j), \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)) \\ + \max(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|, |\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|))\end{gathered}}
\end{aligned}\right\}. \tag{18}$$

Now by considering the weight of every element we have

$$\left.\begin{aligned}
\hat{s}(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}) &= \frac{\begin{gathered}\textstyle\sum_{j=1}^{n} k_j(\min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j), \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) + \min(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j), \\ \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) + \min(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|, |\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|))\end{gathered}}{\begin{gathered}\textstyle\sum_{j=1}^{n} k_j(\max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j), \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) + \max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j), \\ \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j) + \max(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|, |\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|))\end{gathered}} \\
\hat{s}(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}) &= \frac{\begin{gathered}\textstyle\sum_{j=1}^{n} k_j(\min(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j), \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)) + \min(\boldsymbol{Ind}_{N\mathcal{N}_1^{\mathcal{B}-}}(x_j), \\ \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)) + \min(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|, |\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|))\end{gathered}}{\begin{gathered}\textstyle\sum_{j=1}^{n} k_j(\max(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j), \boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) + \max(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j), \\ \boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j) + \max(|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|, |\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|))\end{gathered}}
\end{aligned}\right\}. \tag{19}$$

If we give equal importance to every member, then Equation (19) is reduced to Equation (18).

5.4. Similarity Measures Based on the Matching Functions

We cover the matching function to agreement through the similarity measure of $\mathcal{N}^{\mathcal{B}}FSs$.

Definition 19. *Let $\mathcal{N}_1^{\mathcal{B}} \in \Omega(X)$ and $\mathcal{N}_2^{\mathcal{B}} \in \Omega(X)$, formerly we explain the degree of similarity of $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$ based on the matching function as:*

$$\left.\begin{aligned}
\hat{s}(\mathcal{N}_1^{\mathcal{B}+}, \mathcal{N}_2^{\mathcal{B}+}) &= \frac{\sum_{j=1}^{n}((\boldsymbol{Tru}_{N\mathcal{N}_1^{\mathcal{B}+}}(x_j).\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) + (\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j).\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)) + |\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|.|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|)}{\begin{gathered}\max\langle \textstyle\sum_{j=1}^{n}((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j) + (\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j) + (\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)), \\ \textstyle\sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j) + (\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j) + (\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)))\end{gathered}} \\
\hat{s}(\mathcal{N}_1^{\mathcal{B}-}, \mathcal{N}_2^{\mathcal{B}-}) &= \frac{\sum_{j=1}^{n}((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j).\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)) + (\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j).\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)) + |\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|.|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|)}{\begin{gathered}\max\langle \textstyle\sum_{j=1}^{n}((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j) + (\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j) + (\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)), \\ \textstyle\sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j) + (\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j) + (\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)))\end{gathered}}
\end{aligned}\right\}. \tag{20}$$

Now by considering the weight of every element we have

$$\left.\begin{array}{ll}\hat{s}(\mathcal{N}_1^{\mathcal{B}+},\mathcal{N}_2^{\mathcal{B}+}) & = \dfrac{\sum_{j=1}^{n} k_j(\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j).\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j).\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j))+|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}}(x_j)|.|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}}(x_j)|)}{\begin{array}{c}\max\langle\sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}+}})^2(x_j)),\\ \sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}+}})^2(x_j)))\end{array}} \\ \hat{s}(\mathcal{N}_1^{\mathcal{B}-},\mathcal{N}_2^{\mathcal{B}-}) & = \dfrac{\sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j).\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j).\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j))+|\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}}(x_j)|.|\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}}(x_j)|)}{\begin{array}{c}\max\langle\sum_{j=1}^{n}((\boldsymbol{Tru}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_1^{\mathcal{B}-}})^2(x_j)),\\ \sum_{j=1}^{n} k_j((\boldsymbol{Tru}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Ind}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)+(\boldsymbol{Fal}_{\mathcal{N}_2^{\mathcal{B}-}})^2(x_j)))\end{array}}\end{array}\right\}. \quad (21)$$

(1) *If we give equal importance to every member, then Equation (21) is reduced to Equation (20).*

(2) *If the value of $\hat{s}(\mathcal{N}_1^{\mathcal{B}},\mathcal{N}_2^{\mathcal{B}})$ is larger then its mean $\mathcal{N}_1^{\mathcal{B}}$ and $\mathcal{N}_2^{\mathcal{B}}$ are more closer to each other.*

6. Application

In this Section 5 after defining some similarity measures we proceed towards the main section namely application of the developed model. In this section we provide an algorithm for solving a multiatribute decision making problem related with the HOPE foundation with the help of neutrosophic bipolar fuzzy aggregation operators, neutrosophic bipolar similarity measures under the neutrosophic bipolar fuzzy sets. For detail see [13,42].

Definition 20. *Let $L=\{L_1,L_2,...,L_m\}$ consists of alternatives, and let $P=\{P_1,P_2,...,P_n\}$ containing the attributes and $k=(k_1,k_2,...,k_n)^T$ be the weight vector that describe the importance of attributes such that $k_j\in[0,1]$ and $\sum_{j=1}^{n}k_j=1$. Let us use the neutrosophic bipolar fuzzy sets for L_i as under:*

$$\left.\begin{array}{ll}L_i^+ & =\{\langle P_j,(\boldsymbol{Tru})_{Li}^+(P_j),(\boldsymbol{Ind})_{Li}^+(P_j),(\boldsymbol{Fal})_{Li}^+(P_j)\rangle|P_j\in P\},\ i=1,2,3,...,m\\ L_i^- & =\{\langle P_j,(\boldsymbol{Tru})_{Li}^-(P_j),(\boldsymbol{Ind})_{Li}^-P_j),(\boldsymbol{Fal})_{Li}^-(P_j)\rangle|P_j\in P\},\ i=1,2,3,...,m\end{array}\right\}. \quad (22)$$

such that

$$\begin{array}{rcl}(\boldsymbol{Tru})_{Li}^+(P_j) & \in & [0,3],(\boldsymbol{Ind})_{Li}^+(P_j)\in[0,3],(\boldsymbol{Fal})_{Li}^+(P_j)\in[0,3],\\ 0 & \le & (\boldsymbol{Tru})_{Li}^+(P_j),(\boldsymbol{Ind})_{Li}^+P_j),(\boldsymbol{Fal})_{Li}^+(P_j))\rangle\le 3.\\ (\boldsymbol{Tru})_{Li}^-(P_j) & \in & [-3,0],(\boldsymbol{Ind})_{Li}^-(P_j)\in[-3,0],(\boldsymbol{Fal})_{Li}^-(P_j)\in[-3,0],\\ -3 & \le & (\boldsymbol{Tru})_{Li}^-(P_j),(\boldsymbol{Ind})_{Li}^-P_j),(\boldsymbol{Fal})_{Li}^-(P_j))\rangle\le 0.\end{array}$$

Now we define the positive and negative ideal solutions as under:

$$\left.\begin{array}{ll}L_i^+ & =\{\langle P_j,(\boldsymbol{Tru})_{L^+}^+(P_j),(\boldsymbol{Ind})_{L^+}^+(P_j),(\boldsymbol{Fal})_{L^+}^+(P_j)\rangle|P_j\in P\}\}\\ L_i^- & =\{\langle P_j,(\boldsymbol{Tru})_{L^+}^-(P_j),(\boldsymbol{Ind})_{L^+}^-(P_j),(\boldsymbol{Fal})_{L^+}^-(P_j)\rangle|P_j\in P\}\}\end{array}\right\}, \quad (23)$$

and

$$\left.\begin{array}{ll}L^+ & =\{\langle P_j,(\boldsymbol{Tru})_{L^-}^+(P_j),(\boldsymbol{Ind})_{L^-}^+(P_j),(\boldsymbol{Fal})_{L^-}^+(P_j)\rangle|P_j\in P\}\}\\ L^- & =\{\langle P_j,(\boldsymbol{Tru})_{L^-}^-(P_j),(\boldsymbol{Ind})_{L^-}^-(P_j),(\boldsymbol{Fal})_{L^-}^-(P_j)\rangle|P_j\in P\}\}\end{array}\right\}, \quad (24)$$

where

$$\begin{array}{rcl}(\boldsymbol{Tru})_{L^+}^+(P_j) & = & \max\limits_i\{(\boldsymbol{Tru})_{Li}^+(P_j),\ (\boldsymbol{Tru})_{L^+}^-(P_j)\}=\min\limits_i\{(\boldsymbol{Tru})_{Li}^+(P_j)\},(\boldsymbol{Tru})_{Li}^-(P_j)\\ & = & \max\limits_i\{(\boldsymbol{Tru})_{Li}^-(P_j),\ (\boldsymbol{Tru})_{L^+}^+(P_j)\}=\min\limits_i\{(\boldsymbol{Tru})_{Li}^-(P_j)\}(\boldsymbol{Ind})_{L^+}^+(P_j)\\ & = & \max\limits_i\{(\boldsymbol{Ind})_{Li}^+(P_j),\ (\boldsymbol{Ind})_{L^+}^-(P_j)\}=\min\limits_i\{(\boldsymbol{Ind})_{Li}^+(P_j)\}(\boldsymbol{Ind})_{L^+}^-(P_j)\\ & = & \max\limits_i\{(\boldsymbol{Ind})_{Li}^-(P_j),\ (\boldsymbol{Ind})_{L^+}^+(P_j)\}=\min\limits_i\{(\boldsymbol{Ind})_{Li}^+(P_j)\}.\end{array}$$

$$\begin{aligned}(\boldsymbol{Fal})^{+}_{Li}(P_j) &= \min_i\{(\boldsymbol{Fal})^{+}_{Li}(P_j), (\boldsymbol{Fal})^{-}_{L^+}(P_j)\} = \max_i\{(\boldsymbol{Fal})^{+}_{Li}(P_j)\}.\\ (\boldsymbol{Fal})^{-}_{Li}(P_j) &= \min_i\{(\boldsymbol{Fal})^{-}_{Li}(P_j), (\boldsymbol{Fal})^{+}_{L^+}(P_j)\} = \max_i\{(\boldsymbol{Fal})^{-}_{Li}(P_j)\}.\end{aligned}$$

Now using Equation (15), we find the degree of similarity for L^+, L_i, and L^-, L_i, as under:

$$\left.\begin{aligned}\hat{s}_1(L^+,L_i^+) &= 1-\begin{bmatrix}\frac{1}{2}\sum_{j=1}^{n}k_j(|(\boldsymbol{Tru})^{+}_{L^+}(x_j)-(\boldsymbol{Tru})^{+}_{Li}(x_j)|^{\alpha}\\ +|(\boldsymbol{Ind})^{+}_{L^+}(x_j)-(\boldsymbol{Ind})^{+}_{Li}(x_j)|^{\alpha}\\ +|(\boldsymbol{Fal})^{+}_{L^+}(x_j)-(\boldsymbol{Fal})^{+}_{Li}(x_j)|^{\alpha})\end{bmatrix}^{\frac{1}{\alpha}}\\ \hat{s}_1(L^+,L_i^-) &= 1-\begin{bmatrix}\frac{1}{2}\sum_{j=1}^{n}k_j(|(\boldsymbol{Tru})^{-}_{L^+}(x_j)-(\boldsymbol{Tru})^{-}_{Li}(x_j)|^{\alpha}\\ +|(\boldsymbol{Ind})^{-}_{L^+}(x_j)-(\boldsymbol{Ind})^{-}_{Li}(x_j)|^{\alpha}\\ +|(\boldsymbol{Fal})^{-}_{L^+}(x_j)-(\boldsymbol{Fal})^{-}_{Li}(x_j)|^{\alpha})\end{bmatrix}^{\frac{1}{\alpha}}\end{aligned}\right\}, \tag{25}$$

and

$$\left.\begin{aligned}\hat{s}_1(L^-,L_i^+) &= 1-\begin{bmatrix}\frac{1}{2}\sum_{j=1}^{n}k_j(|(\boldsymbol{Tru})^{+}_{L^-}(x_j)-(\boldsymbol{Tru})^{+}_{Li}(x_j)|^{\alpha}\\ +|(\boldsymbol{Ind})^{+}_{L^-}(x_j)-(\boldsymbol{Ind})^{+}_{Li}(x_j)|^{\alpha}\\ +|(\boldsymbol{Fal})^{+}_{L^-}(x_j)-(\boldsymbol{Fal})^{+}_{Li}(x_j)|^{\alpha})\end{bmatrix}^{\frac{1}{\alpha}}\\ \hat{s}_1(L^-,L_i^-) &= 1-\begin{bmatrix}\frac{1}{2}\sum_{j=1}^{n}k_j(|(\boldsymbol{Tru})^{-}_{L^-}(x_j)-(\boldsymbol{Tru})^{-}_{Li}(x_j)|^{\alpha}\\ +|(\boldsymbol{Ind})^{-}_{L^-}(x_j)-(\boldsymbol{Ind})^{-}_{Li}(x_j)|^{\alpha}\\ +|(\boldsymbol{Fal})^{-}_{L^-}(x_j)-(\boldsymbol{Fal})^{-}_{Li}(x_j)|^{\alpha})\end{bmatrix}^{\frac{1}{\alpha}}\end{aligned}\right\}. \tag{26}$$

Using Equations (25) and (26), calculate di of L_i as under:

$$\left.\begin{aligned}d_i^+ &= \tfrac{s_1(L^+,L_i^+)}{s_1(L^+,L_i^+)+s_1(L^-,L_i^+)},\ i=1,2,\ldots,n.\\ d_i^- &= \tfrac{s_1(L^+,L_i^-)}{s_1(L^+,L_i^-)+s_1(L^-,L_i^-)},\ i=1,2,\ldots,n.\end{aligned}\right\}. \tag{27}$$

If the value of d_i is greater, then the alternative L_i is better.

Also using Equations (17), (19) and (21), we find the degree of similarity for L^+, L_i, and L^-, L_i, as under:

(1) Based on Equation (17), we define the following: We define the following:

$$\left.\begin{aligned}\hat{s}_1(L^+,L_i^+) &= 1-\left[\frac{\begin{matrix}\sum_{j=1}^{n}k_j(\left|(\boldsymbol{Tru}_{L^+}(x_j)-\boldsymbol{Tru}_{L_i^+}(x_j))\right|^{\alpha}\\ +\left|(\boldsymbol{Ind}_{L^+}(x_j)-\boldsymbol{Ind}_{L_i^+}(x_j))\right|^{L}\\ +\left|(\boldsymbol{Fal}_{L^+}(x_j)-\boldsymbol{Fal}_{L_i^+}(x_j))\right|^{L}\end{matrix}}{\begin{matrix}\sum_{j=1}^{n}k_j(\left|(\boldsymbol{Tru}_{L^+}(x_j)-\boldsymbol{Tru}_{L_i^+}(x_j))\right|^{\alpha}\\ +\left|(\boldsymbol{Ind}_{L^+}(x_j)-\boldsymbol{Ind}_{L_i^+}(x_j))\right|^{L}\\ +\left|(\boldsymbol{Fal}_{L^+}(x_j)-\boldsymbol{Fal}_{L_i^+}(x_j))\right|^{L})\end{matrix}}\right]^{\frac{1}{\alpha}}\\ \hat{s}_3(L^+,L_i^-) &= 1-\left[\frac{\begin{matrix}\sum_{j=1}^{n}k_j(\left|(\boldsymbol{Tru}_{L^-}(x_j)-\boldsymbol{Tru}_{L_i^-}(x_j))\right|^{\alpha}\\ +\left|(\boldsymbol{Ind}_{L^-}(x_j)-\boldsymbol{Ind}_{L_i^-}(x_j))\right|^{L}\\ +\left|(\boldsymbol{Fal}_{L^-}(x_j)-\boldsymbol{Fal}_{L_i^-}(x_j))\right|^{L}\end{matrix}}{\begin{matrix}\sum_{j=1}^{n}k_j(\left|(\boldsymbol{Tru}_{L^-}(x_j)-\boldsymbol{Tru}_{L_i^-}(x_j))\right|^{\alpha}\\ +\left|(\boldsymbol{Ind}_{L^-}(x_j)-\boldsymbol{Ind}_{L_i^-}(x_j))\right|^{L}\\ +\left|(\boldsymbol{Fal}_{L^-}(x_j)-\boldsymbol{Fal}_{L_i^-}(x_j))\right|^{L})\end{matrix}}\right]^{\frac{1}{\alpha}}\end{aligned}\right\}. \tag{28}$$

(2) Based on Equation (19), we define the following: We define the following:

$$\left.\begin{array}{ll}\hat{s}_2(L^+,L_i^+) & = \frac{\sum_{j=1}^{n} k_j(\min(\textbf{\textit{Tru}}_{L^+}(x_j),\textbf{\textit{Tru}}_{L_i^+}(x_j))+\min(\textbf{\textit{Ind}}_{L^+}(x_j),\textbf{\textit{Ind}}_{L_i^+}(x_j))+\min(|\textbf{\textit{Fal}}_{L^+}(x_j)|,|\textbf{\textit{Fal}}_{L_i^+}(x_j)|))}{\sum_{j=1}^{n} k_j(\max(\textbf{\textit{Tru}}_{L^+}(x_j),\textbf{\textit{Tru}}_{L_i^+}(x_j))+\max(\textbf{\textit{Ind}}_{L^+}(x_j),\textbf{\textit{Ind}}_{L_i^+}(x_j)+\max(|\textbf{\textit{Fal}}_{L^+}(x_j)|,|\textbf{\textit{Fal}}_{L_i^+}(x_j)|))} \\ \hat{s}_2(L^-,L_i^-) & = \frac{\begin{array}{c}\sum_{j=1}^{n} k_j(\min(\textbf{\textit{Tru}}_{L^-}(x_j),\textbf{\textit{Tru}}_{L_i^-}(x_j)) \\ +\min(\textbf{\textit{Ind}}_{L^-}(x_j),\textbf{\textit{Ind}}_{L_i^-}(x_j)) \\ +\min(|\textbf{\textit{Fal}}_{L^-}(x_j)|,|\textbf{\textit{Fal}}_{L_i^-}(x_j)|))\end{array}}{\begin{array}{c}\sum_{j=1}^{n} k_j(\max(\textbf{\textit{Tru}}_{L^-}(x_j),\textbf{\textit{Tru}}_{L_i^-}(x_j)) \\ +\max(\textbf{\textit{Ind}}_{L^-}(x_j),\textbf{\textit{Ind}}_{L_i^-}(x_j) \\ +\max(|\textbf{\textit{Fal}}_{L^-}(x_j)|,|\textbf{\textit{Fal}}_{L_i^-}(x_j)|))\end{array}}\end{array}\right\}. \quad (29)$$

(3) Based on Equation (21), we define the following: We define the following:

$$\left.\begin{array}{ll}\hat{s}_3(L^+,L_i^+) & = \frac{\begin{array}{c}\sum_{j=1}^{n} k_j(\min((\textbf{\textit{Tru}})_{L^+}^+(x_j),(\textbf{\textit{Tru}})_{Li}^+(x_j)) \\ +\min((\textbf{\textit{Ind}})_{L^+}^+(x_j),(\textbf{\textit{Ind}})_{Li}^+(x_j)) \\ +\min(|(\textbf{\textit{Fal}})_{L^+}^+(x_j)|,|(\textbf{\textit{Fal}})_{Li}^+(x_j)|))\end{array}}{\begin{array}{c}\sum_{j=1}^{n} k_j(\max((\textbf{\textit{Tru}})_{L^+}^+(x_j),(\textbf{\textit{Tru}})_{Li}^+(x_j)) \\ +(\max((\textbf{\textit{Ind}})_{L^+}^+(x_j),(\textbf{\textit{Ind}})_{Li}^+(x_j)) \\ +\max(|(\textbf{\textit{Fal}})_{L^+}^+(x_j)|,|(\textbf{\textit{Fal}})_{Li}^+(x_j)|)\end{array}} \\ \hat{s}_3(L^+,L_i^-) & = \frac{\begin{array}{c}\sum_{j=1}^{n} k_j(\min((\textbf{\textit{Tru}})_{L^+}^-(x_j),(\textbf{\textit{Tru}})_{Li}^-(x_j)) \\ +\min((\textbf{\textit{Ind}})_{L^+}^-(x_j),(\textbf{\textit{Ind}})_{Li}^-(x_j)) \\ +\min(|(\textbf{\textit{Fal}})_{L^+}^-(x_j)|,|(\textbf{\textit{Fal}})_{Li}^-(x_j)|))\end{array}}{\begin{array}{c}\sum_{j=1}^{n} k_j(\max((\textbf{\textit{Tru}})_{L^+}^-(x_j),(\textbf{\textit{Tru}})_{Li}^-(x_j)) \\ +(\max((\textbf{\textit{Ind}})_{L^+}^-(x_j),(\textbf{\textit{Ind}})_{Li}^-(x_j)) \\ +\max(|(\textbf{\textit{Fal}})_{L^+}^-(x_j)|,|(\textbf{\textit{Fal}})_{Li}^-(x_j)|)\end{array}}\end{array}\right\}. \quad (30)$$

Then use (27).

7. Numerical Example

Now we provide a daily life example which shows the applicability of the algorithm provided in Section 6.

Example 1. *The HOPE foundation is an international organization which provides the financial support to the health sector of children of many families in round about 22 different countries in southwest Missouri. This organization provides the support when other organization does not play their role. Every day a child is diagnosed with a severe illness, sustains a debilitating injury, and a family loses the battle with an illness. With these emergencies come unexpected expenses. Here we discuss a problem related with HOPE foundation as:*

HOPE foundation is planning to build a children hospital and they are planning to fit a suitable air conditioning system in the hospital. Different companies offers them different systems. Companies offer three feasible alternatives $L_i = (i = 1,2,3)$*, by observing the hospital' physical structures. Assume that* P_1 *and* P_2*, are the two attributes which are helpful in the installation of air conditioning system with the weight vector as* $k = (0.4, 0.6)^T$ *for the attributes. Now using neutrosophic bipolar fuzzy sets for the alternatives* $L_i = (i = 1,2,3)$ *by examining the different characteristics as under:*

$$\begin{aligned} L_1^+ &= \{\langle P_1, 0.3, 0.4, 0.7\rangle, \langle P_2, 0.8, 0.8, 0.6\rangle\}, \\ L_1^- &= \{\langle P_1, -0.3, -0.2, -0.1\rangle, \langle P_2, -0.4, -0.6, -0.8\rangle\}. \end{aligned}$$

$$\begin{aligned} L_2^+ &= \{\langle P_1, 0.4, 0.6, 0.2\rangle, \langle P_2, 0.3, 0.9, 0.2\rangle\}, \\ L_2^- &= \{\langle P_1, -0.1, -0.3, -0.4\rangle, \langle P_2, -0.8, -0.7, -0.1\rangle\}. \end{aligned}$$

$$\begin{aligned} L_3^+ &= \{\langle P_1, 0.3, 0.5, 0.7\rangle, \langle P_2, 0.2, 0.30.6\rangle\}, \\ L_3^- &= \{\langle P_1, -0.5, -0.1, -0.4\rangle, \langle P_2, -0.3, -0.2, -0.8\rangle\}. \end{aligned}$$

where $L_1^+ = \{\langle P_1, 0.3, 0.4, 0.7\rangle, \langle P_2, 0.8, 0.8, 0.6\rangle\}$ *means that the alternative* L_1 *has the positive preferences which is desirable:* 0.3, 0.8 *as a truth function for past,* 0.4, 0.8 *as a indeterminacy function for future and* 0.7, 0.6 *as a falsity function for present time with respect to the attributes* P_1 *and* P_2 *respectively.*

Similarly $L_1^- = \{\langle P_1, -0.3, -0.2, -0.1\rangle, \langle P_2, -0.4, -0.6, -0.8\rangle\}$ means that the alternative L_1 has the negative preferences which is unacceptable: $-0.3, -0.4$ as a truth function for past, $-0.2, -0.6$ as a indeterminacy function for future and $-0.1, -0.8$ as a falsity function for present time with respect to the attributes P_1 and P_2 respectively.

(1) By Equations (23) and (24) we first calculate L^+ and L^- of the alternatives $L_i = (i = 1, 2, 3)$, as

$$\begin{aligned} L^+ &= \{\langle P_1, 0.4, 0.6, 0.7\rangle, \langle P_2, 0.5, 0.9, 0.6\rangle\}, \\ L^- &= \{\langle P_1, 0.3, 0.4, 0.2\rangle, \langle P_2, 0.2, 0.3, 0.2\rangle\}, \end{aligned}$$

and

$$\begin{aligned} L^+ &= \{\langle P_1, -0.1, -0.1, -0.1\rangle, \langle P_2, -0.3, -0.2, -0.1\rangle\}, \\ L^- &= \{\langle P_1, -0.5, -0.3, -0.4\rangle, \langle P_2, -0.8, -0.7, -0.8\rangle\}. \end{aligned}$$

Then by using Equations (25)–(27), (suppose that $\alpha = 2$ and $k = 1$), we have

$$\begin{aligned} \hat{s}_1(L^+, L_1^+) &= 0.8267, \hat{s}_1(L^+, L_2^+) = 0.775, \hat{s}_1(L^+, L_3^+) = 0.5152, \\ \hat{s}_1(L^+, L_1^-) &= -0.5732, \hat{s}_1(L^+, L_2^-) = -0.8721, \hat{s}_1(L^+, L_3^-) = -0.7776. \end{aligned}$$

$$\begin{aligned} \hat{s}_1(L^-, L_1^+) &= 0.3876, \hat{s}_1(L^-, L_2^+) = 0.5, \hat{s}_1(L^-, L_3^+) = 0.5417, \\ \hat{s}_1(L^-, L_1^-) &= -0.1038, \hat{s}_1(L^-, L_2^-) = -0.2449, \hat{s}_1(L^-, L_3^-) = -0.1119, \end{aligned}$$

and

$$\begin{aligned} \hat{s}_1(L^+, L_1^+) &= -0.2609, \hat{s}_1(L^+, L_2^+) = -0.1157, \hat{s}_1(L^+, L_3^+) = -0.2439, \\ \hat{s}_1(L^+, L_1^-) &= -0.1485, \hat{s}_1(L^+, L_2^-) = -0.075, \hat{s}_1(L^+, L_3^-) = -0.0243. \end{aligned}$$

$$\begin{aligned} \hat{s}_1(L^-, L_1^+) &= -0.6229, \hat{s}_1(L^-, L_2^+) = -0.7146, \hat{s}_1(L^-, L_3^+) = -0.7958, \\ \hat{s}_1(L^-, L_1^-) &= 0.6062, \hat{s}_1(L^-, L_2^-) = 0.3636, \hat{s}_1(L^-, L_3^-) = 0.4803. \end{aligned}$$

Now by Equation (27), we have

$$\left.\begin{array}{c} d_1^+ = 0.7207, d_2^+ = 0.1393, d_3^+ = 0.9093, \\ L_1 > L_2 > L_3 \end{array}\right\}, \tag{31}$$

$$\left.\begin{array}{c} d_1^- = -0.3244, d_2^- = -0.2598, d_3^- = -0.0532, \\ L_3 > L_1 > L_2 \end{array}\right\}, \tag{32}$$

and

$$\left.\begin{array}{c} d_1^+ = 0.2813, d_2^+ = 0.4031, d_3^+ = 0.4728, \\ L_3 > L_2 > L_1 \end{array}\right\}, \tag{33}$$

$$\left.\begin{array}{c} d_1^- = 0.06184, d_2^- = 0.1190, d_3^- = 0.1942, \\ L_3 > L_2 > L_1 \end{array}\right\}. \quad (34)$$

(2) Now by Equations (28) and (29) (suppose that $\alpha = 3$), we have

$$\begin{aligned} \hat{s}_2(L^+, L_1^+) &= 0.9051, \hat{s}_2(L^+, L_2^+) = 0.7283, \hat{s}_2(L^+, L_3^+) = 0.6873, \\ \hat{s}_2(L^+, L_1^-) &= -1.9845, \hat{s}_2(L^+, L_2^-) = -2.338, \hat{s}_2(L^+, L_3^-) = -1.3894. \end{aligned}$$

$$\begin{aligned} \hat{s}_2(L^-, L_1^+) &= 0.6940, \hat{s}_2(L^-, L_2^+) = 0.4952, \hat{s}_2(L^-, L_3^+) = 0.577, \\ \hat{s}_2(L^-, L_1^-) &= -1.0988, \hat{s}_2(L^-, L_2^-) = -1.0717, \hat{s}_2(L^-, L_3^-) = -1.004, \end{aligned}$$

and

$$\begin{aligned} \hat{s}_2(L^+, L_1^+) &= -0.6210, \hat{s}_2(L^+, L_2^+) = -0.6086, \hat{s}_2(L^+, L_3^+) = -0.4944, \\ \hat{s}_2(L^+, L_1^-) &= 0.3714, \hat{s}_2(L^+, L_2^-) = 0.5139, \hat{s}_2(L^+, L_3^-) = 0.3358. \end{aligned}$$

$$\begin{aligned} \hat{s}_2(L^-, L_1^+) &= -2.3840, \hat{s}_2(L^-, L_2^+) = -1.968, \hat{s}_2(L^-, L_3^+) = -2.2632, \\ \hat{s}_2(L^-, L_1^-) &= 0.6972, \hat{s}_2(L^-, L_2^-) = 0.5752, \hat{s}_2(L^-, L_3^-) = 0.6691. \end{aligned}$$

Now again using Equation (27), we have

$$\left.\begin{array}{c} d_1^+ = 0.5660, d_2^+ = 0.5952, d_3^+ = 0.5436, \\ L_2 > L_1 > L_3 \end{array}\right\}, \quad (35)$$

$$\left.\begin{array}{c} d_1^- = 0.6436, d_2^- = 0.6856, d_3^- = 0.5805, \\ L_2 > L_1 > L_3 \end{array}\right\}, \quad (36)$$

and

$$\left.\begin{array}{c} d_1^+ = 0.2066, d_2^+ = 0.2362, d_3^+ = 0.179, \\ L_2 > L_1 > L_3 \end{array}\right\}, \quad (37)$$

$$\left.\begin{array}{c} d_1^- = 0.3475, d_2^- = 0.4719, d_3^- = 0.3341, \\ L_2 > L_1 > L_3 \end{array}\right\}. \quad (38)$$

(3) Thus, by Equations (27), (30) and (31), we have

$$\begin{aligned} \hat{s}_3(L^+, L_1^+) &= 0.4285, \hat{s}_3(L^+, L_2^+) = 0.5675, \hat{s}_3(L^+, L_3^+) = 0.7027, \\ \hat{s}_3(L^+, L_1^-) &= -0.6468, \hat{s}_3(L^+, L_2^-) = -0.6486, \hat{s}_3(L^+, L_3^-) = -0.6316, \end{aligned}$$

and

$$\begin{aligned} \hat{s}_3(L^-, L_1^+) &= 0.4848, \hat{s}_3(L^-, L_2^+) = 0.1538, \hat{s}_3(L^-, L_3^+) = 0.6153, \\ \hat{s}_3(L^-, L_1^-) &= -1.375, \hat{s}_3(L^-, L_2^-) = -1.0625, \hat{s}_2(L^-, L_3^-) = -1.4375. \end{aligned}$$

By Equations (30)–(32) we have

$$\begin{aligned} \hat{s}_3(L^+, L_1^+) &= -0.2727, \hat{s}_3(L^+, L_2^+) = -0.3913, \hat{s}_3(L^+, L_3^+) = -0.3461, \\ \hat{s}_3(L^+, L_1^-) &= 2.6666, \hat{s}_3(L^+, L_2^-) = 2.6666, \hat{s}_3(L^+, L_3^-) = 2.5555. \end{aligned}$$

$$\begin{aligned}\hat{s}_3(L^-,L_1^+) &= -1.060, \hat{s}_3(L^-,L_2^+) = -1.3461, \hat{s}_3(L^-,L_3^+) = -1.4000,\\ \hat{s}_3(L^-,L_1^-) &= 1.4585, \hat{s}_3(L^-,L_2^-) = 1.7500, \hat{s}_3(L^-,L_3^-) = 5217.\end{aligned}$$

By Equations (30)–(32), we have

$$\left.\begin{array}{c} d_1^+ = 0.4691, d_2^+ = 0.7868, d_3^+ = 0.5331, \\ L_2 > L_3 > L_1 \end{array}\right\}, \tag{39}$$

$$\left.\begin{array}{c} d_1^- = 0.3199, d_2^- = 0.3790, d_3^- = 0.3018, \\ L_2 > L_1 > L_3 \end{array}\right\}, \tag{40}$$

and

$$\left.\begin{array}{c} d_1^+ = 0.2046, d_2^+ = 0.2252, d_3^+ = 0.1982, \\ L_2 > L_1 > L_3 \end{array}\right\}, \tag{41}$$

$$\left.\begin{array}{c} d_1^- = 0.3475, d_2^- = 0.6037, d_3^- = 0.6267, \\ L_2 > L_3 > L_1 \end{array}\right\}. \tag{42}$$

From the Equations (35)–(42), we have that the alternative L_2 (feasible alternative) is the best one obtained by all the similarity measures. Thus we conclude that air-conditioning system L_2 is better to installed in the hospital after considering its negative and the positive preferences for past, future and present time.

8. Comparison Analysis

There are a lot of different techniques used so for in decision making problems. For example Chen et al. [23] used fuzzy sets, Atanassov [26] used intutionistic fuzzy sets, Dubios et al. [9], used bipolar fuzzy sets, Zavadskas et al. [37] used neutrosophic sets, Zhan et al. [25], used neutrosophic cubic sets, Ali et al. [33] used bipolar neutrosophic soft sets and so many others discuss decision making problems with respect to the different versions of fuzzy sets. Beg et al., and Xu [38,39,41] discussed similarity measures for fuzzy sets, intutionistic fuzzy sets respectively. In this paper by applying bipolarity to neutrosophic sets allow us to distinguish between the negative and the positive preferences with respect to the past, future and present time which is the unique future of our model. Negative preferences denote what is unacceptable while positive preferences are less restrictive and express what is desirable with respect to the past, future and present time. If we consider only one time frame from the set {past, future and present} one can see our model coincide with bipolar fuzzy sets in decision making as Dubios et al. [9] and Xu [41].

9. Conclusions

We define neutrosophic bipolar fuzzy sets, aggregation operators for neutrosophic bipolar fuzzy sets, similarity measures for neutrosophic bipolar fuzzy sets and produce a real life application in decision making problems. This model can easily used in many directions such as,

(1) Try to solve traffic optimization in transport networks based on local routing using neutrosophic bipolar fuzzy sets.
(2) A hybrid clustering method based on improved artificial bee colony and fuzzy C-Means algorithm using neutrosophic bipolar fuzzy sets.
(3) Hybrid multiattribute group decision making based on neutrosophic bipolar fuzzy sets information and GRA method.
(4) Signatures theory by using neutrosophic bipolar fuzzy sets.
(5) Risk analysis using neutrosophic bipolar fuzzy sets.

Author Contributions: Conceptualization, M.G.; Methodology, R.M.H., M.G. and F.S.; Validation, F.S., M.G.; Formal Analysis, R.M.H. and M.G; Investigation, R.M.H. and M.G.; Resources, F.S. and M.G.; Data Curation, F.S. and M.G.; Writing—Original Draft Preparation, F.S., M.G., and R.M.H.; Visualization, M.G. and R.M.H.; Supervision, F.S.; Project Administration, M.G.; Funding Acquisition, F.S.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Acknowledgments: We, the authors of this project are deeply grateful for the valuable suggestions and comments provided by the respected reviewers to improve the quality and worth or this research. We also acknowledge the professional and devoted attitude of the editor in chief Jose Carlos and associate editor Sunny Sun for providing us the opportunity to present our work in this journal.

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Neutrosophic Duplets of $\{Z_{p^n}, \times\}$ and $\{Z_{pq}, \times\}$ and Their Properties

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Received: 31 July 2018; Accepted: 15 August 2018; Published: 17 August 2018

Abstract: The notions of neutrosophy, neutrosophic algebraic structures, neutrosophic duplet and neutrosophic triplet were introduced by Florentin Smarandache. In this paper, the neutrosophic duplets of Z_{p^n}, Z_{pq} and $Z_{p_1p_2\ldots p_n}$ are studied. In the case of Z_{p^n} and Z_{pq}, the complete characterization of neutrosophic duplets are given. In the case of $Z_{p_1\ldots p_n}$, only the neutrosophic duplets associated with p_is are provided; $i = 1, 2, \ldots, n$. Some open problems related to neutrosophic duplets are proposed.

Keywords: neutrosophic duplets; semigroup; neutrosophic triplet groups

1. Introduction

Real world data, which are predominately uncertain, indeterminate and inconsistent, were represented as neutrosophic set by Smarandache [1]. Neutrosophy deals with the existing neutralities and indeterminacies of the problems. Neutralities in neutrosophic algebraic structures have been studied by several researchers [1–8]. Wang et al. [9] proposed Single-Valued Neutrosophic Set (SVNS) to overcome the difficulty faced in relating neutrosophy to engineering discipline and real world problems. Neutrosophic sets have evolved further as Double Valued Neutrosophic Set (DVNS) [10] and Triple Refined Indeterminate Neutrosophic Set (TRINS) [11]. Neutrosophic sets are useful in dealing with real-world indeterminate data, which Intuitionistic Fuzzy Set (IFS) [12] and Fuzzy sets [13] are incapable of handling accurately [1].

The current trends in neutrosophy and related theories of neutrosophic triplet, related triplet group, neutrosophic duplet, and duplet set was presented by Smarandache [14]. Neutrosophic duplets and neutrosophic triplets have been of interest and many have studied them [15–24]. Neutrosophic duplet semigroup were studied in [19] and the neutrosophic triplet group was introduced in [8]. Neutrosophic duplets and neutrosophic duplet algebraic structures were introduced by Smarandache.

In the case of neutrosophic duplets, we see $ax = a$ and $x = neut(a)$, where, as in L-fuzzy sets [25] as per definition is a mapping from $A : X \to L$, L may be semigroup or a poset or a lattice or a Boolean σ-ring; however, neutrosophic duplets are not mapping, more so in our paper algebraic properties of them are studied for Z_n for specific values of n. However, in the case of all structures, the semigroup or lattice or Boolean σ-ring or a poset, there are elements which are neutrosophic duplets. Here, we mainly analyze neutrosophic duplets in the case of Z_n only number theoretically.

In this paper, we investigate the neutrosophic duplets of $\{Z_{p^n}, \times\}$, where p is a prime (odd or even) and $n \geq 2$. Similarly, neutrosophic duplets in the case of Z_{pq} and $Z_{p_1p_2\ldots p_n}$ are studied. It is noted that the major difference between the neutrals of neutrosophic triplets and that of neutrosophic duplets is that in the former case they are idempotents and in the latter case they are units. Idempotents in the neutrosophic duplets are called trivial neutrosophic duplets.

This paper is organized as five sections, Section 1 is introductory in nature and Section 2 provides the important results of this paper. Neutrosophic duplets in the case of Z_{p^n}; p an odd prime are studied in Section 3. In Section 4, neutrosophic duplets of Z_{pq} and $Z_{p_1 p_2 \ldots p_n}$, and their properties are analyzed. Section 5 discusses the conclusions, probable applications and proposes some open problems.

2. Results

The basic definition of neutrosophic duplet is recalled from [8].

Consider U to be the universe of discourse, and D a set in U, which has a well-defined law #.

Definition 1. *Consider $\langle a, neut(a) \rangle$, where a, and $neut(a)$ belong to D. It is said to be a neutrosophic duplet if it satisfies the following conditions:*

1. *$neut(a)$ is not the same as the unitary element of D in relation with the law # (if any);*
2. *a# $neut(a)$ = neut(a) # a = a; and*
3. *$anti(a) \notin D$ for which a # anti(a) = anti(a) # a = neut(a).*

Here, the neutrosophic duplets of $\{Z_{p^n}, \times\}$, p is a prime (odd or even) and $n \geq 2$ are analyzed number theoretically. Similarly, neutrosophic duplets in the case of Z_{pq} and $Z_{p_1 p_2 \ldots p_n}$ are studied in this paper.

The results proved by this study are:

1. The neutrals of all nontrivial neutrosophic duplets are units of $\{Z_{p^n}, \times\}$, $\{Z_{pq}, \times\}$ and $\{Z_{p_1 p_2 \ldots p_n}, \times\}$.
2. If p is a prime in anyone of the semigroups ($\{Z_{p^n}, \times\}$ or $\{Z_{pq}, \times\}$ or $\{Z_{p_1 p_2 \ldots p_n}, \times\}$) as mentioned in 1, then mp has only p number of neutrals, for the appropriate m.
3. The neutrals of any mp^t for a prime p; $(m, p) = 1$ are obtained and they form a special collection.

3. Neutrosophic Duplets of $\{Z_{p^n}, \times\}$ and its Properties

Neutrosophic duplets and neutrosophic duplet algebraic structures were introduced by Florentin Smarandache in 2016. Here, we investigate neutrosophic duplets of $\{Z_{p^n}, \times\}$, where p is a prime (odd or even) and $n \geq 2$. First, neutrosophic duplets in the case of Z_{2^4} and Z_{3^3} and their associated number theoretic properties are explored to provide a better understanding of the theorems proved. Then, several number theoretical properties are derived.

Example 1. *Let $S = \{Z_{16}, \times\}$ be the semigroup under $\times$ modulo 16. Z_{16} has no idempotents. The units of Z_{16} are $\{1,3,5,7,9,11,13,15\}$. The elements which contribute to the neutrosophic duplets are $\{2,4,6,8,10,12,14\}$. The neutrosophic duplet sets under usual product modulo 16 are:*

$$\{\{2,1\},\{2,9\}\}, \{\{4,1\},\{4,5\},\{4,9\},\{4,13\}\},$$
$$\{\{6,1\},\{6,9\}\}, \{\{8,1\},\{8,3\},\{8,5\},\{8,7\},\{8,9\},\{8,11\},\{8,13\},\{8,15\}\},$$
$$\{\{10,1\},\{10,9\}\}, \{\{12,1\},\{12,5\},\{12,9\},\{12,13\}\}, \{\{14,1\},\{14,9\}\}$$

The observations made from this example are:

1. *Every non-unit of Z_{16} is a neutrosophic duplet.*
2. *Every non-unit divisible by 2, viz. $\{2,6,10,14\}$, has only $\{1,9\}$ as their neutrals.*
3. *Every non-unit divisible by 4 are 4 and 12, which has $\{1,5,9,13\}$ as neutrals.*

The biggest number which divides 16 is 8 and all units act as neutrals in forming neutrosophic duplets. Thus, $A = \{1,3,5,7,9,11,13,15\}$, which forms a group of order 8, yields the 8 neutrosophic duplets; $8 \times i = 8$ for all $i \in A$ and A forms a group under multiplication modulo 16; and $\{1,9\}$ and $\{1,5,9,13\}$ are subgroups of A.

In view of this, we have the following theorem.

Theorem 1. *Let $S = \{Z_{2^n}, \times\}$, be the semigroup under product modulo 2^n, $n \geq 2$.*

(i) *The set of units of S are* $A = \{1, 3, 5, \ldots, 2^n - 1\}$, *forms a group under* $\times$ *and* $|A| = 2^{n-1}$.
(ii) *The set of all neutrosophic duplets with* 2^{n-1} *is A; neutrals of* 2^{n-1} *are A.*
(iii) *All elements of the form* $2m \in Z_{2^n}$ *(m an odd number) has only the elements* $\{1, 2^{n-1} + 1\}$ *to contribute to neutrosophic duplets (neutrals are* $1, 2^{n-1} + 1$*).*
(iv) *All elements of the form* $m2^t \in Z_{2^n}; 1 < t < n-1$*; m odd has its neutrals from* $B = \{1, 2^{n-t} + 1,\ 2^{n-t+1} + 1,\ 2^{n-t+2} + 1,\ \ldots, 2^{n-1} + 1, 2^{n-t} + 2^{n-t+1} + 1, \ldots, 2^{n-t} + 2^{n-1} + 1, \ldots,\ 1 + 2^{n-t} + 2^{n-t+1} + \ldots + 2^{n-1}\}$.

Proof.

(i) Given $S = \{Z_{2^n}, \times\}$ where $n \geq 2$ and S is a semigroup under product modulo 2^n. $A = \{1, 3, 5, 7, \ldots, 2^n - 1\}$ is a group under product as every element is a unit in S and closure axiom is true by property of modulo integers and $|A| = 2^{n-1}$. Hence, Claim (i) is true.
(ii) Now, consider the element 2^{n-1}; the set of duplets for 2^{n-1} is A for $2^{n-1} \times 1 = 2^{n-1}$; $2^{n-1} \times 3 = 2^{n-1}[2+1] = 2^n + 2^{n-1} = 2^{n-1}, \ldots, 2^{n-1}(m)$; ($m$ is odd) will give only $m2^{n-1}$. Hence, this proves Claim (ii).
(iii) Consider $2m \in Z_{2^n}$; we see $2m \times 1 = 2m$ and $2m(2^{n-1} + 1) = m2^n + 2m = 2m$. $(2m, 2^{n-1} + 1)$ is a neutrosophic duplet pair; hence, the claim.
(iv) Let $m2^t \in Z_{2^n}$; clearly, $m2^t \times x = m2^t$ for all $x \in B$.
□

Next, we proceed onto describe the duplet pairs in $S = \{Z_{3^3}, \times\}$.

Example 2. *Let* $S = \{Z_{3^3}, \times\}$ *be a semigroup under product modulo* 3^3*. The units of S are* $A = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26\}$*. Clearly, A forms a group under a product. The non-units of S are* $\{3, 6, 9, 12, 15, 18, 21, 24\}$*. Zero can be included for* $0 \times x = 0$ *for all* $x \in S$*, in particular for* $x \in A$*. The duplet pairs related to* 3 *are* $B_1 = \{\{3, 1\}, \{3, 10\}, \{3, 19\}\}$*. The duplet pairs related to* 6 *are* $B_2 = \{\{6, 1\}, \{6, 10\}, \{6, 19\}\}$*. The duplet pairs related to* 9 *are*

$$B_3 = \{\{9, 1\}, \{9, 4\}, \{9, 7\}, \{9, 13\}, \{9, 10\}, \{9, 16\}, \{9, 19\}, \{9, 22\}, \{9, 25\}\}.$$

The neutrosophic duplets of 12 *are* $B_4 = \{\{12, 1\}, \{12, 10\}, \{12, 19\}\}$*. The neutrosophic duplets of* 15 *are* $B_5 = \{\{15, 1\}, \{15, 10\}, \{15, 19\}\}$*. Finally, the neutrosophic duplets of* 18 *are*

$$B_6 = \{\{18, 1\}, \{18, 4\}, \{18, 7\}, \{18, 13\}, \{18, 10\}, \{18, 16\}, \{18, 19\}, \{18, 22\}, \{18, 25\}\}.$$

The neutrosophic duplets associated with 21 *are* $B_7 = \{\{21, 1\}, \{21, 10\}, \{21, 19\}\}$ *and* 24 *are* $B_8 = \{\{24, 1\}, \{24, 10\}, \{24, 19\}\}$*. Now, the trivial duplet of* 0*, which we take is*

$$B_0 = \{\{0, 1\}, \{0, 4\}, \{0, 7\}, \{0, 13\}, \{0, 10\}, \{0, 16\}, \{0, 19\}, \{0, 22\}, \{0, 25\}\}.$$

We see $L = \{B_0 \cup B_1 \cup B_2 \cup \ldots \cup B_8\}$ *forms a semigroup under product modulo* 27 *and* $o(L) = 45$.

We have the following result.

Theorem 2. *Let* $S = \{Z_{p^n}, \times\}$*, where p is an odd prime,* $n \geq 2$ *is a semigroup under* $\times$*, and product modulo is* p^n*. The units of S are denoted by A and non-units of S are denoted by B. The neutrosophic duplets of S associated with B are groups under product and are subgroups of A. The neutrals of* $tp^s = b \in B$ *are of the form* $D = \{1, 1 + p^{n-s}, 1 + p^{n-s+1}, 1 + p^{n-s+2}, \ldots, 1 + p^{n-1}, 1 + p^{n-s} + p^{n-s+1}, 1 + p^{n-s} + p^{n-s+2}, \ldots, 1 + p^{n-1} + p^{n-s}, \ldots 1 + p^{n-s} + \ldots + p^{n-1}\}$*;* $1 \leq t < m, p/m; 1 < s < n$.

Proof. Let $tp^s \in Z_{p^n}$ all elements which act as neutrosophic duplets for tp^s are from the set D. For any $x \in D$ and $tp^s \in Z_{p^s}$, we see $xtp^s = tp^s$; hence, the claim. □

It is important to note that $S = \{Z_{p^n}, \times\}$ has no non-trivial neutrosophic triplets as Z_{p^n} has no non-trivial idempotents.

Next, we proceed to finding the neutrosophic duplets of Z_{pq}; p and q are distinct primes.

4. Neutrosophic Duplets of Z_{pq} and $Z_{p_1 p_2 \ldots p_n}$

In this section, we study the neutrosophic duplets of Z_{pq} where p and q are primes. Further, we see Z_{pq} also has neutrosophic triplets. The neutrosophic triplets in the case of Z_{pq} have already been characterized in [23]. We find the neutrosophic duplets of Z_{2p}, p a prime. We find the neutrosophic duplets and neutrosophic triplets groups of Z_{26} in the following.

Example 3. *Let $S = \{Z_{26}, \times\}$ be the semigroup under product modulo 26. The idempotents of S are 13 and 14. We see 13 is just a trivial neutrosophic triplet, however only 14 contributes to non-trivial neutrosophic triplets. We now find the neutrosophic duplets of Z_{26}. The units of Z_{26} are $A = \{1, 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 25\}$ and they act as neutrals of the duplets. The non-units which contribute for neutrosophic duplets are $B = \{2, 4, 6, 8, 10, 12, 13, 14, 16, 18, 20, 22, 24\}$. 0 is the trivial duplet as $0 \times x = 0$ for all $x \in A$. Consider $2 \in B$ the pairs of duplets are $\{2, 1\}, 2 \times 14 = 2$ but 14 cannot be taken as $anti(2) = 20$ and $anti(2)$ exists so 2 is not a neutrosophic duplet for $(2, 14, 20)$ is a neutrosophic triplet group.*

Consider $4 \in B; \{4, 1\}$ is a trivial neutrosophic duplet. Then, $4 \times 14 = 4$ and $(4, 14, 16)$ are again a neutrosophic triplet as $anti(4) = 16$ so 4 is not a neutrosophic duplet. Thus, 16 and 20 are also not neutrosophic duplets. Consider $6 \in B$; we see $\{6, 1\}$ is a non-trivial neutrosophic duplet. In addition, $(6, 14, 10)$ are neutrosophic triplet groups so 6 and 10 are not non-trivial neutrosophic duplets. Consider $8 \in B$, $(8, 14, 18)$ is a neutrosophic triplet group. hence 8 and 18 are not neutrosophic duplets. Then, $(12, 14, 12)$ is also a neutrosophic triplet group. Thus, 12 is not a neutrosophic duplet. Let $22 \in B$ be such that $(22, 14, 24)$ is a neutrosophic triplet group, hence 22 and 24 are not neutrosophic duplets.

Consider $13 \in B$; we see the neutrals are $\{1, 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 25\}$. We see the collection of neutrosophic duplets associated with $13 \in Z_{26}$ happens to yield a semigroup under product if 13 is taken as the trivial neutrosophic duplets, as it is an idempotent in Z_{26}, and, in all pairs, it is treated as semigroup of order 13, where $(13, 1)$ and $(13, 13)$ are trivial neutrosophic duplets.

In view of this, we have the following theorem.

Theorem 3. *Let $S = \{Z_{2p}, \times\}$ be a semigroup under product modulo $2p$; p an odd prime. This S has only p and $p + 1$ to be the idempotents and only p contributes for a neutrosophic duplet collection with all units of Z_{2p} and the collection $B = \{(p, x) | x \in Z_{2p}\}$, x is a unit in Z_{2p} forms a commutative semigroup of order p which includes 1 and p which result in the trivial duplets pair $(p, 1)$ and (p, p).*

Proof. Given $S = \{Z_{2p}, \times\}$ is a semigroup under $\times$ and p is an odd prime. We see from [23] p and $p + 1$ are idempotents of Z_{2p}. It is proven in [23] that $p + 1$ acts for the neutrosophic triplet group of Z_{2p} (formed by elements $2, 4, 6, \ldots, 2p - 2$) as the only neutral. (p, p, p) is a trivial neutrosophic triplet. However, Z_{2p} has no neutrosophic duplet other than those related with p alone and $p \times x = p$ for all x belonging to the collection of all units of Z_{2p} including 1. If x is a unit in Z_{2p}, two things are essential: x is odd and $x \neq p$. Since x is odd, we see $x = 2y + 1$ and $p(x) = p(2y + 1) = 2yp + p = p$, hence (p, x) is a neutrosophic duplet. The units of Z_{2p} are $(p - 1)$ in number. Further, (p, p) and $(p, 1)$ form trivial neutrosophic duplets. Thus, the collection of all neutrosophic duplets $B = \{(p, x)\}$, x is a unit and $x = p$ is also taken to form the semigroup of order p and is commutative as the collection of all odd numbers forms a semigroup under product modulo $2p$; hence, the claim. □

It is important and interesting to note that, unlike Z_{p^n}, p is a prime and $n \geq 2$. We see Z_{2p} has both non-trivial neutrosophic triplet groups which forms a classical group [23] as well as has a neutrosophic duplet which forms a semigroup of order p.

Next, we study the case when Z_{pq} is taken where both p and q are odd primes first by an example.

Example 4. *Let* $S = \{Z_{15}, \times\}$ *be a semigroup under product. The idempotents of* Z_{15} *are* 10 *and* 6. *However,* 10 *does not contribute to non-trivial neutrosophic triplet groups other than* $\{5, 10, 5\}$, $\{10, 10, 10\}$. *The neutrosophic triplet groups associated with* 6 *are* $(3, 6, 12)$, $(12, 6, 3)$, $(9, 6, 9)$ *and* $(6, 6, 6)$. *The neutrosophic duplets of* Z_{15} *are contributed by* $\{5\}$, $\{10\}$ *and* $\{3, 12, 6, 9\}$ *in a unique way.*

$$D_1 = \{\{5,1\}, \{5,4\}, \{5,7\}, \{5,13\}, \{5,10\}\},$$
$$D_2 = \{\{10,13\}, \{10,7\}, \{10,1\}, \{10,4\}, \{10,10\}\},$$
$$D_3 = \{\{3,11\}, \{3,1\}, \{3,6\}, \{12,11\}, \{12,1\}, \{12,6\}, \{6,11\}, \{6,1\}, \{6,6\}, \{9,11\}, \{9,1\}, \{9,6\}\}$$

All three collections of duplets put together is not closed under $\times$*; however,* D_2 *and* D_3 *form a semigroup under product modulo* 15. *If we want to make* D_1 *a semigroup, we should adjoin the trivial duplets* $\{0, 4\}$, $\{0, 7\}$, $\{0, 13\}$, $\{0, 1\}$, $\{0, 6\}$, $\{0, 10\}$ *as well as* D_2. *Further, we see* $D_1 \cup D_2 \cup D_3$ *is not closed under product.*

Thus, the study of Z_{pq} where p and q are odd primes happens to be a challenging problem. We give the following examples in the case when $p = 5$ and $q = 7$.

Example 5. *Let* $S = \{Z_{35}, \times\}$ *be a semigroup of order* 35. *The idempotents of* Z_{35} *are* 15 *and* 21. *The neutrosophic triplets associated with* 15 *are* $\{(15, 15, 15), (5, 15, 10), (25, 15, 30), (20, 15, 20), (30, 15, 25), (10, 15, 5)\}$*, a cyclic group of order six. The cyclic group contributed by the neutrosophic triplet groups associated with* 21 *is as follows:* $\{(21, 21, 21), (7, 21, 28), (28, 21, 7), (14, 21, 14)\}$*, which is of order four. The neutrosophic duplets are tabulated in Table 1. Similarly, the neutrosophic duplets associated with* $S = \{Z_{105}, \times\}$ *are tabulated in Table 2.*

Table 1. Neutrosophic Duplets of $\{Z_{35}, \times\}$.

Neutrals for duplets 5, 10, 15, 20, 25, 30	**Neutrals for duplets** 7, 14, 21, 28
1, 8, 15, 22, 24	1, 6, 11, 16, 21, 26, 31

Table 2. Neutrosophic Duplets of $\{Z_{105}, \times\}$.

Neutrals for duplets 3, 6, 9, 12, 18, 21, 24, 27, 33, 36, 39, 48, 51, 54, 57, 66, 69, 78, 81, 87, 93, 96, 99, 102	**Neutrals for duplets** 5, 10, 20, 25, 40, 50, 55, 65, 80, 85, 95, 100
1, 36, 71	1, 22, 43, 64, 84

Neutrals for duplets 7, 14, 28, 49, 56, 77, 91, 98	**Neutrals for duplets** 15, 30, 45, 60, 75, 90
1, 16, 31, 46, 61, 76, 91	1, 8, 15, 22, 29, 36, 43, 50, 57, 64, 71, 78, 85, 92, 99

Neutrals for duplets 21, 42, 63, 84	**Neutrals for duplets** 35, 70
1, 6, 11 16, 21, 26, 31, 36, 41, 46, 51, 56, 61, 66, 71, 76, 81, 86, 91, 96, 101	1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49, 52, 55, 58, 61, 64, 67, 70, 73, 76, 79, 82, 85, 88, 91, 94, 97, 100, 103

Theorem 4. *Let* $\{Z_n, \times\}$ *be a semigroup under product modulo n;* $x \in Z_n \setminus \{0\}$ *has a neutral* $y \in Z_n \setminus \{0\}$ *or is a non-trivial neutrosophic duplet if and only if x is not unit in* Z_n.

Proof. $x \in Z_n \setminus \{0\}$ is a neutrosophic duplet if $x \times y = x(mod\, n)$ and y is called the neutral of x. If $x^2 = x$, then we call the pair (x, x) as trivial neutrosophic duplet pair. We see $x \times y = x$, if x is a unit in Z_n, then there exists a $z \in Z_n$ such that $z \times (x \times y) = z \times x$, so that $y = 1$ as $z \times x = 1(mod\, n)$; so $y = 1$ gives trivial neutrosophic duplets. Thus, x is not a unit if it has to form a non-trivial neutrosophic duplet pair; $x \times y = x$ and $y \neq 1$ then if x is a unit we arrive at contradiction; hence, the theorem. □

Theorem 5. *Let $S = \{Z_{pq}, \times\}$ be a semigroup under product modulo pq, p and q distinct odd primes. There is p number of neutrosophic duplets for every $p, 2p, 3p, \ldots, (q-1)p$. Similarly, there is q number of neutrosophic duplets associated with every $q, 2q, \ldots (p-1)q$. The neutrals of sq and tp is given by $1 + nq$ for $1 \leq t \leq q-1, 0 \leq n \leq p-1$ and that of sq is given by $1 + mp$; $1 \leq s \leq p-1, 0 \leq m \leq q-1$.*

Proof. Given $\{Z_{pq}, \times\}$ is a semigroup under product modulo pq (p and q two distinct odd primes). The neutrals associated with any $tp; 1 \leq t \leq q-1$ is given by the sequence $\{1+q, 2q+1, 3q+1, \ldots, (p-1)q+1\}$ for every $tp \in \{p, 2p, \ldots, (q-1)p\}$. We see, if $tp \in Z_{pq}$,

$$\begin{aligned} &tp \times (1+nq) = tp + tpnq \\ &= tp + tnpq = tp(mod\, pq). \end{aligned}$$

A similar argument for sq completes the proof; hence, the claim. □

Theorem 6. *Let $S = \{Z_{p_1 p_2 \ldots p_n}, \times\}$ be the semigroup under product modulo $p_1 p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are n distinct primes. The duplets are contributed by the non-units of S. The neutrosophic duplets associated with $A_i = \{p_i, 2p_i, \ldots, (p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_n - 1)p_i\}$ are $\{1 + (p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_n)t\}$ where $t = 1, 2, \ldots, p_i - 1$; and $i = 1, 2, \ldots, n$. Thus, every element x_i of A_i has only $p_i - 1$ number of elements which neutralizes x_i; thus, using each x_i, we have $p_i - 1$ neutrosophic duplets.*

Proof. Given $S = \{Z_{p_1 p_2 \ldots p_n}, \times\}$ is a semigroup under product modulo $p_1 \ldots p_n$, where p_is are distinct primes, $i = 1, 2, \ldots, n$. Considering $A_i = \{p_i, 2p_i, \ldots, (p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_n - 1)p_i\}$, we have to prove that, for any sp_i, $sp_i \times [1 + (p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_n)t] = sp_i$; $1 \leq t \leq p_{i-1}$.

Clearly,

$$sp_i \times [1 + (p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_n)t] = sp_i + sp_i[(p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_n)t]$$

$$= sp_i + st[(p_1 p_2 \ldots p_{i-1} p_i p_{i+1} \ldots p_n)] = sp_i$$

as $p_1 p_2 \ldots p_n = 0(mod\, (p_1 p_2 \ldots p_n))$. Hence, the claim. □

Thus, for varying t and varying s given in the theorem, we see

$$\{sp_i, (1 + (p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_n)t)\}$$

is a neutrosophic duplet pair $1 \leq t \leq p_i - 1$; $1 \leq s \leq p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_n$ and $i = 1, 2, \ldots, n$.

5. Discussions and Conclusions

This paper studies the neutrosophic duplets in the case Z_{p^n}, Z_{pq} and $Z_{p_1 p_2 \ldots p_n}$. In the case of Z_{p^n} and Z_{pq}, a complete characterization of them is given; however, in the case $Z_{p_1 \ldots p_n}$, only the neutrosophic duplets associated with p_is are provided; $i = 1, 2, \ldots n$. Further, the following problems are left open:

1. For Z_{pq}, p and q odd primes, how many neutrosophic duplet pairs are there?
2. For $Z_{p_1 \ldots p_n}$, what are the neutrals of $p_i p_j, p_i p_j p_k, \ldots, p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_n$?
3. The study of neutrosophic duplets of $Z_{p_1^{t_1} p_2^{t_2} \ldots p_n^{t_n}}$; $p_1, \ldots, p_n$ are distinct primes and $t_i \geq 1; 1 \leq i \leq n$ is left open.

For future research, one can apply the proposed neutrosophic duplets to SVNS, DVNS or TRINS. These neutrosophic duplets can be applied in problems where neutral elements for a given a in Z_{p^n} or Z_{pq} happens to be many. However, the concept of $anti(a)$ does not exist in the case of neutrosophic duplets. Finally, these neutrosophic duplet collections form a semigroup only when all the trivial neutrosophic duplet pairs $(0, a)$ for all appropriate a are taken. These neutrosophic duplets from Z_{p^n} and Z_{pq} can be used to model suitable problems where the $anti(a)$ under study does not exist and many neutrals are needed. This study can be taken up for further development.

Acknowledgments: The authors would like to thank the reviewers for their reading of the manuscript and many insightful comments and suggestions.

Funding: This research received no external funding.

Author Contributions: The contributions of the authors are roughly equal.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

SVNS	Single Valued Neutrosophic Sets
DVNS	Double Valued Neutrosophic Sets
TRINS	Triple Refined Indeterminate Neutrosophic Sets
IFS	Intuitionistic Fuzzy Sets

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A Neutrosophic Set Based Fault Diagnosis Method Based on Multi-Stage Fault Template Data

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Received: 26 July 2018; Accepted: 13 August 2018; Published: 17 August 2018

Abstract: Fault diagnosis is an important issue in various fields and aims to detect and identify the faults of systems, products, and processes. The cause of a fault is complicated due to the uncertainty of the actual environment. Nevertheless, it is difficult to consider uncertain factors adequately with many traditional methods. In addition, the same fault may show multiple features and the same feature might be caused by different faults. In this paper, a neutrosophic set based fault diagnosis method based on multi-stage fault template data is proposed to solve this problem. For an unknown fault sample whose fault type is unknown and needs to be diagnosed, the neutrosophic set based on multi-stage fault template data is generated, and then the generated neutrosophic set is fused via the simplified neutrosophic weighted averaging (SNWA) operator. Afterwards, the fault diagnosis results can be determined by the application of defuzzification method for a defuzzying neutrosophic set. Most kinds of uncertain problems in the process of fault diagnosis, including uncertain information and inconsistent information, could be handled well with the integration of multi-stage fault template data and the neutrosophic set. Finally, the practicality and effectiveness of the proposed method are demonstrated via an illustrative example.

Keywords: neutrosophic set; fault diagnosis; normal distribution; defuzzification; simplified neutrosophic weighted averaging operator

1. Introduction

Fault diagnosis aims to identify and repair faults in systems, products, and processes, and has been widely applied to various fields, for instance, military [1,2], economic [3,4], and medicine [5,6], and plays a significant part in the prevention of accidents during the normal operation of equipment [7,8]. Owing to the complexity and uncertainty of the actual environment, fault information is usually imprecise, incomplete, and uncertain, and it is thus, difficult to cope with [9–12]. The challenge is to devise a fault diagnosis process to reduce the impact of such imprecision, incompletion, and uncertainty as much as possible. Furthermore, the fault information obtained from multiple sources may be different or even conflicting [13]. In such cases, it is important to check conflicts between the information and to aggregate the information into consistent information.

A great deal of research work has been performed in the field of fault diagnosis, some of which has resulted in the application of efficient approaches to exactly and expeditiously diagnose certain types of faults. Nevertheless, most of these methods fail to diagnose multiple types of faults [14–16]. To solve this problem, some methods based on Bayes theory were proposed [17–19], though efficient aggregation results could only be obtained when the proper and qualified a priori and conditional probabilities were obtainable in the methods based on Bayes theory [20]. As a development of the Bayes theory, the Depmster–Shafer evidence theory was proposed to deal with uncertainty problems [21–24]. Reference [25] describes the integration of the fuzzy set theory and evidence theory to improve the accuracy of various diagnoses. In addition, there have been several research works based on the use of acoustic signals [26–28] for the fault diagnosis of rotating machines. Lee et al. [29] presented a power transformer fault diagnosis method based on set pair analysis (SPA) and association rules. He et al. [30] proposed a novel fault diagnosis method based on the relevance vector machine (RVM) to deal with small data samples. Vibration signal-based fault diagnosis methods [31–33] have also proposed in recent years.

However, uncertain factors in the process of fault diagnosis have not been well handled. In order to deal with uncertain problems under fuzzy information and incoherent information, Smarandache defined the concept of a neutrosophic set [34–37], which is a set of elements that exist in a non-standard unit interval, such as the realness degree, uncertainty degree, or false degree, as a summarization of concepts of the classic set [38], fuzzy set (FS) [39], intuitionistic fuzzy set (IFS) [40,41] and interval valued intuitionistic fuzzy set (IVIFS) [42]. To facilitate the application of the neutrosophic set to practical problems, Wang et al. [43] proposed the concepts of the interval neutrosophic set (INS) and single valued neutrosophic set (SVNS), and Ye [44] defined the concept of the simplified neutrosophic set (SNS). In order to fuse the neutrosophic information to solve realistic problems under a neutrosophic environment, some researchers proposed neutrosophic aggregation operators. For instance, Liu and Wang [45] introduced a single-valued neutrosophic normalised weighted Bonferroni mean operator based on the SVNS. Furthermore, Peng et al. [46] developed simplified neutrosophic information aggregation operators, such as the simplified neutrosophic weighted averaging (SNWA) operator and the simplified neutrosophic weighted geometric (SNWG) operator.

Several methods based on the neutrosophic set have been proposed for fault diagnosis. For instance, Ye proposed cotangent similarity measures for SVNSs based on a cotangent function for the fault diagnosis of steam turbines [47] and the dimension root similarity measure of SVNSs for the fault diagnosis of hydraulic turbines [48], which are all used for fault diagnosis under a single-valued neutrosophic environment. Kong et al. proposed the misfire fault diagnosis method for the fault diagnosis of gasoline engines [49]. Zhang et al. proposed a single-valued neutrosophic (SVN) multi-granulation rough set over a two universe model for the diagnosis of steam turbine faults [50].

There is still a requirement to deal with the uncertainty, imprecision, and incompletion of information and to improve the accuracy of fault diagnosis results with reduced calculations [51–53]. Nevertheless, the complex relationships among fault types and various features of faults in fault diagnosis problems leads to difficulty in fault diagnosis. In addition, with changes in time, the unsteadiness of the actual environment causes uncertainty in fault template data collected at different stages. The uncertainty of multi-stage fault template data, however, fails to be dealt with well. In order to solve this problem, a neutrosophic set based fault diagnosis method based on multi-stage fault template data is proposed in this paper. An unknown fault sample whose fault type is unknown is diagnosed by generating its neutrosophic sets based on multi-stage fault template data, and then the SNWA operator is applied to fuse the multi-stage neutrosophic sets of the unknown fault sample under each feature and to fuse the neutrosophic sets of all features of the unknown fault sample again. Afterward, the fault diagnosis results are determined by the application of the defuzzification method to defuzzy the neutrosophic set of each fault type. This proposed method has several main traits. Firstly, in comparison to some traditional fault methods, for instance, the method based on the relevance vector machine [30], the multi-stage fault template data can deal with the uncertainty of collected data due to the unsteadiness of the actual environment. Afterwards, compared with the method based on random fuzzy variables [54], the application of the neutrosophic set gives consideration to the uncertainty of the fault types and the unknown fault sample, which reflects and handles the uncertainty of fault information well. Compared with former neutrosophic set based methods for fault diagnosis [47–50], the generation of a neutrosophic set based on multi-stage fault template data in this paper can deal with uncertain information better and diagnose the faults efficiently.

The rest of this paper is arranged as follows: Section 2 briefly introduces the concepts of the neutrosophic set, SNS, and the SNWA operator. The proposed method for fault diagnosis is listed step by step in Section 3. In Section 4, a numerical example is used to demonstrate the reasonableness of this proposed method, and to interpret the proposed method. Some summary remarks are shown in Section 5.

2. Preliminaries

The neutrosophic set, introduced by Smarandache [34], is an extension of the classical FS [39], IFS [40], and IVIFS [42]. It is an efficient tool for dealing with the problem with uncertain information. The neutrosophic set concept is defined as follows [43]:

Definition 1. *Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set (A) in X is characterized by a truth-membership function (T_A), an indeterminacy-membership function (I_A) and a falsity-membership function (F_A). $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]0^-,1^+[$. That is,*

$$\begin{aligned} &T_A : X \mapsto]0^-,1^+[\\ &I_A : X \mapsto]0^-,1^+[\\ &F_A : X \mapsto]0^-,1^+[. \end{aligned} \tag{1}$$

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^+$.

In order to promote the application of the neutrosophic set in practical problems, the notion of SNS [44] was proposed as a subclass of the neutrosophic set. The definition of SNS is as follows [44]:

Definition 2. *Let X be a space of points, with a generic element in X denoted by x. A neutrosophic set (A) in X is characterized by a truth-membership function ($T_A(x)$), a indeterminacy-membership function ($I_A(x)$) and a falsity-membership function ($F_A(x)$). If $T_A(x) : X \to [0,1]$, $I_A(x) : X \to [0,1]$ and $F_A(x) : X \to [0,1]$ satisfied:*

$$\begin{aligned} &x \in X \mapsto T_A(x) \in [0,1] \\ &x \in X \mapsto I_A(x) \in [0,1] \\ &x \in X \mapsto F_A(x) \in [0,1] \qquad \text{and} \\ &0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3. \end{aligned} \tag{2}$$

Then an SNS A in X can be denoted as

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\} \tag{3}$$

Which is called an SNS. In particular, if X includes only one element, $N = \langle T_A(x), I_A(x), F_A(x)\rangle$ is called a SNN and is denoted by $\alpha = \langle \mu, \pi, \nu\rangle$. The numbers μ, π, ν denote, respectively, the degree of membership, the degree of indeterminacy-membership, and the degree of non-membership.

For any two SNSs ($A = \langle T_A(x), I_A(x), F_A(x)\rangle, B = \langle T_B(x), I_B(x), F_B(x)\rangle$), the operational relations are defined as the following [44]:

$$\begin{aligned} &A+B = \langle T_A(x)+T_B(x)-T_A(x)T_B(x), I_A(x) + I_B(x) - I_A(x)I_B(x), F_A(x) + F_B(x) - F_A(x)F_B(x)\rangle, \\ &A \times B = \langle T_A(x)T_B(x), I_A(x)I_B(x), F_A(x)F_B(x)\rangle, \\ &\lambda A = \langle 1-(1-T_A(x))^\lambda, 1-(1-I_A(x))^\lambda, 1-(1-F_A(x))^\lambda\rangle, \lambda > 0, \\ &A^\lambda = \langle T_A(x)^\lambda, T_A(x)^\lambda, T_A(x)^\lambda\rangle, \lambda > 0. \end{aligned} \tag{4}$$

Peng et al. [46] developed some simplified neutrosophic information aggregation operators, such as the SNWA operator, which is based on the conception of SNS. It is defined as follows [46]:

Definition 3. *Let* $\alpha_i = \langle \mu_i, \pi_i, \nu_i \rangle, i = 1, 2, \ldots, n$ *be a collection of SNNs. Then,*

$$\begin{aligned} &SNWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = w_1\alpha_1 + w_2\alpha_2 + \ldots + w_n\alpha_n \\ &= \langle 1 - \prod_{i=1}^{n}(1-\mu_i)^{w_i}, \prod_{i=1}^{n}(\pi_i)^{w_i}, \prod_{i=1}^{n}(\nu_i)^{w_i} \rangle, \qquad i = 1, 2, \ldots, n. \end{aligned} \tag{5}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ *is the weight vector of* $\alpha_i (i = 1, 2, \ldots, n)$*, with* $w_i \in [0, 1]$ *and* $\sum_{i=1}^{n} w_i = 1$.

3. The Proposed Method

The characteristics of the actual environment in which a system, product, or process is used, for instance, the temperature, location and air, are unstable over time in the fault diagnosis process, even if the equipment works under the same conditions normally, which leads to uncertainty in the data collected at different stages. These factors have obvious impacts on fault diagnosis results. Thus, the uncertainty of fault information must be dealt with to achieve more efficient diagnosis results. In the face of this problem, a neutrosophic set based fault diagnosis method based on multi-stage fault template data is proposed to diagnose the unknown fault sample in this paper. Consider an unknown fault sample (S) with n features ($C = \{C_1, C_2, \ldots, C_n\}$), whose data have been collected under each feature. The aim of this fault diagnosis method is to identify the fault type of the unknown fault sample (S). The flow-process diagram of the proposed method is shown in Figure 1, and the detailed procedures are elaborated step by step in the following text.

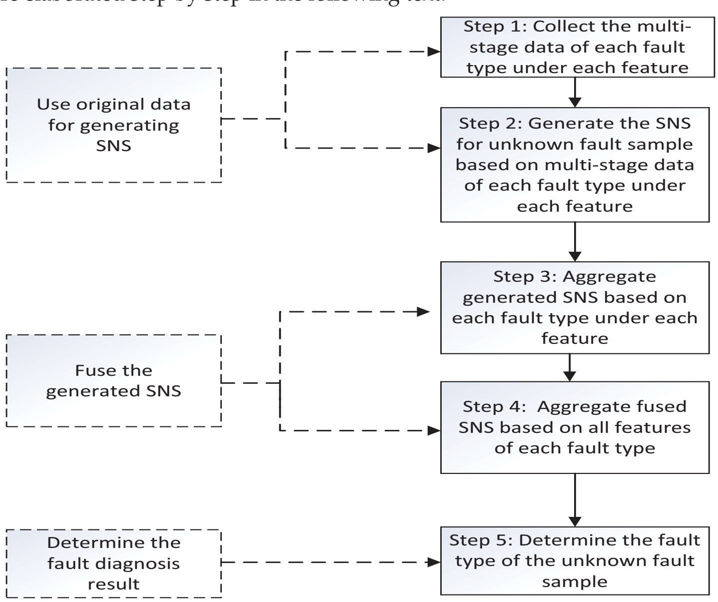

Figure 1. Block diagram of the proposed method.

Step 1 Collect the multi-stage data of fault types under each feature. Suppose that there are m fault types ($F = \{F_1, F_2, \ldots, F_m\}$) with n features ($C = \{C_1, C_2, \ldots, C_n\}$). Firstly, collect the multi-stage data of each fault type under each feature. Each stage's data for each fault type under each feature are obtained by continuously collecting within the time interval (T). Suppose that data from k stages of every fault type under every feature are obtained. The multi-stage data of each fault type under each feature are shown as follows:

$$\begin{array}{c} \\ \vdots \\ F_i \\ \vdots \end{array} \begin{array}{c} \cdots \quad C_j \quad \cdots \\ \left[\begin{array}{ccc} \ddots & \vdots & \ddots \\ \ddots & k\ stages\ data\ of\ F_i\ under\ C_j & \ddots \\ \ddots & \vdots & \ddots \end{array} \right] \end{array}$$

where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Step 2 Generate the SNS for an unknown fault sample (S) based on the multi-stage data of each fault type under each feature. For each stage's data for each fault type under every feature, and for the data of every feature of the unknown fault sample (S), a normal distribution model is established which is obtained by using the arithmetic average (m) and variance (σ^2) of a stage's data as the arithmetic average and standard deviation of the normal distribution model, denoted as $N(m,\ \sigma^2)$. Then, k normal distribution models and k normal distribution figures are generated according to k stages of data of each fault type under each feature. In addition, a normal distribution model is generated based on the data of the unknown fault sample under each feature. The normal distribution figures generated from the data of C_j of unknown fault sample S and k stages of data for C_j of F_i are shown in Figure 2. As the figure shows, each stage's data collected drift to a certain extent in a certain range. In particular, there are distinct differences between the fault type's data collected in the fourth stage and the data of the unknown fault sample.

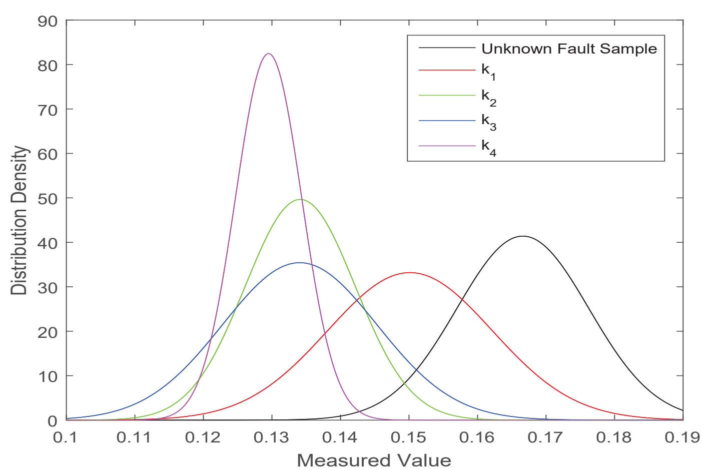

Figure 2. Distribution of S under C_j and F_i under C_j.

The normal distribution function indicates the distribution probability density of the data. The membership degree of SNS is defined as the ratio of the maximum value of the vertical coordinate of the intersection point between the unknown fault sample and the fault type and the peak value of the unknown fault sample. The two normal distribution curves (Figure 3) and the definition of the membership degree (μ) are as follows:

$$\mu = \frac{y_h}{y_m}, \tag{6}$$

where y_h represents the maximum value of the vertical coordinate of the intersection point of distribution between the unknown fault sample (S) and the fault type (F_i), and y_m represents the peak value of the unknown fault sample's distribution.

As the figure shown, the intersection points of distribution between the unknown fault sample and F_i are marked with X, and the peak point of S' distribution is marked with X in the same way. Then, from the Equation (6), the membership degree is generated.

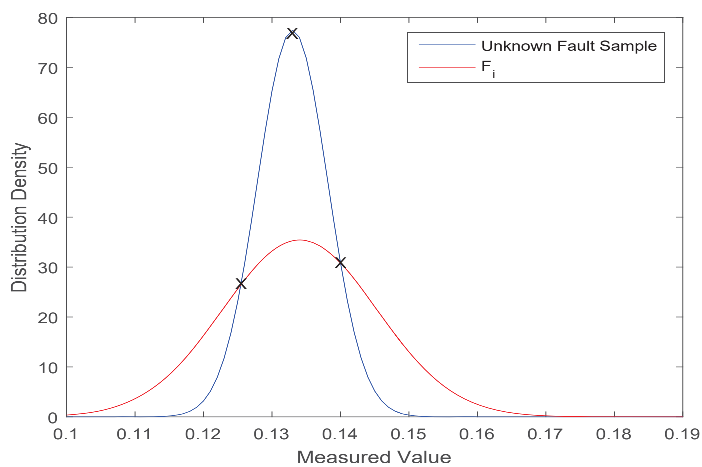

Figure 3. Generation of the membership degree.

In this paper, it is assumed that the non-membership degree and the membership degree are interdependent. The indeterminacy-membership degree indicates the uncertainty degree of neutrosophic information. Entropy represents the uncertainty of the information and has been widely used in many fields. Shannon introduced the quantitative and qualitative model of communication as a statistical process that underlies information theory [55], which is a formalism that was originally applied to digital communication. The indeterminacy-membership degree and non-membership degree are defined as follows:

$$\begin{aligned} &(1) \quad \nu = 1 - \mu, \\ &(2) \quad \pi = \mu log_2(\frac{1}{\mu}) + \nu log_2(\frac{1}{\nu}), \quad \mu \neq 0, \nu \neq 0. \end{aligned} \tag{7}$$

The indeterminacy-membership degree (π) represents the Shannon entropy of the membership degree (μ) and the non-membership degree (ν), and π equals 0 if μ or ν equal 0. Hence, the SNS can be obtained. The generated SNS is shown in Table 1:

Table 1. The generated simplified neutrosophic set (SNS) for S based on multi-stage data.

Fault Type	Stage	Feature			
		C_1	C_2	$\cdots$	C_n
F_1	1	$(\mu_{11}^1, \pi_{11}^1, \nu_{11}^1)$	$(\mu_{12}^1, \pi_{12}^1, \nu_{12}^1)$	$\cdots$	$(\mu_{1n}^1, \pi_{1n}^1, \nu_{1n}^1)$
	2	$(\mu_{11}^2, \pi_{11}^2, \nu_{11}^2)$	$(\mu_{12}^2, \pi_{12}^2, \nu_{12}^2)$	$\cdots$	$(\mu_{1n}^2, \pi_{1n}^2, \nu_{1n}^2)$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	k	$(\mu_{11}^k, \pi_{11}^k, \nu_{11}^k)$	$(\mu_{12}^k, \pi_{12}^k, \nu_{12}^k)$	$\cdots$	$(\mu_{1n}^k, \pi_{1n}^k, \nu_{1n}^k)$
F_2	1	$(\mu_{21}^1, \pi_{21}^1, \nu_{21}^1)$	$(\mu_{22}^1, \pi_{22}^1, \nu_{22}^1)$	$\cdots$	$(\mu_{2n}^1, \pi_{2n}^1, \nu_{2n}^1)$
	2	$(\mu_{21}^2, \pi_{21}^2, \nu_{21}^2)$	$(\mu_{22}^2, \pi_{22}^2, \nu_{22}^2)$	$\cdots$	$(\mu_{2n}^2, \pi_{2n}^2, \nu_{2n}^2)$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	k	$(\mu_{21}^k, \pi_{21}^k, \nu_{21}^k)$	$(\mu_{22}^k, \pi_{22}^k, \nu_{22}^k)$	$\cdots$	$(\mu_{2n}^k, \pi_{2n}^k, \nu_{2n}^k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
F_m	1	$(\mu_{m1}^1, \pi_{m1}^1, \nu_{m1}^1)$	$(\mu_{m2}^1, \pi_{m2}^1, \nu_{m2}^1)$	$\cdots$	$(\mu_{mn}^1, \pi_{mn}^1, \nu_{mn}^1)$
	2	$(\mu_{m1}^2, \pi_{m1}^2, \nu_{m1}^2)$	$(\mu_{m2}^2, \pi_{m2}^2, \nu_{m2}^2)$	$\cdots$	$(\mu_{mn}^2, \pi_{mn}^2, \nu_{mn}^2)$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	k	$(\mu_{m1}^k, \pi_{m1}^k, \nu_{m1}^k)$	$(\mu_{m2}^k, \pi_{m2}^k, \nu_{m2}^k)$	$\cdots$	$(\mu_{mn}^k, \pi_{mn}^k, \nu_{mn}^k)$

Step 3 Aggregate the generated SNS based on each fault type under each feature. In this paper, it is assumed that the weights of data from k stages collected under the same working conditions are equal. The k SNNs of each fault type under each feature are fused via the SNWA operator, as shown in Equation (5). For instance,

$$\alpha_{11} = SNWA(\alpha_{11}^1, \alpha_{11}^2, \ldots, \alpha_{11}^k). \tag{8}$$

Then, the fused SNS matrix (A) is as follows:

$$A = \begin{array}{c} \\ \vdots \\ F_i \\ \vdots \end{array} \begin{array}{c} \cdots \quad C_j \quad \cdots \\ \begin{bmatrix} \ddots & \vdots & \ddots \\ \ddots & (\mu_{ij}, \pi_{ij}, \nu_{ij}) & \ddots \\ \ddots & \vdots & \ddots \end{bmatrix} \end{array}$$

where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Step 4 Aggregate the fused SNS based on all features of each fault type. If the weights of n features are equal, n SNNs of each fault type are fused via the SNWA operator, as shown in Equation (5). For instance,

$$\alpha_1 = SNWA(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}). \tag{9}$$

Then, the fused SNS matrix (F) is as follows:

$$F = \begin{array}{c} F_1 \\ F_2 \\ \vdots \\ F_m \end{array} \begin{bmatrix} (\mu_1, \pi_1, \nu_1) \\ (\mu_2, \pi_2, \nu_2) \\ \vdots \\ (\mu_m, \pi_m, \nu_m) \end{bmatrix}.$$

Step 5 Determine the fault type of the unknown fault sample. Considering the fuzziness of the unknown fault sample and the fault types, direct application of the defuzzification method can intuitively reflect the results of the fault diagnosis and reduce the amount of calculation in the process of fault diagnosis. The crisp number of each SNN is defuzzied and calculated as follows [56]:

$$C_i = \mu_i + (\pi_i)(\frac{\mu_i}{\mu_i + \nu_i}). \tag{10}$$

C_i is the degree to which the information extracted from the data of untested fault supports each fault type. As a result, the ranking order of all the fault types can be determined according to the descending order of their crisp numbers (C_i).

4. Illustrative Example and Discussion

In this section, an example of a motor rotor is used to demonstrate the validity and accuracy rate of the proposed method.

The experimental equipment is a multi-functional flexible rotor test-bed. The vibration displacement sensor and acceleration sensor were placed in the horizontal and vertical directions of the rotor support pedestal, respectively, to collect the rotor vibration signals, and the signals were transmitted to the upper computer through the acquisition box. Then, using the data analysis software under the *LabVIEW* environment, the vibration acceleration spectrum of the rotor and the average amplitude of vibration displacement in the time domain were obtained as the fault feature signals. An unknown fault sample, S_1, was used. When the rotor was running normally, the amplitude of each vibration frequency did not exceed 0.1 m/s^2. When the fault occurred, the frequency and augmentation of the amplitudes of different faults were distinct. The vibration energy of three kinds of fault types were mostly concentrated at $1 - 3X$. Therefore, S_1 was determined to have four features:

1. C_1: The vibration amplitude when the acceleration frequency of the rotor is the basic frequency, $1X$.
2. C_2: The vibration amplitude when the acceleration frequency of the rotor is the frequency $2X$.
3. C_3: The vibration amplitude when the acceleration frequency of the rotor is the frequency $3X$.
4. C_4: The average amplitude of vibration displacement in the time-domain.

The data in this paper originated from ref. [57]. The data of S_1 under each feature was collected. For instance, the data of S_1 under C_1 was as follows:

$$S_{1C_1}\ Data = [0.1421\quad 0.1426\quad 0.1422\quad 0.1422\quad 0.1423\quad 0.1433\quad 0.144\quad 0.1439\quad 0.1437\quad 0.1436\quad 0.1432\quad 0.1434\quad 0.1437\quad 0.1428\quad 0.1424\quad 0.1427\quad 0.1431\quad 0.1425\quad 0.1428\quad 0.1421\quad 0.1424\quad 0.142\quad 0.1422\quad 0.1426\quad 0.1431\quad 0.1428\quad 0.1426\quad 0.1424\quad 0.1422\quad 0.1416\quad 0.1424\quad 0.1429\quad 0.1424\quad 0.1423\quad 0.1421\quad 0.142\quad 0.142\quad 0.1423\quad 0.1425\quad 0.1426].$$

Step 1 Collect the multi-stage data of each fault type under each feature. There are three fault types set up on the test-bed:

1. F_1: Rotor imbalance.
2. F_2: Rotor misalignment.
3. F_3: Support base loosening.

For each feature of each fault type, data from five stages were collected, and for each stage's data, forty consecutive observation values were collected continuously within a time interval of 16 s.

The data in this paper originated from Reference [57]. For instance, the first stage's data of F_1 under C_1 was as follows:

$$F_{1C_1}\ First\ Stage's\ Data = [0.1663\quad 0.1590\quad 0.1568\quad 0.1485\quad 0.1723\quad 0.2006\quad 0.1903\quad 0.1908\quad 0.1986\quad 0.1843\quad 0.1785\quad 0.1610\quad 0.1579\quad 0.1511\quad 0.1532\quad 0.1647\quad 0.1628\quad 0.1646\quad 0.1634\quad 0.1642\quad 0.1648\quad 0.1640\quad 0.1674\quad 0.0661\quad 0.1659\quad 0.1650\quad 0.1633\quad 0.1632\quad 0.1604\quad 0.1542\quad 0.1555\quad 0.1562\quad 0.1540\quad 0.1564\quad 0.1557\quad 0.1542\quad 0.1546\quad 0.1571\quad 0.1537\quad 0.1536].$$

Step 2 Generate the SNS for the unknown fault sample based on the multi-stage data from each fault type under each feature. Each stage's data collected is used to establish the normal distribution model. The generated normal distributions of fault types and the unknown fault sample are listed in Table 2. For instance, the normal distribution of S_{1C_1} data and F_{1C_1} with five stages of data is shown in Figure 4. As the figure shows, each stage's data collected drift to a certain extent in a certain range. In particular, there were distinct differences between the fault types collected in each stage and the data of unknown fault samples. Therefore, it is significant to collect data in multiple stages and to use its integration with the neutrosophic set to deal with the uncertainty of fault information.

Table 2. Multiple distributions of fault types and the unknown fault sample.

Fault Type	Stage	Feature			
		C_1	C_2	C_3	C_4
F_1	1	$N(0.1619, 0.0200)$	$N(0.1538, 0.0112)$	$N(0.1163, 0.0098)$	$N(4.3057, 0.1124)$
	2	$N(0.1596, 0.0073)$	$N(0.1509, 0.0052)$	$N(0.1095, 0.0021)$	$N(4.4143, 0.0226)$
	3	$N(0.1644, 0.0009)$	$N(0.1468, 0.0024)$	$N(0.1063, 0.0037)$	$N(4.2626, 0.6336)$
	4	$N(0.1617, 0.0006)$	$N(0.1519, 0.0316)$	$N(0.1117, 0.0022)$	$N(4.3138, 0.0249)$
	5	$N(0.1598, 0.0010)$	$N(0.1428, 0.0025)$	$N(0.1182, 0.0017)$	$N(4.3319, 0.0347)$
F_2	1	$N(0.1696, 0.0096)$	$N(0.3266, 0.0108)$	$N(0.2772, 0.0250)$	$N(4.9825, 0.1882)$
	2	$N(0.1742, 0.0045)$	$N(0.3278, 0.0083)$	$N(0.2726, 0.0095)$	$N(4.5844, 0.1226)$
	3	$N(0.1932, 0.0138)$	$N(0.3384, 0.0115)$	$N(0.2217, 0.0339)$	$N(4.4358, 0.4015)$
	4	$N(0.1916, 0.0037)$	$N(0.3350, 0.0063)$	$N(0.2131, 0.0053)$	$N(5.0105, 0.6455)$
	5	$N(0.1804, 0.0031)$	$N(0.3187, 0.0041)$	$N(0.2255, 0.0135)$	$N(4.5631, 0.0678)$
F_3	1	$N(0.3387, 0.0071)$	$N(0.3413, 0.0207)$	$N(0.1501, 0.0120)$	$N(9.8483, 0.0709)$
	2	$N(0.3296, 0.0026)$	$N(0.3511, 0.0090)$	$N(0.1341, 0.0080)$	$N(9.7652, 0.0953)$
	3	$N(0.3247, 0.0074)$	$N(0.3409, 0.0135)$	$N(0.1341, 0.0113)$	$N(9.7802, 0.0608)$
	4	$N(0.3265, 0.0049)$	$N(0.3357, 0.0098)$	$N(0.1330, 0.0052)$	$N(9.8739, 0.1267)$
	5	$N(0.3275, 0.0023)$	$N(0.3503, 0.0060)$	$N(0.1295, 0.0048)$	$N(9.7856, 0.1010)$
S_1	1	$N(0.1427, 0.0006)$	$N(0.1109, 0.0316)$	$N(0.1337, 0.0022)$	$N(4.0938, 0.0249)$

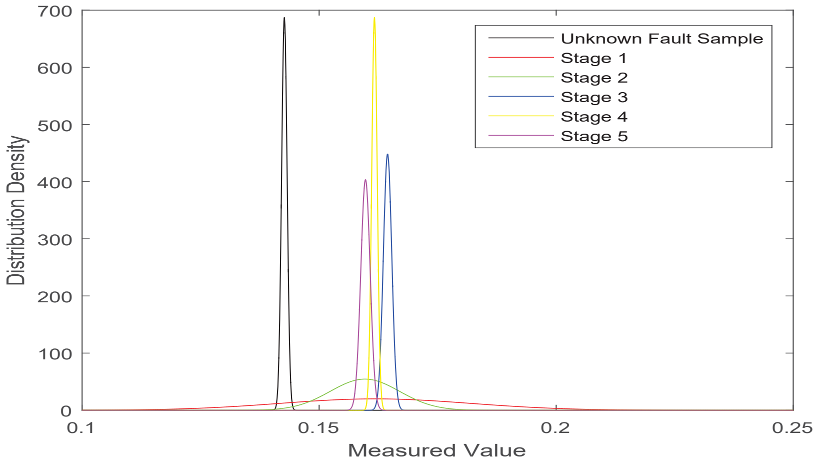

Figure 4. Distribution of S_1 under C_1 and F_1 under C_1.

Then, μ, π, ν are calculated with Equations (6) and (7). For instance, the distribution of S_{1C_1} *Data* was $N(0.1427,\ 0.0006)$, the normal distribution of F_{1C_1}'s first stage of data was $N(0.1619,\ 0.0200)$, and the membership degree of SNN generated from the two distributions is shown in Figure 5. As the figure shows, the intersection points of distribution between the unknown fault sample (S_1) and F_{1C_1}'s first stage data are marked with X, and the peak point of S_1's distribution is marked with X in the same way. Then, from the Equations (6) and (7), the SNN was generated and denoted as $(0.0197,\ 0.0969,\ 0.9803)$. The generated SNSs are listed in Table 3.

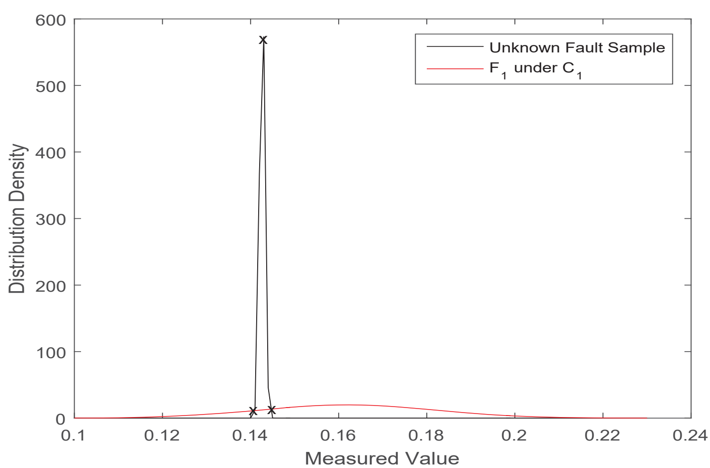

Figure 5. Generation of membership degree.

Table 3. The generated SNS for S_1 based on the multi-stage data from every fault type under every feature.

Fault Type	Stage	Feature			
		C_1	C_2	C_3	C_4
F_1	1	(0.0197, 0.0969, 0.9803)	(0.7400, 0.5731, 0.2600)	(0.0973, 0.3191, 0.9027)	(0.0841, 0.2888, 0.9159)
	2	(0.0092, 0.0521, 0.9908)	(0.6576, 0.6426, 0.3424)	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)
	3	(0.0000, 0.0000, 1.0000)	(0.6382, 0.6545, 0.3618)	(0.0000, 0.0000, 1.0000)	(0.0388, 0.1641, 0.9612)
	4	(0.0000, 0.0000, 1.0000)	(0.8108, 0.4851, 0.1892)	(0.0000, 0.0000, 1.0000)	(0.0001, 0.0006, 0.9999)
	5	(0.0000, 0.0000, 1.0000)	(0.7177, 0.5951, 0.2823)	(0.0004, 0.0032, 0.9996)	(0.0003, 0.0026, 0.9997)
F_2	1	(0.0021, 0.0152, 0.9979)	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)
	2	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	(0.0010, 0.0082, 0.9990)
	3	(0.0001, 0.0010, 0.9999)	(0.0000, 0.0000, 1.0000)	(0.0038, 0.0249, 0.9962)	(0.0486, 0.1944, 0.9514)
	4	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	(0.0164, 0.0836, 0.9836)
	5	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)
F_3	1	(0.0000, 0.0000, 1.0000)	(0.0001, 0.0008, 0.9999)	(0.1118, 0.3502, 0.8882)	(0.0000, 0.0000, 1.0000)
	2	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	(0.2525, 0.5650, 0.7475)	(0.0000, 0.0000, 1.0000)
	3	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	(0.1847, 0.4785, 0.8153)	(0.0000, 0.0000, 1.0000)
	4	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	(0.3815, 0.6648, 0.6185)	(0.0000, 0.0000, 1.0000)
	5	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	(0.4364, 0.6850, 0.5636)	(0.0000, 0.0000, 1.0000)

Step 3 Aggregate the generated SNSs based on each fault type under each feature. Fuse the five stages of SNNs for each fault type under each feature with the SNWA operator, Equation (5). It is assumed that the weights (w) of the five SNNs are $[0.20,\ 0.20,\ 0.20,\ 0.20,\ 0.20]$. For example, the SNNs based on the fault type F_1 under feature C_1 could be fused as follows:

$$\begin{aligned}\alpha_{11} &= SNWA(\alpha_{11}^1, \alpha_{11}^2, \alpha_{11}^3, \alpha_{11}^4, \alpha_{11}^5)\\ &= SNWA((0.0197,\ 0.0969,\ 0.9803), (0.0092,\ 0.0521,\ 0.9908),\\ &\quad (0,\ 0,\ 1), (0,\ 0,\ 1), (0,\ 0,\ 1))\\ &= (0.0058,\ 0.0000,\ 0.9942).\end{aligned}$$

The others are shown in Table 4.

Table 4. The results of fusing the five SNNs of each fault type under each feature.

Fault Type	C_1	C_2	C_3	C_4
F_1	(0.0058, 0.0000, 0.9942)	(0.7200, 0.5868, 0.2800)	(0.0203, 0.0000, 0.9797)	(0.0252, 0.0008, 0.9748)
F_2	(0.0004, 0.0000, 0.9996)	(0.0000, 0.0000, 1.0000)	(0.0008, 0.0000, 0.9992)	(0.0134, 0.0038, 0.9866)
F_3	(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	(0.2836, 0.5332, 0.7164)	(0.0000, 0.0000, 1.0000)

Step 4 Aggregate the fused SNSs based on all features of each fault type. Fusing the SNNs is based on the four features of each fault type by the SNWA operator, Equation (5). In addition, it is supposed the weights (w) of the four SNNs are $[0.25,\ 0.25,\ 0.25,\ 0.25]$. For example, the SNNs based on fault type F_1 could be fused as follows:

$$\begin{aligned}\alpha_1 &= SNWA(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14})\\ &= SNWA((0.0058,\ 0,\ 0.9942), (0.72,\ 0.5868,\ 0.28), (0.0203,\ 0,\ 0.9797),\\ &\quad (0.0252,\ 0.0008,\ 0.9748))\\ &= (0.2633,\ 0.0000,\ 0.7367).\end{aligned}$$

The others are shown in Table 5.

Table 5. The results of fusing the SNNs containing four features based on each fault type.

Fault Type	SNS
F_1	(0.2633, 0.0000, 0.7367)
F_2	(0.0030, 0.0000, 0.9970)
F_3	(0.0952, 0.0000, 0.9048)

Step 5 Determine the fault type of the unknown fault sample. Finally, Table 5 can be regarded as an SNN fault diagnosis matrix which can be used to rank the three fault types via the defuzzification method (Equation (10)). The descendant ranks of the crisp numbers of the three fault types are shown in Table 6.

Table 6. The ranks of the crisp numbers of three fault types.

Fault Type	Crisp Number	Rank
F_1	0.263335	1
F_2	0.003040	3
F_3	0.095221	2

The above ranking results show that the fault type diagnosed by the proposed method is F_1, which is consistent with the true fault type.

In addition, taking the distribution of the data of S_1 under a certain feature, for instance, C_3, and the distribution of the first stage's data of each fault type under the identical feature as an example, the distribution figure is shown in Figure 6. As the figure shows, the maximum intersection points of the ordinate of distribution between S_1 and each fault type (F_i) are marked with X, and the peak point of S_1's distribution is marked with X in the same way. Then, from the calculation formula of the membership degree (Equation (6)), it is clear that the membership of S_1 to F_3 is the maximal one, which conflicts with the originally known information that S_1's actual fault type is F_1, and this situation is not rare. Therefore, the integration of multi-stage fault template data and the neutrosophic set is efficient and significant, and it fuses the conflicting information into coordinated information and obtains the correct diagnosis results.

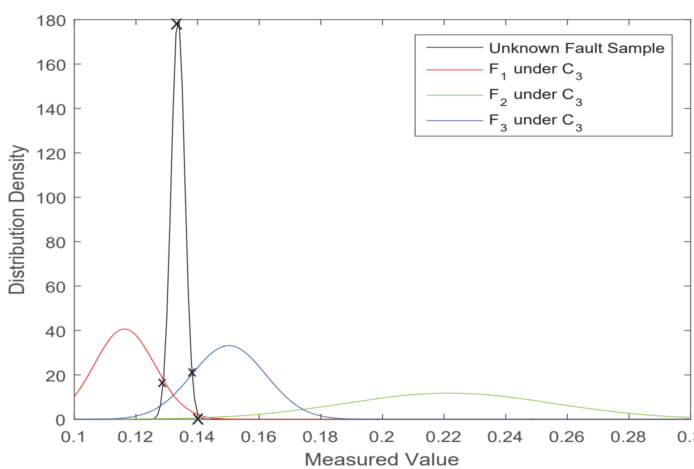

Figure 6. The distribution of S_1 and the three fault types.

Moreover, the proposed method was used to verify the other two unknown fault samples, and these diagnosis results were also correct. The diagnosis result of the three unknown fault samples are shown in Figure 7, where the ordinate indicates the crisp number of the defuzzification result, and the abscissa indicates the fault types. As shown in this figure, the crisp numbers of the unknown fault sample of each fault type are plotted with a line chart. When the crisp number of an unknown fault sample for a certain fault type (F_i) is maximal, the diagnosed fault type of the unknown fault sample is F_i.

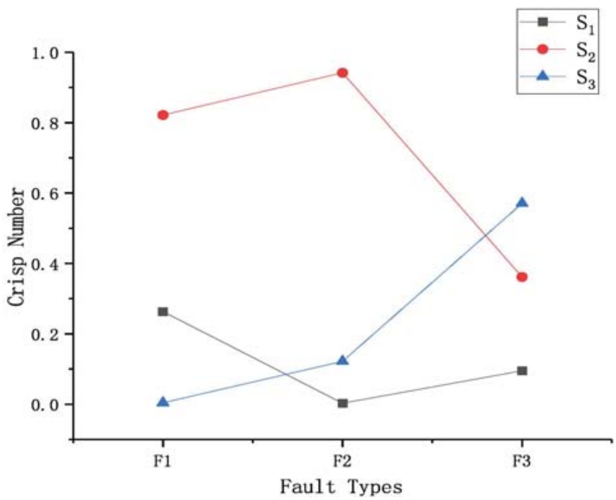

Figure 7. The diagnosis results of the three unknown fault samples.

Compared with Xu's method [54], which was used to diagnose three unknown fault samples (S_1, S_2, S_3), the proposed method was also applied to diagnose identical three unknown fault samples (S_1, S_2, S_3) to demonstrate the reasonableness of this proposed method. The diagnosis results are shown in Table 7.

Table 7. Diagnosis results of the proposed method and Xu's method.

Unknown Fault	Mehod	Rank of Fault Types			Diagnosis Result	Validity
		F1	F2	F3		
S_1	The proposed method	1	3	2	F_1	*Correct*
	Xu' method [54]	1	3	2	F_1	*Correct*
S_2	The proposed method	2	1	3	F_2	*Correct*
	Xu' method [54]	2	1	3	F_2	*Correct*
S_3	The proposed method	3	2	1	F_3	*Correct*
	Xu' method [54]	3	2	1	F_3	*Correct*

From the diagnosis results in Table 7, it is concluded that the similar rankings for all fault types and diagnosis results indicates the practicality and effectiveness of the proposed method. Xu's method [54] only applies to the minimum and maximum mean values of five stages of data, whose boundary rests with the several stages of data collected. However, it is widely admitted that each stage's data would drift to a certain extent over a certain range, and the deviation of data due to the unsteadiness of the actual environment is one of most influencing causes in fault diagnosis results. It is difficult for Xu's method [54] to express and deal with the uncertainty of multi-stage fault template data, which the proposed method coped with appropriately due to the integration of multi-stage fault template data and the neutrosophic set. In adition, the crisp numbers fail to precisely express the information extracted from the data collected due to the unsteadiness of measuring the environment. The neutrosophic set, however, was able to accurately describe the uncertain phenomenon, as it gives consideration to both the uncertainty of fault types and the unknown fault sample. Most kinds of uncertain problems in the process of fault diagnosis, including uncertain information and inconsistent information could be handled well with the integration of multi-stage fault template data and the neutrosophic set.

5. Conclusions

In this paper, to deal with uncertain problems in fault diagnosis, a fault diagnosis method was developed by defuzzying the neutrosophic set obtained from multi-stage data. The focus of this method is the collection of data in multiple stages and the generation of SNS, which was expected to appropriately minimize the uncertainty of fault type information and unknown fault sample information. An illustrative example was provided in this paper, and the results of this example indicate that the proposed method can effectively diagnose the fault type of an unknown fault sample. This neutrosophic set based fault diagnosis method based on multi-stage fault template data not only handles the uncertainty of information collected in fault diagnosis well, but also provides a method for fault diagnosis where there are complicated corresponding relationships between multiple fault types and their features. It is both efficient and convenient when dealing with fault diagnosis problems. Further work will focus on the following directions. An appropriate method for the calculation of features' weights based on the information collected is planned. In addition, for the convenience of calculation, the double aggregation of netrosophic sets may be simplified in future work.

Author Contributions: W.J. and X.D. proposed the method; W.J., Y.Z. and X.D. analyzed the results of experiment; Y.Z. wrote the paper; W.J. and X.D. revised and improved the paper.

Funding: The work is partially supported by National Natural Science Foundation of China (Program No. 61671384,61703338), Natural Science Basic Research Plan in Shaanxi Province of China (Program No. 2018JQ6085), the Seed Foundation of Innovation and Creation for Graduate Students in Northwestern Polytechnical University (Program No. ZZ2017126).

Conflicts of Interest: The authors declare that there is no conflict of interest regarding the publication of this paper.

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Commutative Generalized Neutrosophic Ideals in BCK-Algebras

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Received: 19 July 2018; Accepted: 17 August 2018; Published: 20 August 2018

Abstract: The concept of a commutative generalized neutrosophic ideal in a BCK-algebra is proposed, and related properties are proved. Characterizations of a commutative generalized neutrosophic ideal are considered. Also, some equivalence relations on the family of all commutative generalized neutrosophic ideals in BCK-algebras are introduced, and some properties are investigated.

Keywords: (commutative) ideal; generalized neutrosophic set; generalized neutrosophic ideal; commutative generalized neutrosophic ideal

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1. Introduction

In 1965, Zadeh introduced the concept of fuzzy set in which the degree of membership is expressed by one function (that is, truth or t). The theory of fuzzy set is applied to many fields, including fuzzy logic algebra systems (such as pseudo-BCI-algebras by Zhang [1]). In 1986, Atanassov introduced the concept of intuitionistic fuzzy set in which there are two functions, membership function (t) and nonmembership function (f). In 1995, Smarandache introduced the new concept of neutrosophic set in which there are three functions, membership function (t), nonmembership function (f) and indeterminacy/neutrality membership function (i), that is, there are three components (t, i, f) = (truth, indeterminacy, falsehood) and they are independent components.

Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [2–10]. Moreover, Zhang et al. studied totally dependent-neutrosophic sets, neutrosophic duplet semi-group and cancellable neutrosophic triplet groups (see [11,12]). Song et al. proposed the notion of generalized neutrosophic set and applied it to BCK/BCI-algebras.

In this paper, we propose the notion of a commutative generalized neutrosophic ideal in a BCK-algebra, and investigate related properties. We consider characterizations of a commutative generalized neutrosophic ideal. Using a collection of commutative ideals in BCK-algebras, we obtain a commutative generalized neutrosophic ideal. We also establish some equivalence relations on the family of all commutative generalized neutrosophic ideals in BCK-algebras, and discuss related basic properties of these ideals.

2. Preliminaries

A set X with a constant element 0 and a binary operation $*$ is called a BCI-algebra, if it satisfies $(\forall x, y, z \in X)$:

(I) $((x*y)*(x*z))*(z*y)=0$,
(II) $(x*(x*y))*y=0$,
(III) $x*x=0$,
(IV) $x*y=0,\ y*x=0\ \Rightarrow\ x=y$.

A BCI-algebra X is called a BCK-algebra, if it satisfies $(\forall x\in X)$:

(V) $0*x=0$,

For any BCK/BCI-algebra X, the following conditions hold $(\forall x,y,z\in X)$:

$$x*0=x, \tag{1}$$
$$x\le y\ \Rightarrow\ x*z\le y*z,\ z*y\le z*x, \tag{2}$$
$$(x*y)*z=(x*z)*y, \tag{3}$$
$$(x*z)*(y*z)\le x*y \tag{4}$$

where the relation $\le$ is defined by: $x\le y\iff x*y=0$. If the following assertion is valid for a BCK-algebra X, $\forall x,y\in X$,

$$x*(x*y)=y*(y*x). \tag{5}$$

then X is called a commutative BCK-algebra.

Assume I is a subset of a BCK/BCI-algebra X. If the following conditions are valid, then we call I is an ideal of X:

$$0\in I, \tag{6}$$
$$(\forall x\in X)\,(\forall y\in I)\,(x*y\in I\ \Rightarrow\ x\in I)\,. \tag{7}$$

A subset I of a BCK-algebra X is called a commutative ideal of X if it satisfies (6) and

$$(\forall x,y,z\in X)\,((x*y)*z\in I,\ z\in I\ \Rightarrow\ x*(y*(y*x))\in I)\,. \tag{8}$$

Recall that any commutative ideal is an ideal, but the inverse is not true in general (see [7]).

Lemma 1 ([7]). *Let I be an ideal of a BCK-algebra X. Then I is commutative ideal of X if and only if it satisfies the following condition for all x,y in X:*

$$x*y\in I\ \Rightarrow\ x*(y*(y*x))\in I. \tag{9}$$

For further information regarding BCK/BCI-algebras, please see the books [7,13].

Let X be a nonempty set. A fuzzy set in X is a function $\mu: X\to[0,1]$, and the complement of μ, denoted by μ^c, is defined by $\mu^c(x)=1-\mu(x)$, $\forall x\in X$. A fuzzy set μ in a BCK/BCI-algebra X is called a fuzzy ideal of X if

$$(\forall x\in X)(\mu(0)\ge\mu(x)), \tag{10}$$
$$(\forall x,y\in X)(\mu(x)\ge\min\{\mu(x*y),\mu(y))\}. \tag{11}$$

Assume that X is a non-empty set. A neutrosophic set (NS) in X (see [14]) is a structure of the form:

$$A:=\{\langle x;A_T(x),A_I(x),A_F(x)\rangle\mid x\in X\}$$

where $A_T : X \to [0,1]$, $A_I : X \to [0,1]$, and $A_F : X \to [0,1]$. We shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

A generalized neutrosophic set (GNS) in a non-empty set X is a structure of the form (see [15]):

$$A := \{\langle x; A_T(x), A_{IT}(x), A_{IF}(x), A_F(x)\rangle \mid x \in X, A_{IT}(x) + A_{IF}(x) \leq 1\}$$

where $A_T : X \to [0,1]$, $A_F : X \to [0,1]$, $A_{IT} : X \to [0,1]$, and $A_{IF} : X \to [0,1]$.

We shall use the symbol $A = (A_T, A_{IT}, A_{IF}, A_F)$ for the generalized neutrosophic set

$$A := \{\langle x; A_T(x), A_{IT}(x), A_{IF}(x), A_F(x)\rangle \mid x \in X, A_{IT}(x) + A_{IF}(x) \leq 1\}.$$

Note that, for every GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X, we have (for all x in X)

$$(\forall x \in X)\ (0 \leq A_T(x) + A_{IT}(x) + A_{IF}(x) + A_F(x) \leq 3).$$

If $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a GNS in X, then $\Box A = (A_T, A_{IT}, A_{IT}^c, A_T^c)$ and $\Diamond A = (A_F^c, A_{IF}^c, A_{IF}, A_F)$ are also GNSs in X.

Given a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in a BCK/BCI-algebra X and $\alpha_T, \alpha_{IT}, \beta_F, \beta_{IF} \in [0,1]$, we define four sets as follows:

$$\begin{aligned}
U_A(T, \alpha_T) &:= \{x \in X \mid A_T(x) \geq \alpha_T\},\\
U_A(IT, \alpha_{IT}) &:= \{x \in X \mid A_{IT}(x) \geq \alpha_{IT}\},\\
L_A(F, \beta_F) &:= \{x \in X \mid A_F(x) \leq \beta_F\},\\
L_A(IF, \beta_{IF}) &:= \{x \in X \mid A_{IF}(x) \leq \beta_{IF}\}.
\end{aligned}$$

A GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in a BCK/BCI-algebra X is called a generalized neutrosophic ideal of X (see [15]) if

$$(\forall x \in X) \begin{pmatrix} A_T(0) \geq A_T(x),\ A_{IT}(0) \geq A_{IT}(x) \\ A_{IF}(0) \leq A_{IF}(x), A_F(0) \leq A_F(x) \end{pmatrix}, \tag{12}$$

$$(\forall x, y \in X) \begin{pmatrix} A_T(x) \geq \min\{A_T(x * y), A_T(y)\} \\ A_{IT}(x) \geq \min\{A_{IT}(x * y), A_{IT}(y)\} \\ A_{IF}(x) \leq \max\{A_{IF}(x * y), A_{IF}(y)\} \\ A_F(x) \leq \max\{A_F(x * y), A_F(y)\} \end{pmatrix}. \tag{13}$$

3. Commutative Generalized Neutrosophic Ideals

Unless specified, X will always represent a BCK-algebra in the following discussion.

Definition 1. *A GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X is called a commutative generalized neutrosophic ideal of X if it satisfies the condition* (12) *and*

$$(\forall x, y, z \in X) \begin{pmatrix} A_T(x * (y * (y * x))) \geq \min\{A_T((x * y) * z), A_T(z)\} \\ A_{IT}(x * (y * (y * x))) \geq \min\{A_{IT}((x * y) * z), A_{IT}(z)\} \\ A_{IF}(x * (y * (y * x))) \leq \max\{A_{IF}((x * y) * z), A_{IF}(z)\} \\ A_F(x * (y * (y * x))) \leq \max\{A_F((x * y) * z), A_F(z)\} \end{pmatrix}. \tag{14}$$

Example 1. *Denote $X = \{0, a, b, c\}$. The binary operation $*$ on X is defined in Table 1.*

Table 1. The operation "$*$".

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

We can verify that $(X, *, 0)$ *is a BCK-algebra (see [7]). Define a GNS* $A = (A_T, A_{IT}, A_{IF}, A_F)$ *in* X *by Table 2.*

Table 2. GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$.

X	$A_T(x)$	$A_{IT}(x)$	$A_{IF}(x)$	$A_F(x)$
0	0.7	0.6	0.1	0.3
a	0.5	0.5	0.2	0.4
b	0.3	0.2	0.4	0.6
c	0.3	0.2	0.4	0.6

Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ *is a commutative generalized neutrosophic ideal of* X.

Theorem 1. *Every commutative generalized neutrosophic ideal is a generalized neutrosophic ideal.*

Proof. Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. $\forall x, z \in X$, we have

$$A_T(x) = A_T(x * (0 * (0 * x))) \geq \min\{A_T((x * 0) * z), A_T(z)\} = \min\{A_T(x * z), A_T(z)\},$$

$$A_{IT}(x) = A_{IT}(x * (0 * (0 * x))) \geq \min\{A_{IT}((x * 0) * z), A_{IT}(z)\} = \min\{A_{IT}(x * z), A_{IT}(z)\},$$

$$A_{IF}(x) = A_{IF}(x * (0 * (0 * x))) \leq \max\{A_{IF}((x * 0) * z), A_{IF}(z)\} = \max\{A_{IF}(x * z), A_{IF}(z)\},$$

and

$$A_F(x) = A_F(x * (0 * (0 * x))) \leq \max\{A_F((x * 0) * z), A_F(z)\} = \max\{A_F(x * z), A_F(z)\}.$$

Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal. □

The following example shows that the inverse of Theorem 1 is not true.

Example 2. *Let* $X = \{0, 1, 2, 3, 4\}$ *be a set with the binary operation* $*$ *which is defined in Table 3.*

Table 3. The operation "$*$".

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

We can verify that $(X, *, 0)$ *is a BCK-algebra (see [7]). We define a GNS* $A = (A_T, A_{IT}, A_{IF}, A_F)$ *in* X *by Table 4.*

Table 4. GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$.

X	$A_T(x)$	$A_{IT}(x)$	$A_{IF}(x)$	$A_F(x)$
0	0.7	0.6	0.1	0.3
1	0.5	0.4	0.2	0.6
2	0.3	0.5	0.4	0.4
3	0.3	0.4	0.4	0.6
4	0.3	0.4	0.4	0.6

It is routine to verify that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X, but A is not a commutative generalized neutrosophic ideal of X since

$$A_T(2 * (3 * (3 * 2))) = A_T(2) = 0.3 \ngeq \min\{A_T((2 * 3) * 0), A_T(0)\}$$

and/or

$$A_{IF}(2 * (3 * (3 * 2))) = A_{IF}(2) = 0.4 \nleq \max\{A_{IF}((2 * 3) * 0), A_{IF}(0)\}.$$

Theorem 2. *Suppose that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is commutative if and only if it satisfies the following condition.*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x * y) \leq A_T(x * (y * (y * x))) \\ A_{IT}(x * y) \leq A_{IT}(x * (y * (y * x))) \\ A_{IF}(x * y) \geq A_{IF}(x * (y * (y * x))) \\ A_F(x * y) \geq A_F(x * (y * (y * x))) \end{array} \right). \tag{15}$$

Proof. Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. Taking $z = 0$ in (14) and using (12) and (1) induces (15).

Conversely, let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a generalized neutrosophic ideal of X satisfying the condition (15). Then

$$A_T(x * (y * (y * x))) \geq A_T(x * y) \geq \min\{A_T((x * y) * z), A_T(z)\},$$

$$A_{IT}(x * (y * (y * x))) \geq A_{IT}(x * y) \geq \min\{A_{IT}((x * y) * z), A_{IT}(z)\},$$

$$A_{IF}(x * (y * (y * x))) \leq A_{IF}(x * y) \leq \max\{A_{IF}((x * y) * z), A_{IF}(z)\}$$

and

$$A_F(x * (y * (y * x))) \leq A_F(x * y) \leq \max\{A_F((x * y) * z), A_F(z)\}$$

for all $x, y, z \in X$. Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. □

Lemma 2 ([15]). *Any generalized neutrosophic ideal $A = (A_T, A_{IT}, A_{IF}, A_F)$ of X satisfies:*

$$(\forall x, y, z \in X) \left(x * y \leq z \Rightarrow \left\{ \begin{array}{l} A_T(x) \geq \min\{A_T(y), A_T(z)\} \\ A_{IT}(x) \geq \min\{A_{IT}(y), A_{IT}(z)\} \\ A_{IF}(x) \leq \max\{A_{IF}(y), A_{IF}(z)\} \\ A_F(x) \leq \max\{A_F(y), A_F(z)\} \end{array} \right. \right). \tag{16}$$

We provide a condition for a generalized neutrosophic ideal to be commutative.

Theorem 3. *For any commutative BCK-algebra, every generalized neutrosophic ideal is commutative.*

Proof. Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of a commutative BCK-algebra X. Note that

$$\begin{aligned}((x*(y*(y*x)))*((x*y)*z))*z &= ((x*(y*(y*x)))*z)*((x*y)*z)\\ &\leq (x*(y*(y*x)))*(x*y)\\ &= (x*(x*y))*(y*(y*x)) = 0,\end{aligned}$$

thus, $(x*(y*(y*x)))*((x*y)*z) \leq z, \forall x,y,z \in X$. By Lemma 2 we get

$$\begin{aligned}A_T(x*(y*(y*x))) &\geq \min\{A_T((x*y)*z), A_T(z)\},\\ A_{IT}(x*(y*(y*x))) &\geq \min\{A_{IT}((x*y)*z), A_{IT}(z)\},\\ A_{IF}(x*(y*(y*x))) &\leq \max\{A_{IF}((x*y)*z), A_{IF}(z)\},\\ A_F(x*(y*(y*x))) &\leq \max\{A_F((x*y)*z), A_F(z)\}.\end{aligned}$$

Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. □

Lemma 3 ([15])**.** *If a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X is a generalized neutrosophic ideal of X, then the sets $U_A(T, \alpha_T)$, $U_A(IT, \alpha_{IT})$, $L_A(F, \beta_F)$ and $L_A(IF, \beta_{IF})$ are ideals of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$ whenever they are non-empty.*

Theorem 4. *If a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in X is a commutative generalized neutrosophic ideal of X, then the sets $U_A(T, \alpha_T)$, $U_A(IT, \alpha_{IT})$, $L_A(F, \beta_F)$ and $L_A(IF, \beta_{IF})$ are commutative ideals of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$ whenever they are non-empty.*

The commutative ideals $U_A(T, \alpha_T)$, $U_A(IT, \alpha_{IT})$, $L_A(F, \beta_F)$ and $L_A(IF, \beta_{IF})$ are called *level neutrosophic commutative ideals* of $A = (A_T, A_{IT}, A_{IF}, A_F)$.

Proof. Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X. Thus $U_A(T, \alpha_T)$, $U_A(IT, \alpha_{IT})$, $L_A(F, \beta_F)$ and $L_A(IF, \beta_{IF})$ are ideals of X whenever they are non-empty applying Lemma 3. Suppose that $x, y \in X$ and $x*y \in U_A(T, \alpha_T) \cap U_A(IT, \alpha_{IT})$. Using (15),

$$\begin{aligned}A_T(x*(y*(y*x))) &\geq A_T(x*y) \geq \alpha_T,\\ A_{IT}(x*(y*(y*x))) &\geq A_{IT}(x*y) \geq \alpha_{IT},\end{aligned}$$

and so $x*(y*(y*x)) \in U_A(T, \alpha_T)$ and $x*(y*(y*x)) \in U_A(IT, \alpha_{IT})$. Suppose that $a, b \in X$ and $a*b \in L_A(IF, \beta_{IF}) \cap L_A(F, \beta_F)$. It follows from (15) that $A_{IF}(a*(b*(b*a))) \leq A_{IF}(a*b) \leq \beta_{IF}$ and $A_F(a*(b*(b*a))) \leq A_F(a*b) \leq \beta_F$. Hence $a*(b*(b*a)) \in L_A(IF, \beta_{IF})$ and a*(b*(b*a)) $\in L_A(F, \beta_F)$. Therefore $U_A(T, \alpha_T)$, $U_A(IT, \alpha_{IT})$, $L_A(F, \beta_F)$ and $L_A(IF, \beta_{IF})$ are commutative ideals of X. □

Lemma 4 ([15])**.** *Assume that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a GNS in X and $U_A(T, \alpha_T)$, $U_A(IT, \alpha_{IT})$, $L_A(F, \beta_F)$ and $L_A(IF, \beta_{IF})$ are ideals of X, $\forall \alpha_T$, α_{IT}, β_F, $\beta_{IF} \in [0,1]$. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X.*

Theorem 5. *Let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a GNS in X such that $U_A(T, \alpha_T)$, $U_A(IT, \alpha_{IT})$, $L_A(F, \beta_F)$ and $L_A(IF, \beta_{IF})$ are commutative ideals of X for all α_T, α_{IT}, β_F, $\beta_{IF} \in [0,1]$. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X.*

Proof. Let $\alpha_T, \alpha_{IT}, \beta_F, \beta_{IF} \in [0,1]$ be such that the non-empty sets $U_A(T, \alpha_T)$, $U_A(IT, \alpha_{IT})$, $L_A(F, \beta_F)$ and $L_A(IF, \beta_{IF})$ are commutative ideals of X. Then $U_A(T, \alpha_T)$, $U_A(IT, \alpha_{IT})$, $L_A(F, \beta_F)$ and $L_A(IF, \beta_{IF})$ are ideals of X. Hence $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of X applying Lemma 4. For any $x, y \in X$, let $A_T(x * y) = \alpha_T$. Then $x * y \in U_A(T, \alpha_T)$, and so $x * (y * (y * x)) \in U_A(T, \alpha_T)$ by (9). Hence $A_T(x * (y * (y * x))) \geq \alpha_T = A_T(x * y)$. Similarly, we can show that

$$(\forall x, y \in X)(A_{IT}(x * (y * (y * x))) \geq A_{IT}(x * y)).$$

For any $x, y, a, b, \in X$, let $A_F(x * y) = \beta_F$ and $A_{IF}(a * b) = \beta_{IF}$. Then $x * y \in L_A(F, \beta_F)$ and $a * b \in L_A(IF, \beta_{IF})$. Using Lemma 1 we have $x * (y * (y * x)) \in L_A(F, \beta_F)$ and $a * (b * (b * a)) \in L_A(IF, \beta_{IF})$. Thus $A_F(x * y) = \beta_F \geq A_F(x * (y * (y * x)))$ and $A_{IF}(a * b) = \beta_{IF} \geq A_{IF}((a * b) * b)$. Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X. □

Theorem 6. *Every commutative generalized neutrosophic ideal can be realized as level neutrosophic commutative ideals of some commutative generalized neutrosophic ideal of X.*

Proof. Given a commutative ideal C of X, define a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ as follows

$$A_T(x) = \begin{cases} \alpha_T & \text{if } x \in C, \\ 0 & \text{otherwise,} \end{cases} \qquad A_{IT}(x) = \begin{cases} \alpha_{IT} & \text{if } x \in C, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{IF}(x) = \begin{cases} \beta_{IF} & \text{if } x \in C, \\ 1 & \text{otherwise,} \end{cases} \qquad A_F(x) = \begin{cases} \beta_F & \text{if } x \in C, \\ 1 & \text{otherwise,} \end{cases}$$

where $\alpha_T, \alpha_{IT} \in (0,1]$ and $\beta_F, \beta_{IF} \in [0,1)$. Let $x, y, z \in X$. If $(x * y) * z \in C$ and $z \in C$, then $x * (y * (y * x)) \in C$. Thus

$$\begin{aligned} A_T(x * (y * (y * x))) &= \alpha_T = \min\{A_T((x * y) * z), A_T(z)\}, \\ A_{IT}(x * (y * (y * x))) &= \alpha_{IT} = \min\{A_{IT}((x * y) * z), A_{IT}(z)\}, \\ A_{IF}(x * (y * (y * x))) &= \beta_{IF} = \max\{A_{IF}((x * y) * z), A_{IF}(z)\}, \\ A_F(x * (y * (y * x))) &= \beta_F = \max\{A_F((x * y) * z), A_F(z)\}. \end{aligned}$$

Assume that $(x * y) * z \notin C$ and $z \notin C$. Then $A_T((x * y) * z) = 0$, $A_T(z) = 0$, $A_{IT}((x * y) * z) = 0$, $A_{IT}(z) = 0$, $A_{IF}((x * y) * z) = 1$, $A_{IF}(z) = 1$, and $A_F((x * y) * z) = 1$, $A_F(z) = 1$. It follows that

$$\begin{aligned} A_T(x * (y * (y * x))) &\geq \min\{A_T((x * y) * z), A_T(z)\}, \\ A_{IT}(x * (y * (y * x))) &\geq \min\{A_{IT}((x * y) * z), A_{IT}(z)\}, \\ A_{IF}(x * (y * (y * x))) &\leq \max\{A_{IF}((x * y) * z), A_{IF}(z)\}, \\ A_F(x * (y * (y * x))) &\leq \max\{A_F((x * y) * z), A_F(z)\}. \end{aligned}$$

If exactly one of $(x * y) * z$ and z belongs to C, then exactly one of $A_T((x * y) * z)$ and $A_T(z)$ is equal to 0; exactly one of $A_{IT}((x * y) * z)$ and $A_{IT}(z)$ is equal to 0; exactly one of $A_F((x * y) * z)$ and $A_F(z)$ is equal to 1 and exactly one of $A_{IF}((x * y) * z)$ and $A_{IF}(z)$ is equal to 1. Hence

$$\begin{aligned} A_T(x * (y * (y * x))) &\geq \min\{A_T((x * y) * z), A_T(z)\}, \\ A_{IT}(x * (y * (y * x))) &\geq \min\{A_{IT}((x * y) * z), A_{IT}(z)\}, \\ A_{IF}(x * (y * (y * x))) &\leq \max\{A_{IF}((x * y) * z), A_{IF}(z)\}, \\ A_F(x * (y * (y * x))) &\leq \max\{A_F((x * y) * z), A_F(z)\}. \end{aligned}$$

It is clear that $A_T(0) \geq A_T(x)$, $A_{IT}(0) \geq A_{IT}(x)$, $A_{IF}(0) \leq A_{IF}(x)$ and $A_F(0) \leq A_F(x)$ for all $x \in X$. Therefore $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X.

Obviously, $U_A(T, \alpha_T) = C$, $U_A(IT, \alpha_{IT}) = C$, $L_A(F, \beta_F) = C$ and $L_A(IF, \beta_{IF}) = C$. This completes the proof. □

Theorem 7. *Let $\{C_t \mid t \in \Lambda\}$ be a collection of commutative ideals of X such that*

(1) $X = \bigcup_{t\in\Lambda} C_t$,
(2) $(\forall s, t \in \Lambda)\,(s > t \iff C_s \subset C_t)$

where Λ is any index set. Let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a GNS in X given by

$$(\forall x \in X)\left(\begin{array}{l} A_T(x) = \sup\{t \in \Lambda \mid x \in C_t\} = A_{IT}(x) \\ A_{IF}(x) = \inf\{t \in \Lambda \mid x \in C_t\} = A_F(x) \end{array}\right). \tag{17}$$

Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of X.

Proof. According to Theorem 5, it is sufficient to show that $U(T,t)$, $U(IT,t)$, $L(F,s)$ and $L(IF,s)$ are commutative ideals of X for every $t \in [0, A_T(0) = A_{IT}(0)]$ and $s \in [A_{IF}(0) = A_F(0), 1]$. In order to prove $U(T,t)$ and $U(IT,t)$ are commutative ideals of X, we consider two cases:

(i) $t = \sup\{q \in \Lambda \mid q < t\}$,
(ii) $t \neq \sup\{q \in \Lambda \mid q < t\}$.

For the first case, we have

$$x \in U(T,t) \iff (\forall q < t)(x \in C_q) \iff x \in \bigcap_{q<t} C_q,$$
$$x \in U(IT,t) \iff (\forall q < t)(x \in C_q) \iff x \in \bigcap_{q<t} C_q.$$

Hence $U(T,t) = \bigcap_{q<t} C_q = U(IT,t)$, and so $U(T,t)$ and $U(IT,t)$ are commutative ideals of X. For the second case, we claim that $U(T,t) = \bigcup_{q\geq t} C_q = U(IT,t)$. If $x \in \bigcup_{q\geq t} C_q$, then $x \in C_q$ for some $q \geq t$. It follows that $A_{IT}(x) = A_T(x) \geq q \geq t$ and so that $x \in U(T,t)$ and $x \in U(IT,t)$. This shows that $\bigcup_{q\geq t} C_q \subseteq U(T,t)$ and $\bigcup_{q\geq t} C_q \subseteq U(IT,t)$. Now, suppose $x \notin \bigcup_{q\geq t} C_q$. Then $x \notin C_q$, $\forall q \geq t$. Since $t \neq \sup\{q \in \Lambda \mid q < t\}$, there exists $\varepsilon > 0$ such that $(t-\varepsilon, t) \cap \Lambda = \emptyset$. Thus $x \notin C_q$, $\forall q > t-\varepsilon$, this means that if $x \in C_q$, then $q \leq t-\varepsilon$. So $A_{IT}(x) = A_T(x) \leq t-\varepsilon < t$, and so $x \notin U(T,t) = U(IT,t)$. Therefore $U(T,t) = U(IT,t) \subseteq \bigcup_{q\geq t} C_q$. Consequently, $U(T,t) = U(IT,t) = \bigcup_{q\geq t} C_q$ which is a commutative ideal of X. Next we show that $L(F,s)$ and $L(IF,s)$ are commutative ideals of X. We consider two cases as follows:

(iii) $s = \inf\{r \in \Lambda \mid s < r\}$,
(iv) $s \neq \inf\{r \in \Lambda \mid s < r\}$.

Case (iii) implies that

$$x \in L(IF,s) \iff (\forall s < r)(x \in C_r) \iff x \in \bigcap_{s<r} C_r,$$
$$x \in U(F,s) \iff (\forall s < r)(x \in C_r) \iff x \in \bigcap_{s<r} C_r.$$

It follows that $L(IF,s) = L(F,s) = \bigcap_{s<r} C_r$, which is a commutative ideal of X. Case (iv) induces $(s, s+\varepsilon) \cap \Lambda = \emptyset$ for some $\varepsilon > 0$. If $x \in \bigcup_{s\geq r} C_r$, then $x \in C_r$ for some $r \leq s$, and so $A_{IF}(x) = A_F(x) \leq r \leq s$, that is, $x \in L(IF,s)$ and $x \in L(F,s)$. Hence $\bigcup_{s\geq r} C_r \subseteq L(IF,s) = L(F,s)$. If $x \notin \bigcup_{s\geq r} C_r$, then $x \notin C_r$

for all $r \leq s$ which implies that $x \notin C_r$ for all $r \leq s + \varepsilon$, that is, if $x \in C_r$ then $r \geq s + \varepsilon$. Hence $A_{IF}(x) = A_F(x) \geq s + \varepsilon > s$, and so $x \notin L(A_{IF}, s) = L(A_F, s)$. Hence $L(A_{IF}, s) = L(A_F, s) = \bigcup_{s \geq r} C_r$ which is a commutative ideal of X. This completes the proof. □

Assume thta $f : X \to Y$ is a homomorphism of BCK/BCI-algebras ([7]). For any GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in Y, we define a new GNS $A^f = (A_T^f, A_{IT}^f, A_{IF}^f, A_F^f)$ in X, which is called the *induced GNS*, by

$$(\forall x \in X) \left(\begin{array}{l} A_T^f(x) = A_T(f(x)),\ A_{IT}^f(x) = A_{IT}(f(x)) \\ A_{IF}^f(x) = A_{IF}(f(x)),\ A_F^f(x) = A_F(f(x)) \end{array} \right). \tag{18}$$

Lemma 5 ([15]). *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in Y is a generalized neutrosophic ideal of Y, then the new GNS $A^f = (A_T^f, A_{IT}^f, A_{IF}^f, A_F^f)$ in X is a generalized neutrosophic ideal of X.*

Theorem 8. *Let $f : X \to Y$ be a homomorphism of BCK-algebras. If a GNS $A = (A_T, A_{IT}, A_{IF}, A_F)$ in Y is a commutative generalized neutrosophic ideal of Y, then the new GNS $A^f = (A_T^f, A_{IT}^f, A_{IF}^f, A_F^f)$ in X is a commutative generalized neutrosophic ideal of X.*

Proof. Suppose that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of Y. Then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of Y by Theorem 1, and so $A^f = (A_T^f, A_{IT}^f, A_{IF}^f, A_F^f)$ is a generalized neutrosophic ideal of Y by Lemma 5. For any $x, y \in X$, we have

$$\begin{aligned} A_T^f(x * (y * (y * x))) &= A_T(f(x * (y * (y * x)))) \\ &= A_T(f(x) * (f(y) * (f(y) * f(x)))) \\ &\geq A_T(f(x) * f(y)) \\ &= A_T(f(x * y)) = A_T^f(x * y), \end{aligned}$$

$$\begin{aligned} A_{IT}^f(x * (y * (y * x))) &= A_{IT}(f(x * (y * (y * x)))) \\ &= A_{IT}(f(x) * (f(y) * (f(y) * f(x)))) \\ &\geq A_{IT}(f(x) * f(y)) \\ &= A_{IT}(f(x * y)) = A_{IT}^f(x * y), \end{aligned}$$

$$\begin{aligned} A_{IF}^f(x * (y * (y * x))) &= A_{IF}(f(x * (y * (y * x)))) \\ &= A_{IF}(f(x) * (f(y) * (f(y) * f(x)))) \\ &\leq A_{IF}(f(x) * f(y)) \\ &= A_{IF}(f(x * y)) = A_{IF}^f(x * y), \end{aligned}$$

and

$$\begin{aligned} A_F^f(x * (y * (y * x))) &= A_F(f(x * (y * (y * x)))) \\ &= A_F(f(x) * (f(y) * (f(y) * f(x)))) \\ &\leq A_F(f(x) * f(y)) \\ &= A_F(f(x * y)) = A_F^f(x * y). \end{aligned}$$

Therefore $A^f = (A_T^f, A_{IT}^f, A_{IF}^f, A_F^f)$ is a commutative generalized neutrosophic ideal of X. □

Lemma 6 ([15]). *Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras and let $A = (A_T, A_{IT}, A_{IF}, A_F)$ be a GNS in Y. If the induced GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X is a generalized neutrosophic ideal of X, then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of Y.*

Theorem 9. *Assume thta $f : X \to Y$ is an onto homomorphism of BCK-algebras and $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a GNS in Y. If the induced GNS $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ in X is a commutative generalized neutrosophic ideal of X, then $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of Y.*

Proof. Suppose that $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ is a commutative generalized neutrosophic ideal of X. Then $A^f = (A^f_T, A^f_{IT}, A^f_{IF}, A^f_F)$ is a generalized neutrosophic ideal of X, and thus $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a generalized neutrosophic ideal of Y. For any $a, b, c \in Y$, there exist $x, y, z \in X$ such that $f(x) = a$, $f(y) = b$ and $f(z) = c$. Thus,

$$\begin{aligned} A_T(a*(b*(b*a))) &= A_T(f(x)*(f(y)*(f(y)*f(x)))) = A_T(f(x*(y*(y*x)))) \\ &= A^f_T(x*(y*(y*x))) \geq A^f_T(x*y) \\ &= A_T(f(x)*f(y)) = A_T(a*b), \end{aligned}$$

$$\begin{aligned} A_{IT}(a*(b*(b*a))) &= A_{IT}(f(x)*(f(y)*(f(y)*f(x)))) = A_{IT}(f(x*(y*(y*x)))) \\ &= A^f_{IT}(x*(y*(y*x))) \geq A^f_{IT}(x*y) \\ &= A_{IT}(f(x)*f(y)) = A_{IT}(a*b), \end{aligned}$$

$$\begin{aligned} A_{IF}(a*(b*(b*a))) &= A_{IF}(f(x)*(f(y)*(f(y)*f(x)))) = A_{IF}(f(x*(y*(y*x)))) \\ &= A^f_{IF}(x*(y*(y*x))) \leq A^f_{IF}(x*y) \\ &= A_{IF}(f(x)*f(y)) = A_{IF}(a*b), \end{aligned}$$

and

$$\begin{aligned} A_F(a*(b*(b*a))) &= A_F(f(x)*(f(y)*(f(y)*f(x)))) = A_F(f(x*(y*(y*x)))) \\ &= A^f_F(x*(y*(y*x))) \leq A^f_F(x*y) \\ &= A_F(f(x)*f(y)) = A_F(a*b). \end{aligned}$$

It follows from Theorem 2 that $A = (A_T, A_{IT}, A_{IF}, A_F)$ is a commutative generalized neutrosophic ideal of Y. □

Let $CGNI(X)$ denote the set of all commutative generalized neutrosophic ideals of X and $t \in [0, 1]$. Define binary relations U^t_T, U^t_{IT}, L^t_F and L^t_{IF} on $CGNI(X)$ as follows:

$$\begin{aligned} &(A, B) \in U^t_T \Leftrightarrow U_A(T, t) = U_B(T, t),\ (A, B) \in U^t_{IT} \Leftrightarrow U_A(IT, t) = U_B(IT, t), \\ &(A, B) \in L^t_F \Leftrightarrow L_A(F, t) = L_B(F, t),\ (A, B) \in L^t_{IF} \Leftrightarrow L_A(IF, t) = L_B(IF, t) \end{aligned} \tag{19}$$

for $A = (A_T, A_{IT}, A_{IF}, A_F)$ and $B = (B_T, B_{IT}, B_{IF}, B_F)$ in $CGNI(X)$. Then clearly U^t_T, U^t_{IT}, L^t_F and L^t_{IF} are equivalence relations on $CGNI(X)$. For any $A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)$, let $[A]_{U^t_T}$ (resp., $[A]_{U^t_{IT}}$, $[A]_{L^t_F}$ and $[A]_{L^t_{IF}}$) denote the equivalence class of $A = (A_T, A_{IT}, A_{IF}, A_F)$ modulo U^t_T (resp, U^t_{IT}, L^t_F and L^t_{IF}). Denote by $CGNI(X)/U^t_T$ (resp., $CGNI(X)/U^t_{IT}$, $CGNI(X)/L^t_F$ and $CGNI(X)/L^t_{IF}$) the system of all equivalence classes modulo U^t_T (resp, U^t_{IT}, L^t_F and L^t_{IF}); so

$$CGNI(X)/U^t_T = \{[A]_{U^t_T} \mid A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)\}, \tag{20}$$

$$CGNI(X)/U_{IT}^t = \{[A]_{U_{IT}^t} \mid A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)\}, \tag{21}$$

$$CGNI(X)/L_F^t = \{[A]_{L_F^t} \mid A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)\}, \tag{22}$$

and

$$CGNI(X)/L_{IF}^t = \{[A]_{L_{IF}^t} \mid A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)\}, \tag{23}$$

respectively. Let $CI(X)$ denote the family of all commutative ideals of X and let $t \in [0,1]$. Define maps

$$f_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto U_A(T,t), \tag{24}$$

$$g_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto U_A(IT,t), \tag{25}$$

$$\alpha_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto L_A(F,t), \tag{26}$$

and

$$\beta_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto L_A(IF,t). \tag{27}$$

Then the definitions of f_t, g_t, α_t and β_t are well.

Theorem 10. *Suppose $t \in (0,1)$, the definitions of f_t, g_t, α_t and β_t are as above. Then the maps f_t, g_t, α_t and β_t are surjective from $CGNI(X)$ to $CI(X) \cup \{\emptyset\}$.*

Proof. Assume $t \in (0,1)$.We know that $\mathbf{0}_\sim = (\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}, \mathbf{1}_F)$ is in $CGNI(X)$ where $\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}$ and $\mathbf{1}_F$ are constant functions on X defined by $\mathbf{0}_T(x) = 0$, $\mathbf{0}_{IT}(x) = 0$, $\mathbf{1}_{IF}(x) = 1$ and $\mathbf{1}_F(x) = 1$ for all $x \in X$. Obviously $f_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(T,t)$, $g_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(IT,t)$, $\alpha_t(\mathbf{0}_\sim) = L_{\mathbf{0}_\sim}(F,t)$ and $\beta_t(\mathbf{0}_\sim) = L_{\mathbf{0}_\sim}(IF,t)$ are empty. Let $G(\neq \emptyset) \in CGNI(X)$, and consider functions:

$$G_T : X \to [0,1],\ G \mapsto \begin{cases} 1 & \text{if } x \in G\,, \\ 0 & \text{otherwise,} \end{cases}$$

$$G_{IT} : X \to [0,1],\ G \mapsto \begin{cases} 1 & \text{if } x \in G\,, \\ 0 & \text{otherwise,} \end{cases}$$

$$G_F : X \to [0,1],\ G \mapsto \begin{cases} 0 & \text{if } x \in G\,, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$G_{IF} : X \to [0,1],\ G \mapsto \begin{cases} 0 & \text{if } x \in G\,, \\ 1 & \text{otherwise.} \end{cases}$$

Then $G_\sim = (G_T, G_{IT}, G_{IF}, G_F)$ is a commutative generalized neutrosophic ideal of X, and $f_t(G_\sim) = U_{G_\sim}(T,t) = G$, $g_t(G_\sim) = U_{G_\sim}(IT,t) = G$, $\alpha_t(G_\sim) = L_{G_\sim}(F,t) = G$ and $\beta_t(G_\sim) = L_{G_\sim}(IF,t) = G$. Therefore f_t, g_t, α_t and β_t are surjective. □

Theorem 11. *The quotient sets*

$$CGNI(X)/U_T^t,\ CGNI(X)/U_{IT}^t,\ CGNI(X)/L_F^t \text{ and } CGNI(X)/L_{IF}^t$$

are equipotent to $CI(X) \cup \{\emptyset\}$.

Proof. For $t \in (0,1)$, let f_t^* (resp, g_t^*, α_t^* and β_t^*) be a map from $CGNI(X)/U_T^t$ (resp., $CGNI(X)/U_{IT}^t$, $CGNI(X)/L_F^t$ and $CGNI(X)/L_{IF}^t$) to $CI(X) \cup \{\emptyset\}$ defined by $f_t^*\left([A]_{U_T^t}\right) = f_t(A)$ (resp., $g_t^*\left([A]_{U_{IT}^t}\right) = g_t(A)$, $\alpha_t^*\left([A]_{L_F^t}\right) = \alpha_t(A)$ and $\beta_t^*\left([A]_{L_{IF}^t}\right) = \beta_t(A)$) for all $A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)$. If $U_A(T,t) = U_B(T,t)$, $U_A(IT,t) = U_B(IT,t)$, $L_A(F,t) = L_B(F,t)$ and $L_A(IF,t) = L_B(IF,t)$ for $A = (A_T, A_{IT}, A_{IF}, A_F)$ and $B = (B_T, B_{IT}, B_F, B_{IF})$ in $CGNI(X)$, then $(A,B) \in U_T^t$, $(A,B) \in U_{IT}^t$, $(A,B) \in L_F^t$ and $(A,B) \in L_{IF}^t$. Hence $[A]_{U_T^t} = [B]_{U_T^t}$, $[A]_{U_{IT}^t} = [B]_{U_{IT}^t}$, $[A]_{L_F^t} = [B]_{L_F^t}$ and $[A]_{L_{IF}^t} = [B]_{L_{IF}^t}$. Therefore f_t^* (resp, g_t^*, α_t^* and β_t^*) is injective. Now let $G(\neq \emptyset) \in CGNI(X)$. For $G_\sim = (G_T, G_{IT}, G_{IF}, G_F) \in CGNI(X)$, we have

$$f_t^*\left([G_\sim]_{U_T^t}\right) = f_t(G_\sim) = U_{G_\sim}(T,t) = G,$$

$$g_t^*\left([G_\sim]_{U_{IT}^t}\right) = g_t(G_\sim) = U_{G_\sim}(IT,t) = G,$$

$$\alpha_t^*\left([G_\sim]_{L_F^t}\right) = \alpha_t(G_\sim) = L_{G_\sim}(F,t) = G$$

and

$$\beta_t^*\left([G_\sim]_{L_{IF}^t}\right) = \beta_t(G_\sim) = L_{G_\sim}(IF,t) = G.$$

Finally, for $\mathbf{0}_\sim = (\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}, \mathbf{1}_F) \in CGNI(X)$, we have

$$f_t^*\left([\mathbf{0}_\sim]_{U_T^t}\right) = f_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(T,t) = \emptyset,$$

$$g_t^*\left([\mathbf{0}_\sim]_{U_{IT}^t}\right) = g_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(IT,t) = \emptyset,$$

$$\alpha_t^*\left([\mathbf{0}_\sim]_{L_F^t}\right) = \alpha_t(\mathbf{0}_\sim) = L_{\mathbf{0}_\sim}(F,t) = \emptyset$$

and

$$\beta_t^*\left([\mathbf{0}_\sim]_{L_{IF}^t}\right) = \beta_t(\mathbf{0}_\sim) = L_{\mathbf{0}_\sim}(IF,t) = \emptyset.$$

Therefore, f_t^* (resp, g_t^*, α_t^* and β_t^*) is surjective. □

$\forall t \in [0,1]$, define another relations R^t and Q^t on $CGNI(X)$ as follows:

$$(A,B) \in R^t \Leftrightarrow U_A(T,t) \cap L_A(F,t) = U_B(T,t) \cap L_B(F,t)$$

and

$$(A,B) \in Q^t \Leftrightarrow U_A(IT,t) \cap L_A(IF,t) = U_B(IT,t) \cap L_B(IF,t)$$

for any $A = (A_T, A_{IT}, A_{IF}, A_F)$ and $B = (B_T, B_{IT}, B_{IF}, B_F)$ in $CGNI(X)$. Then R^t and Q^t are equivalence relations on $CGNI(X)$.

Theorem 12. *Suppose* $t \in (0,1)$, *consider the following maps*

$$\varphi_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto f_t(A) \cap \alpha_t(A), \tag{28}$$

and

$$\psi_t : CGNI(X) \to CI(X) \cup \{\emptyset\},\ A \mapsto g_t(A) \cap \beta_t(A) \tag{29}$$

for each $A = (A_T, A_{IT}, A_{IF}, A_F) \in CGNI(X)$. *Then* φ_t *and* ψ_t *are surjective.*

Proof. Assume $t \in (0,1)$. For $\mathbf{0}_\sim = (\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}, \mathbf{1}_F) \in CGNI(X)$,

$$\varphi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap \alpha_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(T,t) \cap L_{\mathbf{0}_\sim}(F,t) = \emptyset$$

and

$$\psi_t(\mathbf{0}_\sim) = g_t(\mathbf{0}_\sim) \cap \beta_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(IT,t) \cap L_{\mathbf{0}_\sim}(IF,t) = \emptyset.$$

For any $G \in CI(X)$, there exists $G_\sim = (G_T, G_{IT}, G_{IF}, G_F) \in CGNI(X)$ such that

$$\varphi_t(G_\sim) = f_t(G_\sim) \cap \alpha_t(G_\sim) = U_{G_\sim}(T,t) \cap L_{G_\sim}(F,t) = G$$

and

$$\psi_t(G_\sim) = g_t(G_\sim) \cap \beta_t(G_\sim) = U_{G_\sim}(IT,t) \cap L_{G_\sim}(IF,t) = G.$$

Therefore φ_t and ψ_t are surjective. □

Theorem 13. *For any* $t \in (0,1)$, *the quotient sets* $CGNI(X)/R^t$ *and* $CGNI(X)/Q^t$ *are equipotent to* $CI(X) \cup \{\emptyset\}$.

Proof. Let $t \in (0,1)$ and define maps

$$\varphi_t^* : CGNI(X)/R^t \to CI(X) \cup \{\emptyset\},\ [A]_{R^t} \mapsto \varphi_t(A)$$

and

$$\psi_t^* : CGNI(X)/Q^t \to CI(X) \cup \{\emptyset\},\ [A]_{Q^t} \mapsto \psi_t(A).$$

If $\varphi_t^*([A]_{R^t}) = \varphi_t^*([B]_{R^t})$ and $\psi_t^*\left([A]_{Q^t}\right) = \psi_t^*\left([B]_{Q^t}\right)$ for all $[A]_{R^t}, [B]_{R^t} \in CGNI(X)/R^t$ and $[A]_{Q^t}, [B]_{Q^t} \in CGNI(X)/Q^t$, then $f_t(A) \cap \alpha_t(A) = f_t(B) \cap \alpha_t(B)$ and $g_t(A) \cap \beta_t(A) = g_t(B) \cap \beta_t(B)$, that is, $U_A(T,t) \cap L_A(F,t) = U_B(T,t) \cap L_B(F,t)$ and $U_A(IT,t) \cap L_A(IF,t) = U_B(IT,t) \cap L_B(IF,t)$. Hence $(A,B) \in R^t$, $(A,B) \in Q^t$. So $[A]_{R^t} = [B]_{R^t}$, $[A]_{Q^t} = [B]_{Q^t}$, which shows that φ_t^* and ψ_t^* are injective. For $\mathbf{0}_\sim = (\mathbf{0}_T, \mathbf{0}_{IT}, \mathbf{1}_{IF}, \mathbf{1}_F) \in CGNI(X)$,

$$\varphi_t^*([\mathbf{0}_\sim]_{R^t}) = \varphi_t(\mathbf{0}_\sim) = f_t(\mathbf{0}_\sim) \cap \alpha_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(\mathbf{0}_T,t) \cap L_{\mathbf{0}_\sim}(\mathbf{1}_F,t) = \emptyset$$

and

$$\psi_t^*\left([\mathbf{0}_\sim]_{Q^t}\right) = \psi_t(\mathbf{0}_\sim) = g_t(\mathbf{0}_\sim) \cap \beta_t(\mathbf{0}_\sim) = U_{\mathbf{0}_\sim}(\mathbf{0}_{IT},t) \cap L_{\mathbf{0}_\sim}(\mathbf{1}_{IF},t) = \emptyset.$$

If $G \in CI(X)$, then $G_\sim = (G_T, G_{IT}, G_{IF}, G_F) \in CGNI(X)$, and so

$$\varphi_t^*([G_\sim]_{R^t}) = \varphi_t(G_\sim) = f_t(G_\sim) \cap \alpha_t(G_\sim) = U_{G_\sim}(G_T,t) \cap L_{G_\sim}(G_F,t) = G$$

and

$$\psi_t^*\left([G_\sim]_{Q^t}\right) = \psi_t(G_\sim) = g_t(G_\sim) \cap \beta_t(G_\sim) = U_{G_\sim}(G_{IT}, t) \cap L_{G_\sim}(G_{IF}, t) = G.$$

Hence φ_t^* and ψ_t^* are surjective, and the proof is complete. □

4. Conclusions

Based on the theory of generalized neutrosophic sets, we proposed the new concept of commutative generalized neutrosophic ideal in a BCK-algebra, and obtained some characterizations. Moreover, we investigated some homomorphism properties related to commutative generalized neutrosophic ideals.

The research ideas of this paper can be extended to a wide range of logical algebraic systems such as pseudo-BCI algebras (see [1,16]). At the same time, the concept of generalized neutrosophic set involved in this paper can be further studied according to the thought in [11,17], which will be the direction of our next research work.

Author Contributions: R.A.B. and Y.B.J. wrote the draft version of the paper, X.Z. wrote the revised version and completed submission of the paper, F.S. as a supervisor gave some guidance.

Funding: This research was funded by the National Natural Science Foundation of China grant number 61573240.

Conflicts of Interest: The authors declare no conflicts of interest.

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A Linguistic Neutrosophic Multi-Criteria Group Decision-Making Method to University Human Resource Management

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Received: 17 July 2018; Accepted: 22 August 2018; Published: 26 August 2018

Abstract: Competition among different universities depends largely on the competition for talent. Talent evaluation and selection is one of the main activities in human resource management (HRM) which is critical for university development. Firstly, linguistic neutrosophic sets (LNSs) are introduced to better express multiple uncertain information during the evaluation procedure. We further merge the power averaging operator with LNSs for information aggregation and propose a LN-power weighted averaging (LNPWA) operator and a LN-power weighted geometric (LNPWG) operator. Then, an extended technique for order preference by similarity to ideal solution (TOPSIS) method is developed to solve a case of university HRM evaluation problem. The main contribution and novelty of the proposed method rely on that it allows the information provided by different decision makers (DMs) to support and reinforce each other which is more consistent with the actual situation of university HRM evaluation. In addition, its effectiveness and advantages over existing methods are verified through sensitivity and comparative analysis. The results show that the proposal is capable in the domain of university HRM evaluation and may contribute to the talent introduction in universities.

Keywords: linguistic neutrosophic sets; multi-criteria group decision-making; power aggregation operator; extended TOPSIS method

1. Introduction

Human resource management (HRM) refers to a process of hiring and developing employees to enhance the core competitiveness of an organization [1]. Acting as the root of national competitiveness, a success in HRM may bring benefit to both the organization and employee well-being; thus, effective HRM has received a higher demand and recognition during the 21st century. Over the past three decades, theory and research on HRM has made considerable progress in various fields, such as tourism industries, health services and universities [2–5]. For example, Zhang et al. [5] investigated a case of HRM for teaching quality assessment using a multi-criteria group decision-making (MAGDM) framework. This framework aimed to improve the teaching quality of college teachers and further enhance the competitiveness of colleges and universities. Apart from the classroom teaching quality evaluation problems in universities, talent introduction also plays a significant role in universities' HRM. Particularly, selecting or evaluating these applicants by inappropriate methods may lead to a failure in HRM and even influence the overall efficiency of the university. Since various applicants and influential criteria are usually involved in the evaluation procedures of HRM by several decision makers (DMs), the evaluation should be recognized as a multi-criteria group decision-making (MCGDM) problem.

The theory of fuzzy set (FS) can handle uncertainty and fuzziness. The neutrosophic set (NS) [6] was initially proposed to express membership, nonmembership and indeterminacy, which is a generalization of FS [7]. Later, many extensions emerged to tackle real engineering and scientific problems [8], among which the popularly used forms are the simplified neutrosophic set (SNS) [9] and the single-valued trapezoidal neutrosophic set (SVTNS) [10–12]. These extensions have been successfully applied in various domains, including green product development [13], outsourcing provider selection [14], clustering analysis [15,16].

However, on some real occasions, people may tend to provide their evaluation information using natural languages rather than the above extensions which are too complex to obtain. For example, people can give some linguistic terms like "excellent", "medium" or "poor" to evaluate the performance of a company staff based on various criteria. Moreover, it may be also difficult for a single person to evaluate all alternatives under each influential aspect due to the high complexity of decision environments. Therefore, the linguistic MCGDM under fuzzy environments has received extensive research attention and gained many excellent results [17]. Up to now, various extensions have been studied in depth to describe linguistic information, such as hesitant fuzzy linguistic term set and some of its extended forms [18–22], linguistic intuitionistic fuzzy set (LIFS) [23,24], Z-number [25], and probabilistic linguistic term set [26,27] etc. However, the drawback of these extensions for linguistic MCGDM is that they cannot cover the inconsistent linguistic decision information which will appear with increasing complexity of the internal and external decision-making environments. Another example is that when one DM was asked to give some evaluations on a teacher from overseas under the aspect teaching skill, the DM may describe his or her bad judgments on the teaching attitude but the good or neutral aspects of the teacher's teaching capacity and teaching method as well. An example of that can be seen from the evaluation: "The teacher is rather average in writing and oral language, and he is able to tailor his teaching method to different students. But my only complaint is that the teacher is a little strict in teaching attitude". It can be noted that the above evaluation includes positive, neutral and negative information all at once. Therefore, this poses a great challenge for linguistic MCGDM methods on how to capture such inconsistent information.

To tackle the above problem, Fang and Ye [28] proposed the linguistic neutrosophic set (LNS), which was generalized from the concept of LIFS [23,24]. By contrast, one LNS is represented by three independent functions of truth-membership, indeterminacy-membership, and falsity-membership in the form of linguistic terms. Thus, the LNS has its prominent advantages in depicting inconsistent and indeterminate linguistic information, and several scholars have extended the LNS in several aspects, such as aggregation operators and similarity (or distance) measures. Li et al. [29] introduced a linguistic neutrosophic geometric Heronian mean (LNGHM) operator and a linguistic neutrosophic prioritized geometric Horonian mean (LNGHM) operator. Fan et al. [30] merged the LNSs with Bonferroni mean operator and proposed a linguistic neutrosophic number normalized weighted Bonferroni mean (LNNNWBM) operator and a linguistic neutrosophic number normalized weighted geometric Bonferroni mean (LNNNWGBM) operator. Shi and Ye [31] introduced two cosine similarity measures of LNSs to tackle MCGDM problems. Liang et al. [32] defined several distance measures of LNS and presented an extended TOPSIS method under the LNS environment.

To facilitate the mathematical operation, several quantification tools of natural language have been introduced, such as 2-type [33], triangular (or trapezoidal) fuzzy number [34,35], cloud model [36] and symbol model [37,38]. These models have greatly contributed to the ease of computation for linguistic information; however, they cannot cover all types of problems and have some limitations to be addressed. To tackle the limitations of prior research, Wang et al. [39] introduced a series of linguistic scale functions (LSFs) for converting linguistic information into real numbers. Through this model, flexibility of modeling information has been greatly enhanced by considering different semantic situations and loss and distortion of information has been mitigated to a great extent. Thus, we apply the LSFs to tackle linguistic neutrosophic information in this paper.

The power averaging (PA) operator, proposed by Yager [40], has been used as one effective information aggregation tool in solving MCDM [41–43] problems since its appearance. Unlike other common aggregation tools, such as weighted averaging [44] and ordered weighted averaging [45,46], which implicate the independent hypothesis among inputs. The PA operator allows the information between inputs to support and reinforce each other. In the HRM evaluation problems, it is very suitable for PA operator to integrate evaluation information of different teams of DMs, as these DMs are not completely independent and the PA operator can measure their support degree among one another.

TOPSIS method was first presented by Huang and Yoon [47]. It considered that the better scheme would be closer to ideal solution [48]. Due to the inevitable vagueness inherent in decision information, fuzzy TOPSIS and its extensions have been deployed [49–51] in real world applications. Considering the advantages of this method, an extended TOPSIS technique is introduced to evaluate alternatives.

As discussed above, our study developed an integrated method by combining PA operator with LNSs and constructing an extended TOPSIS technique to tackle the university HRM evaluation problem. The novelties and contributions of the proposal are listed as following. (1) New algorithms for LNNs based on LSFs is defined, which can reflect differences between various semantics. (2) Based on LSFs and the new operations, a generalized distance measure for LNNs is introduced, which can be reduced to Hamming distance and Euclidean distance of LNNs. The proposed distance measure is more flexible than prior studies because of the application of LSFs and novel operations. (3) Considering the fact that DMs in case of university HRM evaluation may support each other, this paper merges the PA operator with LNSs to tackle information fusion. The proposed method can improve the adaptability of LNNs in real decision.

The context in the rest of this paper is as follows: Section 2 defines some operations and distance measurements of LNSs. Section 3 proposes two aggregation operators for LNSs and investigates their properties. Next, the detailed procedures for a linguistic MCGDM problem are given in Section 4. Then, a case of university HRM evaluation problem verifies the feasibility and validity of our method in Section 5. Finally, Section 6 presents the conclusion and future work.

2. New Operations and Distance Measure for LNNs

After introducing the concepts of linguistic term set (LTS) and LNS, this section defines some new operations and a distance measure for LNNs based on the Archimedean t-norm and t-conorm. For better representation, some preliminaries about LSFs and the Archimedean t-norm and t-conorm are provided in Appendix A and Appendix B, respectively.

2.1. Linguistic Neutrosophic Set

$H = \{h_\tau | \tau = 0, 1, \cdots, 2t, t \in N^*\}$ is a discrete term set, which is finite and totally ordered. Herein, N^* presents a positive integers' set, h_τ is the value of a linguistic variable. Thus, the linguistic variable h_τ in H meets the following two properties [34]: (1) The LTS is ordered: $h_\tau < h_\upsilon$ if and only if $\tau < \upsilon$, where $(h_\tau, h_\upsilon \in H)$; and (2) With existing of a negation operator $neg(h_\tau) = h_{(2t-\tau)}$ $(\tau, \upsilon = 0, 1, \cdots, 2t)$.

In order to preserve as much of the given information and avoid information loss, Xu [52] extended $H = \{h_\tau | \tau = 0, 1, \cdots, 2t\}$ into a continuous LTS $\overline{H} = \{h_\tau | 1 \le \tau \le L\}$, which satisfies the properties of discrete term set H. When $h_\tau \in \overline{H}$, h_τ is called the original linguistic term; otherwise, h_τ is called the virtual linguistic term.

Definition 1 ([28,29]). *Let X be a universe of discourse and $\overline{H} = \{h_\alpha | h_0 \le h_\alpha \le h_{2t}, \alpha \in [0, 2t]\}$, and the LNSs can be defined as follows:*

$$\widetilde{a} = \left\{\left\langle x, h_{T_{\widetilde{a}}}(x), h_{I_{\widetilde{a}}}(x), h_{F_{\widetilde{a}}}(x)\right\rangle | x \in X\right\}, \tag{1}$$

where $0 \le T_{\widetilde{a}} + I_{\widetilde{a}} + F_{\widetilde{a}} \le 6t$ and the values $h_{T_{\widetilde{a}}}(x), h_{I_{\widetilde{a}}}(x), h_{F_{\widetilde{a}}}(x) \in \overline{H}$ represent the degrees of truth-membership, indeterminacy-membership, and falsity-membership, respectively.

Noteworthy, if there contains only one element in X, $\widetilde{a}$ is called a LNN, for notational simplicity, it can be denoted by $\widetilde{a} = \langle h_{T_{\widetilde{a}}}, h_{I_{\widetilde{a}}}, h_{F_{\widetilde{a}}} \rangle$.

2.2. New Operations for LNNs

According to the LSFs in Appendix A and the Archimedean t-norm and t-conorm presented in Appendix B, some novel operations for LNNs are defined as follows.

Definition 2. *Let* $\widetilde{a} = \langle h_{T_{\widetilde{a}}}, h_{I_{\widetilde{a}}}, h_{F_{\widetilde{a}}} \rangle$ *and* $\widetilde{b} = \left\langle h_{T_{\widetilde{b}}}, h_{I_{\widetilde{b}}}, h_{F_{\widetilde{b}}} \right\rangle$ *be two arbitrary LNNs, and* $\zeta \geq 0$*; then the operations for LNNs are defined as follows:*

(1) $\widetilde{a} \oplus \widetilde{b} = \left\langle f^{*-1}\left(\frac{f^*(h_{T_{\widetilde{a}}}) + f^*(h_{T_{\widetilde{b}}})}{1 + f^*(h_{T_{\widetilde{a}}}) f^*(h_{T_{\widetilde{b}}})}\right), f^{*-1}\left(\frac{f^*(h_{I_{\widetilde{a}}}) + f^*(h_{I_{\widetilde{b}}})}{1 + (1 - f^*(h_{I_{\widetilde{a}}}))(1 - f^*(h_{I_{\widetilde{b}}}))}\right), f^{*-1}\left(\frac{f^*(h_{F_{\widetilde{a}}}) + f^*(h_{F_{\widetilde{b}}})}{1 + (1 - f^*(h_{F_{\widetilde{a}}}))(1 - f^*(h_{F_{\widetilde{b}}}))}\right)\right\rangle$;

(2) $\widetilde{a} \otimes \widetilde{b} = \left\langle f^{*-1}\left(\frac{f^*(h_{T_{\widetilde{a}}}) + f^*(h_{T_{\widetilde{b}}})}{1 + (1 - f^*(h_{T_{\widetilde{a}}}))(1 - f^*(h_{T_{\widetilde{b}}}))}\right), f^{*-1}\left(\frac{f^*(h_{I_{\widetilde{a}}}) + f^*(h_{I_{\widetilde{b}}})}{1 + f^*(h_{I_{\widetilde{a}}}) f^*(h_{I_{\widetilde{b}}})}\right), f^{*-1}\left(\frac{f^*(h_{F_{\widetilde{a}}}) + f^*(h_{F_{\widetilde{b}}})}{1 + f^*(h_{F_{\widetilde{a}}}) f^*(h_{F_{\widetilde{b}}})}\right)\right\rangle$;

(3) $\zeta\widetilde{a} = \left\langle f^{*-1}\left(\frac{(1 + f^*(h_{T_{\widetilde{a}}}))^\zeta - (1 - f^*(h_{T_{\widetilde{a}}}))^\zeta}{(1 + f^*(h_{T_{\widetilde{a}}}))^\zeta + (1 - f^*(h_{T_{\widetilde{a}}}))^\zeta}\right), f^{*-1}\left(\frac{2(f^*(h_{I_{\widetilde{a}}}))^\zeta}{(2 - f^*(h_{I_{\widetilde{a}}}))^\zeta + (f^*(h_{I_{\widetilde{a}}}))^\zeta}\right), f^{*-1}\left(\frac{2(f^*(h_{F_{\widetilde{a}}}))^\zeta}{(2 - f^*(h_{F_{\widetilde{a}}}))^\zeta + (f^*(h_{F_{\widetilde{a}}}))^\zeta}\right)\right\rangle$;

(4) $\widetilde{a}^\zeta = \left\langle f^{*-1}\left(\frac{2(f^*(h_{T_{\widetilde{a}}}))^\zeta}{(2 - f^*(h_{T_{\widetilde{a}}}))^\zeta + (f^*(h_{T_{\widetilde{a}}}))^\zeta}\right), f^{*-1}\left(\frac{(1 + f^*(h_{I_{\widetilde{a}}}))^\zeta - (1 - f^*(h_{I_{\widetilde{a}}}))^\zeta}{(1 + f^*(h_{I_{\widetilde{a}}}))^\zeta + (1 - f^*(h_{I_{\widetilde{a}}}))^\zeta}\right), f^{*-1}\left(\frac{(1 + f^*(h_{F_{\widetilde{a}}}))^\zeta - (1 - f^*(h_{F_{\widetilde{a}}}))^\zeta}{(1 + f^*(h_{F_{\widetilde{a}}}))^\zeta + (1 - f^*(h_{F_{\widetilde{a}}}))^\zeta}\right)\right\rangle$; *and*

(5) $neg(\widetilde{a}) = \langle h_{F_{\widetilde{a}}}, 1 - h_{I_{\widetilde{a}}}, h_{T_{\widetilde{a}}} \rangle$.

Example 1. *Let* $H = \{h_0, h_1, h_2, h_3, h_4, h_5, h_6\}$ $=\{$*very poor, poor, slightly poor, fair, slightly good, good, very good*$\}$, $\widetilde{a} = \langle h_3, h_2, h_2 \rangle$, $\widetilde{b} = \langle h_2, h_3, h_3 \rangle$, *and* $\zeta = 2$, *if* $a = 1.4$, *and* $f_1(h_x) = \theta_x = \frac{x}{2t}$ $(x = 0, 1, \cdots, 2t)$. *The calculated results are as follows:*

(1) $\widetilde{a} \oplus \widetilde{b} = \langle h_{4.29}, h_{3.75}, h_{3.75} \rangle$;

(2) $\widetilde{a} \otimes \widetilde{b} = \langle h_{3.75}, h_{4.29}, h_{4.29} \rangle$;

(3) $2\widetilde{a} = \langle h_{4.8}, h_{0.46}, h_{0.46} \rangle$; *and*

(4) $\widetilde{a}^2 = \langle h_{1.2}, h_{3.6}, h_{3.6} \rangle$.

Theorem 1. *Let* $\widetilde{a}$, $\widetilde{b}$, *and* $\widetilde{c}$ *be three LNNs, and* $\zeta \geq 0$*; then the following equations are true:*

(1) $\widetilde{a} \oplus \widetilde{b} = \widetilde{b} \oplus \widetilde{a}$;

(2) $\left(\widetilde{a} \oplus \widetilde{b}\right) \oplus \widetilde{c} = \widetilde{a} \oplus \left(\widetilde{b} \oplus \widetilde{c}\right)$;

(3) $\widetilde{a} \otimes \widetilde{b} = \widetilde{b} \otimes \widetilde{a}$;

(4) $\left(\widetilde{a} \otimes \widetilde{b}\right) \otimes \widetilde{c} = \widetilde{a} \otimes \left(\widetilde{b} \otimes \widetilde{c}\right)$;

(5) $\zeta\widetilde{a} \oplus \zeta\widetilde{b} = \zeta\left(\widetilde{b} \oplus \widetilde{a}\right)$; *and*

(6) $\left(\widetilde{a} \otimes \widetilde{b}\right)^\zeta = \widetilde{a}^\zeta \otimes \widetilde{b}^\zeta$.

Theorem 1 holds according to Definition 2, so the proof is omitted here.

2.3. Distance between Two LNNs

Definition 3. *Let* $\widetilde{a} = \langle h_{T_{\widetilde{a}}}, h_{I_{\widetilde{a}}}, h_{F_{\widetilde{a}}} \rangle$ *and* $\widetilde{b} = \left\langle h_{T_{\widetilde{b}}}, h_{I_{\widetilde{b}}}, h_{F_{\widetilde{b}}} \right\rangle$ *be two arbitrary LNNs,* f^* *is a LSF. Then, the generalized distance measure between* $\widetilde{a}$ *and* $\widetilde{b}$ *is defined as follows:*

$$d\left(\widetilde{a}, \widetilde{b}\right) = \frac{1}{3}\left(\left|f^*(h_{T_{\widetilde{a}}}) - f^*\left(h_{T_{\widetilde{b}}}\right)\right|^\lambda + \left|f^*(h_{I_{\widetilde{a}}}) - f^*\left(h_{I_{\widetilde{b}}}\right)\right|^\lambda + \left|f^*(h_{F_{\widetilde{a}}}) - f^*\left(h_{F_{\widetilde{b}}}\right)\right|^\lambda\right)^{\frac{1}{\lambda}}. \quad (2)$$

When $\lambda = 1$, the above distance measure can be reduced to the Hamming distance; when $\lambda = 2$, it can be reduced to the Euclidean distance. We can see that Equation (2) is a generalized form of distance measure.

Theorem 2. *Let $\widetilde{a} = \langle h_{T_{\widetilde{a}}}, h_{I_{\widetilde{a}}}, h_{F_{\widetilde{a}}} \rangle$, $\widetilde{b} = \left\langle h_{T_{\widetilde{b}}}, h_{I_{\widetilde{b}}}, h_{F_{\widetilde{b}}} \right\rangle$ and $\widetilde{c} = \langle h_{T_{\widetilde{c}}}, h_{I_{\widetilde{c}}}, h_{F_{\widetilde{c}}} \rangle$ be three arbitrary LNNs, then, the following properties are required for the generalized distance measure in Definition 3.*

(1) $d(\widetilde{a}, \widetilde{b}) \geq 0$;
(2) $d(\widetilde{a}, \widetilde{a}) = 0$;
(3) $d(\widetilde{a}, \widetilde{b}) = d(\widetilde{b}, \widetilde{a})$; *and*
(4) $d(\widetilde{a}, \widetilde{c}) \leq d(\widetilde{a}, \widetilde{b}) + d(\widetilde{b}, \widetilde{c})$.

Theorem 2 is proved in the Appendix C for better representation.

3. Linguistic Neutrosophic Aggregation Operators

Yager [40] introduced the PA operator to allow input arguments to support each other. Thus, the traditional PA operator are first reviewed; then, the LNPWA and LNPWG operators are proposed in an environment featuring LNNs.

Definition 4 ([40])**.** *Let $a_j (j = 1, 2, \cdots, n)$ be a collection of positive values and Ω be the set of all given values; then the PA operator is the mapping $PA : \Omega^n \to \Omega$, which can be defined as follows:*

$$PA(a_1, a_2, \cdots, a_n) = \sum_{j=1}^{n} \frac{1 + G(a_j)}{\sum_{j=1}^{n} (1 + G(a_j))} a_j, \tag{3}$$

where

$$G(a_j) = \sum_{i=1, i \neq j}^{n} Sup(a_j, a_i), \tag{4}$$

$Sup(a_j, a_i)$ represents the support for a_j from a_i, and meets the following properties:

(1) $Sup(a_i, a_j) \in [0, 1]$;
(2) $Sup(a_i, a_j) = Sup(a_j, a_i)$; *and*
(3) $Sup(a_i, a_j) \geq Sup(a_l, a_r)$, *when* $d(a_i, a_j) < d(a_l, a_r)$, *and* $d(a_i, a_j)$ *is the distance between* a_i *and* a_j.

3.1. Linguistic Neutrosophic Power Weighted Averaging Operator

This subsection extends the traditional PA operator to LNN. Then, a LNPWA operator is proposed and discussed.

Definition 5. *Let $\widetilde{a}_j = \langle h_{T_{\widetilde{a}}j}, h_{I_{\widetilde{a}}j}, h_{F_{\widetilde{a}}j} \rangle$ $(j = 1, 2, \ldots, n)$ be a set of LNNs. Then, the LNPWA operator can be defined as*

$$LNPWA(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) = \bigoplus_{j=1}^{n} \frac{w_j (1 + G(\widetilde{a}_j)) \widetilde{a}_j}{\sum_{j=1}^{n} w_j (1 + G(\widetilde{a}_j))}, \tag{5}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\widetilde{a}_j$, $w_i \in [0, 1]$, and $\sum_{i=1}^{n} w_i = 1$, $G(a_j) = \sum_{i=1, i \neq j}^{n} w_i Sup(\widetilde{a}_j, \widetilde{a}_i)$, $Sup(\widetilde{a}_j, \widetilde{a}_i)$ is the support for $\widetilde{a}_j$ from $\widetilde{a}_i$, which also satisfies the similar properties in Definition 4.

Theorem 3. *Let* $\tilde{a}_j = \langle h_{T_{\tilde{a}}j}, h_{I_{\tilde{a}}j}, h_{F_{\tilde{a}}j} \rangle$ $(j = 1, 2, \ldots, n)$ *be a set of LNNs, and* $w = (w_1, w_2, \ldots, w_n)^T$ *is the weight vector of* $\tilde{a}_j$, $w_i \in [0,1]$, *and* $\sum_{i=1}^{n} w_i = 1$. *Then, the aggregated result using Equation (5) is also a LNN. For notational simplicity, we assume that* $\zeta_j = w_j(1 + G(\tilde{a}_j)) / \sum_{j=1}^{n} w_j(1 + G(\tilde{a}_j))$.

$$LNPWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left\langle f^{*-1}\left(\frac{\prod_{j=1}^{n}\left(1+f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j} - \prod_{j=1}^{n}\left(1-f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{n}\left(1+f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j} + \prod_{j=1}^{n}\left(1-f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}}\right), f^{*-1}\left(\frac{2\prod_{j=1}^{n}\left(f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{n}\left(2-f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j} + \prod_{j=1}^{n}\left(f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}}\right), f^{*-1}\left(\frac{2\prod_{j=1}^{n}\left(f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{n}\left(2-f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j} + \prod_{j=1}^{n}\left(f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}}\right)\right\rangle. \quad (6)$$

"Appendix D" details the proof of Theorem 3.

The traditional PA operator has the properties of idempotency, monotonicity, and boundedness. It can be proved that the LNPWA operator also satisfies these properties.

Theorem 4. *Let* $\tilde{a}_j = \langle h_{T_{\tilde{a}}j}, h_{I_{\tilde{a}}j}, h_{F_{\tilde{a}}j} \rangle$ $(j = 1, 2, \ldots, n)$ *be a set of LNNs, and* $w = (w_1, w_2, \ldots, w_n)^T$ *is the weight vector of* $\tilde{a}_j$, $w_i \in [0,1]$, *and* $\sum_{i=1}^{n} w_i = 1$. *If* $Sup(\tilde{a}_j, \tilde{a}_i) = 0$ *or* $Sup(\tilde{a}_j, \tilde{a}_i) = k$ $(k \in [0,1])$ *for all* $\tilde{a}_i$ *and* $\tilde{a}_j$. *Hence, the LNPWA operator reduces to the linguistic neutrosophic weighted averaging (LNWA) operator.*

$$LNWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \bigoplus_{j=1}^{n} w_j \tilde{a}_j = \left\langle \begin{array}{l} f^{*-1}\left(\frac{\prod_{j=1}^{n}\left(1+f^*\left(h_{T_{\tilde{a}}}\right)\right)^{w_j} - \prod_{j=1}^{n}\left(1-f^*\left(h_{T_{\tilde{a}}}\right)\right)^{w_j}}{\prod_{j=1}^{n}\left(1+f^*\left(h_{T_{\tilde{a}}}\right)\right)^{w_j} + \prod_{j=1}^{n}\left(1-f^*\left(h_{T_{\tilde{a}}}\right)\right)^{w_j}}\right), \\ f^{*-1}\left(\frac{2\prod_{j=1}^{n}\left(f^*\left(h_{I_{\tilde{a}}}\right)\right)^{w_j}}{\prod_{j=1}^{n}\left(2-f^*\left(h_{I_{\tilde{a}}}\right)\right)^{w_j} + \prod_{j=1}^{n}\left(f^*\left(h_{I_{\tilde{a}}}\right)\right)^{w_j}}\right), \quad f^{*-1}\left(\frac{2\prod_{j=1}^{n}\left(f^*\left(h_{F_{\tilde{a}}}\right)\right)^{w_j}}{\prod_{j=1}^{n}\left(2-f^*\left(h_{F_{\tilde{a}}}\right)\right)^{w_j} + \prod_{j=1}^{n}\left(f^*\left(h_{F_{\tilde{a}}}\right)\right)^{w_j}}\right) \end{array} \right\rangle \quad (7)$$

The proof for Theorem 4 is similar to the proof for Theorem 3; thus, it is omitted here.

3.2. Linguistic Neutrosophic Power Weighted Geometric Operator

Definition 6. *Let* $\tilde{a}_j = \langle h_{T_{\tilde{a}}j}, h_{I_{\tilde{a}}j}, h_{F_{\tilde{a}}j} \rangle$ $(j = 1, 2, \ldots, n)$ *be a set of LNNs. Then, the LNPWG operator can be defined as*

$$LNPWG(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \bigotimes_{j=1}^{n} (\tilde{a}_j)^{\frac{w_j(1+G(\tilde{a}_j))}{\sum_{j=1}^{n} w_j(1+G(\tilde{a}_j))}}, \quad (8)$$

where $w = (w_1, w_2, \ldots, w_n)^T$ *is the weight vector of* $\tilde{a}_j$, $w_i \in [0,1]$, *and* $\sum_{i=1}^{n} w_i = 1$, $G(a_j) = \sum_{i=1, i \neq j}^{n} w_i Sup(\tilde{a}_j, \tilde{a}_i)$, $Sup(\tilde{a}_j, \tilde{a}_i)$ *is the support for* $\tilde{a}_j$ *from* $\tilde{a}_i$ *and also satisfies the properties in Definition 4.*

Theorem 5. *Let* $\tilde{a}_j = \langle h_{T_{\tilde{a}}j}, h_{I_{\tilde{a}}j}, h_{F_{\tilde{a}}j} \rangle$ $(j = 1, 2, \ldots, n)$ *be a set of LNNs, and* $w = (w_1, w_2, \ldots, w_n)^T$ *is the weight vector of* $\tilde{a}_j$, $w_i \in [0,1]$, *and* $\sum_{i=1}^{n} w_i = 1$. *Then, the aggregated result using Equation (8) is still a LNN, For notational simplicity, we assume that* $\zeta_j = w_j(1 + G(\tilde{a}_j)) / \sum_{j=1}^{n} w_j(1 + G(\tilde{a}_j))$.

$$LNPWG(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) = \left\langle f^{*-1}\left(\frac{2\prod_{j=1}^{n}\left(f^*\left(h_{T_{\widetilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{n}\left(2-f^*\left(h_{T_{\widetilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{n}\left(f^*\left(h_{T_{\widetilde{a}}}\right)\right)^{\zeta_j}}\right), f^{*-1}\left(\frac{\prod_{j=1}^{n}\left(1+f^*\left(h_{I_{\widetilde{a}}}\right)\right)^{\zeta_j}-\prod_{j=1}^{n}\left(1-f^*\left(h_{I_{\widetilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{n}\left(1+f^*\left(h_{I_{\widetilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{n}\left(1-f^*\left(h_{I_{\widetilde{a}}}\right)\right)^{\zeta_j}}\right), f^{*-1}\left(\frac{\prod_{j=1}^{n}\left(1+f^*\left(h_{F_{\widetilde{a}}}\right)\right)^{\zeta_j}-\prod_{j=1}^{n}\left(1-f^*\left(h_{F_{\widetilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{n}\left(1+f^*\left(h_{F_{\widetilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{n}\left(1-f^*\left(h_{F_{\widetilde{a}}}\right)\right)^{\zeta_j}}\right)\right\rangle. \tag{9}$$

The proof of Theorem 5 is also omitted duo to the same way as Theorem 3.

4. MCGDM Method Based on the LNPWA and LNPWG Operators

In this part, a MCGDM method based on the LNPWA and LNPWG operators is developed to solve university HRM evaluation problems.

For a MCGDM problem with a finite set of m alternatives, let $D = \{D_1, D_2, \ldots, D_s\}$ be the set of DMs, $A = \{A_1, A_2, \ldots, A_m\}$ be the set of alternatives, and $C = \{C_1, C_2, \ldots, C_n\}$ be the set of criteria. Assume that the weight vector of the criteria is $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_n)^T$, such that $\varpi_j \in [0,1]$ and $\sum_{j=1}^{n} \varpi_j = 1$. Analogously, the weight vector of the DMs is specified as $w = (w_1, w_2, \ldots, w_s)^T$, where $w_k \geq 0$, and $\sum_{k=1}^{s} w_k = 1$. The evaluation values provided by the DMs are transformed into LNNs, and $\widetilde{a}_{ij}^k = \left\langle h^k_{T_{\widetilde{a}}ij}, h^k_{I_{\widetilde{a}}ij}, h^k_{F_{\widetilde{a}}ij} \right\rangle$, $(k = 1, 2, \ldots, s; j = 1, 2, \ldots, n; i = 1, 2, \ldots, m)$ represents the evaluation value of DM $D_k(k = 1, 2, \ldots, s)$ for alternative $\widetilde{a}_i(i = 1, 2, \ldots, m)$ on criteria $C_j(j = 1, 2, \ldots, n)$.

The detailed procedures of the MCGDM method involve the following steps:

Step 1: Normalize the decision matrices.

In general, criteria can be divided into two categories: benefit type and cost type. Using operation (5) in Definition 2, the cost criteria can be transformed into benefit ones as follows:

$$r_{ij}^k = \begin{cases} \left\langle h^k_{T_{\widetilde{a}}ij}, h^k_{I_{\widetilde{a}}ij}, h^k_{F_{\widetilde{a}}ij} \right\rangle, & for\ benifit\ criterion\ c_j \\ \left\langle h^k_{F_{\widetilde{a}}ij}, 1 - h^k_{I_{\widetilde{a}}ij}, h^k_{T_{\widetilde{a}}ij} \right\rangle, & otherwise \end{cases}, \tag{10}$$

Step 2: Obtain the weighted decision matrices.

Using operations in Definition 2, the weighted decision matrices can be constructed by multiplying the given criteria weight vector into the decision matrices.

Step 3: Calculate the supports.

Utilizing the distance measure defined in Definition 3, the support degrees can be obtained by Equation (11):

$$Sup(r_{ij.}^{k_1}, r_{ij}^{k_2}) = 1 - d(r_{ij}^{k_1}, r_{ij}^{k_2})(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n; k_1, k_2 = 1, 2, \cdots, s) \tag{11}$$

Step 4: Calculate the weights associated with $r_{ij}^{k_1}(k_1 = 1, 2, \cdots, s)$.

$$\eta_{ij,}^{k_1} = w_{k_1}\left(1 + G(\widetilde{r}_{k_1})\right) / \sum_{k_1=1}^{q} w_{k_1}\left(1 + G(\widetilde{r}_{k_1})\right) \tag{12}$$

where $G(r_{k_1}) = \sum_{k_2=1, k_2 \neq k_1}^{s} w_{k_2} Sup(r_{k_1}, r_{k_2})$, and w_{k_2} is interpreted as the weight of DM D_{k_2}.

Step 5: Obtain the comprehensive evaluation information.

Using Equation (5) or Equation (9), the normalized evaluation information provided by DMs can be aggregated, and the integrated decision matrix $R = \left[r_{ij}\right]_{m\times n}$ can be obtained.

Step 6: Determine the ideal decision vectors of all alternative decisions.

After aggregating the DMs' evaluation information into the decision matrix $R = \left[r_{ij}\right]_{m\times n}$, which is as follow:

$$R = \left[r_{ij}\right]_{m\times n} = \begin{matrix} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \begin{matrix} \begin{matrix} C_1 & C_2 & \cdots & C_n \end{matrix} \\ \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix} \end{matrix}, \tag{13}$$

We can determine the ideal alternative vector A^* among all the alternatives below:

$$A^* = \begin{matrix} C_1 & C_2 & \cdots & C_n \\ (\langle h_{T_{2t}}, h_{I_0}, h_{F_0}\rangle, & \langle h_{T_{2t}}, h_{I_0}, h_{F_0}\rangle, & \ldots, & \langle h_{T_{2t}}, h_{I_0}, h_{F_0}\rangle) \end{matrix}. \tag{14}$$

Similarly, the negative ideal alternative vector A_c^* can be obtained by the negation of A^*, which has the maximum separation from A^*, as follows:

$$A_c^* = \begin{matrix} C_1 & C_2 & \cdots & C_n \\ (\langle h_{T_0}, h_{I_{2t}}, h_{F_{2t}}\rangle, & \langle h_{T_0}, h_{I_{2t}}, h_{F_{2t}}\rangle, & \ldots, & \langle h_{T_0}, h_{I_{2t}}, h_{F_{2t}}\rangle) \end{matrix}. \tag{15}$$

In addition, we can obtain the left maximum separation from A^* denoted as A^{*-}:

$$A^{*-} = \begin{matrix} C_1 & C_2 & \cdots & C_n \\ \left(\left\langle h_{T_{A^{*-1}}}, h_{I_{A^{*-1}}}, h_{F_{A^{*-1}}}\right\rangle, & \left\langle h_{T_{A^{*-2}}}, h_{I_{A^{*-2}}}, h_{F_{A^{*-i2}}}\right\rangle, & \ldots, & \left\langle h_{T_{A^{*-n}}}, h_{I_{A^{*-n}}}, h_{F_{A^{*-n}}}\right\rangle\right) \end{matrix}, \tag{16}$$

where $h_{T_{A^{*-j}}} = \min\limits_i\left\{h_{T_{A^{*-j}}}\right\}$, $h_{I_{A^{*-j}}} = \max\limits_i\left\{h_{I_{A^{*-j}}}\right\}$, and $h_{F_{A^{*-j}}} = \max\limits_i\left\{h_{F_{A^{*-j}}}\right\}$.

In the same way, we can also obtain the right maximum separation from A^* denoted as A^{*+}:

$$A^{*+} = \begin{matrix} C_1 & C_2 & \cdots & C_n \\ \left(\left\langle h_{T_{A^{*+1}}}, h_{I_{A^{*+1}}}, h_{F_{A^{*+1}}}\right\rangle, & \left\langle h_{T_{A^{*+2}}}, h_{I_{A^{*+2}}}, h_{F_{A^{*+2}}}\right\rangle, & \ldots, & \left\langle h_{T_{A^{*+n}}}, h_{I_{A^{*+n}}}, h_{F_{A^{*+n}}}\right\rangle\right) \end{matrix}, \tag{17}$$

where $h_{T_{A^{*+j}}} = \max\limits_i\left\{h_{T_{A^{*+j}}}\right\}$, $h_{I_{A^{*+j}}} = \min\limits_i\left\{h_{I_{A^{*+j}}}\right\}$, and $h_{F_{A^{*+j}}} = \min\limits_i\left\{h_{F_{A^{*+j}}}\right\}$.

Step 7: Calculate the separations of each alternative decision vector from the ideal decision vector.

Utilizing the distance measure in Definition 3, we can calculate the separations between each alternative vector and the ideal decision vectors of all alternative decisions, they are respectively represented as follows:

$$d(A_i, A^*) = \sum_{j=1}^{n} \frac{1}{3}\left(\left|f^*\left(h_{T_{\alpha j}}\right) - f^*\left(h_{T_{2t}}\right)\right|^{\lambda} + \left|f^*\left(h_{I_{\alpha j}}\right) - f^*\left(h_{I_0}\right)\right|^{\lambda} + \left|f^*\left(h_{F_{\alpha j}}\right) - f^*\left(h_{F_0}\right)\right|^{\lambda}\right)^{\frac{1}{\lambda}}, \tag{18}$$

$$d(A_i, A_c^*) = \sum_{j=1}^{n} \frac{1}{3}\left(\left|f^*\left(h_{T_{\alpha j}}\right) - f^*\left(h_{T_0}\right)\right|^{\lambda} + \left|f^*\left(h_{I_{\alpha j}}\right) - f^*\left(h_{I_{2t}}\right)\right|^{\lambda} + \left|f^*\left(h_{F_{\alpha j}}\right) - f^*\left(h_{F_{2t}}\right)\right|^{\lambda}\right)^{\frac{1}{\lambda}}, \tag{19}$$

$$d(A_i, A^{*-}) = \sum_{j=1}^{n} \frac{1}{3}\left(\left|f^*\left(h_{T_{\alpha j}}\right) - f^*\left(h_{T_{A^{*-j}}}\right)\right|^{\lambda} + \left|f^*\left(h_{I_{\alpha j}}\right) - f^*\left(h_{I_{A^{*-j}}}\right)\right|^{\lambda} + \left|f^*\left(h_{F_{\alpha j}}\right) - f^*\left(h_{F_{A^{*-j}}}\right)\right|^{\lambda}\right)^{\frac{1}{\lambda}}, \tag{20}$$

$$d(A_i, A^{*+}) = \sum_{j=1}^{n} \frac{1}{3}\left(\left|f^*\left(h_{T_{\alpha j}}\right) - f^*\left(h_{T_{A^*-j}}\right)\right|^{\lambda} + \left|f^*\left(h_{I_{\alpha j}}\right) - f^*\left(h_{I_{A^*-j}}\right)\right|^{\lambda} + \left|f^*\left(h_{F_{\alpha j}}\right) - f^*\left(h_{F_{A^*-j}}\right)\right|^{\lambda}\right)^{\frac{1}{\lambda}}. \tag{21}$$

Step 8: Calculate the relative closeness of each alternative decision.

The relative closeness of each alternative decision can be obtained using the following formula:

$$I_i = \frac{d(A_i, A_c^*) + d(A_i, A^{*-}) + d(A_i, A^{*+})}{d(A_i, A^*) + d(A_i, A_c^*) + d(A_i, A^{*-}) + d(A_i, A^{*+})} \tag{22}$$

Step 9: Rank all the alternatives.

According to the relative closeness of each alternative decision I_i, we can rank all the alternatives. The larger the value of I_i, the better the alternative A_i is.

5. A Case of Human Resource Management Problem

5.1. Problem Definition

The present study focuses on a case of HRM problem in a Chinese university to test the proposed MCGDM method. Specifically, the school of management in the university plans to introduce talents from home and abroad to strengthen discipline construction and try to realize the goal of building a high-level innovative university. Three teams of DMs are assembled as a committee and will take the whole responsibility for this recruitment process, these teams are university presidents D_1, deans of management school D_2, and human resource officers D_3, respectively. After strict first interview, six candidates $A_i (i = 1, 2, \ldots, 6)$ remain for the second review. Before the evaluation procedures, an appropriate evaluation index system should be constructed through literature review and expert consultation. In the literature research, Abdullah et al. [1] and Chou et al. [53] identified three dimensions and eight criteria for the HRM evaluation problem; the three dimensions used in their work were infrastructures, input and output. Zhang et al. [5] constructed an evaluation index system of classroom teaching quality; dimensions included in their work were usage of teaching attitude, teaching capacity, teaching content, teaching method and teaching effect. We can see that different evaluation index systems serve for different purposes of HRM evaluation in various industries. This study mainly tackles the HRM evaluation for talent introduction in universities which exists in real-life decision environments. According to Ref. [54], experts agree on the four criteria included in the evaluation index system for the evaluation of HRM, they are teaching skill (C_1), morality (C_2), education background (C_3) and research capability (C_4), respectively. A brief description of each criterion is shown as follows.

Teaching skill is an overall reflect of one teacher's classroom teaching quality which includes several sub-attributes, such as teaching attitude, teaching capacity, teaching content, teaching method and teaching effect.

Morality refers to the teachers' morality in this study. It is a kind of professional morality of teachers which takes up the first place of education and can greatly affects the education's level and quality as a whole. More specifically, the teachers' morality includes the moral consciousness, moral relations and moral activity of the teachers in universities.

Education background is an overview of a person's learning environment and learning ability. It includes the person's educational level, graduate school, major courses, academic achievements, and some other highlights.

Research capability denotes the scientific research ability that is required for scientific research or the research competence someone shows during the process of scientific research. The former is closer to the potential, including someone's abilities in logical thinking, writing and oral language, etc., whereas the latter emphasizes someone's practical scientific research capacity.

With the reform of education and fierce competition among universities, the current form of university education needs more and more modern teachers with the above four abilities. Therefore, this study applies the above four criteria for the case of HRM evaluation, and the six candidates $A_i(i = 1, 2, \ldots, 6)$ are evaluated by the three teams of DMs under each criterion. The weight vector of criteria was assigned by DMs as $\varpi = (0.3, 0.12, 0.31, 0.27)^T$, and the weight vector of DMs was $w = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$. In addition, the LTS was denoted as $H = \{h_0, h_1, \cdots, h_6\} = \{extremely\ poor, very\ poor, poor, medium, good, very\ good, extremely\ good\}$. By interviewing the DMs one by one anonymously, all of their linguistic assessments for each alternative under each criterion are collected together. During this process, DMs in each group are isolated and don't negotiate with each other at all. Consequently, the decision information is provided independently in the form of linguistic terms. Take the evaluation value $\widetilde{a}_{11}^1 = \langle h_5, h_3, h_2 \rangle$ as an example, which represents the evaluation value of DM D_1 for alternative A_1 under criterion C_1. Since the criterion C_1 (teaching skill) includes various aspects, such as teaching attitude, teaching capacity, teaching content, teaching method and teaching effect, the group of DMs D_1 may hold inconsistent linguistic judgments for alternative A_1 with respect to C_1. After collecting all the linguistic assessments for alternative A_1, the linguistic neutrosophic information $\widetilde{a}_{11}^1 = \langle h_5, h_3, h_2 \rangle$ is obtained by calculating the weighted mean values of all the labels of linguistic terms with respect to active, neutral and passive information, respectively. Similarly, the overall evaluation information provided by the teams of DMs can be represented in the form of LNNs in Tables 1–3.

Table 1. Evaluation information of D_1.

D_1	C_1	C_2	C_3	C_4
A_1	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_2 \rangle$
A_2	$\langle h_5, h_3, h_1 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_0, h_3, h_0 \rangle$
A_3	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_0 \rangle$
A_4	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_0 \rangle$
A_5	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_2 \rangle$	$\langle h_0, h_3, h_2 \rangle$
A_6	$\langle h_6, h_3, h_2 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_0, h_3, h_2 \rangle$

Table 2. Evaluation information of D_2.

D_2	C_1	C_2	C_3	C_4
A_1	$\langle h_6, h_3, h_0 \rangle$	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_0 \rangle$
A_2	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_0 \rangle$
A_3	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_0, h_0 \rangle$
A_4	$\langle h_6, h_3, h_2 \rangle$	$\langle h_6, h_3, h_2 \rangle$	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_2 \rangle$
A_5	$\langle h_5, h_5, h_0 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_6, h_3, h_0 \rangle$	$\langle h_0, h_3, h_2 \rangle$
A_6	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_6, h_3, h_2 \rangle$	$\langle h_5, h_3, h_1 \rangle$

Table 3. Evaluation information of D_3.

D_3	C_1	C_2	C_3	C_4
A_1	$\langle h_6, h_3, h_0 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_6, h_3, h_2 \rangle$	$\langle h_5, h_3, h_0 \rangle$
A_2	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_2 \rangle$
A_3	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_6, h_3, h_0 \rangle$	$\langle h_5, h_3, h_0 \rangle$
A_4	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_6, h_3, h_2 \rangle$	$\langle h_0, h_3, h_2 \rangle$
A_5	$\langle h_5, h_3, h_2 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_6, h_3, h_2 \rangle$	$\langle h_5, h_3, h_2 \rangle$
A_6	$\langle h_5, h_3, h_2 \rangle$	$\langle h_0, h_3, h_2 \rangle$	$\langle h_5, h_3, h_0 \rangle$	$\langle h_5, h_3, h_0 \rangle$

5.2. Evaluation Steps of the Proposed Method

The following steps describe the procedures of evaluation for all candidates, and the ranking order of the six alternatives can be obtained. For simplicity of calculation, we chose the LSF f_1^*.

Step 1: Normalize the decision matrices.

It is obvious that all the four criteria are of the benefit type; then, there is no need for normalization.

Step 2: Obtain the weighted decision matrices.

Using operation in Definition 2, the weighted decision matrices can be constructed in Tables 4–6:

Table 4. Weighted evaluation information of D_1.

D_1	C_1	C_2	C_3	C_4
A_1	$\langle h_{2.0696}, h_{5.0201}, h_{4.579} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_0 \rangle$	$\langle h_{1.8772}, h_{5.1166}, h_{4.7165} \rangle$
A_2	$\langle h_{2.0696}, h_{5.0201}, h_{3.9304} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_0 \rangle$	$\langle h_0, h_{5.1166}, h_0 \rangle$
A_3	$\langle h_{2.0696}, h_{5.0201}, h_{4.579} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_0 \rangle$	$\langle h_0, h_{5.1166}, h_0 \rangle$
A_4	$\langle h_{2.0696}, h_{5.0201}, h_{4.579} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_{4.5335} \rangle$	$\langle h_0, h_{5.1166}, h_0 \rangle$
A_5	$\langle h_{2.0696}, h_{5.0201}, h_{4.579} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_{5.4224} \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_{4.5335} \rangle$	$\langle h_0, h_{5.1166}, h_{4.7165} \rangle$
A_6	$\langle h_6, h_{5.0201}, h_{4.579} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_0 \rangle$	$\langle h_0, h_{5.1166}, h_{4.7165} \rangle$

Table 5. Weighted evaluation information of D_2.

D_2	C_1	C_2	C_3	C_4
A_1	$\langle h_6, h_{5.0201}, h_0 \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_{5.4224} \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_{4.5335} \rangle$	$\langle h_{1.8772}, h_{5.1166}, h_0 \rangle$
A_2	$\langle h_{2.0696}, h_{5.0201}, h_0 \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_0 \rangle$	$\langle h_{1.8772}, h_{5.1166}, h_0 \rangle$
A_3	$\langle h_{2.0696}, h_{5.0201}, h_0 \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_{4.5335} \rangle$	$\langle h_{1.8772}, h_0, h_0 \rangle$
A_4	$\langle h_6, h_{5.0201}, h_{4.579} \rangle$	$\langle h_6, h_{5.6051}, h_{5.4224} \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_{4.5335} \rangle$	$\langle h_{1.8772}, h_{5.1166}, h_{4.7165} \rangle$
A_5	$\langle h_{2.0696}, h_{5.6974}, h_0 \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_6, h_{4.9881}, h_0 \rangle$	$\langle h_0, h_{5.1166}, h_{4.7165} \rangle$
A_6	$\langle h_{2.0696}, h_{5.0201}, h_{4.579} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_6, h_{4.9881}, h_{4.5335} \rangle$	$\langle h_{1.8772}, h_{5.1166}, h_{4.1228} \rangle$

Table 6. Weighted evaluation information of D_3.

D_3	C_1	C_2	C_3	C_4
A_1	$\langle h_6, h_{5.0201}, h_0 \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_6, h_{4.9881}, h_{4.5335} \rangle$	$\langle h_{1.8772}, h_{5.1166}, h_0 \rangle$
A_2	$\langle h_{2.0696}, h_{5.0201}, h_{4.579} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_{4.5335} \rangle$	$\langle h_{1.8772}, h_{5.1166}, h_{4.7165} \rangle$
A_3	$\langle h_{2.0696}, h_{5.0201}, h_{4.579} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_6, h_{4.9881}, h_0 \rangle$	$\langle h_{1.8772}, h_{5.1166}, h_0 \rangle$
A_4	$\langle h_{2.0696}, h_{5.0201}, h_{4.579} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_6, h_{4.9881}, h_{4.5335} \rangle$	$\langle h_0, h_{5.1166}, h_{4.7165} \rangle$
A_5	$\langle h_{2.0696}, h_{5.0201}, h_{4.579} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_6, h_{4.9881}, h_{4.5335} \rangle$	$\langle h_{1.8772}, h_{5.1166}, h_{4.7165} \rangle$
A_6	$\langle h_{2.0696}, h_{5.0201}, h_{4.579} \rangle$	$\langle h_0, h_{5.6051}, h_{5.4224} \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_0 \rangle$	$\langle h_{1.8772}, h_{5.1166}, h_0 \rangle$

Step 3: Calculate the supports.

Utilizing the distance measure defined in Definition 3 and Equation (11), the supports can be obtained. Here, we assume that $\lambda = 2$ in the distance measure.

$$\sup\left(r_{ij}^1, r_{ij}^2\right) = \sup\left(r_{ij}^2, r_{ij}^1\right) = \begin{bmatrix} 0.6647 & 0.6988 & 0.7481 & 0.738 \\ 0.7816 & 1 & 1 & 0.8957 \\ 0.7456 & 1 & 0.7481 & 0.7157 \\ 0.7816 & 0.5848 & 1 & 0.738 \\ 0.7428 & 0.6988 & 0.6689 & 1 \\ 0.7816 & 1 & 0.6689 & 0.8906 \end{bmatrix},$$

$$\sup\left(r_{ij}^1, r_{ij}^3\right) = \sup\left(r_{ij}^3, r_{ij}^1\right) = \begin{bmatrix} 0.6647 & 1 & 0.6689 & 0.738 \\ 0.964 & 1 & 0.7481 & 0.718 \\ 1 & 1 & 0.7852 & 1 \\ 1 & 1 & 0.7852 & 0.718 \\ 1 & 0.6988 & 0.7852 & 0.8957 \\ 0.7816 & 0.695 & 1 & 0.718 \end{bmatrix}, \text{ and}$$

$$\sup\left(r_{ij}^2, r_{ij}^3\right) = \sup\left(r_{ij}^3, r_{ij}^2\right) = \begin{bmatrix} 1 & 0.6988 & 0.7852 & 1 \\ 0.7456 & 1 & 0.7481 & 0.738 \\ 0.7456 & 1 & 0.6689 & 0.7157 \\ 0.7816 & 0.5848 & 0.7852 & 0.8957 \\ 0.7428 & 1 & 0.7481 & 0.8957 \\ 1 & 0.695 & 0.6689 & 0.771 \end{bmatrix}$$

Step 4: Calculate the weights associated with $r_{ij}^{k_1}$ $(k_1 = 1, 2, \cdots, s)$.

The weights can be calculated by Equation (12) as follows:

$$\eta_{ij}^1 = \begin{bmatrix} 0.317 & 0.3406 & 0.3295 & 0.3208 \\ 0.3394 & 0.3333 & 0.3393 & 0.3367 \\ 0.3394 & 0.3333 & 0.3382 & 0.3402 \\ 0.3385 & 0.3437 & 0.3384 & 0.3252 \\ 0.3395 & 0.3188 & 0.3323 & 0.3357 \\ 0.323 & 0.3407 & 0.3414 & 0.3349 \end{bmatrix}, \eta_{ij}^2 = \begin{bmatrix} 0.3415 & 0.3188 & 0.3382 & 0.3396 \\ 0.3238 & 0.3333 & 0.3393 & 0.3381 \\ 0.3212 & 0.3333 & 0.3295 & 0.3197 \\ 0.323 & 0.3126 & 0.3384 & 0.3381 \\ 0.3211 & 0.3406 & 0.3295 & 0.3357 \\ 0.3385 & 0.3407 & 0.3172 & 0.3388 \end{bmatrix}, \text{and}$$

$$\eta_{ij}^3 = \begin{bmatrix} 0.3415 & 0.3406 & 0.3323 & 0.3396 \\ 0.3368 & 0.3333 & 0.3213 & 0.3252 \\ 0.3394 & 0.3333 & 0.3323 & 0.3402 \\ 0.3385 & 0.3437 & 0.3232 & 0.3367 \\ 0.3395 & 0.3406 & 0.3382 & 0.3286 \\ 0.3385 & 0.3186 & 0.3414 & 0.3263 \end{bmatrix}$$

Step 5: Obtain the comprehensive evaluation information.

Using Equation (5) or Equation (9), the integrated decision matrix $R = [r_{ij}]_{m \times n}$ are calculated below:

(i) When using Equation (5), the results are listed in Table 7.

Table 7. Comprehensive evaluation information by LNPWA operator.

D_2	C_1	C_2	C_3	C_4
A_1	$\langle h_6, h_{5.0201}, h_0 \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_6, h_{4.9881}, h_0 \rangle$	$\langle h_{1.8772}, h_{5.1166}, h_0 \rangle$
A_2	$\langle h_{2.0696}, h_{5.0201}, h_0 \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_0 \rangle$	$\langle h_{1.2689}, h_{5.1166}, h_0 \rangle$
A_3	$\langle h_{2.0696}, h_{5.0201}, h_0 \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_6, h_{4.9881}, h_0 \rangle$	$\langle h_{1.8772}, h_0, h_0 \rangle$
A_4	$\langle h_6, h_{5.0201}, h_{4.579} \rangle$	$\langle h_6, h_{5.6051}, h_0 \rangle$	$\langle h_6, h_{4.9881}, h_{4.5335} \rangle$	$\langle h_{1.2689}, h_{5.1166}, h_0 \rangle$
A_5	$\langle h_{2.0696}, h_{5.2356}, h_0 \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_6, h_{4.9881}, h_0 \rangle$	$\langle h_{0.6358}, h_{5.1166}, h_{4.7165} \rangle$
A_6	$\langle h_{2.0696}, h_{5.0201}, h_{4.579} \rangle$	$\langle h_{0.5864}, h_{5.6051}, h_0 \rangle$	$\langle h_6, h_{4.9881}, h_0 \rangle$	$\langle h_{1.2721}, h_{5.1166}, h_0 \rangle$

(ii) When using Equation (9), the results are listed in Table 8.

Table 8. Comprehensive evaluation information by LNPWG operator.

D_2	C_1	C_2	C_3	C_4
A_1	$\langle h_{4.5387}, h_{5.0201}, h_{1.8471} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_{2.6567} \rangle$	$\langle h_{3.1737}, h_{4.9881}, h_{3.4741} \rangle$	$\langle h_{1.8772}, h_{5.1166}, h_{1.9671} \rangle$
A_2	$\langle h_{2.0696}, h_{5.0201}, h_{3.2402} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_{2.1327}, h_{4.9881}, h_{1.8396} \rangle$	$\langle h_0, h_{5.1166}, h_{1.9918} \rangle$
A_3	$\langle h_{2.0696}, h_{5.0201}, h_{3.5544} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_0 \rangle$	$\langle h_{3.1737}, h_{4.9881}, h_{1.8834} \rangle$	$\langle h_{1.8772}, h_{4.182}, h_0 \rangle$
A_4	$\langle h_{3.0839}, h_{5.0201}, h_{4.579} \rangle$	$\langle h_{1.7569}, h_{5.6051}, h_{2.6119} \rangle$	$\langle h_{3.1414}, h_{4.9881}, h_{4.5335} \rangle$	$\langle h_0, h_{5.1166}, h_{3.6868} \rangle$
A_5	$\langle h_{2.0696}, h_{5.3227}, h_{3.5549} \rangle$	$\langle h_{0.8573}, h_{5.6051}, h_{2.6553} \rangle$	$\langle h_{4.5051}, h_{4.9881}, h_{3.4741} \rangle$	$\langle h_0, h_{5.1166}, h_{4.7165} \rangle$
A_6	$\langle h_{3.0839}, h_{5.0201}, h_{4.579} \rangle$	$\langle h_0, h_{5.6051}, h_{2.6553} \rangle$	$\langle h_{3.1201}, h_{4.9881}, h_{1.8174} \rangle$	$\langle h_0, h_{5.1166}, h_{3.3929} \rangle$

Step 6: Determine the ideal decision vectors of all alternative decisions.

(i) When using Equation (5), we can determine the ideal alternative vectors among all the alternatives respectively as follows:

$A^* = (\langle h_6, h_0, h_0\rangle, \langle h_6, h_0, h_0\rangle, \langle h_6, h_0, h_0\rangle, \langle h_6, h_0, h_0\rangle)$,
$A_c^* = (\langle h_0, h_6, h_6\rangle, \langle h_0, h_6, h_6\rangle, \langle h_0, h_6, h_6\rangle, \langle h_0, h_6, h_6\rangle)$,
$A^{*-} = (\langle h_{2.0696}, h_{5.2356}, h_{4.579}\rangle, \langle h_{0.5864}, h_{5.6051}, h_0\rangle, \langle h_{2.1327}, h_{4.9881}, h_{4.5335}\rangle, \langle h_{0.6358}, h_{5.1166}, h_{4.7165}\rangle)$, and
$A^{*+} = (\langle h_6, h_{5.0201}, h_0\rangle, \langle h_6, h_{5.6051}, h_0\rangle, \langle h_6, h_{4.9881}, h_0\rangle, \langle h_{1.8772}, h_0, h_0\rangle)$.

(ii) When using Equation (9), the results are:

$A^* = (\langle h_6, h_0, h_0\rangle, \langle h_6, h_0, h_0\rangle, \langle h_6, h_0, h_0\rangle, \langle h_6, h_0, h_0\rangle)$,
$A_c^* = (\langle h_0, h_6, h_6\rangle, \langle h_0, h_6, h_6\rangle, \langle h_0, h_6, h_6\rangle, \langle h_0, h_6, h_6\rangle)$,
$A^{*-} = (\langle h_{2.0696}, h_{5.3227}, h_{4.579}\rangle, \langle h_0, h_{5.6051}, h_{2.6119}\rangle, \langle h_{2.1327}, h_{4.9881}, h_{4.5335}\rangle, \langle h_0, h_{5.1166}, h_{4.7165}\rangle)$, and
$A^{*+} = (\langle h_{4.5387}, h_{5.0201}, h_{1.8417}\rangle, \langle h_{1.7569}, h_{5.6051}, h_0\rangle, \langle h_{4.4051}, h_{4.9881}, h_{1.8174}\rangle, \langle h_{1.8772}, h_{4.182}, h_0\rangle)$.

Step 7: Calculate the separations of each alternative decision vector from the ideal decision vector.

The separations between each alternative and the ideal decision vector by the LNPWA and LNPGA operators are shown in Tables 9 and 10, respectively.

Table 9. Separations by the LNPWA operator.

Distance	$d(A_i, A^*)$	$d(A_i, A_c^*)$	$d(A_i, A^{*-})$	$d(A_i, A^{*+})$	I_i
A_1	2.1903	2.1162	1.6257	1.7055	0.7132
A_2	2.3229	2.0653	1.5863	1.7175	0.698
A_3	1.3743	2.8968	2.3562	0	0.7926
A_4	2.3229	2.0653	1.5863	1.7175	0.698
A_5	2.9288	0.561	0	2.3562	0.499
A_6	2.3222	2.0656	1.5864	1.7174	0.6981

Table 10. Separations by the LNPWG operator.

Distance	$d(A_i, A^*)$	$d(A_i, A_c^*)$	$d(A_i, A^{*-})$	$d(A_i, A^{*+})$	I_i
A_1	2.2863	1.5118	1.1097	0.7259	0.5942
A_2	2.711	1.3681	0.9082	0.9641	0.5445
A_3	1.9575	2.1815	1.7206	0	0.6659
A_4	2.9016	0.8254	0.3432	1.4138	0.4709
A_5	3.0628	0.5194	0	1.7205	0.4224
A_6	2.8615	0.9176	0.4412	1.3295	0.4844

Step 8: Calculate the relative closeness of each alternative decision.

The results of relative closeness of each alternative decision are shown in the last column of Tables 9 and 10.

Step 9: Rank all the alternatives.

According to the relative closeness of each alternative decision I_i, we can rank all the alternatives. When using LNPWA operator, the ranking result is $A_3 \succ A_1 \succ A_6 \succ A_2 = A_4 \succ A_5$, whereas when using LNPWG operator, the result turns out $A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$. There is a subtle distinction between the results obtained by the LNPWA and LNPWG operators, but the alternative A_3 remains the most performant and competitive candidate.

5.3. Sensitivity Analysis and Discussion

The aim of sensitivity analysis is to investigate the effects of different semantics and the distance parameter λ on the final ranking results of alternatives. To do so, the calculated results are shown in Tables 11 and 12 and Figures 1 and 2, respectively.

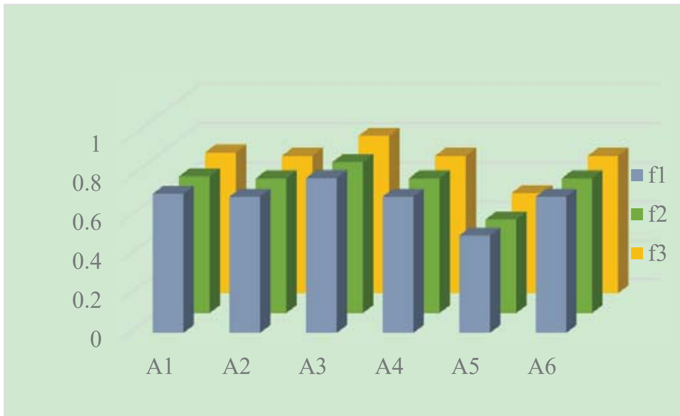

Figure 1. Ranking results by the LNPWA operator.

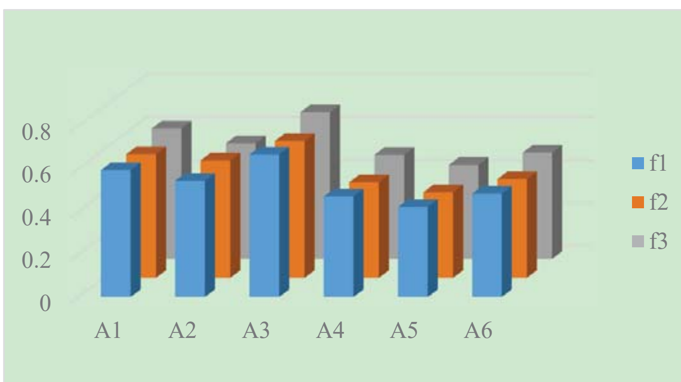

Figure 2. Ranking results by the LNPWG operator.

Table 11. Results of different LSFs f^* ($\lambda = 2$).

		Alternatives						Ranking Results
		A_1	A_2	A_3	A_4	A_5	A_6	
f_1^*	LNPWA	0.713	0.698	0.793	0.698	0.499	0.698	$A_3 \succ A_1 \succ A_6 \succ A_2 = A_4 \succ A_5$
	LNPWG	0.594	0.544	0.666	0.471	0.422	0.484	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$
f_2^*	LNPWA	0.7	0.69	0.773	0.69	0.48	0.69	$A_3 \succ A_1 \succ A_6 \succ A_2 \succ A_4 \succ A_5$
	LNPWG	0.578	0.549	0.64	0.447	0.401	0.462	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$
f_3^*	LNPWA	0.721	0.704	0.806	0.704	0.514	0.704	$A_3 \succ A_1 \succ A_6 \succ A_2 = A_4 \succ A_5$
	LNPWG	0.608	0.54	0.684	0.486	0.439	0.496	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$

It can be seen from Table 11 and Figures 1 and 2 that the alternative A_3 remained to be the best one, and A_5 was consistently identified as the worst choice no matter how the aggregation operator or semantics change. When using the LNPWA operator, the ranking result remains $A_3 \succ A_1 \succ A_6 \succ A_2 = A_4 \succ A_5$. The difference in semantics slightly influenced the values of I_i, but did not result in different ranking orders. Similarly, when using the LNPWG operator, the ranking result always is $A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$. It is clear that the ranking results varied when using different aggregation operators. This may be caused by the distinct inherent characteristic of these two operators, since the LNPWA operator is based on the arithmetic averaging, whereas the LNPWG operator is based on the geometric averaging. This demonstrates that the ranking results have stability by our proposed method in some degree.

The following Table 12 the influence of the distance parameter λ on the final ranking results of alternatives when the semantics were fixed as $f^* = f_1^*$. It can be seen that the ranking results kept the same as $A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$ when using the LNPWG operator. However, results by the LNPWA operator change among $A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$, $A_3 \succ A_1 \succ A_6 \succ A_2 = A_4 \succ A_5$ and $A_1 \succ A_6 \succ A_2 = A_4 \succ A_3 \succ A_5$. Thus, we can conclude that the differences in the aggregation operators and the parameter λ could influence the evaluation results, DMs should choose appropriate parameter λ and aggregation operators according to their own inherent characteristics.

Table 12. Results of different parameter $\lambda(f^* = f_1^*)$.

λ	$f^* = f_1^*$	
	Ranking by LNPWA operator	Ranking by LNPWG operator
1	$A_3 \succ A_1 \succ A_6 \succ A_2 = A_4 \succ A_5$	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$
2	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$
3	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$
4	$A_3 \succ A_1 \succ A_6 \succ A_2 = A_4 \succ A_5$	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$
5	$A_3 \succ A_1 \succ A_2 = A_6 = A_4 \succ A_5$	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$
6	$A_3 \succ A_1 \succ A_6 \succ A_2 = A_4 \succ A_5$	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$
7	$A_1 \succ A_3 \succ A_6 \succ A_2 = A_4 \succ A_5$	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$
8	$A_1 \succ A_6 \succ A_2 = A_4 \succ A_3 \succ A_5$	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$
9	$A_1 \succ A_6 \succ A_2 = A_4 \succ A_3 \succ A_5$	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$
10	$A_1 \succ A_6 \succ A_2 = A_4 \succ A_3 \succ A_5$	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$

5.4. Comparison Analysis and Discussion

This subsection conducts a comparative study to validate the practicality and advantages of the proposed method in the LNS contexts, and the results are shown in Table 13. Brief descriptions about the comparative methods are as follows.

(1) Weighted arithmetic and geometric averaging operators of LNNs [28]: the concept of LNNs was first proposed by Fang and Ye [28]. In their study, two aggregation operators including the LNN-weighted arithmetic averaging (LNNWAA) operator and LNN-weighted geometric averaging (LNNWGA) operator are utilized to derive collective evaluations. Then, based on their proposed score function and accuracy function of LNNs, the ranking order of alternatives is obtained.

(2) Bonferroni mean operators of LNNs [30]: the LNNNWBM operator and LNNNWGBM operator are proposed to aggregate evaluations to obtain the collective LNN for each alternative. Subsequently, the results are derived by expected value.

(3) An extended TOPSIS method [32]: a weighted model based on maximizing deviation is used to determine criteria weights. Subsequently, an extended TOPSIS method with LNNs is proposed to rank alternatives.

Table 13. Comparison results with the existing methods.

MCGDM	Ranking Results
Proposed method by LNPWA operator	$A_3 \succ A_1 \succ A_6 \succ A_2 = A_4 \succ A_5$
Proposed method by LNPWG operator	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$
LNNWAA operator [28]	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_5 \succ A_4$
LNNWGA operator [28]	$A_3 \succ A_1 \succ A_2 \succ A_6 \succ A_4 \succ A_5$
LNNNWBM operator [30] ($p = q = 1$)	$A_1 \succ A_3 \succ A_6 \succ A_2 \succ A_5 \succ A_4$
LNNNWGBM operator [30] ($p = q = 1$)	$A_1 \succ A_3 \succ A_6 \succ A_2 \succ A_4 \succ A_5$
An extended TOPSIS method [32] ($\lambda = 2$)	$A_3 \succ A_1 \succ A_6 \succ A_2 = A_4 \succ A_5$

As shown in Table 13, different methods resulted in different ranking results, but the optimal candidate remained to be A_3, despite the results obtained by the Bonferroni mean operators of LNNs [30]. The main reasons for these differences may be as follows: (1) The operations for LNNs between this study and the comparative methods are remarkably different. The operations in the existing methods [28,30,32] just considered the linguistic variables' labels which may cause information loss and distortion. (2) Different aggregation operators and ranking rules might also cause different ranking results. Specifically, the LNNWAA and LNNWGA operators defined in [28] were respectively based on the arithmetic mean and geometric mean operators, whereas the Bonferroni mean operators of LNNs [30] implicated the interactive hypothesis among inputs. Unlike the existing aggregation tools, the proposed PA operator for LNNs allows the information provided by different DMs to support and reinforce each other, and it is a nonlinear weighted average operator.

From above discussions, the unique features of the proposal and its main advantages over others can be simply summarized below.

(1) The comparative methods [28,30,32] dealt with the LNNs only considering the labels of linguistic variables while ignoring the differences in various semantics. It has been contended that the same linguistic variable possesses different meanings for different people and has diverse meanings for the same person under various situations [55]. Therefore, directly using the labels of linguistic variables may lead to information loss during information aggregation. To cover this challenge, this study redefines the operations for LNNs based on the LSFs and Archimedean t-norm and t-conorm, which increases the flexibility and accuracy of linguistic information transformation.

(2) The extended TOPSIS method [32] only considered two relatively positive and negative ideal solutions to determine the values of correlation coefficient for each alternative. By contrast, this study takes both the relatively and absolutely positive and negative ideal solutions into account. Therefore, the ranking result by this proposed method may be somewhat more comprehensive than the existing method [32].

(3) For information fusion, all the existing methods [28,30,32] failed to consider the support degree among different DMs during the aggregation processes. Although it is true that different aggregation operators cater to different practical decision situations, the proposed PA operators within LNN contexts are more feasible in dealing with the university HRM evaluation problem in this study.

6. Conclusions and Future Work

Talent introduction plays an important role in the long-term development of a university. This is closely related to the university's discipline development and comprehensive strength. Therefore, there is a need for proper HRM evaluation that uses group decision-making methods efficiently in order to utilize human resources. This study recognized the HRM evaluation procedures as a complex MCGDM problems within the LNNs' circumstances. Through merging the PA operator with LNSs, we developed two aggregation operators (LNPWA and LNPWG) for information fusion. Then, we made some modifications in the classical TOPSIS method to determine the ranking order of alternatives. The strengths of the proposed method have been discussed via comparative analysis.

Nevertheless, this study also holds several limitations which can suggest several avenues for future research. First, the information fusion process adds to the computational complexity of the obtained results because the proposed LNPWA and LNPWG operators are both nonlinear weighted average operators, where the weights associated with each DM should be calculated by their input arguments. Fortunately, the pressure from complex computation can be remarkably eased with the assistance of programming software. Second, with the rapid development of information technology, it is also possible to extend the current results for other management systems under the network-based environments [56,57].

By analyzing the achieved results, the practical implications of our research may be summarized in two aspects. On the one hand, this study proposes a novel linguistic neutrosophic MCGDM method which contributes to expanding the theoretical depth of university HRM. It may offer comprehensive supports for decision-making of modern universities' talent introduction. In addition, the developed method can also be further expanded to solving group decision-making problems in other fields, such as tourism. On the other hand, this study further explores the application of linguistic MCGDM methods in HRM. The obtained knowledge can be very helpful to improve the performance of the human resource of universities accordingly.

Author Contributions: R.-x.L., J.-q.W. and Z.-b.J. conceived and worked together to achieve this work; and R.-x.L. and Z.-b.J. wrote the paper.

Funding: This research was funded by Fundamental Research Funds for the Central Universities of Central South University grant number (No. 502211710), and the APC was also funded by (No. 502211710).

Acknowledgments: The authors would like to thank the editors and anonymous reviewers for their great help on this study.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Linguistic Scale Function

By means of literature review, we can gather the following choices acting as LSFs.

(1) The LSF f_1 is based on the subscript function $sub(h_\tau) = \tau$:

$$f_1(h_x) = \theta_x = \frac{x}{2t}\ (x = 0, 1, \cdots, 2t),\ \theta_x \in [0, 1]. \tag{A1}$$

The above function is divided on average. It is commonly used for its simple form and easy calculation, but it lacks a reasonable theoretical basis [58].

(2) The LSF f_2 is based on the exponential scale:

$$f_2(h_y) = \theta_y = \begin{cases} \frac{\alpha^t - \alpha^{t-y.}}{2\alpha^t - 2}\ (y = 0, 1, \cdots, t) \\ \frac{\alpha^t + \alpha^{y-t} - 2}{2\alpha^t - 2}\ (y = t+1, t+2, \cdots, 2t) \end{cases} \tag{A2}$$

Here, the absolute deviation between any two adjacent linguistic labels decreases with the increase of y in the interval $[0, t]$, and increases with the increase of y in the interval $[t+1, 2t]$.

(3) The LSF f_3 is based on prospect theory:

$$f_3(h_z) = \theta_z = \begin{cases} \frac{t^\beta - (t-z).^\beta}{2t^\beta}\ (z = 0, 1, \cdots, t) \\ \frac{t^\gamma + (z-t)^\gamma}{2t^\gamma}\ (z = t+1, t+2, \cdots, 2t) \end{cases} \tag{A3}$$

Here, $\beta, \gamma \in [0, 1]$, and when $\beta = \gamma = 1$, the LSF f_3 is reduced to f_1. Moreover, the absolute deviation between any two adjacent linguistic labels increases with the increase of y in the interval $[0, t]$, and decreases with the increase of y in the interval $[t+1, 2t]$.

Each of the above LSFs f_1, f_2, and f_3 can be expanded to a strictly monotonically increasing and continuous function: $f^* : \overline{S} \to R^+$ $(R^+ = \{r | r \geq 0, r \in R\})$, which satisfies $f^*(s_\tau) = \theta_\tau$. Therefore, the inverse function of f^*, denoted as f^{*-1}, exists due to its monotonicity.

Appendix B. The Archimedean *T*-norm and *T*-conorm

According to Reference [59], a *t*-norm $T(x,y)$ is called Archimedean *t*-norm if it is continuous and $T(x,x) < x$, for all $x \in (0,1)$. An Archimedean *t*-norm is called a strict Archimedean *t*-norm if it is strictly increasing in every variable for $x, y \in (0,1)$. In addition, a *t*-conorm $S(x,y)$ is called Archimedean *t*-conorm if it is continuous and $S(x,x) > x$, for all $x \in (0,1)$. An Archimedean *t*-conorm is called a strict Archimedean *t*-conorm if it is strictly increasing in every variable for $x, y \in (0,1)$.

In this study, we apply one well-known Archimedean *t*-norm and *t*-conorm [60], as $S(x,y) = (x+y)/(1+xy)$ and $T(x,y) = xy/[1+(1-x)(1-y)]$, respectively.

Appendix C. The Proof of Theorem 2

Proof. It is clear that properties (1)–(3) in Theorem 2 hold. The proof of property (4) in Theorem 2 is shown below.

First, the distances $d(\tilde{a},\tilde{c})$, $d(\tilde{a},\tilde{b})$ and $d(\tilde{b},\tilde{c})$ can be easily determined respectively as follows:

$$d(\tilde{a},\tilde{c}) = \frac{1}{3}\left(\left|f^*(h_{T_{\tilde{a}}}) - f^*(h_{T_{\tilde{c}}})\right|^\lambda + \left|f^*(h_{I_{\tilde{a}}}) - f^*(h_{I_{\tilde{c}}})\right|^\lambda + \left|f^*(h_{F_{\tilde{a}}}) - f^*(h_{F_{\tilde{c}}})\right|^\lambda\right)^{\frac{1}{\lambda}},$$

$$d\left(\tilde{a},\tilde{b}\right) = \frac{1}{3}\left(\left|f^*(h_{T_{\tilde{a}}}) - f^*\left(h_{T_{\tilde{b}}}\right)\right|^\lambda + \left|f^*(h_{I_{\tilde{a}}}) - f^*\left(h_{I_{\tilde{b}}}\right)\right|^\lambda + \left|f^*(h_{F_{\tilde{a}}}) - f^*\left(h_{F_{\tilde{b}}}\right)\right|^\lambda\right)^{\frac{1}{\lambda}}, \text{ and}$$

$$d\left(\tilde{b},\tilde{c}\right) = \frac{1}{3}\left(\left|f^*\left(h_{T_{\tilde{b}}}\right) - f^*(h_{T_{\tilde{c}}})\right|^\lambda + \left|f^*\left(h_{I_{\tilde{b}}}\right) - f^*(h_{I_{\tilde{c}}})\right|^\lambda + \left|f^*\left(h_{F_{\tilde{b}}}\right) - f^*(h_{F_{\tilde{c}}})\right|^\lambda\right)^{\frac{1}{\lambda}}.$$

Since $|a+b| \leq |a| + |b|$, then $\left|f^*(h_{T_{\tilde{a}}}) - f^*(h_{T_{\tilde{c}}})\right| = \left|f^*(h_{T_{\tilde{a}}}) - f^*\left(h_{T_{\tilde{b}}}\right) + f^*\left(h_{T_{\tilde{b}}}\right) - f^*(h_{T_{\tilde{c}}})\right|$, and $\left|f^*(h_{T_{\tilde{a}}}) - f^*\left(h_{T_{\tilde{b}}}\right) + f^*\left(h_{T_{\tilde{b}}}\right) - f^*(h_{T_{\tilde{c}}})\right| \leq \left|f^*(h_{T_{\tilde{a}}}) - f^*\left(h_{T_{\tilde{b}}}\right)\right| + \left|f^*\left(h_{T_{\tilde{b}}}\right) - f^*(h_{T_{\tilde{c}}})\right|$.

Thus, $\left|f^*(h_{T_{\tilde{a}}}) - f^*(h_{T_{\tilde{c}}})\right| \leq \left|f^*(h_{T_{\tilde{a}}}) - f^*\left(h_{T_{\tilde{b}}}\right)\right| + \left|f^*\left(h_{T_{\tilde{b}}}\right) - f^*(h_{T_{\tilde{c}}})\right|$.

Similarly, we can obtain $\left|f^*(h_{I_{\tilde{a}}}) - f^*(h_{I_{\tilde{c}}})\right| \leq \left|f^*(h_{I_{\tilde{a}}}) - f^*\left(h_{I_{\tilde{b}}}\right)\right| + \left|f^*\left(h_{I_{\tilde{b}}}\right) - f^*(h_{I_{\tilde{c}}})\right|$, and $\left|f^*(h_{F_{\tilde{a}}}) - f^*(h_{F_{\tilde{c}}})\right| \leq \left|f^*(h_{F_{\tilde{a}}}) - f^*\left(h_{F_{\tilde{b}}}\right)\right| + \left|f^*\left(h_{F_{\tilde{b}}}\right) - f^*(h_{F_{\tilde{c}}})\right|$.

Then

$$\frac{1}{3}\left(\left|f^*(h_{T_{\tilde{a}}}) - f^*(h_{T_{\tilde{c}}})\right|^\lambda + \left|f^*(h_{I_{\tilde{a}}}) - f^*(h_{I_{\tilde{c}}})\right|^\lambda + \left|f^*(h_{F_{\tilde{a}}}) - f^*(h_{F_{\tilde{c}}})\right|^\lambda\right)^{\frac{1}{\lambda}} \leq$$
$$\frac{1}{3}\left(\left|f^*(h_{T_{\tilde{a}}}) - f^*\left(h_{T_{\tilde{b}}}\right)\right|^\lambda + \left|f^*(h_{I_{\tilde{a}}}) - f^*\left(h_{I_{\tilde{b}}}\right)\right|^\lambda + \left|f^*(h_{F_{\tilde{a}}}) - f^*\left(h_{F_{\tilde{b}}}\right)\right|^\lambda\right)^{\frac{1}{\lambda}} +$$
$$\frac{1}{3}\left(\left|f^*\left(h_{T_{\tilde{b}}}\right) - f^*(h_{T_{\tilde{c}}})\right|^\lambda + \left|f^*\left(h_{I_{\tilde{b}}}\right) - f^*(h_{I_{\tilde{c}}})\right|^\lambda + \left|f^*\left(h_{F_{\tilde{b}}}\right) - f^*(h_{F_{\tilde{c}}})\right|^\lambda\right)^{\frac{1}{\lambda}}$$

Thus, property (4) in Theorem 2 holds. □

Appendix D. The Proof of Theorem 3

For ease of computation, we assume that $\zeta_j = w_j(1 + G(\tilde{a}_j)) / \sum_{j=1}^{n} w_j(1 + G(\tilde{a}_j))$. In the following steps, Equation (5) will be proven using mathematical induction on n.

(1) Utilizing the operations for LNNs defined in Definition 2, when $n = 2$, we have

$$\begin{aligned}
&LNPWA(\tilde{a}_1, \tilde{a}_2) = \zeta_1 \tilde{a}_1 \oplus \zeta_2 \tilde{a}_2 = \\
&\left\langle f^{*-1}\left(\frac{\left(1+f^*\left(h_{T_{\tilde{a}1}}\right)\right)^{\zeta_1}\left(1+f^*\left(h_{T_{\tilde{a}2}}\right)\right)^{\zeta_2}-\left(1-f^*\left(h_{T_{\tilde{a}1}}\right)\right)^{\zeta_1}\left(1-f^*\left(h_{T_{\tilde{a}2}}\right)\right)^{\zeta_2}}{\left(1+f^*\left(h_{T_{\tilde{a}1}}\right)\right)^{\zeta_1}\left(1+f^*\left(h_{T_{\tilde{a}2}}\right)\right)^{\zeta_2}+\left(1-f^*\left(h_{T_{\tilde{a}1}}\right)\right)^{\zeta_1}\left(1-f^*\left(h_{T_{\tilde{a}2}}\right)\right)^{\zeta_2}}\right),\right. \\
&f^{*-1}\left(\frac{2\left(f^*\left(h_{I_{\tilde{a}1}}\right)\right)^{\zeta_1}\left(f^*\left(h_{I_{\tilde{a}2}}\right)\right)^{\zeta_2}}{\left(2-f^*\left(h_{I_{\tilde{a}1}}\right)\right)^{\zeta_1}\left(2-f^*\left(h_{I_{\tilde{a}2}}\right)\right)^{\zeta_2}+\left(f^*\left(h_{I_{\tilde{a}1}}\right)\right)^{\zeta_1}\left(f^*\left(h_{I_{\tilde{a}2}}\right)\right)^{\zeta_2}}\right), \\
&\left. f^{*-1}\left(\frac{2\left(f^*\left(h_{F_{\tilde{a}1}}\right)\right)^{\zeta_1}\left(f^*\left(h_{F_{\tilde{a}2}}\right)\right)^{\zeta_2}}{\left(2-f^*\left(h_{F_{\tilde{a}1}}\right)\right)^{\zeta_1}\left(2-f^*\left(h_{F_{\tilde{a}2}}\right)\right)^{\zeta_2}+\left(f^*\left(h_{F_{\tilde{a}1}}\right)\right)^{\zeta_1}\left(f^*\left(h_{F_{\tilde{a}2}}\right)\right)^{\zeta_2}}\right)\right\rangle.
\end{aligned} \tag{A4}$$

That is

$$\begin{aligned}
&LNPWA(\tilde{a}_1, \tilde{a}_2) = \zeta_1 \tilde{a}_1 \oplus \zeta_2 \tilde{a}_2 = \\
&\left\langle f^{*-1}\left(\frac{\prod_{j=1}^{2}\left(1+f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}-\prod_{j=1}^{2}\left(1-f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{2}\left(1+f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{2}\left(1-f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}}\right),\right. \\
&\left. f^{*-1}\cdot\left(\frac{2\prod_{j=1}^{2}\left(f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{2}\left(2-f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{2}\left(f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}}\right), f^{*-1}\left(\frac{2\prod_{j=1}^{2}\left(f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{2}\left(2-f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{2}\left(f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}}\right)\right\rangle
\end{aligned} \tag{A5}$$

Thus, when $n = 2$, Equation (5) is true.

(2) Suppose that when $n = k$, Equation (5) is true. That is,

$$\begin{aligned}
&LNPWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) = \\
&\left\langle f^{*-1}\left(\frac{\prod_{j=1}^{k}\left(1+f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}-\prod_{j=1}^{k}\left(1-f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{k}\left(1+f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{k}\left(1-f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}}\right),\right. \\
&\left. f^{*-1}\left(\frac{2\prod_{j=1}^{k}\left(f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{k}\left(2-f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{k}\left(f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}}\right), f^{*-1}\left(\frac{2\prod_{j=1}^{k}\left(f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{k}\left(2-f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{k}\left(f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}}\right)\right\rangle.
\end{aligned} \tag{A6}$$

Then, when $n = k + 1$, the following result can be obtained:

$$\begin{aligned}
&LNPWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_{k+1}) = LNPWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_k) \oplus \zeta_{k+1}\tilde{a}_{k+1} \\
&= \left\langle f^{*-1}\left(\frac{\prod_{j=1}^{k}\left(1+f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}-\prod_{j=1}^{k}\left(1-f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{k}\left(1+f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{k}\left(1-f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}}\right),\right. \\
&\left. f^{*-1}\left(\frac{2\prod_{j=1}^{k}\left(f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{k}\left(2-f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{k}\left(f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}}\right), f^{*-1}\left(\frac{2\prod_{j=1}^{k}\left(f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{k}\left(2-f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{k}\left(f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}}\right)\right\rangle \oplus \\
&\left\langle \begin{array}{l} f^{*-1}\left(\frac{\left(1+f^*\left(h_{T_{\tilde{a}k+1}}\right)\right)^{\zeta_{k+1}}-\left(1-f^*\left(h_{T_{\tilde{a}k+1}}\right)\right)^{\zeta_{k+1}}}{\left(1+f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta}+\left(1-f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta}}\right), \quad f^{*-1}\left(\frac{2\left(f^*\left(h_{I_{\tilde{a}k+1}}\right)\right)^{\zeta_{k+1}}}{\left(2-f^*\left(h_{I_{\tilde{a}k+1}}\right)\right)^{\zeta_{k+1}}+\left(f^*\left(h_{I_{\tilde{a}k+1}}\right)\right)^{\zeta_{k+1}}}\right), \\ f^{*-1}\left(\frac{2\left(f^*\left(h_{F_{\tilde{a}k+1}}\right)\right)^{\zeta_{k+1}}}{\left(2-f^*\left(h_{F_{\tilde{a}k+1}}\right)\right)^{\zeta_{k+1}}+\left(f^*\left(h_{F_{\tilde{a}k+1}}\right)\right)^{\zeta_{k+1}}}\right) \end{array} \right\rangle \\
&= \left\langle f^{*-1}\left(\frac{\prod_{j=1}^{k+1}\left(1+f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}-\prod_{j=1}^{k+1}\left(1-f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{k+1}\left(1+f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{k+1}\left(1-f^*\left(h_{T_{\tilde{a}}}\right)\right)^{\zeta_j}}\right),\right. \\
&\left. f^{*-1}\left(\frac{2\prod_{j=1}^{k+1}\left(f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{k+1}\left(2-f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{k+1}\left(f^*\left(h_{I_{\tilde{a}}}\right)\right)^{\zeta_j}}\right), f^{*-1}\left(\frac{2\prod_{j=1}^{k+1}\left(f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}}{\prod_{j=1}^{k+1}\left(2-f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}+\prod_{j=1}^{k+1}\left(f^*\left(h_{F_{\tilde{a}}}\right)\right)^{\zeta_j}}\right)\right\rangle
\end{aligned} \tag{A7}$$

Then, when $n = k + 1$, Equation (5) is true. Therefore, Equation (5) is true for all n.

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Probabilistic Single-Valued (Interval) Neutrosophic Hesitant Fuzzy Set and Its Application in Multi-Attribute Decision Making

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Received: 21 August 2018; Accepted: 14 September 2018; Published: 19 September 2018

Abstract: The uncertainty and concurrence of randomness are considered when many practical problems are dealt with. To describe the aleatory uncertainty and imprecision in a neutrosophic environment and prevent the obliteration of more data, the concept of the probabilistic single-valued (interval) neutrosophic hesitant fuzzy set is introduced. By definition, we know that the probabilistic single-valued neutrosophic hesitant fuzzy set (PSVNHFS) is a special case of the probabilistic interval neutrosophic hesitant fuzzy set (PINHFS). PSVNHFSs can satisfy all the properties of PINHFSs. An example is given to illustrate that PINHFS compared to PSVNHFS is more general. Then, PINHFS is the main research object. The basic operational relations of PINHFS are studied, and the comparison method of probabilistic interval neutrosophic hesitant fuzzy numbers (PINHFNs) is proposed. Then, the probabilistic interval neutrosophic hesitant fuzzy weighted averaging (PINHFWA) and the probability interval neutrosophic hesitant fuzzy weighted geometric (PINHFWG) operators are presented. Some basic properties are investigated. Next, based on the PINHFWA and PINHFWG operators, a decision-making method under a probabilistic interval neutrosophic hesitant fuzzy circumstance is established. Finally, we apply this method to the issue of investment options. The validity and application of the new approach is demonstrated.

Keywords: probabilistic single-valued (interval) neutrosophic hesitant fuzzy set; multi-attribute decision making; aggregation operator

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1. Introduction

In real life, uncertainty widely exists, like an expert system, information fusion, intelligent computations and medical diagnoses. When some decision problems need to be solved, establishing mathematical models of uncertainty plays an important role. Especially when dealing with big data problems, the uncertainty must be considered. Therefore, to describe the uncertainty of the problems, Zadeh [1] presented the fuzzy set theory. Next, many new types of fuzzy set theory have been developed, including the intuitionistic fuzzy set [2], hesitant fuzzy set (HFS) [3], dual hesitant fuzzy set (DHFS) [4], interval-valued intuitionistic fuzzy set (IVIFS) [5,6], necessary and possible hesitant fuzzy sets [7] and dual hesitant fuzzy probability [8]. The fuzzy set theory is a useful tool to figure out uncertain information [9]. In addition, Fuzzy set theory has also been applied to algebraic systems [10–13].

Simultaneously, in actual productions, statistical uncertainty needs to be considered. The probabilistic method is not always effective when we deal with epistemic uncertain problems [14]. Thus, those problems makes researchers attempt to combine fuzzy set theory with probability theory as a new fuzzy concept. For example, (1) probability theory as a method of knowledge representation [15–18]; (2) increase the probability value when processing fuzzy decision making problems [19–21]; (3) through the combination of stochastic simulation with nonlinear programming, the fuzzy values can be generated [22,23]. In [24], Hao et al. lists a detailed summary. In the probabilistic fuzzy circumstances, probabilistic data will be lost easily. Thus, under the fuzzy linguistic environments [25–27], Pang et al. [28] established a new type of probabilistic fuzzy linguistic term set and successfully solved these issues. In some practical issues, it is necessary to fully consider the ambiguity and probability. In 2016, Xu and Zhou [29] produced the hesitant probabilistic fuzzy set (HPFS). Then, Hao et al. [24] researched a new probabilistic dual hesitant fuzzy set (PDHFS) and applied it to the uncertain risk evaluation issues.

In [30], Smarandache introduced the neutrosophic set (NS) as a new type of fuzzy set. The NS A includes three independent members: truth membership $T_A(x) \in [0,1]$, indeterminacy membership $I_A(x) \in [0,1]$ and falsity membership $F_A(x) \in [0,1]$. NS theory has been widely used in algebraic systems [31–36]. Next, some new types of NS were introduced, like single-valued NS (SVNS) [37] and interval NS (INS) [38]. Ye utilized SVNS theory applied to different types of decision making (DM) issues [39–41]. In [42], Ye presented a simplified neutrosophic set (SNS). Xu and Xia utilized HFS theory for actual life productions [43–46]. Next, in a hesitant fuzzy environment, a group DM method was introduced by Xu et al. [47]. However, there are some types of questions that are difficult to solve by HFS. Thus, Zhu [4] introduced a DHFS theory. Then, Ye [48] established a correlation coefficient of DHFS. When decision makers are making decisions, DHFS theory cannot express the doubts of decision makers, completely. Next, in 2005, a single-valued neutrosophic hesitant fuzzy set (SVNHFS) was established by Ye [49], and interval neutrosophic hesitant fuzzy set (INHFS) was introduced by Liu [50]. Recently, neutrosophic fuzzy set theory has been widely researched and applied [51–55].

The aleatory uncertainty needs to be considered under the probabilistic neutrosophic hesitant fuzzy environments. Recently, fuzzy random variables have been used to describe probability information in uncertainty. However, in the above NS theories, the probabilities is not considered. Thus, if a neutrosophic multi-attribute decision making (MADM) problem under the probabilistic surroundings needs to be solved, the probabilities as a part of a fuzzy system will be lost. Until now, this problem has not given an effective solution. Peng et al. [56] proposed a new method: the probability multi-valued neutrosophic set (PMVNS). The PMVNS theory successfully solves multi-criteria group decision-making problems without loss of information. Then, we offer the notion of probabilistic SVNHFS (the probabilistic interval neutrosophic hesitant fuzzy set (PINHFS)) based on fuzzy set, HFS, PDHFS, NS and IVNHFS. To solve the MADM problems under the probabilistic interval neutrosophic hesitant fuzzy circumstance, the concept of PINHFS is used. By comparison, we find that the application of PINHFS is wider than that of the probabilistic single-valued neutrosophic hesitant fuzzy set (PSVNHFS), and it is closer to real life. Thus, we can study the case of the interval.

The rest of the paper is organized as follows: Section 2 briefly describes some basic definitions. In Section 3, the concepts of PSVNHFS and PINHFS are introduced, respectively. Next, PINHFS is the main research object. The comparison method of probabilistic interval neutrosophic hesitant fuzzy numbers (PINHFNs) is proposed. In Section 4, the basic operation laws of PINHFN are investigated. The probabilistic interval neutrosophic hesitant fuzzy weighted averaging (PINHFWA) and the probability interval neutrosophic hesitant fuzzy weighted geometric (PINHFWG) operators are established, and some basic properties are studied in Section 5. In Section 6, a MAMD method based on the PINHFWA and PINHFWG operators is proposed. Section 7 gives an illustrative example according to our method. To explain that PINHFS comparedto PSVNHFS is more extensive, in Section 8, the PSVNHFS being a special case of PINHFS, the probabilistic single-valued neutrosophic hesitant fuzzy weighted averaging (PSVNHFWA) and probabilistic single-valued neutrosophic hesitant fuzzy weighted geometric (PSVNHFWG) operators are introduced and a numerical example given to illustrate. Last, we summarize the conclusion and further research work.

2. Preliminaries

Let us review some fundamental definitions of HFS, SVNHFS and INHFS in this section.

Definition 1. *([3]) Let X be a non-empty finite set; an HFS A on X is defined in terms of a function* $h_A(x)$ *that when applied to X returns a finite subset of [0, 1], and we can express HFSs by:*

$$A = \{\langle x, h_A(x)\rangle | x \in X\},$$

where $h_A(x)$ *is a set of some different values in [0, 1], representing the possible membership degrees of the element* $x \in X$ *to A. We call* $h_A(x)$ *a hesitant fuzzy element (HFE), denoted by h, which reads* $h = \{\lambda | \lambda \in h\}$.

Definition 2. *([49]) Let X be a fixed set; an SVNHFS on X is defined as:*

$$N = \{\langle x, \tilde{t}(x), \tilde{i}(x), \tilde{f}(x)\rangle | x \in X\}$$

in which $\tilde{t}(x)$, $\tilde{i}(x)$ *and* $\tilde{f}(x)$ *are three sets of some values in [0, 1], denoting the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees and falsity-membership hesitant degrees of the element* $x \in X$ *to the set N, respectively, with the conditions* $0 \leq \delta, \gamma, \eta \leq 1$ *and* $0 \leq \delta^+ + \gamma^+ + \eta^+ + \leq 3$, *where* $\delta \in \tilde{t}(x), \gamma \in \tilde{i}(x), \eta \in \tilde{f}(x)$, $\delta^+ \in \tilde{t}(x) = \bigcup_{\delta \in \tilde{t}(x)} max\delta$, $\gamma^+ \in \tilde{i}(x) = \bigcup_{\gamma \in \tilde{i}(x)} max\gamma$, $\eta^+ \in \tilde{f}(x) = \bigcup_{\eta \in \tilde{f}(x)} max\eta$ *for* $x \in X$.

Definition 3. *([50]) Let X be a non-empty finite set; an interval neutrosophic hesitant fuzzy set (INHFS) on X is represented by:*

$$A = \{(x, T_A(x), I_A(x), F_A(x)) | x \in X\},$$

where $T_A(x) = \{\tilde{\alpha} | \tilde{\alpha} \in T_A(x)\}$, $I_A(x) = \{\tilde{\beta} | \tilde{\beta} \in I_A(x)\}$ *and* $T_A(x) = \{\tilde{\gamma} | \tilde{\gamma} \in F_A(x)\}$ *are three sets of some interval values in real unit interval [0, 1], which denotes the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees and falsity-membership hesitant fuzzy degrees of element* $x \in X$ *to the set A and satisfies these limits:* $\tilde{\alpha} = [\alpha^L, \alpha^U] \subseteq [0,1]$, $\tilde{\beta} = [\beta^L, \beta^U] \subseteq [0,1]$, $\tilde{\gamma} = [\gamma^L, \gamma^U] \subseteq [0,1]$ *and* $0 \leq sup\tilde{\alpha}^+ + sup\tilde{\beta}^+ + sup\tilde{\gamma}^+ \leq 3$, *where* $\tilde{\alpha}^+ = \bigcup_{\tilde{\alpha} \in T_A(x)} max\{\tilde{\alpha}\}$, $\tilde{\beta}^+ = \bigcup_{\tilde{\beta} \in I_A(x)} max\{\tilde{\beta}\}$ *and* $\tilde{\gamma}^+ = \bigcup_{\tilde{\gamma} \in F_A(x)} max\{\tilde{\gamma}\}$ *for* $x \in X$.

3. The Probabilistic Single-Valued (Interval) Neutrosophic Hesitant Fuzzy Set

In this section, the concepts of PSVNHFS and PINHFS are introduced. Since PINHFS is more general than PSVNHFS, the situation of PINHFS is mainly discussed.

Definition 4. *Let X be a fixed set. A probabilistic single-valued neutrosophic hesitant fuzzy set (PSVNHFS) on X is defined by the following mathematical symbol:*

$$NP = \{\langle x, \tilde{t}(x)|P^{\tilde{t}}(x), \tilde{i}(x)|P^{\tilde{i}}(x), \tilde{f}(x)|P^{\tilde{f}}(x)\rangle | x \in X\}. \quad (1)$$

The components $\tilde{t}(x)|P^{\tilde{t}}(x)$, $\tilde{i}(x)|P^{\tilde{i}}(x)$ *and* $\tilde{f}(x)|P^{\tilde{f}}(x)$ *are three sets of some possible elements where* $\tilde{t}(x)$, $\tilde{i}(x)$ *and* $\tilde{f}(x)$ *represent the possible truth-membership hesitant degrees, indeterminacy-membership* *hesitant degrees and falsity-membership hesitant degrees to the set X of x, respectively.* $P^{\tilde{t}}(x)$, $P^{\tilde{i}}(x)$ *and* $P^{\tilde{f}}(x)$ *are the corresponding probabilistic information for these three types of degrees. There is:*

$$0 \leq \alpha, \beta, \gamma \leq 1, 0 \leq \delta^{+} + \gamma^{+} + \eta^{+} + \leq 3; P_a^{\tilde{t}} \in [0,1], P_b^{\tilde{i}} \in [0,1], P_c^{\tilde{f}} \in [0,1]; \sum_{a=1}^{\#\tilde{t}} P_a^{\tilde{t}} = 1, \sum_{b=1}^{\#\tilde{i}} P_b^{\tilde{i}} = 1, \sum_{c=1}^{\#\tilde{f}} P_c^{\tilde{f}} = 1.$$

where $\alpha \in \tilde{t}(x)$, $\beta \in \tilde{i}(x)$, $\gamma \in \tilde{f}(x)$. $\alpha^{+} \in \tilde{t}^{+}(x) == \bigcup_{\alpha \in \tilde{t}(x)} max\alpha$, $\beta^{+} \in \tilde{i}^{+}(x) = \bigcup_{\beta \in \tilde{i}(x)} max\beta$, $\gamma^{+} \in \tilde{f}^{+}(x) = \bigcup_{\gamma \in \tilde{f}(x)} max\gamma$, $P_a^{\tilde{t}} \in P^{\tilde{t}}$, $P_b^{\tilde{i}} \in P^{\tilde{i}}$, $P_c^{\tilde{f}} \in P^{\tilde{f}}$. *The symbols* $\#\tilde{t}$, $\#\tilde{i}$ *and* $\#\tilde{f}$ *are the total numbers of elements in the components* $\tilde{t}(x)|P^{\tilde{t}}(x)$, $\tilde{i}(x)|P^{\tilde{i}}(x)$ *and* $\tilde{f}(x)|P^{\tilde{f}}(x)$, *respectively.*

For convenience, we call $\tilde{n}p = \langle \tilde{t}(x)|P^{\tilde{t}}(x), \tilde{i}(x)|P^{\tilde{i}}(x), \tilde{f}(x)|P^{\tilde{f}}(x)\rangle$ a probabilistic single-valued neutrosophic hesitant fuzzy number (PSVNHFN). It is defined by the mathematical symbol: $\tilde{n} = \{\tilde{t}|P^{\tilde{t}}, \tilde{i}|P^{\tilde{i}}, \tilde{f}|P^{\tilde{f}}\}$.

Next, a numerical example about investment options is used to explain the PSVNHFS.

Example 1. *OF four investment selections* A_h, *select the only investment option of an investment company. The investment corporation wants to have an effective evaluation and to choose the best investment opportunity; thus, the decision maker needs to use the PSVNHFS theory. According to the practical situation, there are three main attributes: (1)* C_1 *is the hazard of investment; (2)* C_2 *is the future outlook; (3)* C_3 *is the environment index. Thus, the data on these four options are represented by SVNHFS, as illustrated in Tables 1–4. Every table is called a probabilistic single-valued neutrosophic hesitant fuzzy decision matrix (PSVNHFDM).*

Table 1. A probabilistic single-valued neutrosophic hesitant fuzzy decision matrix (PSVNHFDM) D_1 with respect to A_1.

Attributes	Investment Selection A_1
C_1	$\{\{0.3|0.2, 0.4|0.3, 0.5|0.5\}, \{0.1|1\}, \{0.3|0.6, 0.4|0.4\}\}$
C_2	$\{\{0.5|0.5, 0.6|0.5\}, \{0.2|0.2, 0.3|0.8\}, \{0.3|0.4, 0.4|0.6\}\}$
C_3	$\{\{0.2|0.1, 0.3|0.9\}, \{0.1|0.3, 0.2|0.7\}, \{0.5|0.2, 0.6|0.8\}\}$

Table 2. PSVNHFDM D_2 with respect to A_2.

Attributes	Investment Selection A_2
C_1	$\{\{0.6|0.1, 0.7|0.9\}, \{0.1|0.4, 0.2|0.6\}, \{0.2|0.5, 0.3|0.5\}\}$
C_2	$\{\{0.6|0.2, 0.7|0.8\}, \{0.1|1\}, \{0.3|1\}\}$
C_3	$\{\{0.6|0.3, 0.7|0.7\}, \{0.1|0.6, 0.2|0.4\}, \{0.1|0.7, 0.2|0.3\}\}$

Table 3. PSVNHFDM D_3 with respect to A_3.

Attributes	Investment Selection A_3
C_1	$\{\{0.5|0.5, 0.6|0.5\}, \{0.4|1\}, \{0.2|0.2, 0.3|0.8\}\}$
C_2	$\{\{0.6|1\}, \{0.3|1\}, \{0.4|1\}\}$
C_3	$\{\{0.5|0.6, 0.6|0.4\}, \{0.1|1\}, \{0.3|1\}\}$

Table 4. PSVNHFDM D_4 with respect to A_4.

Attributes	Investment Selection A_4
C_1	$\{\{0.7|0.4, 0.8|0.6\}, \{0.1|1\}, \{0.1|0.1, 0.2|0.9\}\}$
C_2	$\{\{0.6|0.6, 0.7|0.4\}, \{0.1|1\}, \{0.2|1\}\}$
C_3	$\{\{0.3|0.9, 0.5|0.1\}, \{0.2|1\}, \{0.1|0.1, 0.2|0.8, 0.3|0.1\}\}$

In general, in the real world, if the three types of hesitant degrees of the PSVNHFS are interval values, this is a special case of INHFS. This kind of interval is more able to express the problems that people encounter when making choices in real life. However, the PSVNHFS is not an effective tool to solve this problem. Thus, we need to propose a new method to solve this problem. Then, the SVNHFS can be used as a special case of the probabilistic interval neutrosophic hesitant fuzzy circumstance. Thus, the probabilistic interval neutrosophic hesitant fuzzy set (PINHFS) is proposed and studied. The advantages of this are: SVNHFS can be studied in a wider range; the scope of application is also broader and closer to real life. Hence, we will give the concept of PINHFS. Simultaneously, in the rest of this paper, we take PINHFS as an example to conduct research.

Definition 5. *Let X be a fixed set, a probabilistic interval neutrosophic hesitant fuzzy set (PINHFS) on X is defined by the following mathematical symbol:*

$$N = \{\langle x, T(x)|P^T(x), I(x)|P^I(x), F(x)|P^F(x)\rangle | x \in X\}.$$

The components $T(x)|P^T(x)$, $I(x)|P^I(x)$ and $F(x)|P^F(x)$ are three sets of possible elements where $T(x)$, $I(x)$ and $F(x)$ are three sets of some interval values in the real unit interval [0, 1], which denotes the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees and falsity-membership hesitant fuzzy degrees of element $x \in X$ to the set N, respectively. $P^T(x)$, $P^I(x)$ and $P^F(x)$ are the corresponding probabilistic information for these three types of degrees. There is:

$$\tilde{\alpha} = [\alpha^L, \alpha^U] \subseteq [0,1], \tilde{\beta} = [\beta^L, \beta^U] \subseteq [0,1], \tilde{\gamma} = [\gamma^L, \gamma^U] \subseteq [0,1]; 0 \leq sup\tilde{\alpha}^+ + sup\tilde{\beta}^+ + sup\tilde{\gamma}^+ \leq 3;$$
$$P_a^T \in [0,1], P_b^I \in [0,1], P_c^F \in [0,1], \sum_{a=1}^{\#T} P_a^T = 1, \sum_{b=1}^{\#I} P_b^I = 1, \sum_{c=1}^{\#F} P_c^F = 1;$$

where $\tilde{\alpha} \in T(x)$, $\tilde{\beta} \in I(x)$ and $\tilde{\gamma} \in F(x)$. $\tilde{\alpha}^+ = \bigcup_{\tilde{\alpha} \in T_A(x)} max\{\tilde{\alpha}\}$, $\tilde{\beta}^+ = \bigcup_{\tilde{\beta} \in I_A(x)} max\{\tilde{\beta}\}$, and $\tilde{\gamma}^+ = \bigcup_{\tilde{\gamma} \in F_A(x)} max\{\tilde{\gamma}\}$. $P_a^T \in P^T$, $P_b^I \in P^I$, $P_c^F \in P^F$. The symbols #T, #I and $\#\tilde{f}$ are the total numbers of elements in the components $T(x)|P^T(x)$, $I(x)|P^I(x)$ and $F(x)|P^F(x)$, respectively.

For convenience, we call $n = \langle T(x)|P^T(x), I(x)|P^I(x), F(x)|P^F(x)\rangle$ a probabilistic interval neutrosophic hesitant fuzzy number (PINHFN). It is defined by the mathematical symbol: $n = \{T|P^T, I|P^I, F|P^F\}$

If $\alpha^L = \alpha^U$, $\beta^L = \beta^U$, $\gamma^L = \gamma^U$, the PINHFS is transformed into the PSVNHFS.

Therefore, we know PINHFS is more general than PSVNHFS. PSVNHFS can satisfy all the properties of PINHFS. Thus, this paper mainly studies PINHFS.

Definition 6. *For a PINHFN n, where $a = 1,2,...,\#T$, $b = 1,2,...,\#I$, $c = 1,2,...,\#F$, the score function $s(n)$ is defined as:*

$$s(n) = \frac{\sum_{a=1}^{\#T}(\alpha_a^L + \alpha_a^U)P_a^T + \sum_{b=1}^{\#I}(2 - (\beta_b^L + \beta_b^U))P_b^I + \sum_{c=1}^{\#F}(2 - (\gamma_c^L + \beta_c^U))P_c^F}{6}, \tag{2}$$

where #T, #I and $\#\tilde{f}$ are the total numbers of elements in the components $T(x)|P^T(x)$, $I(x)|P^I(x)$ and $F(x)|P^F(x)$, respectively.

Definition 7. *For a PINHFN n, where $a = 1,2,...,\#T$, $b = 1,2,...,\#I$, $c = 1,2,...,\#F$, the deviation function $d(n)$ is defined as:*

$$d(n) = \frac{\sum_{a=1}^{\#T}(\alpha_a^L + \alpha_a^U - 2s(n))^2 \cdot P_a^T + \sum_{b=1}^{\#I}(2 - \beta_b^L - \beta_b^U - 2s(n))^2 \cdot P_b^I + \sum_{c=1}^{\#F}(2 - \gamma_c^L - \beta_c^U - 2s(n))^2 \cdot P_c^F}{4} \tag{3}$$

where #T, #I and $\#\tilde{f}$ are the total numbers of elements in the components $T(x)|P^T(x)$, $I(x)|P^I(x)$ and $F(x)|P^F(x)$, respectively.

Definition 8. *Let n_1 and n_2 be two PINHFNs, the comparison of the method for n_1 and n_2 is as follows:*

(1) *If $s(n_1) > s(n_2)$, then $n_1 > n_2$;*
(2) *If $s(n_1) = s(n_2), d(n_1) > d(n_2)$, then $n_1 > n_2$;*
(3) *If $s(n_1) = s(n_2), d(n_1) = d(n_2)$, then $n_1 = n_2$.*

4. Some Basic Operations of PINHFNs

Definition 9. *Let $n_1 = \{T_1|P^{T_1}, I_1|P^{I_1}, F_1|P^{F_1}\}$ and $n_2 = \{T_2|P^{T_2}, I_2|P^{I_2}, F_2|P^{F_2}\}$ be two PINHFNs, then:*

(1) $$(n_1)^c = \bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1} \{\tilde{\gamma}_1|P_1^{F_1}, [1-\beta_1^U, 1-\beta_1^L]|P_1^{I_1}, \tilde{\alpha}_1|P_1^{T_1}\},$$

(2) $$n_1 \cap n_2 = \bigcap_{\substack{\tilde{\alpha}_1\in T_1,\tilde{\beta}\in I_1,\tilde{\gamma}_1\in F_1,\\ \tilde{\eta}_2\in T_2,\tilde{\theta}\in I_2,\tilde{\mu}_2\in F_2}} \{\{\tilde{\alpha}_1 \cap \tilde{\eta}_2|\frac{P_2^{T_1}P_2^{T_2}}{\sum P_1^{T_1}P_2^{T_2}}\}, \{\tilde{\beta}_1 \cup \tilde{\theta}_2|\frac{P_1^{I_1}P_2^{I_2}}{\sum P_1^{I_1}P_2^{I_2}}\}, \{\tilde{\gamma}_1 \cup \tilde{\mu}_2|\frac{P_1^{F_1}P_2^{F_2}}{\sum P_1^{F_1}P_2^{F_2}}\}\},$$

(3) $$n_1 \cup n_2 = \bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}\in I_1,\tilde{\gamma}_2\in F_1,\tilde{\eta}_2\in T_2,\tilde{\theta}\in I_2,\tilde{\mu}_2\in F_2} \{\{\tilde{\alpha}_1 \cup \tilde{\eta}_2|\frac{P_1^{T_1}P_2^{T_2}}{\sum P_1^{T_1}P_2^{T_2}}\}, \{\tilde{\beta}_1 \cap \tilde{\theta}_2|\frac{P_1^{I_1}P_2^{I_2}}{\sum P_{b'}^{I_1}P_2^{I_2}}\}, \{\tilde{\gamma}_1 \cap \tilde{\mu}_2|\frac{P_1^{F_1}P_2^{F_2}}{\sum P_1^{F_1}P_2^{F_2}}\}\},$$

(4) $$(n_1)^\lambda = \bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1} \{\{[(\alpha_1^L)^\lambda, (\alpha_1^U)^\lambda]|P_1^{T_1}\}, \{[1-(1-\beta_1^L)^\lambda, 1-(1-\beta_1^U)^\lambda]|P_1^{I_1}\}, \{[1-(1-\gamma_1^L), 1-(1-\gamma_1^U)]^\lambda|P_1^{F_1}\}\},$$

(5) $$\lambda(n_1) = \bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1} \{\{[1-(1-\lambda_1^L)^\lambda, 1-(1-\lambda_1^L)^\lambda]|P_1^{T_1}\}, \{[(\beta_1^L)^\lambda, (\beta_1^U)^\lambda]|P_1^{I_1}\}, \{[(\gamma_1^L)^\lambda, (\gamma_1^U)^\lambda]|P_1^{F_1}\}\},$$

(6) $$n_1 \oplus n_2 = \bigcup_{\substack{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1,\\ \tilde{\eta}_2\in T_2,\tilde{\theta}_2\in I_2,\tilde{\mu}_2\in F_2}} \{\{[\alpha_1^L + \eta_2^L - \alpha_2^L\eta_2^L,\ \alpha_1^U + \eta_2^U - \alpha_2^U\eta_2^U]|P_1^{T_1}P_2^{T_2}\}, \{[\beta_1^L\theta_2^L,\ \beta_1^U\theta_2^U]|P_1^{I_1}P_2^{I_2}\}, \{[\gamma_1^L\mu_2^L, \gamma_1^U\mu_2^U]|P_1^{F_1}P_{c^2}^{F_2}\}\},$$

(7) $$n_1 \otimes n_2 = \bigcup_{\substack{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1,\\ \tilde{\eta}_2\in T_2,\tilde{\theta}_2\in I_2,\tilde{\mu}_2\in F_2}} \{\{[\alpha_1^L\eta_2^L, \alpha_1^U\eta_2^U]|P_1^{T_1}P_2^{T_2}\}, \{[\beta_1^L + \theta_2^L - \beta_1^L\theta_2^L,\ \beta_1^U + \theta_2^U - \beta_1^U\theta_2^U|P_1^{I_1}P_2^{I_2}\}, \{[\gamma_1^L + \mu_2^L - \gamma_1^L\mu_2^L,\ \gamma_1^U + \mu_2^U - \gamma_1^U\mu_2^U]|P_1^{F_1}P_{c^2}^{F_2}\}\},$$

where $P_1^{T_1}$; $P_1^{I_1}$ and $P_1^{F_1}$ are hesitant probabilities of $\tilde{\alpha}_1 \in T_1$, $\tilde{\beta}_1 \in I_1$ and $\tilde{\gamma}_1 \in F_1$, respectively. $P_2^{T_2}$; $P_2^{I_2}$ and $P_1^{F_2}$ are corresponding hesitant probabilities of $\tilde{\eta}_2 \in T_2$, $\tilde{\theta}_2 \in I_2$ and $\tilde{\mu}_2 \in F_2$.

Theorem 1. *Let n_1 and n_2 be two PINHFNs, then $(n_1)^c$, $n_1 \cap n_2$, $n_1 \cup n_2$, $(n_1)^\lambda$, $\lambda(n_1)$, $n_1 \oplus n_2$ and $n_1 \otimes n_2$ are PINHFNs.*

Proof. By Definition 5, Definition 9, it is easy to prove the result. □

Theorem 2. *Let* $n_1 = (T_1|P^{T_1}, I_1|P^{I_1}, F_1|P^{F_1})$, $n_2 = (T_2|P^{T_2}, I_2|P^{I_2}, F_2|P^{F_2})$ *and* $n_3 = (T_3|P^{T_3}, I_3|P^{I_3}, F_3|P^{F_3})$ *be three PINHFNs,* $\lambda, \lambda_1, \lambda_2 \geq 0$, *then:*

(1) $n_1 \oplus n_2 = n_2 \oplus n_1$; $n_1 \otimes n_2 = n_2 \otimes n_1$,
(2) $(n_1 \oplus n_2) \oplus n_3 = n_1 \oplus (n_2 \oplus n_3)$; $(n_1 \otimes n_2) \otimes n_3 = n_1 \otimes (n_2 \otimes n_3)$,
(3) $\lambda(n_1 \oplus n_2) = \lambda(n_1) \oplus \lambda(n_2)$,
(4) $(n_1 \otimes n_2)^\lambda = (n_1)^\lambda \otimes (n_2)^\lambda$,
(5) $(n_1)^{\lambda_1+\lambda_2} = (n_1)^{\lambda_1} \otimes (n_1)^{\lambda_2}$; $(\lambda_1 + \lambda_2)n_1 = \lambda_1(n_1) \oplus \lambda_2(n_1)$.

Proof. If $P_1^{T_1}$; $P_1^{I_1}$ and $P_1^{F_1}$ are probabilities of $\tilde{\alpha}_1 \in T_1$, $\tilde{\beta}_1 \in I_1$ and $\tilde{\gamma}_1 \in F_1$, respectively. $P_2^{T_2}$, $P_2^{I_2}$ and $P_2^{F_2}$ are corresponding probabilities of $\tilde{\eta}_2 \in T_2$, $\tilde{\theta}_2 \in I_2$ and $\tilde{\mu}_2 \in F_2$. $P_3^{T_3}$, $P_3^{I_3}$ and $P_3^{F_3}$ are corresponding probabilities of $\tilde{\xi}_3 \in T_3$, $\tilde{\sigma}_3 \in I_3$ and $\tilde{\phi}_3 \in F_3$, then we have:

(1) By Definition 9, we can get that (1) is true.

(2)

$$\begin{aligned}(n_1 \oplus n_2) \oplus n_3 = &\bigcup_{\substack{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1,\\ \tilde{\eta}_2\in T_2,\tilde{\theta}_2\in I_2,\tilde{\mu}_2\in F_2\\ \tilde{\xi}_3\in T_3,\tilde{\sigma}_3\in I_3,\tilde{\phi}_3\in F_3}} \{\{[\alpha_1^L + (\eta_2^L + \xi_3^L - \eta_2^L\xi_3^L) - \alpha_1^L(\eta_2^L + \xi_3^L - \eta_2^L\xi_3^L),\\
&\alpha_1^U + (\eta_2^U + \xi_3^U - \eta_2^U\xi_3^U) - \alpha_1^U(\eta_2^U + \xi_3^U - \eta_2^U\xi_3^U)]|P_1^{T_1}(P_2^{T_2}P_3^{T_3})\},\\
&\{[\beta_1^L(\theta_2^L\sigma_3^L), \beta_1^U(\theta_2^U\sigma_3^U)]|P_1^{I_1}(P_2^{I_2}P_3^{I_3})\},\\
&\{[\lambda_1^L(\mu_2^L\phi_3^L), \lambda_1^U(\mu_2^U\phi_3^U)]|P_1^{F_1}(P_2^{F_2}P_3^{F_3})\}\}\\
= &n_1 \oplus (n_2 \oplus n_3).\end{aligned}$$

Similarly, we can obtain $(n_1 \otimes n_2) \otimes n_3 = n_1 \otimes (n_2 \otimes n_3)$.

(3)

$$\begin{aligned}\lambda(n_1 \oplus n_2) = &\bigcup_{\substack{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1,\\ \tilde{\eta}_2\in T_2,\tilde{\theta}_2\in I_2,\tilde{\mu}_2\in F_2}} \{\{[1-(1-(\alpha_1^L + \eta_2^L - \alpha_1^L\eta_2^L))^\lambda, 1-(1-(\alpha_1^U + \eta_2^U - \alpha_1^U\eta_2^U))^\lambda]|P_1^{T_1}PT_{22}\}\\
&\{[(\beta_1^L)^\lambda(\theta_2^L)^\lambda, (\beta_1^U)^\lambda(\theta_2^U)^\lambda]|P_1^{I_1}P_2^{I_2}\}, \{[(\gamma_1^L)^\lambda(\mu_2^L)^\lambda, (\gamma_1^U)^\lambda(\mu_2^U)^\lambda]|P_1^{F_1}P_2^{F_2}\}\}\\
= &\bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1} \{\{[1-(1-\alpha_1^L)^\lambda, 1-(1-\alpha_1^U)^\lambda]|P_1^{T_1}\}, \{[(\beta_1^L)^\lambda, (\beta_1^U)^\lambda|P_1^{I_1}\}, \{[(\gamma_1^L)^\lambda, (\gamma_1^U)^\lambda]|P_1^{F_1}\}\}\\
&\oplus \bigcup_{\tilde{\eta}_2\in T_2,\tilde{\theta}_2\in I_2,\tilde{\mu}_2\in F_2} \{\{[1-(1-\eta_2^L)^\lambda, 1-(1-\eta_2^U)^\lambda]|P_2^{T_2}\}, \{[(\theta_2^L)^\lambda, (\theta_2^U)^\lambda]|P_2^{I_2}\}, \{[(\mu_2^L)^\lambda, (\mu_2^U)^\lambda]|P_2^{F_2}\}\}\\
= &\lambda(n_1) \oplus \lambda(n_2).\end{aligned}$$

(4)

$$(n_1 \otimes n_2)^\lambda = \bigcup_{\substack{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1, \\ \tilde{\eta}_2 \in T_2, \tilde{\theta}_2 \in I_2, \tilde{\mu}_2 \in F_2}} \{\{[(\alpha_1^L \eta_2^L)^\lambda, (\alpha_1^U \eta_2^U)^\lambda] | P_1^{T_1} P_2^{T_2})\},$$
$$\{[1-(1-(\beta_1^L+\theta_2^L-\beta_1^L\theta_2^L))^\lambda, 1-(1-(\beta_1^U+\theta_2^U-\beta_1^U\theta_2^U))^\lambda] | P_1^{I_1} P_2^{I_2}\},$$
$$\{[1-(1-(\gamma_1^L+\mu_2^L-\gamma_1^L\mu_2^L))^\lambda, 1-(1-(\gamma_1^U+\mu_2^U-\gamma_1^U\mu_2^U))^\lambda] | P_1^{F_1} P_2^{F_2}\}\}$$

$$= \bigcup_{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1} \{\{[(\alpha_1^L)^\lambda, (\alpha_1^U)^\lambda] | P_1^{T_1}\}, \{[1-(1-\beta_1^L)^\lambda, 1-(1-\beta_1^U)^\lambda] | P_1^{I_1}\},$$
$$\{[1-(1-\gamma_1^L)^\lambda, 1-(1-\gamma_1^U)^\lambda] | P_1^{F_1}\}\}$$

$$\otimes \bigcup_{\tilde{\eta}_2 \in T_2, \tilde{\theta}_2 \in I_2, \tilde{\mu}_2 \in F_2} \{\{[(\eta_2^L)^\lambda, (\eta_2^U)^\lambda] | P_2^{T_2}\}, \{[1-(1-\theta_2^L)^\lambda, 1-(1-\theta_2^U)^\lambda] | P_2^{I_2}\},$$
$$\{[1-(1-\mu_2^L)^\lambda, 1-(1-\mu_2^U)^\lambda] | P_2^{F_2}\}\}$$

$$= (n_1)^\lambda \otimes (n_2)^\lambda.$$

(5)

$$(n_1)^{\lambda_1+\lambda_2} = \bigcup_{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1} \{\{[(\alpha_1^L)^{\lambda_1+\lambda_2}, (\alpha_1^U)^{\lambda_1+\lambda_2}] | P_1^{T_1}\}, \{[1-(1-\beta_1^L)^{\lambda_1+\lambda_2}, 1-(1-\beta_1^U)^{\lambda_1+\lambda_2}] | P_1^{I_1}\},$$
$$\{[1-(1-\gamma_1^L)^{\lambda_1+\lambda_2}, 1-(1-\gamma_1^U)^{\lambda_1+\lambda_2}] | P_1^{F_1}\}\}$$

$$= \bigcup_{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1} \{\{\alpha_{a'}^{\lambda_1} | P_{a'}^{\tilde{t}_1}\}, \{(1-(1-\beta_{b'})^{\lambda_1}) | P_{b'}^{\tilde{i}_1}\}, \{(1-(1-\gamma_{c'})^{+\lambda_1}) | P_{c'}^{\tilde{f}_1}\}\}$$
$$\otimes \bigcup_{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1} \{\{[(\alpha_1^L)^{\lambda_2}, (\alpha_1^U)^{\lambda_2}] | P_1^{T_1}\}, \{[1-(1-\beta_1^L)^{\lambda_2}, 1-(1-\beta_1^U)^{\lambda_2}] | P_1^{I_1}\},$$
$$\{[1-(1-\gamma_1^L)^{\lambda_2}, 1-(1-\gamma_1^U)^{\lambda_2}] | P_1^{F_1}\}\}$$

$$= (n_1)^{\lambda_1} \otimes (n_1)^{\lambda_2}.$$

Similarly, we have $(\lambda_1+\lambda_2)n_1 = \lambda_1(n_1) \oplus \lambda_2(n_1)$. □

Theorem 3. *Let n_1 and n_2 be two PINHFNs, $\lambda \geq 0$, then:*

(1) $((n_1)^c)^\lambda = (\lambda(n_1))^c$,
(2) $\lambda(n_1)^c = ((n_1)^\lambda)^c$,
(3) $(n_1)^c \oplus {n_2}^c = (n_1 \otimes n_2)^c$,
(4) $(n_1)^c \otimes (n_2)^c = (n_1 \oplus n_2)^c$.

Proof. $P_1^{T_1}$; $P_1^{I_1}$ and $P_1^{F_1}$ are hesitant probabilities of $\tilde{\alpha}_1 \in T_1$, $\tilde{\beta}_1 \in I_1$ and $\tilde{\gamma}_1 \in F_1$, respectively. $P_2^{T_2}$, $P_2^{I_2}$ and $P_2^{F_2}$ are corresponding hesitant probabilities of $\tilde{\eta}_2 \in T_2$, $\tilde{\theta}_2 \in I_2$ and $\tilde{\mu}_2 \in F_2$. Then:

(1)

$$((n_1)^c)^\lambda = (\bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1}\{\{[\gamma_1^L,\gamma_1^U]|P_1^{F_1}\},\{[1-\beta_1^U,1-\beta_1^U]|P_1^{I_1}\},\{\alpha_1^L,\alpha_1^U]|P_1^{T_1}\}\})^\lambda$$

$$= \bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1}\{\{[(\gamma_1^L)^\lambda,(\gamma_1^U)^\lambda]|P_1^{F_1}\},\{[1-(\beta_1^U)^\lambda,1-(\beta_1^U)^\lambda]|P_1^{I_1}\},$$
$$[1-(1-\alpha_1^L)^\lambda,1-(1-\alpha_1^U)^\lambda]|P_1^{\tilde{I}_1}\}$$

$$= (\lambda(\bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1}\{\{[\alpha_1^L,\alpha_1^U]|P_1^{T_1}\},[\beta_1^L,\beta_1^U]|P_1^{I_1}\},[\gamma_1^L,\gamma_1^U]|P_1^{F_1}\}\}))^c$$
$$= (\lambda(n_1))^c.$$

(2)

$$\lambda(n_1)^c = \lambda(\bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1}\{\{[\gamma_1^L,\gamma_1^U]|P_1^{F_1}\},[1-\beta_1^U,1-\beta_1^U]|P_1^{I_1}\},\{[\alpha_1^L,\alpha_1^L]|P_1^{T_1}\}\})$$

$$= \bigcup_{\alpha_{a'}\in\tilde{t}_1,\beta_{b'}\in\tilde{i}_1,\gamma_{c'}\in\tilde{f}_1}\{\{1-(1-\gamma_1^L)^\lambda,1-(1-\gamma_1^U)^\lambda]|P_1^{F_1}\},\{[(1-\beta_1^U)^\lambda,(1-\beta_1^L)^\lambda]|P_1^{I_1}\},$$
$$\{[(\alpha_1^L)^\lambda,(\alpha_1^U)^\lambda]|P_1^{T_1}\}\}$$

$$= (\bigcup_{\alpha_{a'}\in\tilde{t}_1,\beta_{b'}\in\tilde{i}_1,\gamma_{c'}\in\tilde{f}_1}\{\{[(\alpha_1^L)^\lambda,(\alpha_1^U)^\lambda]|P_1^{T_1}\},\{[1-(1-\beta_1^L)^\lambda,1-(1-\beta_1^U)^\lambda]|P_1^{I_1}\},$$
$$\{[1-(1-\gamma_1^L)^\lambda,1-(1-\gamma_1^U)^\lambda]|P_1^{F_1}\}\})^c$$
$$= ((n_1)^\lambda)^c.$$

(3)

$$(n_1)^c\oplus(n_2)^c = (\bigcup_{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1}\{\{[\gamma_1^L,\gamma_1^U]|P_1^{F_1}\},\{[1-\beta_1^U,1-\beta_1^L]|P_1^{I_1}\},\{[\alpha_1^L,\alpha_1^U]|P_1^{T_1}\}\})$$
$$\oplus(\bigcup_{\tilde{\eta}_2\in T_2,\tilde{\theta}_2\in I_2,\tilde{\mu}_2\in F_2}\{[\mu_2^L,\mu_2^U]|P_2^{F_2}\},\{[1-\theta_2^U,1-\theta_2^L]|P_2^{I_2}\},\{[\eta_2^L,\eta_2^U]|P_2^{T_2}\}\})$$

$$= \bigcup_{\substack{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1,\\ \tilde{\eta}_2\in T_2,\tilde{\theta}_2\in I_2,\tilde{\mu}_2\in F_2}}\{\{[\gamma_1^L+\mu_2^L-\gamma_1^L\mu_2^L,\gamma_1^U+\mu_2^U-\gamma_1^U\mu_2^U]|P_1^{F_1}P_2^{F_2}\},$$
$$\{[(1-\beta_2^L)(1-\theta_2^L),(1-\beta_2^U)(1-\theta_2^U)]|P_1^{I_1}P_2^{I_2}\},\{[\alpha_1^L\eta_2^L,\alpha_1^U\eta_2^U]|P_1^{T_1}P_2^{T_2}\}\}$$

$$= (\bigcup_{\substack{\tilde{\alpha}_1\in T_1,\tilde{\beta}_1\in I_1,\tilde{\gamma}_1\in F_1,\\ \tilde{\eta}_2\in T_2,\tilde{\theta}_2\in I_2,\tilde{\mu}_2\in F_2}}\{\{[\alpha_1^L\eta_2^L,\alpha_1^U\eta_2^U]|P_1^{T_1}P_2^{T_2}\},\{[\beta_1^L+\theta_2^L-\beta_1^L\theta_2^L,\beta_1^U+\theta_2^U-\beta_1^U\theta_2^U]|P_1^{I_1}P_2^{I_2}\},$$
$$\{[\gamma_1^L+\mu_2^L-\gamma_1^L\mu_2^L,\gamma_1^U+\mu_2^U-\gamma_1^U\mu_2^U]|P_1^{F_1}P_2^{F_2}\}\})^c$$
$$= (n_1\otimes n_2)^c.$$

(4)

$$(n_1)^c \otimes (n_2)^c = \Big(\bigcup_{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1} \{\{[\gamma_1^L, \gamma_1^U] | P_1^{F_1}\}, \{[1-\beta_1^U, 1-\beta_1^L] | P_1^{I_1}\}, \{[\alpha_1^L, \alpha_1^U] | P_1^{T_1}\}\} \Big)$$
$$\otimes \Big(\bigcup_{\tilde{\eta}_2 \in T_2, \tilde{\theta}_2 \in I_2, \tilde{\mu}_2 \in F_2} \{\{[\mu_2^L, \mu_2^U] | P_2^{F_2}\}, \{[1-\theta_2^U, 1-\theta_2^L] | P_2^{I_2}\}, \{[\eta_2^L, \eta_2^U] | P_2^{T_2}\}\} \Big)$$
$$= \bigcup_{\substack{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1, \\ \tilde{\eta}_2 \in T_2, \tilde{\theta}_2 \in I_2, \tilde{\mu}_2 \in F_2}} \{\{[\gamma_1^L \mu_2^L, \gamma_1^U \mu_2^U] | P_1^{F_1} P_2^{F_2}\}, \{[1-\beta_1^U \theta_2^U, 1-\beta_1^L \theta_2^L] | P_1^{I_1} P_2^{I_2}\},$$
$$\{[\alpha_1^L + \eta_2^L - \alpha_1^L \eta_2^L, \alpha_1^U + \eta_2^U - \alpha_1^U \eta_2^U] | P_1^{T_1} P_2^{T_2}\}\}$$
$$= \Big(\bigcup_{\substack{\tilde{\alpha}_1 \in T_1, \tilde{\beta}_1 \in I_1, \tilde{\gamma}_1 \in F_1, \\ \tilde{\eta}_2 \in T_2, \tilde{\theta}_2 \in I_2, \tilde{\mu}_2 \in F_2}} \{\{[\alpha_1^L + \eta_2^L - \alpha_1^L \eta_2^L, \alpha_1^U + \eta_2^U - \alpha_1^U \eta_2^U] | P_1^{T_1} P_2^{T_2}\}, \{[\beta_1^L \theta_2^L, \beta_1^U \theta_2^U] | P_1^{I_1} P_2^{I_2}\},$$
$$\{[\gamma_1^L \mu_2^L, \gamma_1^U \mu_2^U] | P_1^{F_1} P_2^{F_2}\}\} \Big)^c$$
$$= (n_1 \oplus n_2)^c.$$

□

The PSVNHFS also satisfies the above properties, and the process of the proof is omitted.

5. The Basic Aggregation Operators for PINHFSs

Definition 10. *Let n_j ($x = 1, 2, \cdots, X$) be a non-empty collection of PINHFNs, then a probabilistic interval neutrosophic hesitant fuzzy weighted averaging (PINHFWA) operator can be indicated as:*

$$PINHFWA(n_1, n_2, \cdots, n_X) = \bigoplus_{j=1}^{X} w_j(n_j)$$
$$= \bigcup \{\{[1 - \prod_{j=1}^{X}(1-\alpha_j^L)^{w_j}, 1 - \prod_{j=1}^{X}(1-\alpha_j^U)^{w_j}] | \prod_{j=1}^{X} P_j^{T_j}\}, \tag{4}$$
$$\{[\prod_{j=1}^{X}(\beta_j^L)^{w_j}, \prod_{j=1}^{X}(\beta_j^U)^{w_j}] | \prod_{j=1}^{X} P_j^{I_j}\}, \{[\prod_{j=1}^{X}(\gamma_j^L)^{w_j}, \prod_{j=1}^{X}(\gamma_j^L)^{w_j}] | \prod_{j=1}^{X} P_j^{F_j}\}\},$$

where $[\alpha_j^L, \alpha_j^U] = \tilde{\alpha}_j \in T_j$, $[\beta_j^L, \beta_j^U] = \tilde{\beta}_j \in I_j$, $[\gamma_j^L, \gamma_j^U] = \tilde{\gamma} \in F_j$, $P_j^{T_j}$, $P_j^{I_j}$ and $P_j^{F_j}$ are corresponding hesitant probabilities of $\tilde{\alpha}_j \in T_j$, $\tilde{\beta}_j \in I_j$ and $\tilde{\gamma}_j \in F_j$. $j = 1, 2, \cdots, X$, w_j is the weight of n_j and $\sum_{j=1}^{X} w_j = 1$. If all wights are $\frac{1}{X}$, then the PINHFWA operator reduces to the probabilistic interval neutrosophic hesitant fuzzy averaging (PINHFA) operator:

$$PINHFA(n_1, n_2, \cdots, n_X) = \bigoplus_{j=1}^{X} \frac{1}{X}(n_j)$$
$$= \bigcup \{\{[1 - \prod_{j=1}^{X}(1-\alpha_j^L)^{\frac{1}{X}}, 1 - \prod_{j=1}^{X}(1-\alpha_j^L)^{\frac{1}{X}}] | \prod_{j=1}^{X} P_j^{T_j}\}, \tag{5}$$
$$\{[\prod_{j=1}^{X}(\beta_j^L)^{\frac{1}{X}}, \prod_{j=1}^{X}(\beta_j^U)^{\frac{1}{X}}] | \prod_{j=1}^{X} P_j^{I_j}\}, \{[\prod_{j=1}^{X}(\gamma_j^L)^{\frac{1}{X}}, \prod_{j=1}^{X}(\gamma_j^U)^{\frac{1}{X}}] | \prod_{j=1}^{X} P_j^{F_j}\}\}.$$

Theorem 4. *(Monotonicity) Let* $n_j = \{\{\tilde{\alpha}_j|P_J^{T_j}\}, \{\tilde{\beta}_j|P_j^{I_j}\}, \{\tilde{\gamma}_j|P_j^{F_j}\}\}$ *and* $m_j = \{\{\tilde{\eta}_j|P_j^{T_j^*}\}, \{\tilde{\theta}_2|P_j^{I_j^*}\}, \{\tilde{\mu}_j|P_j^{F_j^*}\}\}$ *be two collections of PINHFNs;* $w_j (j = 1, 2, \cdots, X)$ *is weight, and* $\sum_{j=1}^{n} w_j = 1$*. If* $P_j^{T_j} = P_j^{T_j^*}$, $P_j^{I_j} = P_j^{I_j^*}$, $P_j^{F_j} = P_j^{F_j^*}$ *and* $\alpha_j^L \le \eta_j^L$, $\alpha_j^U \le \eta_j^U$, $\beta_j^L \ge \theta_j^L$, $\beta_j^U \ge \theta_j^U$, $\gamma_j^L \ge \mu_j^L$, $\gamma_j^U \ge \mu_j^U$*, then:*

$$PINHFWA(n_1, n_2, \cdots, n_X) \le PINHFWA(m_1, m_2, \cdots, m_X). \quad (6)$$

Proof. Since $\alpha_j^L \le \eta_j^L$, $\alpha_j^U \le \eta_j^U$, $\beta_j^L \ge \theta_j^L$, $\beta_j^U \ge \theta_j^U$, $\gamma_j^L \ge \mu_j^L$, $\gamma_j^U \ge \mu_j^U$ for all j, we have:

$$1 - \prod(1 - \alpha_j^L)^{w_j} \le 1 - \prod(1 - \eta_j^L)^{w_j}, 1 - \prod(1 - \alpha_j^U)^{w_j} \le 1 - \prod(1 - \eta_j^U)^{w_j};$$
$$\prod(\beta_j^L)^{w_j} \ge \prod(\theta_j^L)^{w_j}, \prod(\beta_j^U)^{w_j} \ge \prod(\theta_j^U)^{w_j};$$
$$\prod(\gamma_j^L)^{w_j} \ge \prod(\mu_j^L)^{w_j}, \prod(\gamma_j^U)^{w_j} \ge \prod(\mu_j^U)^{w_j}.$$

Simultaneously, we have $P_j^{T_j} = P_j^{T_j^*}$, $P_j^{I_j} = P_j^{I_j^*}$, $P_j^{F_j} = P_j^{F_j^*}$, so we can obtain:

$$(1 - \prod(1 - \alpha_j^L)^{w_j}) \prod P_j^{T_j} - \prod(\beta_j^L)^{w_j} \prod P_j^{I_j} - \prod(\gamma_j^L)^{w_j} \prod P_j^{F_j}\} \le$$
$$(1 - \prod(1 - \eta_j^L)^{w_j}) \prod P_j^{T_j^*} - \prod(\theta_j^L)^{w_j} \prod P_j^{I_j^*} \prod(\mu_j^L)^{w_j} \prod P_j^{F_j^*},$$
$$(1 - \prod(1 - \alpha_j^U)^{w_j}) \prod P_j^{T_j} - \prod(\beta_j^U)^{w_j} \prod P_j^{I_j} - \prod(\gamma_j^U)^{w_j} \prod P_j^{F_j}\} \le$$
$$(1 - \prod(1 - \eta_j^U)^{w_j}) \prod P_j^{T_j^*} - \prod(\theta_j^U)^{w_j} \prod P_j^{I_j^*} \prod(\mu_j^U)^{w_j} \prod P_j^{F_j^*},$$

then by the score function 6 and Definition 8, we have $PINHFWA(n_1, n_2, \cdots, n_X) \le PINHFWA(m_1, m_2, \cdots, m_X)$. □

Theorem 5. *(Boundedness) Let* $n_j = \{\{\tilde{\alpha}_j|P_j^{T_j}\}, \{\tilde{\beta}_j|P^{I_j}\}, \{\tilde{\gamma}_j|P^{F_j}\}\}$ *be a PINHFN* $(j = 1, 2, \cdots, X)$, $\tilde{\alpha}_j \in T_j$, $\tilde{\beta}_b \in I_j$, $\tilde{\gamma}_j \in F_j$, $P_j^{T_j}$; $P_j^{I_j}$ *and* $P_j^{F_j}$ *are hesitant probabilities of* $\tilde{\alpha}_j$, $\tilde{\beta}_j$ *and* $\tilde{\gamma}_j$*, respectively.* w_j $(j = 1, 2, \cdots, X)$ *is a weight, and* $\sum_{j=1}^{X} w_j = 1$*. If:*

$$N^- = \{\{[min\{\alpha_j^L\}, min\{\alpha_j^U\}]|min\{P_j^{T_j}\}\}, \{[max\{\beta_j^L\}, max\{\beta_j^U\}]|max\{P_j^{I_j}\}\}, \{[max\{\gamma_j^L\}, max\{\gamma_j^L\}]|max\{P_j^{F_j}\}\}\},$$
$$N^+ = \{\{[max\{\alpha_j^L\}, max\{\alpha_j^U\}]|max\{P_j^{T_j}\}\}, \{[min\{\beta_j^L\}, min\{\beta_j^U\}]|min\{P_j^{I_j}\}\}, \{[min\{\gamma_j^L\}, min\{\gamma_j^L\}]|min\{P_j^{F_j}\}\}\}.$$

Then:

$$PINHFWA(N^-) \le PINHFWA(n_1, n_2, \cdots, n_X) \le PINHFWA(N^+) \quad (7)$$

Proof. For all PINHFNs n_l, we have:

$$min\{\alpha_j^L\} \le \alpha_j^L \le max\{\alpha_j^L\}, min\{\alpha_j^U\} \le \alpha_j^U \le max\{\alpha_j^U\};$$
$$min\{\beta_j^L\} \le \beta_j^L \le max\{\beta_j^L\}, min\{\beta_j^U\} \le \beta_j^U \le max\{\beta_j^U\};$$
$$min\{\gamma_j^L\} \le \gamma_j^L \le max\{\gamma_j^L\}, min\{\gamma_j^U\} \le \gamma_j^U \le max\{\gamma_j^U\};$$
$$min\{P_j^{T_j}\} \le P_j^{T_j} \le max\{P_j^{T_j}\}, min\{P_j^{I_j}\} \le P_j^{I_l} \le max\{P_j^{I_j}\},$$
$$min\{P_j^{F_j}\} \le P_j^{F_j} \le max\{P_j^{F_j}\}.$$

Thus,

$$\begin{aligned}
1-\prod(1-\alpha_j^L)^{w_j} &\geq 1-\prod(1-min\{\alpha_j^L\})^{w_j} = 1-(1-min\{\alpha_l^L\})^{\Sigma w_j} = min\{\alpha_j^L\},\\
1-\prod(1-\alpha_j^U)^{w_j} &\geq 1-\prod(1-min\{\alpha_j^U\})^{w_j} = 1-(1-min\{\alpha_l^U\})^{\Sigma w_j} = min\{\alpha_j^U\},\\
\prod(\beta_j^L)^{w_j} &\leq \prod(max\{\beta_j^L\})^{w_j} = (max\{\beta_j^L\})^{\Sigma w_j} = max\{\beta_j^L\},\\
\prod(\beta_j^U)^{w_j} &\leq \prod(max\{\beta_j^U\})^{w_j} = (max\{\beta_j^U\})^{\Sigma w_j} = max\{\beta_j^U\},\\
\prod(\gamma_j^L)^{w_j} &\leq \prod(max\{\gamma_j^L\})^{w_j} = (max\{\gamma_j^L\})^{\Sigma w_j} = max\{\gamma_j^L\},\\
\prod(\gamma_j^U)^{w_j} &\leq \prod(max\{\gamma_j^U\})^{w_j} = (max\{\gamma_j^U\})^{\Sigma w_j} = max\{\gamma_j^U\}.
\end{aligned}$$

Next, by Definition 10, we have:

$$\begin{aligned}
NHPFWA(N^-) = \bigcup\{\{[min\{\alpha_j^L\}, min\{\alpha_j^U\}]|\prod min\{P_j^{T_j}\}, \{[max\{\beta_j^L\}, max\{\beta_j^U\}]|\prod max\{P_j^{I_j}\}\},\\
\{[max\{\gamma_j^L\}, [max\{\gamma_j^U\}]|\prod max\{P_j^{F_j}\}\}\}.
\end{aligned}$$

By score function 6 and Definition 8, we can obtain $PINHFWA(N^-) \leq PINHFWA(n_1, n_2, \cdots, n_X)$. Similarly, we have $PINHFWA(n_1, n_2, \cdots, n_X) \leq PINHFWA(N^+)$. □

Theorem 6. *(Idempotency) If $n_j = \{\{[\alpha^L, \alpha^U]|P_1\}, \{[\beta^L, \beta^U]|P_2\}, \{[\gamma^L, \gamma^U]|P_3\}\}$, $j = 1, 2, \cdots, X$, w_j is the weight of n_j, $\sum_{j=1}^{X} w_j = 1$, then:*

$$PINHFWA(n_1, n_2, \cdots, n_X) = \{\{[\alpha^L, \alpha^U]|P_1\}, \{[\beta^L, \beta^U]|P_2\}, \{[\gamma^L, \gamma^U]|P_3\}\}. \tag{8}$$

Proof. Since $n_j = \{\{[\alpha^L, \alpha^U]|P_1\}, \{[\beta^L, \beta^U]|P_2\}, \{[\gamma^L, \gamma^U]|P_3\}\}$, thus we have:

$$\begin{aligned}
&1-\prod(1-\alpha^L)^{w_j} = 1-(1-\alpha^L)^{\Sigma w_j} = \alpha^L, 1-\prod(1-\alpha^U)^{w_j} = 1-(1-\alpha^U)^{\Sigma w_j} = \alpha^U;\\
&\prod(\beta^L)^{w_j} = (\beta^L)^{\Sigma w_j} = \beta^L, \prod(\beta^U)^{w_j} = (\beta^U)^{\Sigma w_j} = \beta^U,\\
&\prod(\gamma^L)^{w_j} = (\gamma^L)^{\Sigma w_j} = \gamma^L, \prod(\gamma^U)^{w_j} = (\gamma^U)^{\Sigma w_j} = \gamma^U,\\
&\prod(P_1)^{w_j} = (P_1)^{\Sigma w_j} = P_1, \prod(P_2)^{w_j} = (P_2)^{\Sigma w_j} = P_2, \prod(P_3)^{w_j} = (P_3)^{\Sigma w_j} = P_3.
\end{aligned}$$

It is easy to get:

$$PINHFWA(\tilde{np}_1, \tilde{np}_2, \cdots, \tilde{np}_X) = \{\{[\alpha^L, \alpha^U]|P_1\}, \{[\beta^L, \beta^U]|P_2\}, \{[\gamma^L, \gamma^U]|P_3\}\}.$$

□

Theorem 7. *(Commutativity) If $A = \{n_1, n_2, \cdots, n_X\}$ is a collection and $B = \{m_1, m_2, \cdots, m_X\}$ is a new permutation of A, then:*

$$PINHFWA(n_1, n_2, \cdots, n_X) = PINHFWA(m_1, m_2, \cdots, m_X).$$

Proof. By Definition 10, it is easy to prove it. □

Definition 11. *Let n_j $(j = 1,2,\cdots,X)$ be a non-empty collection of PINHFNs; a probability interval neutrosophic hesitant fuzzy weighted geometric (PINHFWG) operator can be indicated as:*

$$\begin{aligned} &PINHFWG(n_1,n_2,\cdots,n_X) = \bigotimes_{j=1}^{X} w_j(n_j) \\ &= \bigcup\{\{[\prod_{j=1}^{X}(\alpha_j^L)^{w_j}, \prod_{j=1}^{X}(\alpha_j^U)^{w_j}] | \prod_{j=1}^{X} P_j^{T_j}\}, \{[1-\prod_{j=1}^{X}(1-\beta_j^L)^{w_j}, 1-\prod_{j=1}^{X}(1-\beta_j^U)^{w_j}] | \prod_{j=1}^{X} P_j^{I_j}\}, \\ &\quad \{[1-\prod_{j=1}^{X}(1-\gamma_j^L)^{w_j}, 1-\prod_{j=1}^{X}(1-\gamma_j^U)^{w_j}] | \prod_{j=1}^{X} P_j^{F_j}\}\}, \end{aligned} \tag{9}$$

where $[\alpha_j^L, \alpha_j^U] = \tilde{\alpha}_j \in T_j, [\beta_j^L, \beta_j^U] = \tilde{\beta}_j \in I_j, [\gamma_j^L, \gamma_j^U] = \tilde{\gamma} \in F_j$, $P_j^{T_j}$, $P_j^{I_j}$ and $P_j^{F_j}$ are corresponding hesitant probabilities of $\tilde{\alpha}_j$, $\tilde{\beta}_j$ and $\tilde{\gamma}_j$. $j = 1,2,\cdots,X$, w_j is the weight of n_j and $\sum_{j=1}^{X} w_j = 1$. If all wights are $\frac{1}{X}$, then the PINHFWG operator converts to the probabilistic interval neutrosophic hesitant fuzzy geometric (PINHFG) operator:

$$\begin{aligned} &PINHFG(n_1,n_2,\cdots,n_X) = \bigotimes_{j=1}^{X} \frac{1}{X}(n_j) \\ &= \bigcup\{\{[\prod_{j=1}^{X}(\alpha_j^L)^{\frac{1}{X}}, \prod_{j=1}^{X}(\alpha_j^U)^{\frac{1}{X}}] | \prod_{j=1}^{X} P_j^{T_j}\}, \{[1-\prod_{j=1}^{X}(1-\beta_j^L)^{\frac{1}{X}}, 1-\prod_{j=1}^{X}(1-\beta_j^U)^{\frac{1}{X}}] | \prod_{j=1}^{X} P_j^{I_j}\}, \\ &\quad \{[1-\prod_{j=1}^{X}(1-\gamma_j^L)^{\frac{1}{X}}, 1-\prod_{j=1}^{X}(1-\gamma_j^U)^{\frac{1}{X}}] | \prod_{j=1}^{X} P_j^{F_j}\}\}. \end{aligned} \tag{10}$$

Theorem 8. *(Monotonicity) Let $n_j = \{\{\tilde{\alpha}_j|P_J^{T_j}\}, \{\tilde{\beta}_j|P_j^{I_j}\}, \{\tilde{\gamma}_j|P_j^{F_j}\}\}$ and $m_j = \{\{\tilde{\eta}_j|P_j^{T_j^*}\}, \{\tilde{\theta}_2|P_j^{I_j^*}\}, \{\tilde{\mu}_j|P_j^{F_j^*}\}\}$ be two collections of PINHFNs; $w_j(j = 1,2,\cdots,X)$ is weight, and $\sum_{j=1}^{n} w_j = 1$. If $P_j^{T_j} = P_j^{T_j^*}$, $P_j^{I_j} = P_j^{I_j^*}$, $P_j^{F_j} = P_j^{F_j^*}$ and $\alpha_j^L \le \eta_j^L$, $\alpha_j^U \le \eta_j^U$, $\beta_j^L \ge \theta_j^L$, $\beta_j^U \ge \theta_j^U$, $\gamma_j^L \ge \mu_j^L$, $\gamma_j^U \ge \mu_j^U$, then:*

$$PINHFWG(n_1,n_2,\cdots,n_X) \le PINHFWG(m_1,m_2,\cdots,m_X). \tag{11}$$

Proof. This is similar to Theorem 4. □

Theorem 9. *(Boundedness) Let $n_j = \{\{\tilde{\alpha}_j|P_j^{T_j}\}, \{\tilde{\beta}_j|P^{I_j}\}, \{\tilde{\gamma}_j|P^{F_j}\}\}$ be a PINHFN $(j = 1,2,\cdots,X)$, $\tilde{\alpha}_j \in T_j$, $\tilde{\beta}_b \in I_j$, $\tilde{\gamma}_j \in F_j$, $P_j^{T_j}$; $P_j^{I_j}$ and $P_j^{F_j}$ are hesitant probabilities of $\tilde{\alpha}_j$, $\tilde{\beta}_j$ and $\tilde{\gamma}_j$, respectively. w_j $(j = 1,2,\cdots,X)$ is a weight, and $\sum_{j=1}^{X} w_j = 1$. If:*

$$P^- = \{\{[min\{\alpha_j^L\}, min\{\alpha_j^U\}]|min\{P_j^{T_j}\}\}, \{[max\{\beta_j^L\}, max\{\beta_j^U\}]|max\{P_j^{I_j}\}\}, \{[max\{\gamma_j^L\}, max\{\gamma_j^L\}]|max\{P_j^{F_j}\}\}\},$$
$$P^+ = \{\{[max\{\alpha_j^L\}, max\{\alpha_j^U\}]|max\{P_j^{T_j}\}\}, \{[min\{\beta_j^L\}, min\{\beta_j^U\}]|min\{P_j^{I_j}\}\}, \{[min\{\gamma_j^L\}, min\{\gamma_j^L\}]|min\{P_j^{F_j}\}\}\},$$

then:

$$PINHFWG(P^-) \le PINHFWG(n_1,n_2,\cdots,n_X) \le PINHFWG(P^+) \tag{12}$$

Proof. This is similar to Theorem 5. □

Theorem 10. *(Idempotency) If* $n_j = \{\{[\alpha^L, \alpha^U]|P_1\}, \{[\beta^L, \beta^U]|P_2\}, \{[\gamma^L, \gamma^U]|P_3\}\}$, $j = 1, 2, \cdots, X$, w_j *is the weight of* n_j, $\sum_{j=1}^{X} w_j = 1$, *then:*

$$PINHFWG(n_1, n_2, \cdots, n_X) = \{\{[\alpha^L, \alpha^U]|P_1\}, \{[\beta^L, \beta^U]|P_2\}, \{[\gamma^L, \gamma^U]|P_3\}\}. \tag{13}$$

Proof. This is similar to Theorem 6. □

Theorem 11. *(Commutativity) If* $A = \{n_1, n_2, \cdots, n_X\}$ *is a collection and* $B = \{m_1, m_2, \cdots, m_X\}$ *is a new permutation of* A, *then:*

$$PINHFWG(n_1, n_2, \cdots, n_X) = PINHFWG(m_1, m_2, \cdots, m_X).$$

Proof. We can obtain it by Definition 13. □

Lemma 1. *[3] Let* $x_i \geq 0$, $w_i \geq 0$, $i = 1, 2, \cdots, n$ *and* $\sum_{i=1}^{n} w_i = 1$, *then:*

$$\prod_{i=1}^{n} (x_i)^{w_i} \leq \sum_{i=1}^{n} x_i w_i,$$

Theorem 12. *If* $n_j = \{\{\tilde{\alpha}_j | P_j^{T_j}\}, \{\tilde{\beta}_j | P^{I_j}\}, \{\tilde{\gamma}_j | P^{F_j}\}\}$ *is a collection of PINHFNs and* $j = 1, 2, \cdots, X$, w_j *is the weight of* n_j, $w_j \geq 0$ *and* $\sum_{j=1}^{X} w_j = 1$, *then:*

$$PINHFWG(n_1, n_2, \cdots, n_X) \leq PINHFWA(n_1, n_2, \cdots, n_X),$$
$$PINHFG(n_1, n_2, \cdots, n_X) \leq PINHFA(n_1, n_2, \cdots, n_X).$$

Proof. Since $\tilde{\alpha}_j = [\alpha_j^L, \alpha_j^U]$, $\tilde{\beta}_j = [\beta_j^L, \beta_j^U]$, $\tilde{\gamma}_j = [\gamma_j^L, \gamma_j^U]$, $\alpha_j^L, \alpha_j^U \in [0, 1]$. Thus, By Lemma 1, we have:

$$\prod (\alpha_j^L)^{w_j} \leq \sum w_j \alpha_j^L = 1 - \sum w_j (1 - \alpha_j^L) \leq 1 - \prod (1 - \alpha_j^L)^{w_j},$$
$$\prod (\alpha_j^U)^{w_j} \leq \sum w_j \alpha_j^U = 1 - \sum w_j (1 - \alpha_j^U) \leq 1 - \prod (1 - \alpha_j^U)^{w_j}.$$

Thus, we can obtain:

$$\prod (\alpha_j^L)^{w_j} \prod P_j^{T_j} \leq (1 - \prod (1 - \alpha_j^L)^{w_j}) \prod P_j^{T_j},$$
$$\prod (\alpha_j^U)^{w_j} \prod P_j^{T_j} \leq (1 - \prod (1 - \alpha_j^U)^{w_j}) \prod P_j^{T_j}.$$

Similarly, we can also get:

$$\prod (\beta_j^L)^{w_j} \prod P_j^{I_j} \leq (1 - \prod (1 - \beta_j^L)^{w_j}) \prod P_j^{I_j}, \prod (\beta_j^U)^{w_j} \prod P_j^{I_j} \leq (1 - \prod (1 - \beta_j^U)^{w_j}) \prod P_j^{I_j},$$
$$\prod (\gamma_j^L)^{w_j} \prod P_j^{F_j} \leq (1 - \prod (1 - \gamma_j^L)^{w_j}) \prod P_j^{F_j}, \prod (\gamma_j^U)^{w_j} \prod P_j^{F_j} \leq (1 - \prod (1 - \gamma_j^U)^{w_j}) \prod P_j^{F_j}.$$

Next, by the score function 6, we know:

$$PINHFWG(n_1, n_2, \cdots, n_X) \leq PINHFWA(n_1, n_2, \cdots, n_X).$$

Similar to the above process of the proof, we know inequality $PINHFG(n_1, n_2, \cdots, n_X) \leq PINHFA(n_1, n_2, \cdots, n_X)$ is right. □

6. MADM Based on the PINHFWA and PINHFWG Operators

In this section, the PINHFWA and PINHFWG operators are used to solve MADM problems with probabilistic interval neutrosophic hesitant fuzzy circumstances.

Let $A = \{A_1, A_2, \cdots, A_M\}$ be a collection of options and $C = \{C_1, C_2, \cdots, C_N\}$ be a set of attributes. In order to assess A_h ($h = 1, 2, \cdots, M$) with the attribute C_k ($k = 1, 2, \cdots, N$) represented by the PINHFN $n_{hk} = \{T_{hk}|P^{T_{hk}}, I_{hk}|P^{I_{hk}}, F_{hk}|P^{F_{hk}}\}$, next, we can construct a probabilistic interval neutrosophic hesitant fuzzy decision matrix (PINHFDM) $D = (n_{hk})_{M\times N}$ ($h = 1, 2, \cdots, M; k = 1, 2, \cdots, N$). The weight vector of C is $w = (w_1, w_2, \cdots, w_N)$. Then, the evaluation steps can select an optimal option:

- Step 1. Use the PINHFWA or PINHFWG operator to aggregate N PINHFNs for an alternative A_h, $h = 1, 2, \cdots, M$.
- Step 2. Calculate the score values of all PINHFNs; if we get the same for $s(n)$, then we need to compare the deviation values.
- Step 3. Rank and select the optimal option A_h.

7. Illustrative Example

The background of the numerical case comes from Example 1. Therefore, this section is not covered in detail. The weight vector of C is $w = (0.35, 0.25, 0.4)$. Thus, four PINHFDMs are established, illustrated in Tables 5–8.

Table 5. A probabilistic interval neutrosophic hesitant fuzzy decision matrix (PINHFDM) D_1 with respect to A_1.

Attributes	Investment Selection A_1
C_1	$\{\{[0.3,0.4]|0.1,[0.4,0.4]|0.1,[0.4,0.5]|0.8\},\{[0.1,0.2]|1\},\{0.3,0.4]|1\}\}$
C_2	$\{\{[0.4,0.5]|0.5,[0.5,0.6]]|0.5\},\{[0.2,0.3]|1\},\{[0.3,0.3]|0.7,[0.3,0.4]|0.3\}\}$
C_3	$\{\{[0.2,0.3]|1\},\{[0.1,0.2]|1\},\{[0.4,0.5]|0.7,[0.5,0.6]|0.3\}\}$

Table 6. PINHFDM D_2 with respect to A_2.

Attributes	Investment Selection A_2
C_1	$\{\{[0.6,0.7]|1\},\{[0.1,0.2]|1\},\{[0.1,0.2]|0.2,[0.2,0.3]|0.8\}\}$
C_2	$\{\{[0.6,0.7]|1\},\{[0.1,0.1]|1\},\{[0.2,0.3]|1\}\}$
C_3	$\{\{[0.6,0.7]|1\},\{[0.1,0.2]|1\},\{[0.1,0.2]|1\}\}$

Table 7. PINHFDM D_3 with respect to A_3.

Attributes	Investment Selection A_3
C_1	$\{\{[0.3,0.4]|0.3,[0.5,0.6]|0.7\},\{[0.2,0.4]|1\},\{[0.2,0.3]|1\}\}$
C_2	$\{\{[0.5,0.6]|1\},\{[0.2,0.3]|1\},\{[0.3,0.4]|1\}\}$
C_3	$\{\{[0.5,0.6]|1\},\{[0.1,0.2]|0.4,[0.2,0.3]|0.6\},\{[0.2,0.3]|1\}\}$

Table 8. PINHFDM D_4 with respect to A_4.

Attributes	Investment Selection A_4
C_1	$\{\{[0.7,0.8]|1\},\{[0,0.1]|1\},\{[0.1,0.2]|1\}\}$
C_2	$\{\{[0.6,0.7]|1\},\{[0,0.1]|1\},\{[0.2,0.2]|1\}\}$
C_3	$\{\{[0.3,0.5]|1\},\{[0.2,0.3]|1\},\{[0.1,0.2]|0.2,[0.3,0.3]|0.8\}\}$

- Step 1. Select the PINHFWA operator to aggregate all PINHFNs of n_{hk} ($h = 1, 2, 3, 4; k = 1, 2, 3$) to obtain the collective PINHFN n_h ($h = 1, 2, 3, 4$) for the alternative A_h ($h = 1, 2, 3, 4$).

$$\begin{aligned}
n_1 &= PINHFWA(n_{11}, n_{12}, n_{13}) \\
&= \{\{[0.2895, 0.3903]|0.05, [0.3212, 0.4234]|0.05, [0.3268, 0.3903]|0.05, [0.3568, 0.4234]|0.05, \\
&\quad [0.3268, 0.4280]|0.4, [0.3568, 0.4590]|0.4\}, \\
&\quad \{[0.1189, 0.2213]|1\}, \\
&\quad \{[0.3366, 0.407]|0.49, [0.368, 0.4378]|0.21, [0.3366, 0.4373]|0.21, [0.368, 0.4704]|0.09\}; \\
n_2 &= PINHFWA(n_{21}, n_{22}, n_{23}) \\
&= \{\{[0.6, 0.7]|1\}, \{[0.1, 0.1682]|1\}, \{[0.1189, 0.2213]|0.2, [0.1516, 0.2551]|0.8\}\}; \\
n_3 &= PINHFWA(n_{31}, n_{32}, n_{33}) \\
&= \{\{[0.4375, 0.5390]|0.3, [0.5, 0.6]|0.7\}, \{[0.1516, 0.2821]|0.4, [0.2, 0.3318]|0.6\}, \{[0.2213, 0.3224]|1\}\}; \\
n_4 &= PINHFWA(n_{41}, n_{42}, n_{43}) \\
&= \{\{0.5476, 0.6807]|1\}, \{[0, 0.1552]|1\}, \{[0.1189, 0.2]|0.2, [0.1845, 0.2352]|0.8\}\}.
\end{aligned}$$

- Step 2. By (2), count the score values of all PINHFNs n_h $(h = 1, 2, 3, 4)$,

$$n_1 = 0.6104, n_2 = 0.7731, n_3 = 0.6711, \tilde{np}_4 = 0.7789.$$

- Step 3. Rank the PINHFNs by Definition 8; we have:

$$A_4 > A_2 > A_3 > A_1.$$

Thus, we know that A_4 is the best choice.

Next, we will make use of the PINHFWG operator to solve the MADM problem.

- Step 1′. Aggregate PINHFNs n_{hk} $(h = 1, 2, 3, 4; k = 1, 2, 3)$ by taking advantage of the PINHFWG operator to get the collective PINHFN n_h for A_h.

$$\begin{aligned}
n_1 &= PINHFWG(n_{11}, n_{12}, n_{13}) \\
&= \{\{[0.2741, 0.377]|0.05, [0.2898, 0.3946]|0.05, [0.3031, 0.377]|0.05, [0.3205, 0.3946]|0.05, \\
&\quad [0.3031, 0.4076]|0.4, [0.3205, 0.4266]|0.4\} \\
&\quad \{[0.1261, 0.2263]|1\}, \\
&\quad \{[0.3419, 0.4203]|0.49, [0.3881, 0.4698]|0.21, [0.3419, 0.4422]|0.21, [0.3881, 0.4898]|0.09\}\}; \\
n_2 &= PINHFWG(n_{21}, n_{22}, n_{23}) \\
&= \{\{[0.6, 0.7]|1\}, \{[0.1, 0.1761]|1\}, \{[0.1261, 0.2263]|0.2, [0.1614, 0.2616]|0.8\}\}; \\
n_3 &= PINHFWG(n_{31}, n_{32}, n_{33}) \\
&= \{\{[0.4181, 0.5206]|0.3, [0.5, 0.6]|0.7\}, \{[0.1614, 0.3004]|0.4, [0.2000, 0.3368]|0.6\}, \\
&\quad \{[0.2263, 0.3265]|1\}\}; \\
n_4 &= PINHFWG(n_{41}, n_{42}, n_{43}) \\
&= \{\{[0.4799, 0.6411]|1\}, \{[0.0854, 0.1861]|1\}, \{[0.1261, 0.2000]|0.2, [0.2097, 0.2416]|0.8\}\}.
\end{aligned}$$

- Step 2′. By Definition 6, we have:

$$n_1 = 0.595, n_2 = 0.7692, n_3 = 0.6653, n_4 = 0.7372.$$

- Step 3′. Rank A_h $(h = 1, 2, 3, 4)$ on the basis of Step 2′,

$$A_2 > A_4 > A_3 > A_1.$$

Thus, A_2 is the best choice.

8. The Basic Aggregation Operator for PSVNHFS

In this subsection, we construct the PSVNHFWA operator and the PSVNHFWG operator. The comparison method of PIVNHFNs is proposed.

Definition 12. *Let* $\tilde{np}_x$ $(x = 1, 2, \cdots, X)$ *be a non-empty collection of PSVNHFNs , then a PSVNHFWA operator can be indicated as:*

$$\begin{aligned} &PSVNHFWA(\tilde{np}_1, \tilde{np}_2, \cdots, \tilde{np}_X) = \bigoplus_{x=1}^{X} w_x(\tilde{np}_x) \\ &= \bigcup\{\{(1-\prod_{j=1}^{X}(1-\alpha_j)^{w_j})|\prod_{j=1}^{X}P^{\tilde{t}_j}\}, \{\prod_{j=1}^{X}\beta_j^{w_j}|\prod_{k=1}^{X}P^{\tilde{i}_j}\}, \{\prod_{j=1}^{X}\gamma_j^{w_j}|\prod_{j=1}^{X}P^{\tilde{f}_j}\}\}, \end{aligned} \tag{14}$$

where $\alpha_j \in \tilde{t}_j, \beta_j \in \tilde{i}_j, \gamma_j \in \tilde{f}_j$, $j = 1, 2, \cdots, X$, w_j *is the weight of* $\tilde{np}_j$ *and* $\sum_{j=1}^{X} w_j = 1$.

Definition 13. *Let* $\tilde{np}_x$ $(x = 1, 2, \cdots, X)$ *be a non-empty collection of PSVNHFNs, then the PSVNHFWG operator can be indicated as:*

$$\begin{aligned} &PSVNHFWG(\tilde{np}_1, \tilde{np}_2, \cdots, \tilde{np}_X) = \bigotimes_{j=1}^{X} w_j(\tilde{np}_j) \\ &= \bigcup\{\{\prod_{j=1}^{X}(\alpha_j)^{w_j})|\prod_{j=1}^{X}P^{\tilde{t}_j}\}, \{(1-\prod_{j=1}^{X}(1-\beta_j)^{w_j})|\prod_{j=1}^{X}P^{\tilde{i}_j}\}, \{(1-\prod_{j=1}^{X}(1-\gamma_j)^{w_j})|\prod_{j=1}^{X}P^{\tilde{f}_j}\}\}, \end{aligned} \tag{15}$$

where $\alpha_j \in \tilde{t}_j, \beta_j \in \tilde{i}_j, \gamma_j \in \tilde{f}_j$, $j = 1, 2, \cdots, X$, w_j *is the weight of* $\tilde{np}_j$ *and* $\sum_{j=1}^{X} w_j = 1$.

Since the PSVNHFN is a special case of PINHFN, thus the score function $s(\tilde{np})$, deviation function $d(\tilde{np})$ and sorting method can utilize Definition 6, Definition 7 and Definition 8, respectively. In order to solve the MADM problem of the probabilistic single-valued neutrosophic hesitant fuzzy circumstance, the algorithm can use the same method described in Section 6. Next, The application can use Example 1.

- Step 1. Select the PSVNHFWA operator to aggregate all PSVNHFNs of $(\tilde{np})_{hk}$ $(h = 1, 2, 3, 4; k = 1, 2, 3)$ to obtain the PSVNHFN $\tilde{np}_h$ $(h = 1, 2, 3, 4)$ for the option A_h $(h = 1, 2, 3, 4)$.

$$\begin{aligned} \tilde{np}_1 = \{&\{0.3212|0.01, 0.3568|0.015, 0.3966|0.025, 0.3580|0.01, 0.3917|0.015, 0.4293|0.025, 0.3565|0.09, \\ &0.3903|0.1350, 0.4280|0.2250, 0.3914|0.09, 0.4234|0.1350, 0.4590|0.2250\}, \{0.1189|0.06, 0.1569|0.14, 0.1316|0.24, \\ &0.1737|0.56\}, \{0.368|0.048, 0.407|0.032, 0.3955|0.072, 0.4373|0.048, 0.3959|0.192, 0.4378|0.128, 0.4254|0.288, \\ &0.4704|0.192\}\} \\ \tilde{np}_2 = \{&\{0.6|0.006, 0.6435|0.014, 0.6383|0.054, 0.6776|0.126, 0.6278|0.024, 0.6682|0.056, 0.6634|0.216, 0.7|0.504\}, \\ &\{0.1|0.24, 0.132|0.16, 0.1275|0.36, 0.1682|0.24\}, \{0.1677|0.35, 0.2213|0.15, 0.1933|0.35, 0.2551|0.15\}\}; \\ \tilde{np}_3 = \{&\{0.5271|0.3, 0.5675|0.2, 0.5627|0.3, 0.6|0.2\}, \{0.2138|1\}, \{0.2797|0.2, 0.3224|0.8\}\}; \\ \tilde{np}_4 = \{&\{0.5476|0.216, 0.6045|0.024, 0.579|0.144, 0.632|0.016, 0.6074|0.324, 0.6569|0.036, 0.6347|0.216, \\ &0.6807|0.024\}, \{0.132|1\}, \{0.1189|0.01, 0.1569|0.08, 0.1846|0.01, 0.1516|0.09, 0.2|0.72, 0.2352|0.09\}\}. \end{aligned}$$

- Step 2. By (2), count the score values of all $\tilde{np}_h$ $(h = 1, 2, 3, 4)$,

$$s(\tilde{np}_1) = 0.6108, s(\tilde{np}_2) = 0.7839, s(\tilde{np}_3) = 0.6776, s(\tilde{np}_4) = 0.7579.$$

- Step 3. Rank the PSVNHFNs by Definition 8; we have.

$$A_2 > A_4 > A_3 > A_1.$$

Thus, we know that A_2 is the best choice.

Next, we will make use of the PSVNHFWG operator to solve Example 1.

- Step 1′. Aggregate PSVNHFNs $\tilde{np}_h k$ $(h = 1,2,3,4; k = 1,2,3)$ by taking advantage of the PSVNHFWG operator to get the $\tilde{np}_h$ for A_h.

$$\tilde{np}_1 = \{\{0.2898|0.01, 0.3409|0.09, 0.3033|0.01, 0.3568|0.09, 0.3205|0.015, 0.377|0.135, 0.3355|0.015, 0.3946|0.135, 0.3466|0.025, 0.4076|0.225, 0, 3627|0.025, 0.4266|0.225\}, \{0.1261|0.06, 0.1663|0.14, 0.1548|0.24, 0.1937|0.56\}, \{0.3881|0.048, 0.4404|0.192, 0.4113|0.072, 0.4615|0.288, 0.4203|0.032, 0.4698|0.128, 0.4422|0.048, 0.4898|0.192\}\},$$

$$\tilde{np}_2 = \{\{0.6|0.006, 0.6382|0.014, 0.6236|0.024, 0.6632|0.056, 0.6333|0.054, 0.6735|0.126, 0.6581|0.216, 0.7|0.504\}, \{0.1|0.24, 0.1414|0.16, 0.1363|0.36, 0.1761|0.24\}, \{0.1889|0.35, 0.2263|0.15, 0.226|0.35, 0.2616|0.15\}\}.$$

$$\tilde{np}_3 = \{\{0.5233|0.3, 0.5629|0.2, 0.5578|0.3, 0.6|0.2\}, \{0.2666|1\}, \{0.2942|0.2, 0.3265|0.8\}\}.$$

$$\tilde{np}_4 = \{\{0.4799|0.216, 0.5887|0.024, 0.4988|0.144, 0.6119|0.016, 0.5029|0.324, 0.6169|0.036, 0.5226|0.216, 0.6411|0.024\}, \{0.1414|1\}, \{0.1261|0.01, 0.1663|0.08, 0.2097|0.01, 0.1614|0.09, 0.2|0.72, 0.2416|0.09\}\}.$$

- Step 2′. By Formula (2), we have:

$$s(\tilde{np}_1) = 0.5507, s(\tilde{np}_2) = 0.7741, s(\tilde{np}_3) = 0.6568, s(\tilde{np}_4) = 0.7248.$$

- Step 3′. Rank A_h $(h = 1,2,3,4)$ by Definition 8,

$$A_2 > A_4 > A_3 > A_1.$$

Thus, A_2 is the best choice.

In order to demonstrated the effectiveness of our approaches, a comparison was established with other methods. They are shown in Tables 9 and 10.

Table 9. Comparison of the results obtained by different methods under the single-valued neutrosophic hesitant fuzzy circumstance.

Method	Sort of Results	Best Alternative	Worst Alternative
SVNHFWA operator [49]	$A_4 > A_2 > A_3 > A_1$	A_3	A_4
SVNHFWG operator [49]	$A_2 > A_4 > A_3 > A_1$	A_2	A_1
PSVNHFWA operator	$A_2 > A_4 > A_3 > A_1$	A_2	A_1
PSVNHFWG operator	$A_2 > A_4 > A_3 > A_1$	A_2	A_1

Table 10. Comparison of the results obtained by different methods under the interval neutrosophic hesitant fuzzy circumstance.

Method	Sort of Results	Best Alternative	Worst Alternative
GWA operator$(1 \leq \lambda \leq 39)$ [50]	$A_3 > A_1 > A_2 > A_4$	A_3	A_4
PINHFWA operator	$A_4 > A_2 > A_3 > A_1$	A_4	A_1
PINHFWG operator	$A_2 > A_4 > A_3 > A_1$	A_2	A_1

In [49], Ye introduced the single-valued neutrosophic hesitant fuzzy weighted averaging (SVNHFWA) and single-valued neutrosophic hesitant fuzzy weighted geometric (SVNHFWG) operators and applied them to the single-valued neutrosophic hesitant fuzzy circumstance. In [50], Liu proposed the generalized weighted aggregation (GWA) operator and established the MADM method under the interval neutrosophic hesitant fuzzy circumstance. However, probability is not considered in [49,50]. The ranking results are presented in Table 9 and Table 10. According to the Table 9, A_2 is always the best choice, A_1 is always the worst option. According to the Table 10, the best option is A_4 under the group's major points, whereas the best selection is A_2 under the individual major points. A_1 is always the worst choice. Apparently, the SVNHFS, IVHFS and PSVNHFS are special cases of PINHFS. Thus, the PINHFS is is wider than other methods.

9. Conclusions

In this paper, as a generation of fuzzy set theory, a new concept of PSVNHFS (PINHFS) is proposed based on the NHS and INS. The score function and the deviation function are defined. A comparison method is proposed. PSVNHFS is a special case of PINHFS; thus, PINHFS has a wider range of applications. Therefore, this paper mainly discusses the situation of the interval. Then, some basic operation laws of PINHFNs are introduced and investigated. Next, the PINHFWA and PINHFWG operators are presented, and some properties are studied. PSVNHFSs also satisfies the properties mentioned above. We can determine the optimal alternative by utilizing the PINHFWA (PINHFWG) operator. Finally, a numerical example was given. It is proven that the new approach is more flexible and suitable for practical issues. In addition, an example raised in this paper is to explain that PINHFS is more general than PSVNHFS. In the future, others aggregation operators of PINHFNs can be researched, and more practical applications in other areas can be solved, like medical diagnoses.

Author Contributions: All authors have contributed equally to this paper.

Funding: This research was funded by the National Natural Science Foundation of China (Grant Nos 61573240, 61473239).

Acknowledgments: Thanks to the reviewers for their valuable comments.

Conflicts of Interest: The authors declare no conflict of interest.

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On the Classification of Bol-Moufang Type of Some Varieties of Quasi Neutrosophic Triplet Loop (Fenyves BCI-Algebras)

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Received: 4 September 2018 ; Accepted: 20 September 2018 ; Published: 21 September 2018

Abstract: In this paper, Bol-Moufang types of a particular quasi neutrosophic triplet loop (BCI-algebra), chritened Fenyves BCI-algebras are introduced and studied. 60 Fenyves BCI-algebras are introduced and classified. Amongst these 60 classes of algebras, 46 are found to be associative and 14 are found to be non-associative. The 46 associative algebras are shown to be Boolean groups. Moreover, necessary and sufficient conditions for 13 non-associative algebras to be associative are also obtained: p-semisimplicity is found to be necessary and sufficient for a F_3, F_5, F_{42} and F_{55} algebras to be associative while quasi-associativity is found to be necessary and sufficient for F_{19}, F_{52}, F_{56} and F_{59} algebras to be associative. Two pairs of the 14 non-associative algebras are found to be equivalent to associativity (F_{52} and F_{55}, and F_{55} and F_{59}). Every BCI-algebra is naturally an F_{54} BCI-algebra. The work is concluded with recommendations based on comparison between the behaviour of identities of Bol-Moufang (Fenyves' identities) in quasigroups and loops and their behaviour in BCI-algebra. It is concluded that results of this work are an initiation into the study of the classification of finite Fenyves' quasi neutrosophic triplet loops (FQNTLs) just like various types of finite loops have been classified. This research work has opened a new area of research finding in BCI-algebras, vis-a-vis the emergence of 540 varieties of Bol-Moufang type quasi neutrosophic triplet loops. A 'Cycle of Algebraic Structures' which portrays this fact is provided.

Keywords: quasigroup; loop; BCI-algebra; Bol-Moufang; quasi neutrosophic loops; Fenyves identities

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1. Introduction

BCK-algebras and BCI-algebras are abbreviated as two B-algebras. The former was raised in 1966 by Imai and Iseki [1], Japanese mathematicians, and the latter was put forward in the same year by Iseki [2]. The two algebras originated from two different sources: set theory and propositional calculi.

There are some systems which contain the only implicational functor among logical functors, such as the system of weak positive implicational calculus, BCK-system and BCI-system. Undoubtedly, there are common properties among those systems. We know that there are close relationships between the notions of the set difference in set theory and the implication functor in logical systems. For example, we have the following simple inclusion relations in set theory:

$$(A - B) - (A - C) \subseteq C - B, \qquad A - (A - B) \subseteq B.$$

These are similar to the propositional formulas in propositional calculi:

$$(p \to q) \to ((q \to r) \to (p \to r)), \qquad p \to ((p \to q) \to q),$$

which raise the following questions: What are the most essential and fundamental properties of these relationships? Can we formulate a general algebra from the above consideration? How will we find an axiom system to establish a good theory of general algebras? Answering these questions, K.Iseki formulated the notions of two B-algebras in which BCI-algebras are a wider class than BCK-algebras. Their names are taken from BCK and BCI systems in combinatory logic.

BCI-Algebras are very interesting algebraic structures that have generated wide interest among pure mathematicians.

1.1. BCI-algebra, Quasigroups, Loops and the Fenyves Identities

We start with some definitions and examples of some varieties of quasi neutrosophic triplet loop.

Definition 1. *A triple $(X, *, 0)$ is called a BCI-algebra if the following conditions are satisfied for any $x, y, z \in X$:*

1. $((x * y) * (x * z)) * (z * y) = 0$;
2. $x * 0 = x$;
3. $x * y = 0$ *and* $y * x = 0 \Longrightarrow x = y$.

We call the binary operation $*$ on X the multiplication on X, and the constant 0 in X the zero element of X. We often write X instead of $(X, *, 0)$ for a BCI-algebra in brevity. Juxtaposition xy will at times be used for $x * y$ and will have preference over $*$ i.e., $xy * z = (x * y) * z$.

Example 1. *Let S be a set. Let 2^S be the power set of S, $-$ the set difference and $\emptyset$ the empty set. Then $(2^S, -, \emptyset)$ is a BCI-algebra.*

Example 2. *Suppose $(G, \cdot, e)$ is an abelian group with e as the identity element. Define a binary operation $*$ on G by putting $x * y = xy^{-1}$. Then $(G, *, e)$ is a BCI-algebra.*

Example 3. $(\mathbb{Z}, -, 0)$ *and* $(\mathbb{R} - \{0\}, \div, 1)$ *are BCI-algebras.*

Example 4. *Let S be a set. Let 2^S be the power set of S, $\triangle$ the symmetric difference and $\emptyset$ the empty set. Then $(2^S, \triangle, \emptyset)$ is a BCI-algebra.*

The following theorems give necessary and sufficient conditions for the existence of a BCI-algebra.

Theorem 1. *(Yisheng [3])*
Let X be a non-empty set, $*$ a binary operation on X and 0 a constant element of X. Then $(X, *, 0)$ is a BCI-algebra if and only if the following conditions hold:

1. $((x * y) * (x * z)) * (z * y) = 0$;
2. $(x * (x * y)) * y = 0$;
3. $x * x = 0$;
4. $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$.

Definition 2. *A BCI-algebra $(X, *, 0)$ is called a BCK-algebra if $0 * x = 0$ for all $x \in X$.*

Definition 3. *A BCI-algebra $(X, *, 0)$ is called a Fenyves BCI-algebra if it satisfies any of the identities of Bol-Moufang type.*

The identities of Bol-Moufang type are given below:

F_1: $xy * zx = (xy * z)x$
F_2: $xy * zx = (x * yz)x$ (Moufang identity)
F_3: $xy * zx = x(y * zx)$
F_4: $xy * zx = x(yz * x)$ (Moufang identity)
F_5: $(xy * z)x = (x * yz)x$
F_6: $(xy * z)x = x(y * zx)$ (extra identity)
F_7: $(xy * z)x = x(yz * x)$
F_8: $(x * yz)x = x(y * zx)$
F_9: $(x * yz)x = x(yz * x)$
F_{10}: $x(y * zx) = x(yz * x)$
F_{11}: $xy \cdot xz = (xy * x)z$
F_{12}: $xy * xz = (x * yx)z$
F_{13}: $xy * xz = x(yx * z)$ (extra identity)
F_{14}: $xy * xz = x(y * xz)$
F_{15}: $(xy * x)z = (x * yx)z$
F_{16}: $(xy * x)z = x(yx * z)$
F_{17}: $(xy * x)z = x(y * xz)$ (Moufang identity)
F_{18}: $(x * yx)z = x(yx * z)$
F_{19}: $(x * yx)z = x(y * xz)$ (left Bol identity)
F_{20}: $x(yx * z) = x(y * xz)$
F_{21}: $yx * zx = (yx * z)x$
F_{22}: $yx * zx = (y * xz)x$ (extra identity)
F_{23}: $yx * zx = y(xz * x)$
F_{24}: $yx * zx = y(x * zx)$
F_{25}: $(yx * z)x = (y * xz)x$
F_{26}: $(yx * z)x = y(xz * x)$ (right Bol identity)
F_{27}: $(yx * z)x = y(x * zx)$ (Moufang identity)
F_{28}: $(y * xz)x = y(xz * x)$
F_{29}: $(y * xz)x = y(x * zx)$
F_{30}: $y(xz * x) = y(x * zx)$
F_{31}: $yx * xz = (yx * x)z$
F_{32}: $yx * xz = (y * xx)z$
F_{33}: $yx * xz = y(xx * z)$
F_{34}: $yx * xz = y(x * xz)$
F_{35}: $(yx * x)z = (y * xx)z$
F_{36}: $(yx * x)z = y(xx * z)$ (RC identity)
F_{37}: $(yx * x)z = y(x * xz)$ (C identity)
F_{38}: $(y * xx)z = y(xx * z)$
F_{39}: $(y * xx)z = y(x * xz)$ (LC identity)
F_{40}: $y(xx * z) = y(x * xz)$
F_{41}: $xx * yz = (x * xy)z$ (LC identity)
F_{42}: $xx * yz = (xx * y)z$
F_{43}: $xx * yz = x(x * yz)$
F_{44}: $xx * yz = x(xy * z)$
F_{45}: $(x * xy)z = (xx * y)z$
F_{46}: $(x * xy)z = x(x * yz)$ (LC identity)
F_{47}: $(x * xy)z = x(xy * z)$
F_{48}: $(xx * y)z = x(x * yz)$ (LC identity)
F_{49}: $(xx * y)z = x(xy * z)$
F_{50}: $x(x * yz) = x(xy * z)$
F_{51}: $yz * xx = (yz * x)x$
F_{52}: $yz * xx = (y * zx)x$
F_{53}: $yz * xx = y(zx * x)$ (RC identity)
F_{54}: $yz * xx = y(z * xx)$
F_{55}: $(yz * x)x = (y * zx)x$
F_{56}: $(yz * x)x = y(zx * x)$ (RC identity)
F_{57}: $(yz * x)x = y(z * xx)$ (RC identity)
F_{58}: $(y * zx)x = y(zx * x)$
F_{59}: $(y * zx)x = y(z * xx)$
F_{60}: $y(zx * x) = y(z * xx)$

Consequent upon this definition, there are 60 varieties of Fenyves BCI-algebras. Here are some examples of Fenyves' BCI-algebras:

Example 5. *Let us assume the BCI-algebra $(G, *, e)$ in Example 2. Then $(G, *, e)$ is an F_8-algebra, F_{19}-algebra, F_{29}-algebra, F_{39}-algebra, F_{46}-algebra, F_{52}-algebra, F_{54}-algebra, F_{59}-algebra.*

Example 6. *Let us assume the BCI-algebra $(2^S, -, \emptyset)$ in Example 1. Then $(2^S, -, \emptyset)$ is an F_3-algebra, F_5-algebra, F_{21}-algebra, F_{29}-algebra, F_{42}-algebra, F_{46}-algebra, F_{54}-algebra and F_{55}-algebra.*

Example 7. *The BCI-algebra $(2^S, \triangle, \emptyset)$ in Example 4 is associative.*

Example 8. *By considering the direct product of the BCI-algebras $(G, *, e)$ and $(2^S, -, \emptyset)$ of Example 2 and Example 1 respectively, we have a BCI-algebra $\left(G \times 2^S, (*, -), (e, \emptyset)\right)$ which is a F_{29}-algebra and a F_{46}-algebra.*

Remark 1. *Direct products of sets of BCI-algebras will result in BCI-algebras which are F_i-algebra for distinct i's.*

Definition 4. *A BCI-algebra $(X, *, 0)$ is called associative if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$.*

Definition 5. *A BCI-algebra $(X, *, 0)$ is called p-semisimple if $0 * (0 * x) = x$ for all $x \in X$.*

Theorem 2. *(Yisheng [3]) Suppose that $(X, *, 0)$ is a BCI-algebra. Define a binary relation $\leqslant$ on X by which $x \leqslant y$ if and only if $x * y = 0$ for any $x, y \in X$. Then $(X, \leqslant)$ is a partially ordered set with 0 as a minimal element (meaning that $x \leqslant 0$ implies $x = 0$ for any $x \in X$).*

Definition 6. *A BCI-algebra $(X, *, 0)$ is called quasi-associative if $(x * y) * z \leq x * (y * z)$ for all $x, y, z \in X$.*

The following theorems give equivalent conditions for associativity, quasi-associativity and p-semisimplicity in a BCI-algebra:

Theorem 3. *(Yisheng [3])*
Given a BCI-algebra X, the following are equivalent $x, y, z \in X$:

1. *X is associative.*
2. $0 * x = x$.
3. $x * y = y * x \; \forall \, x, y \in X$.

Theorem 4. *(Yisheng [3])*
Let X be a BCI-algebra. Then the following conditions are equivalent for any $x, y, z, u \in X$:

1. *X is p-semisimple*
2. $(x * y) * (z * u) = (x * z) * (y * u)$.
3. $0 * (y * x) = x * y$.
4. $(x * y) * (x * z) = z * y$.
5. $z * x = z * y$ *implies* $x = y$. *(the left cancellation law i.e., LCL)*
6. $x * y = 0$ *implies* $x = y$.

Theorem 5. *(Yisheng [3])*
Given a BCI-algebra X, the following are equivalent for all $x, y \in X$:

1. *X is quasi-associative.*
2. $x * (0 * y) = 0$ *implies* $x * y = 0$.
3. $0 * x = 0 * (0 * x)$.
4. $(0 * x) * x = 0$.

Theorem 6. *(Yisheng [3])*
A triple $(X, *, 0)$ is a BCI-algebra if and only if there is a partial ordering $\leqslant$ on X such that the following conditions hold for any $x, y, z \in X$:

1. $(x * y) * (x * z) \leqslant z * y$;
2. $x * (x * y) \leqslant y$;
3. $x * y = 0$ *if and only if* $x \leqslant y$.

Theorem 7. *(Yisheng [3])*
Let X be a BCI-algebra. X is p-semisimple if and only if one of the following conditions holds for any $x, y, z \in X$:

1. $x * z = y * z$ *implies* $x = y$. *(the right cancellation law i.e., RCL)*
2. $(y * x) * (z * x) = y * z$.
3. $(x * y) * (x * z) = 0 * (y * z)$.

Theorem 8. *(Yisheng [3])*
Let X be a BCI-algebra. X is p-semisimple if and only if one of the following conditions holds for any $x, y \in X$:

1. $x * (0 * y) = y$.
2. $0 * x = 0 \implies x = 0$.

Theorem 9. *(Yisheng [3]) Suppose that* $(X, *, 0)$ *is a BCI-algebra. X is associative if and only if X is p-semisimple and X is quasi-associative.*

Theorem 10. *(Yisheng [3]) Suppose that* $(X, *, 0)$ *is a BCI-algebra. Then* $(x * y) * z = (x * z) * y$ *for all* $x, y, z \in X$.

Remark 2. *In Theorem 9, quasi-associativity in BCI-algebra plays a similar role to that which weak associativity (i.e., the* F_i *identities) plays in quasigroup and loop theory.*

We now move on to quasigroups and loops.

Definition 7. *Let L be a non-empty set. Define a binary operation* $(\cdot)$ *on L . If* $x \cdot y \in L$ *for all* $x, y \in L$, $(L, \cdot)$ *is called a groupoid. If in a groupoid* $(L, \cdot)$, *the equations:*

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

have unique solutions for x and y respectively, then $(L, \cdot)$ *is called a quasigroup. If in a quasigroup* $(L, \cdot)$, *there exists a unique element e called the identity element such that for all* $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ *is called a loop.*

Definition 8. *Let* $(L, \cdot)$ *be a groupoid.*
The left nucleus of L is the set $N_\lambda(L, \cdot) = N_\lambda(L) = \{a \in L : ax \cdot y = a \cdot xy \ \forall\ x, y \in L\}$.
The right nucleus of L is the set $N_\rho(L, \cdot) = N_\rho(L) = \{a \in L : y \cdot xa = yx \cdot a \ \forall\ x, y \in L\}$.
The middle nucleus of L is the set $N_\mu(L, \cdot) = N_\mu(L) = \{a \in L : ya \cdot x = y \cdot ax \ \forall\ x, y \in L\}$.
The nucleus of L is the set $N(L, \cdot) = N(L) = N_\lambda(L, \cdot) \cap N_\rho(L, \cdot) \cap N_\mu(L, \cdot)$.
The centrum of L is the set $C(L, \cdot) = C(L) = \{a \in L : ax = xa \ \forall\ x \in L\}$.
The center of L is the set $Z(L, \cdot) = Z(L) = N(L, \cdot) \cap C(L, \cdot)$.

In the recent past, and up to now, identities of Bol-Moufang type have been studied on the platform of quasigroups and loops by Fenyves [4], Phillips and Vojtechovsky [5], Jaiyeola [6–8], Robinson [9], Burn [10–12], Kinyon and Kunen [13] as well as several other authors.

Since the late 1970s, BCI and BCK algebras have been given a lot of attention. In particular, the participation in the research of polish mathematicians Tadeusz Traczyk and Andrzej Wronski as well as Australian mathematician William H. Cornish, in addition to others, is causing this branch of algebra to develop rapidly. Many interesting and important results are constantly discovered. Now, the theory of BCI-algebras has been widely spread to areas such as general theory which include congruences, quotient algebras, BCI-Homomorphisms, direct sums and direct products, commutative BCK-algebras, positive implicative and implicative BCK-algebras, derivations of BCI-algebras, and ideal theory of BCI-algebras ([1,14–17]).

1.2. BCI-Algebras as a Quasi Neutrosophic Triplet Loop

Consider the following definition.

Definition 9. *(Quasi Neutrosophic Triplet Loops (QNTL), Zhang et al. [18])*
Let $(X, *)$ *be a groupoid.*

1. *If there exist* $b, c \in X$ *such that* $a * b = a$ *and* $a * c = b$, *then a is called an NT-element with (r-r)-property. If every* $a \in X$ *is an NT-element with (r-r)-property, then,* $(X, *)$ *is called a (r-r)-quasi NTL.*
2. *If there exist* $b, c \in X$ *such that* $a * b = a$ *and* $c * a = b$, *then a is called an NT-element with (r-l)-property. If every* $a \in X$ *is an NT-element with (r-l)-property, then,* $(X, *)$ *is called a (r-l)-quasi NTL.*
3. *If there exist* $b, c \in X$ *such that* $b * a = a$ *and* $c * a = b$, *then a is called an NT-element with (l-l)-property. If every* $a \in X$ *is an NT-element with (l-l)-property, then,* $(X, *)$ *is called a (l-l)-quasi NTL.*

4. *If there exist $b, c \in X$ such that $b * a = a$ and $a * c = b$, then a is called an NT-element with (l-r)-property. If every $a \in X$ is an NT-element with (l-r)-property, then, $(X, *)$ is called a (l-r)-quasi NTL.*
5. *If there exist $b, c \in X$ such that $a * b = b * a = a$ and $a * c = b$, then a is called an NT-element with (lr-r)-property. If every $a \in X$ is an NT-element with (lr-r)-property, then, $(X, *)$ is called a (lr-r)-quasi NTL.*
6. *If there exist $b, c \in X$ such that $a * b = b * a = a$ and $c * a = b$, then a is called an NT-element with (lr-l)-property. If every $a \in X$ is an NT-element with (lr-l)-property, then, $(X, *)$ is called a (lr-l)-quasi NTL.*
7. *If there exist $b, c \in X$ such that $a * b = a$ and $a * c = c * a = b$, then a is called an NT-element with (r-lr)-property. If every $a \in X$ is an NT-element with (r-lr)- property, then, $(X, *)$ is called a (r-lr)-quasi NTL.*
8. *If there exist $b, c \in X$ such that $b * a = a$ and $a * c = c * a = b$, then a is called an NT-element with (l-lr)-property. If every $a \in X$ is an NT-element with (l-lr)-property, then, $(X, *)$ is called a (l-lr)-quasi NTL.*
9. *If there exist $b, c \in X$ such that $a * b = b * a = a$ and $a * c = c * a = b$, then a is called an NT-element with (lr-lr)-property. If every $a \in X$ is an NT-element with (lr-lr)-property, then, $(X, *)$ is called a (lr-lr)-quasi NTL.*

Consequent upon Definition 9 and the 60 Fenyves identities F_i, $1 \le i \le 60$, there are 60 varieties of Fenyves quasi neutrosophic triplet loops (FQNTLs) for each of the nine varieties of QNTLs in Definition 9. Thereby making it 540 varieties of Fenyves quasi neutrosophic triplet loops (FQNTLs) in all. A BCI-algebra is a (r-r)-QNT, (r-l)-QNTL and (r-lr)-QNTL. Thus, any F_i BCI-algebra, $1 \le i \le 60$ belongs to at least one of the following varieties of Fenyves quasi neutrosophic triplet loops: (r-r)-QNTL, (r-l)-QNTL and (r-lr)-QNTL which we refer to as (r-r)-FQNTL, (r-l)-FQNTL and (r-lr)-FQNTL respectively. Any associative QNTL will be called quasi neutrosophic triplet group (QNTG).

The variety of quasi neutrosophic triplet loop is a generalization of neutrosophic triplet group (NTG) which was originally introduced by Smarandache and Ali [19]. Neutrosophic triplet set (NTS) is the foundation of neutrosophic triplet group. New results and developments on neutrosophic triplet groups and neutrosophic triplet loop have been reported by Zhang et al. [18,20,21], and Smarandache and Jaiyéọlá [22,23].

It must be noted that triplets are not connected at all with intuitionistic fuzzy set. Neutrosophic set [24] is a generalization of intuitionistic fuzzy set (a generalization of fuzzy set). In Intuitionistic fuzzy set, an element has a degree of membership and a degree of non-membership, and the deduction of the sum of these two from 1 is considered the hesitant degree of the element. These intuitionistic fuzzy set components are dependent (viz. [25–28]). In the neutrosophic set, an element has three independent degrees: membership (truth-t), indeterminacy (i), and non-membership (falsity-f), and their sum is up to 3. However, the current paper utilizes the neutrosophic triplets, which are not defined in intuitionistic fuzzy set, since there is no neutral element in intuitionistic fuzzy sets. In a neutrosophic triplet set $(X, *)$, for each element $x \in X$ there exists a neutral element denoted $neut(x) \in X$ such that $x * neut(x) = neut(x) * x = x$, and an opposite of x denoted $anti(x) \in X$ such that $anti(x) * x = x * anti(x) = neut(x)$. Thus, the triple $(x, neut(x), anti(x))$ is called a neutrosophic triplet which in the philosophy of 'neutrosophy', can be algebraically harmonized with (t, i, f) in neutrosophic set and then extended for neutrosophic hesitant fuzzy [29] set as proposed for (t, i, f)-neutrosophic structures [30]. Unfortunately, such harmonization is not readily defined in intuitionistic fuzzy sets.

Theorem 11. *(Zhang et al. [18]) A (r-lr)-QNTG or (l-lr)-QNTG is a NTG.*

This present study looks at Fenyves identities on the platform of BCI-algebras. The main objective of this study is to classify the Fenyves BCI-algebras into associative and non-associative types. It will also be shown that some Fenyves identities play the roles of quasi-associativity and p-semisimplicity , vis-a-vis Theorem 9 in BCI-algebras.

2. Main Results

We shall first clarify the relationship between a BCI-algebra, a quasigroup and a loop.

Theorem 12.

1. *A BCI algebra X is a quasigroup if and only if it is p-semisimple.*
2. *A BCI algebra X is a loop if and only if it is associative.*
3. *An associative BCI algebra X is a Boolean group.*

Proof. We use Theorem 3, Theorem 7 and Theorem 4.

1. From Theorems 7 and 4, p-semisimplicity is equivalent to the left and right cancellation laws, which consequently implies that X is a quasigroup if and only if it is p-semisimple.
2. One of the axioms that a BCI-algebra satisfies is $x * 0 = x$ for all $x \in X$. So, 0 is already the right identity element. Now, from Theorem 3, associativity is equivalent to $0 * x = x$ for all $x \in X$. So, 0 is also the left identity element of X. The conclusion follows.
3. In a BCI-algebra, $x * x = 0$ for all $x \in X$. And 0 is the identity element of X. Hence, every element is the inverse of itself.

□

Lemma 1. *Let $(X, *, 0)$ be a BCI-algebra.*

1. $0 \in N_\rho(X)$.
2. $0 \in N_\lambda(X), N_\mu(X)$ *implies X is quasi-associative.*
3. *If $0 \in N_\lambda(X)$, then the following are equivalent:*
   (a) *X is p-semisimple.*
   (b) $xy = 0y \cdot x$ *for all* $x, y \in L$.
   (c) $xy = 0x \cdot y$ *for all* $x, y \in L$.
4. *If $0 \in N_\lambda(X)$ or $0 \in N_\mu(X)$, then X is p-semisimple if and only if X is associative.*
5. *If $0 \in N(X)$, then X is p-semisimple if and only if X is associative.*
6. *If $(X, *, 0)$ is a BCK-algebra, then*
   (a) $0 \in N_\lambda(X)$.
   (b) $0 \in N_\mu(X)$ *implies X is a trivial BCK-algebra.*
7. *The following are equivalent:*
   (a) *X is associative.*
   (b) $x \in N_\lambda(X)$ *for all* $x \in X$.
   (c) $x \in N_\rho(X)$ *for all* $x \in X$.
   (d) $x \in N_\mu(X)$ *for all* $x \in X$.
   (e) $0 \in C(X)$.
   (f) $x \in C(X)$ *for all* $x \in X$.
   (g) $x \in Z(X)$ *for all* $x \in X$.
   (h) $0 \in Z(X)$.
   (i) *X is a (lr-r)-QNTL.*
   (j) *X is a (lr-l)-QNTL.*
   (k) *X is a (lr-lr)-QNTL*
8. *If $(X, *, 0)$ is a BCK-algebra and $0 \in C(X)$, then X is a trivial BCK-algebra.*

Proof. This is routine by simply using the definitions of nuclei, centrum, center of a BCI-algebra and QNTL alongside Theorems 3–10 appropriately. □

Remark 3. *Based on Theorem 11, since an associative BCI-algebra is a (r-lr)-QNTG, then, an associative BCI-algebra is a NTG. This corroborates the importance of the study of non-associative BCI-algebra i.e., weak associative laws (F_i-identities) in BCI-algebra, as mentioned earlier in the objective of this work.*

Theorem 13. *Let $(X, *, 0)$ be a BCI-algebra. If X is any of the following Fenyves BCI-algebras, then X is associative.*

1. F_1-algebra
2. F_2-algebra
3. F_4-algebra
4. F_6-algebra
5. F_7-algebra
6. F_9-algebra
7. F_{10}-algebra
8. F_{11}-algebra
9. F_{12}-algebra
10. F_{13}-algebra
11. F_{14}-algebra
12. F_{15}-algebra
13. F_{16}-algebra
14. F_{17}-algebra
15. F_{18}-algebra
16. F_{20}-algebra
17. F_{22}-algebra
18. F_{23}-algebra
19. F_{24}-algebra
20. F_{25}-algebra
21. F_{26}-algebra
22. F_{27}-algebra
23. F_{28}-algebra
24. F_{30}-algebra
25. F_{31}-algebra
26. F_{32}-algebra
27. F_{33}-algebra
28. F_{34}-algebra
29. F_{35}-algebra
30. F_{36}-algebra
31. F_{37}-algebra
32. F_{38}-algebra
33. F_{40}-algebra
34. F_{41}-algebra
35. F_{43}-algebra
36. F_{44}-algebra
37. F_{45}-algebra
38. F_{47}-algebra
39. F_{48}-algebra
40. F_{49}-algebra
41. F_{50}-algebra
42. F_{51}-algebra
43. F_{53}-algebra
44. F_{57}-algebra
45. F_{58}-algebra
46. F_{60}-algebra

Proof.

1. Let X be an F_1-algebra. Then $xy * zx = (xy * z)x$. With $z = y$, we have $xy * yx = (xy * y)x$ which implies $xy * yx = (xy * x)y = (xx * y)y = (0 * y)y = 0 * (y * y)$ (since $0 \in N_\lambda(X)$; this is achieved by putting $y = x$ in the F_1 identity) $= 0 * 0 = 0$. This implies $xy * yx = 0$. Now replacing x with y, and y with x in the last equation gives $yx * xy = 0$ implying that $x * y = y * x$ as required.
2. Let X be an F_2-algebra. Then $xy * zx = (x * yz)x$. With $y = z$, we have $xz * zx = (x * zz)x = (x * 0) * x = x * x = 0$ implying that $xz * zx = 0$. Now replacing x with z, and z with x in the last equation gives $zx * xz = 0$ implying that $x * z = z * x$ as required.
3. Let X be a F_4-algebra. Then, $xy * zx = x(yz * x)$. Put $y = x$ and $z = 0$, then you get $0 * 0x = x$ which means X is p-semisimple. Put $x = 0$ and $y = 0$ to get $0z = 0 * 0z$ which implies that X is quasi-associative (Theorem 5). Thus, by Theorem 9, X is associative.
4. Let X be an F_6-algebra. Then, $(xy * z)x = x(y * zx)$. Put $x = y = 0$ to get $0z = 0 * 0z$ which implies that X is quasi-associative (Theorem 5). Put $y = 0$ and $z = x$, then we have $0 * x = x$. Thus, X is associative.
5. Let X be an F_7-algebra. Then $(xy * z)x = x(yz * x)$. With $z = 0$, we have $xy * x = x(y * x)$. Put $y = x$ in the last equation to get $xx * x = (x * xx)$ implying $0 * x = x$.
6. Let X be an F_9-algebra. Then $(x * yz)x = x(yz * x)$. With $z = 0$, we have $(x * y) * x = x(y * x)$. Put $y = x$ in the last equation to get $(x * x)x = x(x * x)$ implying $0 * x = x$.
7. Let X be an F_{10}-algebra. Then, $x(y * zx) = x(yz * x)$. Put $y = x = z$, then we have $x * 0x = 0$. So, $0x = 0 \Rightarrow x = 0$. which means that X is p-semisimple (Theorem 8(2)). Hence, X has the LCL by Theorem 4. Thence, the F_{10} identity $x(y * zx) = x(yz * x) \Rightarrow y * zx = yz * x$ which means that X is associative.
8. Let X be an F_{11}-algebra. Then $xy * xz = (xy * x)z$. With $y = 0$, we have $x * xz = xx * z$. Put $z = x$ in the last equation to get $x = 0 * x$ as required.
9. Let X be an F_{12}-algebra. Then $xy * xz = (x * yx)z$. With $z = 0$, we have $xy * x = x * yx$. Put $y = x$ in the last equation to get $xx * x = x * xx$ implying $0 * x = x$ as required.
10. Let X be an F_{13}-algebra. Then $xy * xz = x(yx * z)$. With $z = 0$, we have $(x * y)x = x * yx$ which implies $(x * x)y = x * yx$ which implies $0 * y = x * yx$. Put $y = x$ in the last equation to get $0 * x = x$ as required.
11. Let X be an F_{14}-algebra. Then $xy * xz = x(y * xz)$. With $z = 0$, we have $xy * x = x * yx$. Put $y = x$ in the last equation to get $0 * x = x$ as required.
12. Let X be an F_{15}-algebra. Then $(xy * x)z = (x * yx)z$. With $z = 0$, we have $(xy * x) = (x * yx)$. Put $y = x$ in the last equation to get $0 * x = x$ as required.
13. Let X be an F_{16}-algebra. Then $(xy * x)z = x(yx * z)$. With $z = 0$, we have $(xy * x) = (x * yx)$. Put $y = x$ in the last equation to get $0 * x = x$ as required.

14. Let X be an F_{17}-algebra. Then $(xy * x)z = x(y * xz)$. With $z = 0$, we have $(xy * x) = x(y * x)$. Put $y = x$ in the last equation to get $0 * x = x$ as required.
15. Let X be an F_{18}-algebra. Then $(x * yx)z = x(yx * z)$. With $y = 0$, we have $(x * 0x)z = x(0x * z)$. Since $0 \in N_\lambda(X)$ and $0 \in N_\mu(X)$, (these are obtained by putting $x = 0$ and $x = y$ respectively in the F_{18}-identity), the last equation becomes $(x0 * x)z = x(0 * xz) = x0 * xz = x * xz$ which implies $0 * z = x * xz$. Put $x = z$ in the last equation to get $0 * z = z$ as required.
16. This is similar to the proof for F_{10}-algebra.
17. Let X be an F_{22}-algebra. Then $yx * zx = (y * xz)x$. Put $y = x, z = 0$, then $0x = 0 * 0x$ which implies that X is quasi-associative. By Theorem 10, the F_{22} identity implies that $yx * zx = yx * xz$. Substitute $x = 0$ to get $yz = y * 0z$. Now, put $y = z$ in this to get $z * 0z = 0$. So, $0z = 0 \Rightarrow z = 0$. Hence, X is p-semisimple (Theorem 8(2)). Thus, by Theorem 9, X is associative.
18. Let X be an F_{23}-algebra. Then $yx * zx = y(xz * x)$. With $z = 0$, we have $yx * 0x = y(x * x)$ which implies $yx * 0x = y$. Since $0 \in N_\mu(X)$, (this is obtained by putting $z = x$ in the F_{23}-identity), the last equation becomes $(yx * 0) * x = y$ which implies $(yx * x) = y$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
19. Let X be an F_{24}-algebra. Then $yx * zx = y(x * zx)$. With $z = 0$, we have $yx * 0x = y(x * 0x)$. Since $0 \in N_\mu(X)$,(this is obtained by putting $x = 0$ in the F_{24}-identity), the last equation becomes $((yx)0 * x) = y(x0 * x)$ which implies $yx * x = y$. Put $y = x$ in the last equation to get $0 * y = y$ as required.
20. Let X be an F_{25}-algebra. Then $(yx * z)x = (y * xz)x$. Put $x = 0$, then $yz = y * 0z$. Substitute $z = y$, then $y * 0y = 0$. So, $0y = 0 \Rightarrow y = 0$. Hence, X is p-semisimple (Theorem 8(2)). Hence, X has the RCL by Theorem 7. Thence, the F_{25} identity $(yx * z)x = (y * xz)x$ implies $yx * z = y * xz$. Thus, X is associative.
21. Let X be an F_{26}-algebra. Then $(yx * z)x = y(xz * x)$. With $z = 0$, we have $yx * x = y$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
22. Let X be an F_{27}-algebra. Then $(yx * z)x = y(x * zx)$. Put $z = x = y$, then $0x * x = 0$ which implies X is quasi-associative. Put $x = 0$ and $y = z$ to get $z * 0z = 0$. So, $0z = 0 \Rightarrow z = 0$. Hence, X is p-semisimple (Theorem 8(2)). Thus, by Theorem 9, X is associative.
23. Let X be an F_{28}-algebra. Then $(y * xz)x = y(xz * x)$. With $z = 0$, we have $yx * x = y$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
24. The proof of this is similar to the proof for F_{10}-algebra.
25. Let X be an F_{31}-algebra. Then $yx * xz = (yx * x)z$. By Theorem 10, the F_{31} identity becomes F_{25} identity which implies that X is associative.
26. Let X be an F_{32}-algebra. Then $yx * xz = (y * xx)z$. With $z = 0$, we have $yx * x = y$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
27. Let X be an F_{33}-algebra. Then $yx * xz = y(xx * z)$. With $z = 0$, we have $yx * x = y$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
28. Let X be an F_{34}-algebra. Then $yx * xz = y(x * xz)$. With $z = 0$, we have $yx * x = y$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
29. Let X be an F_{35}-algebra. Then $(yx * x)z = (y * xx)z$. With $z = 0$, we have $yx * x = y$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
30. Let X be an F_{36}-algebra. Then $(yx * x)z = y(xx * z)$. With $z = 0$, we have $yx * x = y$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
31. Let X be an F_{37}-algebra. Then $(yx * x)z = y(x * xz)$. With $z = 0$, we have $yx * x = y$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
32. Let X be an F_{38}-algebra. Then, $yz = y * 0z$. Put $z = y$, then $y * 0y = 0$. So, $0y = 0 \Rightarrow y = 0$. Hence, X is p-semisimple (Theorem 8(2)). Now, put $y = x$, then $xz = x * 0z$. Now, substitute $x = 0$ to get $0z = 0 * 0z$ which means that X is quasi-associative. Thus, by Theorem 9, X is associative.
33. Let X be an F_{40}-algebra. By the F_{40} identity, $y * 0z = y(x * xz)$. Put $z = x = y$ to get $0 * 0x = 0$. So, $0x = 0 \Rightarrow x = 0$. Hence, X is p-semisimple (Theorem 8(2)). Thus, X has the LCL by Theorem 4. Thence, the F_{40} identity $y(xx * z) = y(x * xz)$ becomes $0 * z = x * xz$. Substituting $z = x$, we get $0x = x$ which means that X is associative.

34. Let X be an F_{41}-algebra. Then $xx * yz = (x * xy)z$. With $z = 0$, we have $0 * y = x * xy$. Put $y = x$ in the last equation to get $0 * x = x$ as required.
35. Let X be an F_{43}-algebra. Then $xx * yz = x(x * yz)$. With $z = 0$, we have $0 * y = x(x * y)$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
36. Let X be an F_{44}-algebra. Then $xx * yz = x(xy * z)$. With $z = 0$, we have $0 * y = x(x * y)$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
37. Let X be an F_{45}-algebra. Then $(x * xy)z = (xx * y)z$. With $z = 0$, we have $x * xy = 0 * y$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
38. Let X be an F_{47}-algebra. Then $(x * xy)z = x(xy * z)$. With $y = 0$, we have $0 * z = x(x * z)$. Put $x = z$ in the last equation to get $0 * z = z$ as required.
39. Let X be an F_{48}-algebra. Then $(xx * y)z = x(x * yz)$. With $z = 0$, we have $0 * y = x * xy$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
40. Let X be an F_{49}-algebra. Then $(xx * y)z = x(xy * z)$. With $y = 0$, we have $0 * z = x * xz$. Put $x = z$ in the last equation to get $0 * z = z$ as required.
41. This is similar to the proof for F_{10}-algebra.
42. Let X be an F_{51}-algebra. Then $yz * xx = (yz * x)x$. With $z = 0$, we have $y = (y * x)x$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
43. Let X be an F_{53}-algebra. Then $yz * xx = y(zx * x)$ which becomes $yz = y(zx * x)$. Put $z = x$ to get $yx = y * 0x$. Substituting $y = x$, we get $x * 0x = 0$. So, $0x = 0 \Rightarrow x = 0$, which means that X is p-semisimple (Theorem 8(2)). Now, put $y = 0$ in $yx = y * 0x$ to get $0x = 0 * 0x$. Hence, X is quasi-associative. Thus, X is associative.
44. Let X be an F_{57}-algebra. Then $(yz * x)x = y(z * xx)$. With $z = 0$, we have $yx * x = y$. Put $x = y$ in the last equation to get $0 * y = y$ as required.
45. Let X be an F_{58}-algebra. Then $(y * zx)x = y(zx * x)$. Put $y = x = z$ to get $x * 0x = 0$. So, $0x = 0 \Rightarrow x = 0$, which means that X is p-semisimple (Theorem 8(2)). Now, put $z = x, y = 0$ to get $0x = 0 * 0x$. Hence, X is quasi-associative. Thus, X is associative.
46. Let X be an F_{60}-algebra. Then $y(zx * x) = y(z * xx)$. Put $y = x = z$ to get $x * 0x = 0$. So, $0x = 0 \Rightarrow x = 0$, which means that X is p-semisimple (Theorem 8(2)). Hence, X has the LCL by Theorem 4. Thence, the F_{10} identity becomes $zx * x = z * xx$. Now, substitute $z = x$ to get $0x = x$. Thus, X is associative.

□

Corollary 1. *Let $(X, *, 0)$ be a BCI-algebra. If X is any of the following Fenyves' BCI-algebras, then $(X, *)$ is a Boolean group.*

1. *F_1-algebra*
2. *F_2-algebra*
3. *F_4-algebra*
4. *F_6-algebra*
5. *F_7-algebra*
6. *F_9-algebra*
7. *F_{10}-algebra*
8. *F_{11}-algebra*
9. *F_{12}-algebra*
10. *F_{13}-algebra*
11. *F_{14}-algebra*
12. *F_{15}-algebra*
13. *F_{16}-algebra*
14. *F_{17}-algebra*
15. *F_{18}-algebra*
16. *F_{20}-algebra*
17. *F_{22}-algebra*
18. *F_{23}-algebra*
19. *F_{24}-algebra*
20. *F_{25}-algebra*
21. *F_{26}-algebra*
22. *F_{27}-algebra*
23. *F_{28}-algebra*
24. *F_{30}-algebra*
25. *F_{31}-algebra*
26. *F_{32}-algebra*
27. *F_{33}-algebra*
28. *F_{34}-algebra*
29. *F_{35}-algebra*
30. *F_{36}-algebra*
31. *F_{37}-algebra*
32. *F_{38}-algebra*
33. *F_{40}-algebra*
34. *F_{41}-algebra*
35. *F_{43}-algebra*
36. *F_{44}-algebra*
37. *F_{45}-algebra*
38. *F_{47}-algebra*
39. *F_{48}-algebra*
40. *F_{49}-algebra*
41. *F_{50}-algebra*
42. *F_{51}-algebra*
43. *F_{53}-algebra*
44. *F_{57}-algebra*
45. *F_{58}-algebra*
46. *F_{60}-algebra*

Proof. This follows from Theorems 12 and 13. □

Theorem 14. *Let $(X, *, 0)$ be a BCI-algebra.*

1. *Let X be an F_3-algebra. X is associative if and only if $x(x * zx) = xz$ if and only if X is p-semisimple.*
2. *Let X be an F_5-algebra. X is associative if and only if $(xy * x)x = yx$.*
3. *Let X be an F_{21}-algebra. X is associative if and only if $(yx * x)x = x * y$.*
4. *Let X be an F_{42}-algebra. X is associative if and only if X is p-semisimple.*

5. *Let X be an F_{55}-algebra. X is associative if and only if $[(y * x) * x] * x = x * y$.*
6. (a) *X is an F_5-algebra and p-semisimple if and only if X is associative.*
   (b) *Let X be an F_8-algebra. X is associative if and only if $x(y * zx) = yz$.*
7. *Let X be an F_{19}-algebra. X is associative if and only if quasi-associative.*
8. *X is an F_{39}-algebra and obeys $y(x * xz) = zy$ if and only if X is associative.*
9. *Let X be a F_{46}-algebra. X is associative if and only if $0(0 * 0x) = x$.*
10. (a) *X is an F_{52}-algebra and F_{55}-algebra if and only if X is associative.*
    (b) *X is an F_{52}-algebra and obeys $(y * zx)x = zy$ if and only if X is associative.*
    (c) *X is an F_{55}-algebra and p-semisimple if and only if X is associative.*
    (d) *Let X be an F_{52}-algebra. X is associative if and only if X is quasi-associative.*
11. (a) *X is an F_{59}-algebra and F_{55}-algebra if and only if X is associative.*
    (b) *X is an F_{52}-algebra and obeys $(y * zx)x = zy$ if and only if X is associative.*
    (c) *Let X be a F_{56}-algebra. X is associative if and only if X is quasi-associative.*
    (d) *Let X be an F_{59}-algebra. X is associative if and only if X is quasi-associative.*

Proof.

1. Suppose X is a F_3-algebra. Then, $xy * zx = x(y * zx)$. Put $y = x$ to get $0 * zx = x(x * zx)$. Substituting $x = 0$, we have $0z = 0 * 0z$ which means X is quasi-associative. Going by Theorem 9, X is associative if and only if X is p-semisimple. Furthermore, by Theorem 4(3) and $0 * zx = x(x * zx)$, an F_3-algebra X is associative if and only if $xy = x(x * zx)$.
2. Suppose X is associative. Then $0 * x = x$. X is F_5 implies $(xy * z)x = (x * yz)x$. With $z = x$, we have $(xy * x)x = (x * yx)x \Rightarrow (xy * x)x = (x * x)yx \Rightarrow (xy * x)x = 0 * yx \Rightarrow (xy * x)x = yx$ as required. Conversely, suppose $(xy * x)x = yx$. Put $z = x$ in $(xy * z)x = (x * yz)x$ to get $(xy * x)x = (x * yx)x \Rightarrow (xy * x)x = (x * x)yx \Rightarrow (xy * x)x = 0 * yx \Rightarrow yx = 0 * yx$ (since $(xy * x)x = yx$). So, X is associative.
3. Suppose X is associative. Then $x * y = y * x$. X is F_{21} implies $yx * zx = (yx * z)x$. With $z = x$, we have $(yx * x)x = y * x = x * y$ as required. Conversely, suppose $(yx * x)x = x * y$. Put $z = x$ in F_{21} to get $(yx * x)x = y * x$. So, $x * y = y * x$ as required.
4. Suppose X is associative. Then $0 * z = z$. X is F_{42} implies $xx * yz = (xx * y)z$. With $y = 0$, we have $0 * 0z = 0 * z = z$ as required. Conversely, suppose $0 * 0z = z$. Put $y = 0$ in F_{42} to get $0 * 0z = 0 * z$. So, $0 * z = z$ as required.
5. Suppose X is associative. Then $x * y = y * x$. X is F_{55} implies $[(y * z) * x] * x = [y * (z * x)] * x$. With $z = x$, we have $[(y * x) * x] * x = y * x = x * y$ as required. Conversely, suppose $[(y * x) * x] * x = x * y$. Put $z = x$ in F_{55} to get $y * x = [(y * x) * x] * x = x * y$. So, $y * x = x * y$ as required.

The proofs of 6 to 11 follow by using the concerned F_i and F_j identities (plus p-simplicity by Theorem 12 in some cases) to get an F_k which is equivalent to associativity by Theorem 13 or which is not equivalent to associativity by 1 to 5 of Theorem 14. □

3. Summary, Conclusions and Recommendations

In this work, we have been able to construct examples of Fenyves' BCI-algebras. We have also obtained the basic algebraic properties of Fenyves' BCI-algebras. Furthermore, we have categorized the Fenyves' BCI-algebras into a 46 member associative class (as captured in Theorem 13). Members of this class include F_1, F_2, F_4, F_6, F_7, F_9, F_{10}, F_{11}, F_{12}, F_{13}, F_{14}, F_{15}, F_{16}, F_{17}, F_{18}, F_{20}, F_{22}, F_{23}, F_{24}, F_{25}, F_{26}, F_{27}, F_{28}, F_{30}, F_{31}, F_{32}, F_{33}, F_{34}, F_{35}, F_{36}, F_{37}, F_{38}, F_{40}, F_{41}, F_{43}, F_{44}, F_{45}, F_{47}, F_{48}, F_{49}, F_{50}, F_{51}, F_{53}, F_{57}, F_{58}, F_{60}-algebras; and a 14 member non-associative class. Those Fenyves identities that are equivalent to associativity in BCI-algebras are denoted by ✓ in the fifth column of Table 1. For those that belong to the non-associative class, we have been able to obtain conditions under which they would be associative (as reflected in Theorem 14). This class includes F_3, F_5, F_8, F_{19}, F_{21},F_{29}, F_{39} , F_{42}, F_{46}, F_{52}, F_{54}, F_{55}, F_{56}, F_{59}-algebras. In Table 1 which summarizes the results, members of this class are identified by the symbol '‡'.

Other researchers who have studied Fenyves' identities on the platform of loops, namely Phillips and Vojtechovsky [5], Jaiyeola [6], Kinyon and Kunen (2004) found Moufang (F_2, F_4, F_{17}, F_{27}), extra

(F_6, F_{13}, F_{22}), F_9, F_{15}, left Bol (F_{19}), right Bol (F_{26}), Moufang (F_4, F_{27}), F_{30}, F_{35}, F_{36}, C (F_{37}), F_{38}, F_{39}, F_{40}, LC(F_{39}, F_{41}, F_{46}, F_{48}), F_{42}, F_{43}, F_{45}, F_{51}, RC(F_{36}, F_{53}, F_{56}, F_{57}), F_{54}, and F_{60} Fenyves' identities not to be equivalent to associativity in loops. Interestingly, in our study, some of these identities, particularly the extra identity (F_6, F_{13}, F_{22}), F_7, F_9, F_{15}, F_{17}, right Bol (F_{26}), Moufang (F_4, F_{27}), F_{30}, F_{35}, F_{38}, F_{40}, RC (F_{36}, F_{53}, F_{57}), C (F_{37}), LC (F_{41}, F_{48}), F_{43}, F_{45}, F_{51} and F_{60} have been found to be equivalent to associativity in BCI-algebras. In addition, the aforementioned researchers found F_1, F_3, F_5, F_7, F_8, F_{10}, F_{11}, F_{12}, F_{14}, F_{16}, F_{18}, F_{20}, F_{21}, F_{23}, F_{24}, F_{25}, F_{28}, F_{29}, F_{31}, F_{32}, F_{33}, F_{34}, F_{44}, F_{47}, F_{49}, F_{50}, F_{52}, F_{55}, F_{58} and F_{59} identities to be equivalent to associativity in loops. We have also found some (F_7, F_{10}, F_{11}, F_{12}, F_{14}, F_{16}, F_{18},F_{20}, F_{23}, F_{24}, F_{25}, F_{28}, F_{31}, F_{32}, F_{33}, F_{44}, F_{47}, F_{49}, F_{50}, F_{58}) of these identities to be equivalent to associativity in BCI-algebras while some others (F_3, F_5, F_8, F_{20}, F_{21}, F_{29},F_{55}, F_{59}) were not equivalent to associativity in BCI-algebras.

In loop theory, it is well known that:

- A loop is an extra loop if and only if the loop is both a Moufang loop and a C-loop.
- A loop is a Moufang loop if and only if the loop is both a right Bol loop and a left Bol-loop.
- A loop is a C-loop if and only if the loop is both a RC-loop and a LC-loop.

In this work, we have been able to establish (as stated below) somewhat similar results for a few of the Fenyves' identities in a BCI-algebra X:

- X is an F_i-algebra and F_j-algebra if and only if X is associative, for the pairs: $i = 52$, $j = 55$, $i = 59$, $j = 55$.

Fenyves [31], and Phillips and Vojtěchovský [32,33] found some of the 60 F_i identities to be equivalent to associativity in quasigroups and loops (i.e., groups), and others to describe weak associative laws such as extra, Bol, Moufang, central, flexible laws in quasigroups and loops. Their results are summarised in the second, third and fourth columns of Table 1 with the use of ✓. In this paper, we went further to establish that 46 Fenyves' identities are equivalent to associativity in BCI-algebras while 14 Fenyves' identities are not equivalent to associativity in BCI-algebras. These two categories are denoted by ✓ and ‡ in the fifth column of Table 1.

After the works of [31–33], the authors in [34–38] did an extension by investigating and classifying various generalized forms of the identities of Bol-Moufang types in quasigroups and one sided/two sided loops into associative and non-associative categories. This answered a question originally posed in [39] and also led to the study of one of the newly discovered generalized Bol-Moufang types of loop in Jaiyéọlá et al. [40]. While all the earlier mentioned research works on Bol-Moufang type identities focused on quasigroups and loop, this paper focused on the study of Bol-Moufang type identities (Fenyves' identities) in special types of groupoids (BCI-algebra and quasi neutrosophic triplet loops) which are not necessarily quasigroups or loops (as proved in Theorem 12). Examples of such well known varieties of groupoids were constructed by Ilojide et al. [41], e.g., Abel-Grassmann's groupoid.

The results of this work are an initiation into the study of the classification of finite Fenyves' quasi neutrosophic triplet loops (FQNTLs) just like various types of finite loops have been classified (e.g., Bol loops, Moufang loops and FRUTE loops). In fact, a library of finite Moufang loops of small order is available in the GAPS-LOOPS package [42]. It will be intriguing to have such a library of FQNTLs.

Overall, this research work (especially for the non-associative F_i's) has opened a new area of research findings in BCI-algebras and Bol-Moufang type quasi neutrosophic triplet loops as shown in Figure 1.

Table 1. Characterization of Fenyves Identities in Quasigroups, Loops and BCI-Algebras by Associativity.

Fenyves Identity	$F_i \equiv ASS$ Inaloop	$F_i \not\equiv ASS$ Inaloop	Quassigroup $\Rightarrow$ Loop	$F_i + BCI \Rightarrow ASS$
F_1	✓		✓	✓
F_2		✓	✓	✓
F_3	✓		✓	‡
F_4		✓		✓
F_5	✓			‡
F_6		✓	✓	✓
F_7	✓			✓
F_8	✓			‡
F_9		✓		✓
F_{10}	✓			✓
F_{11}	✓		✓	✓
F_{12}	✓		✓	✓
F_{13}		✓	✓	✓
F_{14}	✓			✓
F_{15}		✓		✓
F_{16}	✓			✓
F_{17}		✓	✓	✓
F_{18}	✓		✓	✓
F_{19}		✓		‡
F_{20}	✓			✓
F_{21}	✓		✓	‡
F_{22}		✓	✓	✓
F_{23}	✓			✓
F_{24}	✓			✓
F_{25}	✓			✓
F_{26}		✓		✓
F_{27}		✓	✓	✓
F_{28}	✓		✓	✓
F_{29}	✓			‡
F_{30}		✓		✓
F_{31}	✓		✓	✓
F_{32}	✓		✓	✓
F_{33}	✓			✓
F_{34}	✓			✓
F_{35}		✓		✓
F_{36}		✓		✓
F_{37}		✓		✓
F_{38}		✓	✓	✓

Table 1. *Cont.*

Fenyves Identity	$F_i \equiv ASS$ Inaloop	$F_i \not\equiv ASS$ Inaloop	Quassigroup $\Rightarrow$ Loop	$F_i + BCI$ $\Rightarrow ASS$
F_{39}		✓		‡
F_{40}		✓		✓
F_{41}		✓	✓	✓
F_{42}		✓		‡
F_{43}		✓		✓
F_{44}	✓			✓
F_{45}		✓		✓
F_{46}		✓		‡
F_{47}	✓		✓	✓
F_{48}		✓		✓
F_{49}	✓			✓
F_{50}	✓			✓
F_{51}		✓		✓
F_{52}	✓			‡
F_{53}		✓	✓	✓
F_{54}		✓		‡
F_{55}	✓			‡
F_{56}		✓		‡
F_{57}		✓		✓
F_{58}	✓		✓	✓
F_{59}	✓			‡
F_{60}		✓		✓

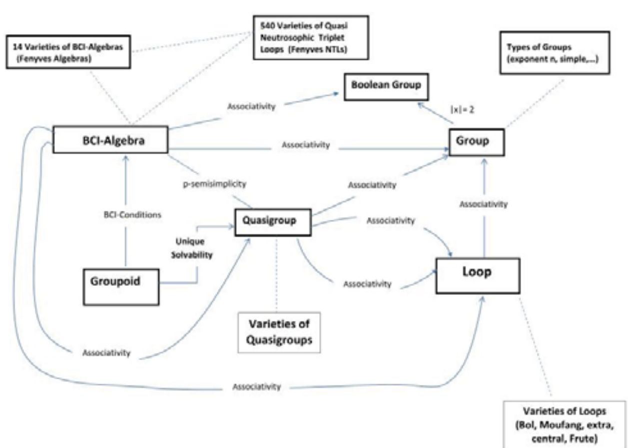

Figure 1. New Cycle of Algebraic Structures.

Author Contributions: Conceptualization, T.G.J. and E.I.; Methodology, T.G.J. and E.I.; Validation, T.G.J., M.O.O., F.S., T.G.J., E.I. and M.O.O. ; Investigation, T.G.J. and E.I.; Resources, T.G.J. and F.S.; Writing—Original Draft Preparation, E.I.; Writing—Review & Editing, T.G.J. and M.O.O.; Supervision, T.G.J., M.O.O. and F.S.; Project Administration,T.G.J. and F.S.; Funding Acquisition, F.S.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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Clustering Neutrosophic Data Sets and Neutrosophic Valued Metric Spaces

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Received: 5 August 2018; Accepted: 21 September 2018; Published: 24 September 2018

Abstract: In this paper, we define the neutrosophic valued (and generalized or G) metric spaces for the first time. Besides, we newly determine a mathematical model for clustering the neutrosophic big data sets using G-metric. Furthermore, relative weighted neutrosophic-valued distance and weighted cohesion measure, is defined for neutrosophic big data set. We offer a very practical method for data analysis of neutrosophic big data although neutrosophic data type (neutrosophic big data) are in massive and detailed form when compared with other data types.

Keywords: G-metric; neutrosophic G-metric; neutrosophic sets; clustering; neutrosophic big data; neutrosophic logic

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1. Introduction and Preliminaries

Neutrosophic Logic is a neonate study area in which each proposition is estimated to have the proportion (percentage) of truth in a subset T, the proportion of indeterminacy in a subset I, and the proportion of falsity in a subset F. We utilize a subset of truth (or indeterminacy, or falsity), instead of a number only, since in many situations we do not have ability to strictly specify the proportions of truth and of falsity but only approximate them; for instance, a proposition is between 25% and 55% true and between 65% and 78% false; even worse: between 33% and 48% or 42 and 53% true (pursuant to several observer), and 58% or between 66% and 73% false. The subsets are not essential intervals, but any sets (open or closed or half open/half-closed intervals, discrete, continuous, intersections or unions of the previous sets, etc.) in keeping with the given proposition. Zadeh initiated the adventure of obtaining meaning and mathematical results from uncertainty situations (fuzzy) [1]. Fuzzy sets brought a new dimension to the concept of classical set theory. Atanassov introduced intuitionistic fuzzy sets including membership and non-membership degrees [2]. Neutrosophy was proposed by Smarandache as a computational approach to the concept of neutrality [3]. Neutrosophic sets consider membership, non-membership and indeterminacy degrees. Intuitionistic fuzzy sets are defined by the degree of membership and non-membership and, uncertainty degrees by the 1-(membership degree plus non-membership degree), while the degree of uncertainty is evaluated independently of the degree of membership and non-membership in neutrosophic sets. Here, membership, non-membership, and degree of uncertainty (uncertainty), such as degrees of accuracy and falsity, can be evaluated according to the interpretation of the places to be used. It depends entirely on the subject area (the universe of discourse). This reveals a difference between neutrosophic set and intuitionistic fuzzy set. In this sense, the concept of neutrosophic is a possible solution and representation of problems in various fields. Two detailed and mathematical fundamental differences between relative truth (IFL) and absolute truth (NL) are:

(i) NL can discern absolute truth (truth in all possible worlds, according to Leibniz) from the relative truth (truth in at least one world) because NL (absolute truth) = 1^+ while IFL (relative truth) = 1. This has practice in philosophy (see the Neutrosophy). The standard interval [0, 1] used in IFL has been extended to the unitary non-standard interval $]^-0, 1^+[$ in NL. Parallel earmarks for absolute or relative falsehood and absolute or relative indeterminacy are permitted in NL.

(ii) There is no limit on T, I, F other than they are subsets of $]^-0, 1^+[$, thus: $^-0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3^+$ in NL. This permissiveness allows dialetheist, paraconsistent, and incomplete information to be described in NL, while these situations cannot be described in IFL since F (falsehood), T (truth), I (indeterminacy) are restricted either to t + i + f = 1 or to $t^2 + f^2 \leq 1$, if T, I, F are all reduced to the points t, i, f respectively, or to sup T + sup I + sup F = 1 if T, I, F are subsets of [0, 1] in IFL.

Clustering data is one of the most significant problems in data analysis. Useful and efficient algorithms are needed for big data. This is even more challenging for neutrosophic data sets, particularly those involving uncertainty. These sets are elements of some decision-making problems, [4–8]. Several distances and similarities are used for decision-making problems [9,10]. Algorithms for the clustering big data sets use the distances (metrics). There are some metrics used in algorithms to analysis neutrosophic data sets: Hamming, Euclidean, etc. In this paper, we examine clustering of neutrosophic data sets via neutrosophic valued distances.

The big data notion is a new label for the giant size of data–both structured and unstructured—that overflows several sectors on a time-to-time basis. It does not mean overall data are significant and the significant aspect is to obtain desired specific data interpretation. Big data can be analyzed for pre-cognition that make possible more consistent decisions and strategic having positions. Doug Laney [11] sort to make the definition of big data the three Vs and Veracity widespread: (1) Velocity: This refers to dynamic data and captures data streams in near real-time. Data streams in at an exceptional speed and must be dealt with in a well-timed mode. (2) Variety: Data comes in all types of formats—from structured, numeric data in traditional databases to formless materials. On the one hand, variety denotes to the various sources and types of organized and formless data. Storing data is made from sources like worksheets and databases. (3) Volume: Organizations gather data from a range of sources, including social media, business operations, and data from the sensor or machine to machine. (4) Veracity: It mentions to the biases, noise, and anomaly in data. That corresponds with the question "Is the data that is being put in storage and extracted meaningful to the problem being examined?".

In this paper, we also focus on K-sets cluster algorithm which is a process of analyzing data with the aim of evaluating neutrosophic big data sets. The K-sets cluster is an unrestrained type of learning that is used when one wants to utilize unlabeled data, [12]. The goal of the algorithm is to find groups of data with the number of groups represented by variable K. The algorithm works iteratively to set-aside each data point obtained to one of the K groups based on the properties obtained. The data points are clustered according to feature similarity. Instead of identifying groups before examining patterns, clustering helps to find and analyze naturally occurring groups. "Choosing K" has the goal of "how the number of groups can be determined". Each center of a congregation is a collection of property values describe the groups that emerged. Analysis of centroid feature weights can be used to qualitatively interpret what kind of group is represented by each cluster. The algorithm finds the clusters and data set labels for a particular pre-chosen K. To have the number of clusters in the data, the user must run the K-means clustering algorithm for a range of K values and compare the results. In general, there is no technique to determine a specific K value, but a precise estimate can be obtained using the following methods. In general, one of the metrics used to compare the results between the different K values as the average distance between the data points and their cluster synthesis. As the number of sets increases, it will always reduce the distance to the data points, while the K increment will always lower this metric as other criteria, and when K is the same as the number of data points, reaching zero will be excessive. Thus, this metric cannot be used as a single purpose. Rather, the average distance to the center as a function of K is plotted where the shear rate falls sharply, it can be used to determine K approximately.

A number of other techniques are available for verification of K, including cross-validation, information criteria, information theoretical jump method, and G-tools algorithm. In addition, monitoring the distribution of data points between groups provides information about how the algorithm splits data for each K. K-sets algorithms base on the measurement of distances of sets. A distance is a measurement of how far apart each pair of elements of a given set is. Distance functions in mathematics and many other computational sciences are important concepts. They have wide usage areas, for example, the goal of quantifying a dissimilarity (or equivalently similarity) between two objects, sets or set of sets in some sense. However, due to the massive, complicated and different type data sets today, definitions of distance functions are required to be more generalized and detailed. For this purpose, we define a novel metric for similarity and distance to give Neutrosophic Valued-Metric Spaces (NVGMS). We present relative weighted measure definition and finally K-sets algorithm after given the definition of NVGMS.

Some readers who are unfamiliar with the topic in this paper need to have a natural example to understand the topic well. There is a need for earlier data in everyday life to give a natural example for the subject first described in this paper. There is no this type of data (we mean neutrosophic big data) in any source, but we will give an example of how to obtain and cluster such a data in Section 6 of the paper. If we encounter a sample of neutrosophic big data in the future, we will present the results with a visual sample as a technical report. In this paper, we have developed a mathematically powerful method for the notion of concepts that are still in its infancy.

1.1. G-Metric Spaces

Metric space is a pair of (A, d), where A is a non-empty set and d is a metric which is defined by a certain distance and the elements of the set A. Some metrics may have different values such as a complex-valued metric [13,14]. Mustafa and Sims defined G-metric by generalizing this definition [15]. Specifically, fixed point theorems on analysis have been used in G-metric spaces [16,17].

Definition 1. *Let A be a non-empty set and d be a metric on A, then if the following conditions hold, the pair (A, d) is called a metric space. Let $x, y, z \in A$*

(1) $d(x, y) \geq 0$, *(non-negativity)*
(2) $d(x, y) = 0 \Leftrightarrow x = y$, *(identity)*
(3) $d(x, y) = d(y, x)$, *(symmetry)*
(4) $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality).*

where $d : A \times A \to R^+ \cup \{0\}$.

Definition 2. *[15] Let A be a non-empty set. A function $G : A \times A \times A \to [0, +\infty)$ is called G-distance if it satisfies the following properties:*

(1) $G(x, y, z) = 0$ *if and only if* $x = y = z$,
(2) $G(x, x, y) \neq 0$ *whenever* $x \neq y$,
(3) $G(x, x, y) \leq G(x, y, z)$ *for any* $x, y, z \in A$, *with* $z \neq y$,
(4) $G(x, y, z) = G(x, z, y) = \ldots$ *(symmetric for all elements),*
(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $a, x, y, z \in A$ *(Rectangular inequality).*

The pair (A, G) is called a G-metric space. Moreover, if G-metric has the following property then it is called symmetric: $G(x, x, y) = G(x, y, y), \forall x, y \in A$.

Example 1. *In 3-dimensional Euclidean metric space, one can assume the G-metric space* (E^3, G) *as the following:*

$$G(x,y,z) = 2(\|x \times y\| + \|z \times y\| + \|x \times z\|)$$

where $x, y, z \in E^3$ *and* $\|. \times .\|$ *represent the norm of the vector product of two vectors in* E^3*. It is obvious that it satisfies all conditions in the Definition 2 because of the norm has the metric properties, and it is symmetric.*

Example 2. *Let (A, d) is a metric space. Then*

$$G(x,y,z) = d(x,y) + d(y,z) - d(x,z)$$

is a G-metric, where $x, y, z \in A$*. The fact that d is a metric indicates that it has triangle inequality. Thus, G is always positive definite.*

Proposition 1. *[17] Let (A, G) be a G-metric space then a metric on A can be defined from a G-metric:*

$$d_G(x,y) = G(x,x,y) + G(x,y,y)$$

1.2. Neutrosophic Sets

Neutrosophy is a generalized form of the philosophy of intuitionistic fuzzy logic. In neutrosophic logic, there is no restriction for truth, indeterminacy, and falsity and they have a unit real interval value for each element neutrosophic set. These values are independent of each other. Sometimes, intuitionistic fuzzy logic is not enough for solving some real-life problems, i.e., engineering problems. So, mathematically, considering neutrosophic elements are becoming important for modelling these problems. Studies have been conducted in many areas of mathematics and other related sciences especially computer science since Smarandache made this philosophical definition, [18,19].

Definition 3. *Let E be a universe of discourse and* $A \subseteq E$*.* $A = \{(x, T(x), I(x), F(x)) : x \in E\}$ *is a neutrosophic set or single valued neutrosophic set (SVNS), where* $T_A, I_A, F_A : A \rightarrow]^{-}0, 1^{+}[$ *are the truth-membership function, the indeterminacy-membership function and the falsity-membership function, respectively. Here,* $^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$*.*

Definition 4. *For the SVNS A in E, the triple* $\langle T_A, I_A, F_A \rangle$ *is called the single valued neutrosophic number (SVNN).*

Definition 5. *Let* $n = \langle T_n, I_n, F_n \rangle$ *be an SVNN, then the score function of n can be given as follow:*

$$s_n = \frac{1 + T_n - 2I_n - F_n}{2} \tag{1}$$

where $s_n \in [-1, 1]$*.*

Definition 6. *Let* $n = \langle T_n, I_n, F_n \rangle$ *be an SVNN, then the accuracy function of n can be given as follow:*

$$h_n = \frac{2 + T_n - I_n - F_n}{3} \tag{2}$$

where $h_n \in [0, 1]$*.*

Definition 7. *Let* n_1 *and* n_2 *be two SVNNs. Then, the ranking of two SVNNs can be defined as follows:*

(I) *If* $s_{n_1} > s_{n_2}$*, then* $n_1 > n_2$*;*
(II) *If* $s_{n_1} = s_{n_2}$ and $h_{n_1} \geq h_{n_2}$*, then* $n_1 \geq n_2$*.*

2. Neutrosophic Valued Metric Spaces

The distance is measured via some operators which are defined in some non-empty sets. In general, operators in metric spaces have zero values, depending on the set and value.

2.1. Operators

Definition 8. [20,21], *Let A be non-empty SVNS and* $x = \langle T_x, I_x, F_x \rangle, y = \langle T_y, I_y, F_y \rangle$ *be two SVNNs. The operations that addition, multiplication, multiplication with scalar* $\alpha \in \mathbb{R}^+$*, and exponential of SVNNs are defined as follows, respectively:*

$$\begin{aligned} x \oplus y &= \langle T_x + T_y - T_x T_y, I_x I_y, F_x F_y \rangle \\ x \odot y &= \langle T_x T_y, I_x + I_y - I_x I_y, F_x + F_y - F_x F_y \rangle \\ \alpha x &= \langle 1 - (1 - T_x)^\alpha, I_x^\alpha, F_x^\alpha \rangle \\ x^\alpha &= \langle T_x^\alpha, 1 - (1 - I_x)^\alpha, 1 - (1 - F_x)^\alpha \rangle \end{aligned}$$

From this definition, we have the following theorems as a result:

Theorem 1. *Let* $x = \langle T_x, I_x, F_x \rangle$ *be an SVNN. The neutral element of the additive operator of the set A is* $0_A = \langle 0, 1, 1 \rangle$.

Proof. Let $x = \langle T_x, I_x, F_x \rangle$ and $0_A = \langle T_0, I_0, F_0 \rangle$ are two SVNN and using Definition 8 we have

$$\begin{aligned} &x \oplus 0_A = \langle T_x + T_0 - T_x T_0, I_x I_0, F_x F_0 \rangle = \langle T_x, I_x, F_x \rangle \\ &\Rightarrow \langle T_0, I_0, F_0 \rangle = \langle 0, 1, 1 \rangle = 0_A \end{aligned}$$

(There is no need to show left-hand side because the operator is commutative in every component).
□

To compare the neutrosophic values based on a neutral element, we shall calculate the score and accuracy functions of a neutral element $0_A = \langle 0, 1, 1 \rangle$, respectively:

$$s_0 = \frac{1 + T_0 - 2I_0 - F_0}{2} = -1 \text{ and } h_0 = \frac{2 + T_0 - I_0 - F_0}{3} = 0$$

Theorem 2. *Let* $x = \langle T_x, I_x, F_x \rangle$ *be an SVNN. The neutral element of the multiplication operator of the A is* $1_A = \langle 1, 0, 0 \rangle$.

Proof. Let $x = \langle T_x, I_x, F_x \rangle$ and $1_A = \langle T_1, I_1, F_1 \rangle$ are two SVNN and using Definition 8 we have

$$\begin{aligned} &x \odot 1_A = \langle T_x T_1, I_x + I_1 - I_x I_1, F_x + F_1 - F_x F_1 \rangle = \langle T_x, I_x, F_x \rangle \\ &\Rightarrow \langle T_1, I_1, F_1 \rangle = \langle 1, 0, 0 \rangle = 1_A \end{aligned}$$

In addition, score and accuracy functions of the neutral element $1_A = \langle 1, 0, 0 \rangle$ are $s_1 = \frac{1+T_1-2I_1-F_1}{2} = 1$ and $h_1 = \frac{2+T_1-I_1-F_1}{3} = 1$, respectively. □

2.2. Neutrosophic Valued Metric Spaces

In this section, we consider the metric and generalized metric spaces in the neutrosophic meaning.

Definition 9. *Ordering in the Definition 6 gives an order relation for elements of the conglomerate SVNN. Suppose that the mapping* $d : X \times X \to A$*, where X and A are SVNS, satisfies:*

(I) $0_A \leq d(x,y)$ *and* $d(x,y) = 0_A \Leftrightarrow s_x = s_y$ and $h_x = h_y$ *for all* $x, y \in X$.
(II) $d(x,y) = d(y,x)$ *for all* $x, y \in X$.

Then d is called a neutrosophic valued metric on X, and the pair (X,d) is called neutrosophic valued metric space. Here, the third condition (triangular inequality) of the metric spaces is not suitable for SVNS because the addition is not ordinary addition.

Theorem 3. *Let (X,d) be a neutrosophic valued metric space. Then, there are relationships among truth, indeterminacy and falsity values:*

(I) $0 < T(x,y) - 2I(x,y) - F(x,y) + 3$ *and if* $s_o = s_d$ then $0 < T(x,y) - I(x,y) - F(x,y) + 2$.
(II) *If* $d(x,y) = 0_A \Leftrightarrow T(x,y) = 0, I(x,y) = F(x,y) = 1$.
(III) $T(x,y) = T(y,x)$, $I(x,y) = I(y,x)$, $F(x,y) = F(y,x)$ *so, each distance function must be symmetric.*

where $T(.,.)$, $I(.,.)$ and $F(.,.)$ are distances within themselves of the truth, indeterminacy and falsity functions, respectively.

Proof.

(I)
$$\begin{aligned} 0_A < d(x,y) &\Leftrightarrow \langle 0,1,1\rangle < \langle T(x,y), I(x,y), F(x,y)\rangle \\ &\Leftrightarrow s_0 < s_d \Leftrightarrow -1 < \tfrac{1 + T(x,y) - 2I(x,y) - F(x,y)}{2} \\ &\Leftrightarrow 0 < T(x,y) - 2I(x,y) - F(x,y) + 3 \end{aligned}$$

(II)
$$\begin{aligned} d(x,y) = d(y,x) &\Leftrightarrow \langle T(x,y), I(x,y), F(x,y)\rangle = \langle T(y,x), I(y,x), F(y,x)\rangle \\ &\Leftrightarrow T(x,y) = T(y,x), I(x,y) = I(y,x), F(x,y) = F(y,x) \end{aligned}$$ □

Example 3. *Let A be non-empty SVNS and $x = \langle T_x, I_x, F_x\rangle, y = \langle T_y, I_y, F_y\rangle$ be two SVNNs. If we define the metric $d : X \times X \to A$, as:*

$$d(x,y) = \langle T(x,y), I(x,y), F(x,y)\rangle = \langle |T_x - T_y|, 1 - |I_x - I_y|, 1 - |F_x - F_y|\rangle$$

then

(I) $0 < |T_x - T_y| - 2(1 - |I_x - I_y|) - (1 - |F_x - F_y|) + 3$
$\Rightarrow 0 < |T_x - T_y| + 2|I_x - I_y| + |F_x - F_y|$
Then it satisfies the first condition.
(II) *Since the properties of the absolute value function, this condition is obvious.*
So, (X,d) is a neutrosophic-valued metric space.

3. Neutrosophic Valued *G*-Metric Spaces

Definition 10. *Let X and A be a non-empty SVNS. A function $G : X \times X \times X \to A$ is called neutrosophic valued G-metric if it satisfies the following properties:*

(1) $G(x,y,z) = 0_A$ *if and only if* $x = y = z$,
(2) $G(x,x,y) \neq 0_A$ *whenever* $x \neq y$,
(3) $G(x,x,y) \leq G(x,y,z)$ *for any* $x,y,z \in X$, *with* $z \neq y$,
(4) $G(x,y,z) = G(x,z,y) = \ldots$ *(symmetric for all elements).*

The pair (*X*, *G*) is called a neutrosophic valued *G*-metric space.

Theorem 4. *Let (X, G) be a neutrosophic valued G-metric space then, it satisfies followings:*

(1) $T(x,x,x) = 0, I(x,x,x) = F(x,x,x) = 1.$
(2) *Assume* $x \neq y$ *, then* $T(x,y,z) \neq 0, I(x,y,z) \neq 1, F(x,y,z) \neq 1.$
(3) $0 < T(x,y,z) - T(x,x,y) + 2(I(x,x,y) - I(x,y,z)) + F(x,x,y) - F(x,y,z)$
(4) $T(x,y,z), I(x,y,z)$ and $F(x,y,z)$ *are symmetric for all elements.*

where $T(.,.,.)$, $I(.,.,.)$ *and* $F(.,.,.)$ *are G-distance functions of truth, indeterminacy and falsity values of the element of the set, respectively.*

Proofs are made in a similar way to neutrosophic valued metric spaces.

Example 4. *Let X be non-empty SVNS and the G-distance function defined by:*

$$G(x,y,z) = \frac{1}{3}(d(x,y) \oplus d(x,z) \oplus d(y,z))$$

where $d(.,.)$ *is a neutrosophic valued metric. The pair (X, G) is obviously a neutrosophic valued G-metric space because of* $d(.,.)$*. Further, it has commutative properties.*

4. Relative Weighted Neutrosophic Valued Distances and Cohesion Measures

The relative distance measure is a method used for clustering of data sets, []. We define the relative weighted distance, which is a more sensitive method for big data sets.

Let $x_i = \langle T_{x_i}, F_{x_i}, I_{x_i} \rangle \in A(\text{non-empty SVNS}), i = 0 \ldots n$ be SVNNs. Then neutrosophic weighted average operator of these SVNNs is defined as:

$$M_a(A) = \sum_{i=1}^{n} \chi_i x_i = \left\langle 1 - \prod_{i=1}^{n} (1 - T_{x_i})^{\chi_i}, \prod_{i=1}^{n} (I_{x_i})^{\chi_i}, \prod_{i=1}^{n} (F_{x_i})^{\chi_i} \right\rangle$$

where χ_i is weighted for the i th data. For a given a neutrosophic data set $W = \{w_1, w_2, w_3, \ldots, w_n\}$ and a neutrosophic valued metric d, we define a relative neutrosophic valued distance for choosing another reference neutrosophic data and compute the relative neutrosophic valued distance as the average of the difference of distances for all the neutrosophic data $w_i \in W$.

Definition 11. *The relative neutrosophic valued distance from a neutrosophic data* w_i *to another neutrosophic data* w_j *is defined as follows:*

$$RD(w_i \| w_j) = \frac{1}{n} \sum_{w_k \in W} \left(d(w_i, w_j) \ominus d(w_i, w_k) \right)$$

Here, since T, I, F values of SVNNs cannot be negative, we can define the expression $d(w_i, w_j) \ominus d(w_i, w_k)$ *as the distance between these two neutrosophic-valued metrics. Furthermore, the distance of metrics is again neutrosophic-valued here so, a related neutrosophic-valued distance can be defined as:*

$$\begin{aligned} d(w_i, w_j) \ominus d(w_i, w_k) &= \langle T(w_i, w_j), I(w_i, w_j), F(w_i, w_j) \rangle \ominus \langle T(w_i, w_k), I(w_i, w_k), F(w_i, w_k) \rangle \\ &= \left\langle 1 - \left| T(w_i, w_j) - (T(w_i, w_k) - 1)^2 \right|, 1 - \left| I(w_i, w_j) - I(w_i, w_k)^2 \right|, 1 - \left| F(w_i, w_j) - F(w_i, w_k)^2 \right| \right\rangle \end{aligned} \tag{3}$$

The difference operator $\ominus$ generally is not a neutrosophic-valued metric (or G-metric). We used some abbreviations for saving space.

$$
\begin{aligned}
RD(w_i \| w_j) &= \tfrac{1}{n} \sum_{w_k \in W} \left(d(w_i, w_j) \ominus d(w_i, w_k) \right) \\
&= d(w_i, w_j) \ominus \tfrac{1}{n} \sum_{w_k \in W} d(w_i, w_k) \\
&= \left\langle T(w_i, w_j), I(w_i, w_j), F(w_i, w_j) \right\rangle \ominus \tfrac{1}{n} \left(d(w_i, w_1) \oplus d(w_i, w_2) \oplus \ldots \oplus d(w_i, w_n) \right) \\
&= \left\langle T(w_i, w_j), I(w_i, w_j), F(w_i, w_j) \right\rangle \\
&\ominus \tfrac{1}{n} \left[\langle T(w_i, w_1), I(w_i, w_1), F(w_i, w_1) \rangle \oplus \ldots \oplus \langle T(w_i, w_1), I(w_i, w_1), F(w_i, w_1) \rangle \right] \\
&= \left\langle T(w_i, w_j), I(w_i, w_j), F(w_i, w_j) \right\rangle \\
&\ominus \tfrac{1}{n} \left[\left\langle \sum_{k \in W} T(w_i, w_k) - \prod_{k \in W} T(w_i, w_k), \prod_{k \in W} I(w_i, w_k), \prod_{k \in W} F(w_i, w_k) \right\rangle \right] \\
&= \left\langle T(w_i, w_j), I(w_i, w_j), F(w_i, w_j) \right\rangle \\
&\ominus \left\langle 1 - \left[1 - \sum_{k \in W} T(w_i, w_k) + \prod_{k \in W} T(w_i, w_k) \right]^{1/n}, \prod_{k \in W} I(w_i, w_k)^{1/n}, \prod_{k \in W} F(w_i, w_k)^{1/n} \right\rangle \\
&= \langle T_1, I_1, F_1 \rangle \ominus \langle T_2, I_2, F_2 \rangle \\
&= \left\langle 1 - \left| T_1 - (T_2 - 1)^2 \right|, 1 - \left| I_1 - I_2{}^2 \right|, 1 - \left| F_1 - F_2{}^2 \right| \right\rangle
\end{aligned}
$$

where T_1, I_1, F_1 and T_2, I_2, F_2 are the first, second, and third elements of SVNN in the previous equation, respectively.

Definition 12. *The relative weighted neutrosophic valued distance from a neutrosophic data w_i to another neutrosophic data w_j is defined as follows:*

$$
\begin{aligned}
RD_\chi(w_i \| w_j) &= \sum_{\substack{w_k \in W \\ i \neq j, j \neq k, i \neq k}} \chi_w \left(d(w_i, w_j) \ominus d(w_i, w_k) \right) \\
&= \chi_{ij} d(w_i, w_j) \ominus \sum_{\substack{w_k \in W \\ i \neq j, j \neq k, i \neq k}} \chi_{ik} d(w_i, w_k) \\
&= \chi_{ij} \left\langle T(w_i, w_j), I(w_i, w_j), F(w_i, w_j) \right\rangle \\
&\ominus \left(\chi_{i1} \langle T(w_i, w_1), I(w_i, w_1), F(w_i, w_1) \rangle \oplus \ldots \oplus \chi_{in} \langle T(w_i, w_n), I(w_i, w_n), F(w_i, w_n) \rangle \right) \\
&= \left\langle 1 - \left(1 - T(w_i, w_j)\right)^{\chi_{ij}}, I(w_i, w_j)^{\chi_{ij}}, F(w_i, w_j)^{\chi_{ij}} \right\rangle \\
&\ominus \left(\begin{array}{c} \langle 1 - (1 - T(w_i, w_1))^{\chi_{i1}}, I(w_i, w_1)^{\chi_{i1}}, F(w_i, w_1)^{\chi_{i1}} \rangle \oplus \ldots \\ \oplus \langle 1 - (1 - T(w_i, w_n))^{\chi_{in}}, I(w_i, w_n)^{\chi_{in}}, F(w_i, w_n)^{\chi_{in}} \rangle \end{array} \right) \\
&= \left\langle 1 - \left(1 - T(w_i, w_j)\right)^{\chi_{ij}}, I(w_i, w_j)^{\chi_{ij}}, F(w_i, w_j)^{\chi_{ij}} \right\rangle \\
&\ominus \left\langle \sum_{\substack{k=1 \\ k \neq i,j}}^{n} \widetilde{T}_{ik} - \prod_{\substack{k=1 \\ k \neq i,j}}^{n} \widetilde{T}_{ik}, \prod_{\substack{k=1 \\ k \neq i,j}}^{n} \widetilde{I}_{ik}, \prod_{\substack{k=1 \\ k \neq i,j}}^{n} \widetilde{F}_{ik} \right\rangle \\
&= \langle T_1, I_1, F_1 \rangle \ominus \langle T_2, I_2, F_2 \rangle \\
&= \left\langle 1 - \left| T_1 - (T_2 - 1)^2 \right|, 1 - \left| I_1 - I_2{}^2 \right|, 1 - \left| F_1 - F_2{}^2 \right| \right\rangle
\end{aligned}
$$

where $\widetilde{T}_{ik} = 1 - (1 - T(w_i, w_k))^{\chi_{ik}}$, $\widetilde{I}_{ik} = I(w_i, w_k)^{\chi_{ik}}$, $\widetilde{F}_{ik} = F(w_i, w_k)^{\chi_{ik}}$.

Definition 13. *The relative weighted neutrosophic valued distance (from a random neutrosophic data w_i) to a neutrosophic data w_j is defined as follows:*

$$\begin{aligned} RD_\chi(w_j) &= \sum_{w_i \in W} \chi_i RD_\chi(w_i \| w_j) \\ &= \sum_{w_i \in W} \chi_i \left[\sum_{w_k \in W} \chi_w \left(d(w_i, w_j) \ominus d(w_i, w_k) \right) \right] \\ &= \sum_{w_i \in W} \chi_i \left[\sum_{w_k \in W} \chi_w \left(\delta(d_{ij}, d_{ik}) \right) \right] \end{aligned}$$

Definition 14. *The relative weighted neutrosophic valued distance from a neutrosophic data set W_1 to another neutrosophic data set W_2 is defined as follows:*

$$RD_\chi(W_1 \| W_2) = \sum_{x \in W_1} \chi_x \sum_{y \in W_2} \chi_y RD_\chi(x \| y)$$

Definition 15. *(Weighted cohesion measure between two neutrosophic data) The difference of the relative weighted neutrosophic-valued distance to w_j and the relative weighted neutrosophic-valued distance from w_i to w_j, i.e.,*

$$\rho_\chi(w_i, w_j) = RD_\chi(w_j) \ominus RD_\chi(w_i \| w_j) \tag{4}$$

is called the weighted neutrosophic-valued cohesion measure between two neutrosophic data w_i and w_j. If $\rho_\chi(w_i, w_j) \geq 0_W$ (resp. $\rho_\chi(w_i, w_j) \leq 0_W$) then w_i and w_j are said to be cohesive (resp. incohesive). So, the relative weighted neutrosophic distance from w_i and w_j is not larger than the relative weighted neutrosophic distance (from a random neutrosophic data) to w_j.

Definition 16. *(Weighted cohesion measure between two neutrosophic data sets) Let w_i and w_j are elements of the neutrosophic data sets U and V, respectively. Then the measure*

$$\rho_\chi(U, V) = \sum_{w_i \in U} \chi_u \sum_{w_j \in V} \chi_v \rho_\chi(w_i, w_j) \tag{5}$$

is called weighted cohesion neutrosophic-valued measure of the neutrosophic data sets U and V.

Definition 17. *(Cluster) The non-empty neutrosophic data set W is called a cluster if it is cohesive, i.e., $\rho(W, W) \geq 0_W$.*

5. Clustering via Neutrosophic Valued *G*-Metric Spaces

In this section, we can cluster neutrosophic big data thank to defined weighted distance definitions in Section 4 and *G*-metric definition.

Definition 18. *The neutrosophic valued weighted G-distance from a neutrosophic data w to a neutrosophic big data set U is defined as follows:*

$$G(w, y, z) = \sum_{y \in U} \chi_u \sum_{z \in U} \chi_u (d(w, y) \oplus d(w, z) \ominus d(y, z)) \tag{6}$$

Algorithm (K-sets algorithm)

Input: A neutrosophic big data set $W = \{w_1, w_2, \ldots, w_n\}$, a neutrosophic distance measure $d(.,.)$, and the number of sets K.
Output: A partition of neutrosophic sets $\{U_1, U_2, \ldots, U_K\}$.

1. Initially, choose arbitrarily K disjoint nonempty sets $U_1, U_2, \ldots, U_K$ as a partition of W.
2. for i from 1 to n do
   begin
   Compute $G(x_i, y_k, z_k)$ for each set U_k.
   Find the set to which the point x_i is closest in terms of G-distance.
   Assign point x_i to that set.
   end
3. Repeat from 2 until there is no further change.

6. Application and Example

We will give an example of the definition of the data that could have this kind of data and fall into the frame to fit this definition. We can call a data set a big data set if it is difficult and/or voluminous to define, analyze and visualize a data set. We give a big neutrosophic data example in accordance with this definition and possible use of G-metric, but it is fictional since there is no real neutrosophic big data example yet. It is a candidate for a good example that one of the current topics, image processing for big data analysis. Imagine a camera on a circuit board that is able to distinguish colors, cluster all the tools it can capture in the image and record that data. The camera that can be used for any color (for example white color vehicle) assigns the following degrees:

(I) The vehicle is at a certain distance at which the color can be detected, and the truth value of the portion of the vehicle is determined.
(II) The rate at which the vehicle can be detected by the camera is assigned as the uncertainty value (the mixed color is the external factors such as the effect of daylight and the color is determined on a different scale).
(III) The rate of not seeing a large part of the vehicle or the rate of out of range of the color is assigned as the value of falsity.

Thus, data of the camera is clustering via G-metric. This result gives that the numbers according to the daily quantities and colors of vehicles passing by are determined. The data will change continuously as long as the road is open, and the camera records the data. There will be a neutrosophic data for each vehicle. So, a Big Neutrosophic Data Clustering will occur.

Here, the weight functions we have defined for the metric can be given 1 value for the main colors (red-yellow-blue). For other secondary or mixed colors, the color may be given a proportional value depending on which color is closer.

A Numerical Toy Example

Take 5 neutrosophic data with their weights are equal to 1 to make a numerical example:

$$W = \{w_1\langle 0.6, 0.6, 0.6\rangle, w_2\langle 0.8, 0.4, 0.5\rangle, w_3\langle 0.5, 0.8, 0.7\rangle, w_4\langle 0.9, 0.5, 0.6\rangle, w_5\langle 0.1, 0.2, 0.7\rangle\}$$

K = 3 disjoint sets can be chosen $U_1 = \{w_1, w_4, w_5\}, U_2 = \{w_2, w_3\}$.

Then

$$d(w_i, w_j) = \begin{bmatrix} \langle 0,1,1\rangle & \langle 0.2,0.8,0.9\rangle & \langle 0.1,0.8,0.9\rangle & \langle 0.3,0.9,1.0\rangle & \langle 0.5,0.6,0.9\rangle \\ \langle 0.2,0.8,0.9\rangle & \langle 0,1,1\rangle & \langle 0.3,0.6,0.8\rangle & \langle 0.1,0.9,0.9\rangle & \langle 0.7,0.8,0.8\rangle \\ \langle 0.1,0.8,0.9\rangle & \langle 0.3,0.6,0.8\rangle & \langle 0,1,1\rangle & \langle 0.4,0.7,0.9\rangle & \langle 0.4,0.4,1.0\rangle \\ \langle 0.3,0.9,1.0\rangle & \langle 0.1,0.9,0.9\rangle & \langle 0.4,0.7,0.9\rangle & \langle 0,1,1\rangle & \langle 0.2,0.8,0.9\rangle \\ \langle 0.5,0.6,0.9\rangle & \langle 0.7,0.8,0.8\rangle & \langle 0.4,0.4,1.0\rangle & \langle 0.2,0.8,0.9\rangle & \langle 0,1,1\rangle \end{bmatrix}$$

where we assume the $d(w_i, w_j)$ as in Example 3. So, we can compute the G-metrics of the data as in Equation (3):

$$\begin{aligned}
G(w_1, U_1) &= G(w_1, w_4, w_5) = \langle 0.99, 0.90, 0.91\rangle \\
G(w_1, U_2) &= G(w_1, w_2, w_3) = \langle 0.79, 0.72, 0.83\rangle \\
G(w_2, U_1) &= G(w_2, w_1, w_4) \oplus G(w_2, w_1, w_5) \oplus G(w_2, w_4, w_5) = \langle 0.9874, 0.6027, 0.6707\rangle \\
G(w_2, U_2) &= G(w_2, w_2, w_3) = \langle 0, 1, 1\rangle \\
G(w_3, U_1) &= G(w_3, w_1, w_4) \oplus G(w_3, w_1, w_5) \oplus G(w_3, w_4, w_5) = \langle 1, 0.4608, 0.6707\rangle \\
G(w_3, U_2) &= G(w_3, w_2, w_3) = \langle 0, 1, 1\rangle \\
G(w_4, U_1) &= G(w_4, w_1, w_5) = \langle 0.81, 0.64, 0.91\rangle \\
G(w_4, U_2) &= G(w_4, w_2, w_3) = \langle 0.97, 0.73, 0.83\rangle
\end{aligned}$$

So, according to the calculations above, w_4 belongs to set U_1 and the other data belong to U_2. Here, we have made the data belonging to the clusters according to the fact that the truth values of the G-metrics are mainly low. If the truth value of G-distance is low, then the data is closer to the set.

7. Conclusions

This paper has introduced many new notions and definitions for clustering neutrosophic big data and geometric similarity metric of the data. Neutrosophic data sets have density. For example, sets having indeterminacy density or neutrosophic density and these are adding the more data and complexity. So, neutrosophic data sets are complex big data sets. Separation and clustering of these sets are evaluated according to weighted distances. Neutrosophic data sets in the last part of the paper, K-sets algorithm has been given for neutrosophic big data sets. We hope that the results in this paper can be applied to other data types like interval neutrosophic big data sets and can be analyzed in other metric spaces such as neutrosophic complex valued G-metric spaces etc. and can help to solve problems in other study areas.

Author Contributions: Conceptualization, F.T. and S.T.; Methodology, F.T.; Validation, F.T., S.T. and F.S.; Investigation, F.T. and S.T.; Resources, F.T., S.T. and F.S.; Writing-Original Draft Preparation, F.T. and S.T.; Writing—Review&Editing, F.T., S.T. and F.S.; Supervision, S.T. and F.S.; Funding Acquisition, F.S.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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Generalized Neutrosophic Soft Expert Set for Multiple-Criteria Decision-Making

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Received: 12 September 2018; Accepted: 22 September 2018; Published: 26 September 2018

Abstract: Smarandache defined a neutrosophic set to handle problems involving incompleteness, indeterminacy, and awareness of inconsistency knowledge, and have further developed it neutrosophic soft expert sets. In this paper, this concept is further expanded to generalized neutrosophic soft expert set (GNSES). We then define its basic operations of complement, union, intersection, AND, OR, and study some related properties, with supporting proofs. Subsequently, we define a GNSES-aggregation operator to construct an algorithm for a GNSES decision-making method, which allows for a more efficient decision process. Finally, we apply the algorithm to a decision-making problem, to illustrate the effectiveness and practicality of the proposed concept. A comparative analysis with existing methods is done and the result affirms the flexibility and precision of our proposed method.

Keywords: aggregation operator; complement; intersection; membership; neutrosophic soft set

1. Introduction

For a proper description of objects in an uncertain and ambiguous environment, indeterminate and incomplete information has to be properly handled. Intuitionistic fuzzy sets were introduced by Atanassov [1], followed by Molodtsov on soft sets [2] and neutrosophy logic [3] and neutrosophic sets [4] were introduced by Smarandache. The term neutro-sophy means knowledge of neutral thought and this neutral represents the main distinction between fuzzy and intuitionistic fuzzy logic and a set. At present, work on the soft set theory is progressing rapidly. Various operations and applications of soft sets have been developed rapidly, including the possibility of fuzzy soft set [5], soft multiset theory [6], multiparameterized soft set [7], soft intuitionistic fuzzy sets [8], Q-fuzzy soft sets [9–11], multi Q-fuzzy sets [12–14], N-soft set [15], Hesitant N-soft set [16], and Fuzzy N-soft set [17], thereby, opening avenues to genetic applications [18,19]. Later, Maji [20] have introduced a more generalized concept—which is a combination of neutrosophic sets and soft sets—and have studied its properties. Alhazaymeh and Hassan [21,22] have studied the concept of vague soft set, which were later extended to vague soft expert set theory [23,24], bipolar fuzzy soft expert set [25], and multi Q-fuzzy soft expert set [26]. Şahin et al. [27] introduced neutrosophic soft expert sets, while Al-Quran and Hassan [28,29] extended it further to neutrosophic vague soft expert set. Neutrosophic set theory has also been applied to multiple attribute decision-making [30–32]. Fuzzy modelling has long been widely applied to physical problems, which include intuitionistic hesitant fuzzy [33], t-concept lattices [34], fuzzy operators [35], medical image retrieval [36], and artificial bee colony [37] and multi criteria decision making [38,39]. Neutrosophic sets have also gained traction with recent publications on neutrosophic triplets [40,41], Q-neutrosophic soft relations [42], Q-neutrosophic soft sets [43], and Q-neutrosophic soft expert set [44].

This paper anticipates the neutrosophic set discussions to handle problems involving incompleteness, indeterminacy, and awareness of inconsistency of knowledge, which is further developed to neutrosophic soft expert sets. We intend to extend the discussion further, by proposing the concept of generalized neutrosophic soft expert set (GNSES) and its basic operations of complement, union, intersection, AND, and OR, along with a definition of GNSES-aggregation operator, to construct an algorithm of a GNSES decision method. Finally we provide an application of the constructed algorithm to solve a decision-making problem.

2. Preliminaries

In this section, we review the basic definitions of a neutrosophic set, neutrosophic soft set, soft expert sets, neutrosophic soft expert sets, and neutrosophic parametrized (NP)-aggregation operator, which are required as preliminaries.

Definition 1. *[4] Let U be a universe of discourse, with a generic element in U denoted by u, then a neutrosophic (NS) set A is an object having the form*

$$A = \{ < u : \ T_A(u),\ I_A(u)\ , F_A(u) >,\ u \in U\}$$

where the functions T, I, F: $U \to\ ]^-0, 1^+[$ define, respectively, the degree of membership (or Truth), the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element $u \in U$ to the set A with the condition.

$$^-0\ \leq\ T_A(u) + I_A(u) + F_A(u)\ \leq\ 3^+$$

Definition 2. *[20] Let U be an initial universe set and E be a set of parameters. Consider $A \subseteq E$. Let $NS(U)$ denote the set of all neutrosophic sets of U. The collection (F, A) is termed to be the neutrosophic soft set over U, where F is a mapping given by $F : A \to NS(U)$.*

Definition 3. *[23] U is an initial universe, E is a set of parameters, X is a set of experts (agents), and $O = \{agree = 1, disagree = 0\}$ a set of opinions. Let $Z = E \times X \times O$ and $A \subseteq Z$. A pair (F, A) is called a soft expert set over U, where F is a mapping given by $F : A \to P(U)$ where $P(U)$ denoted the power set of U.*

Definition 4. *[27] A pair (F, A) is called a neutrosophic soft expert set over U, where F is a mapping given by $F : A \to P(U)$ where $P(U)$ denotes the power neutrosophic set of U.*

Definition 5. *[27] The complement of a neutrosophic soft expert set (F, A) is denoted by $(F, A)^c$, and is defined as $(F, A)^c = (F^c, \neg A)$ where $F^c = \neg A \to P(U)$ is a mapping given by $F^c(x)$ = neutrosophic soft expert complement with $T_{F^c(x)} = F_{F(x)}$, $I_{F^c(x)} = I_{F(x)}$, $F_{F^c(x)} = T_{F(x)}$.*

Definition 6. *[27] The agree-neutrosophic soft expert set $(F, A)_1$ over U is a neutrosophic soft expert subset of* (F, A) defined as

$$(F, A)_1 = \{F_1(m) : m \in E \times X \times \{1\}\}.$$

Definition 7. *[27] The disagree-neutrosophic soft expert set $(F, A)_0$ over U is a neutrosophic soft expert subset of (F, A)*, defined as

$$(F, A)_0 = \{F_0(m) : m \in E \times X \times \{0\}\}.$$

Definition 8. *[27] Let (H, A) and (G, B) be two neutrosophic soft expert sets (NSESs) over the common universe U. Then the union of (H, A) and (G, B) is denoted by "$(H, A)\ \widetilde{\cup}\ (G, B)$", and is defined by*

$(H, A) \tilde{\cup} (G, B) = (K, C)$, *where* $C = A \cup B$ *and the truth-membership, indeterminacy-membership, and falsity-membership of* (K, C) *are as follows:*

$$T_{K(e)}(m) = \begin{cases} T_{H(e)}(m), & if\ e \in A - B \\ T_{G(e)}(m), & if\ e \in B - A \\ \max\left(T_{H(e)}(m), T_{G(e)}(m)\right), & if\ e \in A \cap B \end{cases}$$
$$I_{K(e)}(m) = \begin{cases} I_{H(e)}(m), & if\ e \in A - B \\ I_{G(e)}(m), & if\ e \in B - A \\ \frac{I_{H(e)}(m) + I_{G(e)}(m)}{2}, & if\ e \in A \cap B \end{cases}$$
$$F_{K(e)}(m) = \begin{cases} F_{H(e)}(m), & if\ e \in A - B \\ F_{G(e)}(m), & if\ e \in B - A \\ \min\left(F_{H(e)}(m), F_{G(e)}(m)\right), & if\ e \in A \cap B \end{cases}$$

Definition 9. *[27] Let* (H, A) *and* (G, B) *be two NSESs over the common universe* U*. Then the intersection of* (H, A) *and* (G, B) *is denoted by "*$(H, A) \tilde{\cap} (G, B)$*" and is defined by* $(H, A) \tilde{\cap} (G, B) = (K, C)$*, where* $C = A \cap B$ *and the truth-membership, indeterminacy-membership, and falsity-membership of* (K, C) *are as follows:*

$$T_{K(e)}(m) = \min\left(T_{H(e)}(m), T_{G(e)}(m)\right)$$
$$I_{K(e)}(m) = \frac{I_{H(e)}(m) + I_{G(e)}(m)}{2}$$
$$F_{K(e)}(m) = \max\left(F_{H(e)}(m), F_{G(e)}(m)\right),\ if\ e \in A \cap B.$$

Definition 10. *[45] Let* $\Psi_K \in$ *NP-soft set. Then an NP-aggregation operator of* Ψ_K*, denoted by* Ψ_K^{agg}*, is defined by*

$$\Psi_K^{agg} = \left\{\left(\langle u, \mu_K^{agg}, \vartheta_K^{agg}, \omega_K^{agg}\rangle\right) : u \in U\right\}, \tag{1}$$

which is a neutrosophic set over U*,*

$$\mu_K^{agg}(u) = \frac{1}{|U|}\sum_{\substack{e \in E \\ u \in U}} \mu_K(u).\lambda f_{K(x)}(u),\ \mu_K^{agg} : U \to [0,1] \tag{2}$$

$$\vartheta_K^{agg}(u) = \frac{1}{|U|}\sum_{\substack{e \in E \\ u \in U}} \vartheta_K(u).\lambda f_{K(x)}(u),\ \vartheta_K^{agg} : U \to [0,1] \tag{3}$$

$$\omega_K^{agg} = \frac{1}{|U|}\sum_{\substack{e \in E \\ u \in U}} \omega_K(u).\ \lambda f_{K(x)}(u),\ \omega_K^{agg} : U \to [0,1] \tag{4}$$

and where,

$$\lambda f_{K(x)}(u) = \begin{cases} 1, & x \in f_{K(x)}(u), \\ 0, & otherwise, \end{cases}$$

such that $|U|$ *is the cardinality of* U*.*

3. Generalized Neutrosophic Soft Expert Set

In this section, we introduce the concept of generalized neutrosophic soft expert set (GNSES) and define some of its properties. Throughout this paper, U is an initial universe, E is a set of parameters, X is a set of experts (agents), and $O = \{\text{agree} = 1, \text{disagree} = 0\}$ a set of opinions. Let $Z = E \times X \times O$ and $A \subseteq Z$ and u is a fuzzy set of A; that is, $u : A \to I = [0, 1]$.

Definition 11. *A pair (F^u, A) is called a generalized neutrosophic soft expert set (GNSES) over U, where F^u is a mapping given by*

$$F^u : A \to \mathcal{N}(U) \times I,$$

with $\mathcal{N}(U)$ being the set of all neutrosophic soft expert subsets of U. For any parameter $e \in A$, $F(e)$ is referred as the neutrosophic value set of parameter e, i.e.,

$$F(e) = \left\{ \langle u/T_{F(e)}(u), I_{F(e)}(u), F_{F(e)}(u) \rangle \right\},$$

where $T, I, F : U \to]^-0, 1^+[$ are the membership function of truth, indeterminacy, and falsity, respectively, of the element $u \in U$. For any $u \in U$ and $e \in A$

$$^-0 \ \leq T_{F(e)}(u) + I_{F(e)}(u) + F_{F(e)}(u) \leq 3^+$$

In fact, F^u is a parameterized family of neutrosophic soft expert sets on U, which has the degree of possibility of the approximate value set which is prepresented by $u(e)$ for each parameter e, which can be written as follows:

$$F^u(e) = \left\{ \left(\tfrac{u_1}{F(e)(u_1)}, \tfrac{u_2}{F(e)(u_2)}, \tfrac{u_3}{F(e)(u_3)}, \cdots, \tfrac{u_n}{F(e)(u_n)} \right), u(e) \right\}.$$

Example 1. *Suppose that $U = \{u_1, u_2, u_3\}$ is a set of computers and $E = \{e_1, e_2, e_3\}$ is a set of decision parameters. Let $X = \{p, q, r\}$ be set of experts. Suppose that*

$$\begin{aligned}
F^u(e_1, p, 1) &= \left\{ \left(\tfrac{u_1}{0.4,0.3,0.2}, \tfrac{u_2}{0.6,0.1,0.8}, \tfrac{u_3}{0.5,0.7,0.2} \right), 0.3 \right\} \\
F^u(e_1, q, 1) &= \left\{ \left(\tfrac{u_1}{0.3,0.2,0.5}, \tfrac{u_2}{0.5,0.6,0.2}, \tfrac{u_3}{0.8,0.1,0.4} \right), 0.4 \right\} \\
F^u(e_1, r, 1) &= \left\{ \left(\tfrac{u_1}{0.8,0.4,0.3}, \tfrac{u_2}{0.7,0.3,0.5}, \tfrac{u_3}{0.2,0.6,0.5}, \right), 0.8 \right\} \\
F^u(e_2, p, 1) &= \left\{ \left(\tfrac{u_1}{0.7,0.3,0.6}, \tfrac{u_2}{0.5,0.1,0.4}, \tfrac{u_3}{0.8,0.6,0.3} \right), 0.2 \right\} \\
F^u(e_2, q, 1) &= \left\{ \left(\tfrac{u_1}{0.6,0.7,0.1}, \tfrac{u_2}{0.8,0.4,0.7}, \tfrac{u_3}{0.5,0.1,0.7} \right), 0.6 \right\} \\
F^u(e_2, r, 1) &= \left\{ \left(\tfrac{u_1}{0.5,0.1,0.8}, \tfrac{u_2}{0.9,0.3,0.6}, \tfrac{u_3}{0.4,0.1,0.7} \right), 0.5 \right\} \\
F^u(e_3, p, 1) &= \left\{ \left(\tfrac{u_1}{0.6,0.3,0.2}, \tfrac{u_2}{0.5,0.6,0.7}, \tfrac{u_3}{0.8,0.1,0.4} \right), 0.7 \right\} \\
F^u(e_3, q, 1) &= \left\{ \left(\tfrac{u_1}{0.7,0.3,0.4}, \tfrac{u_2}{0.6,0.2,0.5}, \tfrac{u_3}{0.7,0.4,0.6} \right), 0.4 \right\} \\
F^u(e_3, r, 1) &= \left\{ \left(\tfrac{u_1}{0.8,0.4,0.3}, \tfrac{u_2}{0.5,0.3,0.6}, \tfrac{u_3}{0.1,0.4,0.2} \right), 0.5 \right\} \\
F^u(e_1, p, 0) &= \left\{ \left(\tfrac{u_1}{0.4,0.1,0.2}, \tfrac{u_2}{0.7,0.3,0.5}, \tfrac{u_3}{0.4,0.1,0.6} \right), 0.1 \right\} \\
F^u(e_1, q, 0) &= \left\{ \left(\tfrac{u_1}{0.7,0.3,0.5}, \tfrac{u_2}{0.6,0.2,0.4}, \tfrac{u_3}{0.4,0.5,0.1} \right), 0.3 \right\} \\
F^u(e_1, r, 0) &= \left\{ \left(\tfrac{u_1}{0.6,0.4,0.3}, \tfrac{u_2}{0.7,0.2,0.6}, \tfrac{u_3}{0.4,0.1,0.3} \right), 0.2 \right\} \\
F^u(e_2, p, 0) &= \left\{ \left(\tfrac{u_1}{0.5,0.1,0.7}, \tfrac{u_2}{0.4,0.5,0.1}, \tfrac{u_3}{0.7,0.1,0.4} \right), 0.2 \right\} \\
F^u(e_2, q, 0) &= \left\{ \left(\tfrac{u_1}{0.4,0.3,0.6}, \tfrac{u_2}{0.7,0.2,0.5}, \tfrac{u_3}{0.8,0.1,0.4} \right), 0.6 \right\} \\
F^u(e_2, r, 0) &= \left\{ \left(\tfrac{u_1}{0.3,0.2,0.6}, \tfrac{u_2}{0.4,0.3,0.5}, \tfrac{u_3}{0.5,0.1,0.4} \right), 0.4 \right\} \\
F^u(e_3, p, 0) &= \left\{ \left(\tfrac{u_1}{0.4,0.3,0.6}, \tfrac{u_2}{0.5,0.1,0.6}, \tfrac{u_3}{0.6,0.2,0.5} \right), 0.5 \right\} \\
F^u(e_3, q, 0) &= \left\{ \left(\tfrac{u_1}{0.6,0.2,0.7}, \tfrac{u_2}{0.8,0.1,0.4}, \tfrac{u_3}{0.5,0.3,0.4} \right), 0.7 \right\} \\
F^u(e_3, r, 0) &= \left\{ \left(\tfrac{u_1}{0.5,0.4,0.6}, \tfrac{u_2}{0.6,0.4,0.3}, \tfrac{u_3}{0.7,0.2,0.1} \right), 0.2 \right\}
\end{aligned}$$

The generalized neutrosophic soft expert set (GNSES) is a parameterized family $\{F(e_i),\ i = 1, 2, \ldots\}$ of all neutrosophic sets of U and describes a collection of approximation of an object.

Definition 12. *Let (F^u, A) and (G^η, B) be two generalized neutrosophic soft expert sets (GNSESs) over U. Then (F^u, A) is said to be a generalized neutrosophic soft expert subset of (G^η, B) if* *i.* $B \subseteq A$, *and*

ii. $G^{\eta}(\varepsilon)$ *is a generalized neutrosophic soft expert subset* $F^{u}(\varepsilon)$, *for all* $\varepsilon \in B$,

Example 2. *Consider Example 1. Suppose that A and B are as follows.*

$$A = \{(e_1, p, 1), (e_2, p, 1), (e_2, q, 0), (e_3, r, 1)\} B = \{(e_1, p, 1), (e_2, p, 1), (e_3, r, 1)\}.$$

Since B is a neutrosophic soft expert subset of A, clearly $B \subset A$. *Let* (G^{η}, B) *and* (F^{u}, A) *be defined as follows:*

$$\begin{aligned}
(F^u, A) = \; & \Big\{ \Big[(e_1, p, 1), \Big(\tfrac{u_1}{0.4,0.3,0.2}, \tfrac{u_2}{0.6,0.1,0.8}, \tfrac{u_3}{0.5,0.7,0.2}\Big), 0.3\Big] \\
& \Big[(e_2, p, 1), \Big(\tfrac{u_1}{0.7,0.3,0.6}, \tfrac{u_2}{0.5,0.1,0.4}, \tfrac{u_3}{0.8,0.6,0.3}\Big), 0.2\Big], \\
& \Big[(e_2, q, 0), \Big(\tfrac{u_1}{0.4,0.3,0.6}, \tfrac{u_2}{0.7,0.2,0.5}, \tfrac{u_3}{0.8,0.1,0.4}\Big), 0.6\Big], \\
& \Big[(e_3, r, 1), \Big(\tfrac{u_1}{0.8,0.4,0.3}, \tfrac{u_2}{0.5,0.3,0.6}, \tfrac{u_3}{0.1,0.4,0.2}\Big), 0.5\Big]\Big\}. \\
(G^{\eta}, B) = \; & \Big\{ \Big[(e_1, p, 1), \Big(\tfrac{u_1}{0.4,0.3,0.2}, \tfrac{u_2}{0.6,0.1,0.8}, \tfrac{u_3}{0.5,0.7,0.2}\Big), 0.3\Big], \\
& \Big[(e_2, p, 1), \Big(\tfrac{u_1}{0.7,0.3,0.6}, \tfrac{u_2}{0.5,0.1,0.4}, \tfrac{u_3}{0.8,0.6,0.3}\Big), 0.2\Big], \\
& \Big[(e_3, r, 1), \Big(\tfrac{u_1}{0.8,0.4,0.3}, \tfrac{u_2}{0.5,0.3,0.6}, \tfrac{u_3}{0.1,0.4,0.2}\Big), 0.5\Big]\Big\}.
\end{aligned}$$

Therefore $(G^{\eta}, B) \subseteq (F^{u}, A)$.

Definition 13. *Two GNSESs* (F^{u}, A) *and* (G^{η}, B) *over U are said to be equal if* (F^{u}, A) *is a GNSES subset of* (G^{η}, B) *and* (G^{η}, B) *is a GNSES subset of* (F^{u}, A).

Definition 14. *An agree-GNSESs* $(F^{u}, A)_1$ *over U is a GNSES subset of* (F^{u}, A) *defined as follows.*

$$(F^u, A)_1 = \{F_1(\propto) : \propto \in E \times X \times \{1\}\}.$$

Example 3. *Consider Example 1. The agree-GNSES* $(F^{u}, Z)_1$ *over U is*

$$\begin{aligned}
(F^u, Z)_1 = \; & \Big\{ \Big[(e_1, p, 1), \Big(\tfrac{u_1}{0.4,0.3,0.2}, \tfrac{u_2}{0.6,0.1,0.8}, \tfrac{u_3}{0.5,0.7,0.2},\Big), 0.3\Big], \\
& \Big[(e_1, q, 1), \Big(\tfrac{u_1}{0.3,0.2,0.5}, \tfrac{u_2}{0.5,0.6,0.2}, \tfrac{u_3}{0.8,0.1,0.4},\Big), 0.4\Big], \\
& \Big[(e_1, r, 1), \Big(\tfrac{u_1}{0.8,0.4,0.3}, \tfrac{u_2}{0.7,0.3,0.5}, \tfrac{u_3}{0.2,0.6,0.5},\Big), 0.8\Big], \\
& \Big[(e_2, p, 1), \Big(\tfrac{u_1}{0.7,0.3,0.6}, \tfrac{u_2}{0.5,0.1,0.4}, \tfrac{u_3}{0.8,0.6,0.3},\Big), 0.2\Big], \\
& \Big[(e_2, q, 1), \Big(\tfrac{u_1}{0.6,0.7,0.1}, \tfrac{u_2}{0.8,0.4,0.7}, \tfrac{u_3}{0.5,0.1,0.7},\Big), 0.6\Big], \\
& \Big[(e_2, r, 1), \Big(\tfrac{u_1}{0.5,0.1,0.8}, \tfrac{u_2}{0.9,0.3,0.6}, \tfrac{u_3}{0.4,0.1,0.7},\Big), 0.5\Big], \\
& \Big[(e_3, p, 1), \Big(\tfrac{u_1}{0.6,0.3,0.2}, \tfrac{u_2}{0.5,0.6,0.7}, \tfrac{u_3}{0.8,0.1,0.4},\Big), 0.7\Big], \\
& \Big[(e_3, q, 1), \Big(\tfrac{u_1}{0.7,0.3,0.4}, \tfrac{u_2}{0.6,0.2,0.5}, \tfrac{u_3}{0.7,0.4,0.6},\Big), 0.4\Big], \\
& \Big[(e_3, r, 1), \Big(\tfrac{u_1}{0.8,0.4,0.3}, \tfrac{u_2}{0.5,0.3,0.6}, \tfrac{u_3}{0.1,0.4,0.2},\Big), 0.5\Big]\Big\}.
\end{aligned}$$

Definition 15. *A disagree-GNSESs* $(F^{u}, A)_0$ *over U is a GNSES subset of* (F^{u}, A) *is defined as follows:*

$$(F^u, A)_0 = \{F_0(\propto) : \propto \in E \times X \times \{0\}\}.$$

Example 4. *Consider Example 1. The disagree-GNSES $(F^u, Z)_0$ over U is*

$$
\begin{aligned}
(F^u, Z)_0 = \Big\{ & \Big[(e_1, p, 0), \Big(\tfrac{u_1}{0.4,0.1,0.2}, \tfrac{u_2}{0.7,0.3,0.5}, \tfrac{u_3}{0.4,0.1,0.6},\Big), 0.1\Big], \\
& \Big[(e_1, q, 0), \Big(\tfrac{u_1}{0.7,0.3,0.5}, \tfrac{u_2}{0.6,0.2,0.4}, \tfrac{u_3}{0.4,0.5,0.1},\Big), 0.3\Big], \\
& \Big[(e_1, r, 0), \Big(\tfrac{u_1}{0.6,0.4,0.3}, \tfrac{u_2}{0.7,0.2,0.6}, \tfrac{u_3}{0.4,0.1,0.3},\Big), 0.2\Big], \\
& \Big[(e_2, p, 0), \Big(\tfrac{u_1}{0.5,0.1,0.7}, \tfrac{u_2}{0.4,0.5,0.1}, \tfrac{u_3}{0.7,0.1,0.4},\Big), 0.2\Big], \\
& \Big[(e_2, q, 0), \Big(\tfrac{u_1}{0.4,0.3,0.6}, \tfrac{u_2}{0.7,0.2,0.5}, \tfrac{u_3}{0.8,0.1,0.4},\Big), 0.6\Big], \\
& \Big[(e_2, r, 0), \Big(\tfrac{u_1}{0.3,0.2,0.6}, \tfrac{u_2}{0.4,0.3,0.5}, \tfrac{u_3}{0.5,0.1,0.4},\Big), 0.4\Big], \\
& \Big[(e_3, p, 0), \Big(\tfrac{u_1}{0.4,0.3,0.6}, \tfrac{u_2}{0.5,0.1,0.6}, \tfrac{u_3}{0.6,0.2,0.5},\Big), 0.5\Big], \\
& \Big[(e_3, q, 0), \Big(\tfrac{u_1}{0.6,0.2,0.7}, \tfrac{u_2}{0.8,0.1,0.4}, \tfrac{u_3}{0.5,0.3,0.4},\Big), 0.7\Big], \\
& \Big[(e_3, r, 0), \Big(\tfrac{u_1}{0.5,0.4,0.6}, \tfrac{u_2}{0.6,0.4,0.3}, \tfrac{u_3}{0.7,0.2,0.1},\Big), 0.2\Big]\Big\}.
\end{aligned}
$$

Definition 16. *The complement of a GNSES (F^u, A), denoted by $(F^u, A)^c$, is defined as $(F^u, A)^c = (F^{u(c)}, \neg A)$ where $F^{u(c)} : \neg A \to \mathcal{N}(U) \times I$ is a mapping given by*

$$
F^{u(c)}(\propto) = \left\{ \begin{array}{c} T_{F(\propto)^{(c)}} = F_{F(\propto)}, \\ I_{F(\propto)^{(c)}} = \overline{1} - I_{F(\propto)}, \\ F_{F(\propto)^{(c)}} = T_{F(\propto)}, \\ u^c(\propto) = \overline{1} - u(\propto) \end{array} \right\} \text{for each } \propto \in E.
$$

Example 5. *Consider Example 1. By using the definition of GNSES complement, the complement of F^u denoted by $F^{u(c)}$, is as follows:*

$$
\begin{aligned}
(F^{u(c)}, Z) = \Big\{ & \Big[(\neg e_1, p, 1), \Big(\tfrac{u_1}{0.2,0.7,0.4}, \tfrac{u_2}{0.8,0.9,0.6}, \tfrac{u_3}{0.2,0.3,0.5},\Big), 0.7\Big], \\
& \Big[(\neg e_1, q, 1), \Big(\tfrac{u_1}{0.5,0.8,0.3}, \tfrac{u_2}{0.2,0.4,0.5}, \tfrac{u_3}{0.4,0.9,0.8},\Big), 0.6\Big], \\
& \Big[(\neg e_1, r, 1), \Big(\tfrac{u_1}{0.3,0.6,0.8}, \tfrac{u_2}{0.5,0.7,0.7}, \tfrac{u_3}{0.5,0.4,0.2},\Big), 0.2\Big], \\
& \Big[(\neg e_2, p, 1), \Big(\tfrac{u_1}{0.6,0.7,0.7}, \tfrac{u_2}{0.4,0.9,0.5}, \tfrac{u_3}{0.3,0.4,0.8},\Big), 0.8\Big], \\
& \Big[(\neg e_2, q, 1), \Big(\tfrac{u_1}{0.1,0.3,0.6}, \tfrac{u_2}{0.7,0.6,0.8}, \tfrac{u_3}{0.7,0.9,0.5},\Big), 0.4\Big], \\
& \Big[(\neg e_2, r, 1), \Big(\tfrac{u_1}{0.8,0.9,0.5}, \tfrac{u_2}{0.6,0.7,0.9}, \tfrac{u_3}{0.7,0.9,0.4},\Big), 0.5\Big], \\
& \Big[(\neg e_3, p, 1), \Big(\tfrac{u_1}{0.2,0.7,0.6}, \tfrac{u_2}{0.7,0.4,0.5}, \tfrac{u_3}{0.4,0.9,0.8},\Big), 0.3\Big], \\
& \Big[(\neg e_3, q, 1), \Big(\tfrac{u_1}{0.4,0.7,0.7}, \tfrac{u_2}{0.5,0.8,0.6}, \tfrac{u_3}{0.6,0.6,0.7},\Big), 0.6\Big], \\
& \Big[(\neg e_3, r, 1), \Big(\tfrac{u_1}{0.3,0.6,0.8}, \tfrac{u_2}{0.6,0.7,0.5}, \tfrac{u_3}{0.2,0.6,0.1},\Big), 0.5\Big], \\
& \Big[(\neg e_1, p, 0), \Big(\tfrac{u_1}{0.2,0.9,0.4}, \tfrac{u_2}{0.5,0.7,0.7}, \tfrac{u_3}{0.6,0.9,0.4},\Big), 0.9\Big], \\
& \Big[(\neg e_1, q, 0), \Big(\tfrac{u_1}{0.5,0.7,0.7}, \tfrac{u_2}{0.4,0.8,0.6}, \tfrac{u_3}{0.1,0.5,0.4},\Big), 0.7\Big], \\
& \Big[(\neg e_1, r, 0), \Big(\tfrac{u_1}{0.3,0.6,0.6}, \tfrac{u_2}{0.6,0.8,0.7}, \tfrac{u_3}{0.3,0.9,0.4},\Big), 0.8\Big], \\
& \Big[(\neg e_2, p, 0), \Big(\tfrac{u_1}{0.7,0.9,0.5}, \tfrac{u_2}{0.1,0.5,0.4}, \tfrac{u_3}{0.4,0.9,0.7},\Big), 0.8\Big], \\
& \Big[(\neg e_2, q, 0), \Big(\tfrac{u_1}{0.6,0.7,0.4}, \tfrac{u_2}{0.5,0.8,0.7}, \tfrac{u_3}{0.4,0.9,0.8},\Big), 0.4\Big], \\
& \Big[(\neg e_2, r, 0), \Big(\tfrac{u_1}{0.6,0.8,0.3}, \tfrac{u_2}{0.5,0.7,0.4}, \tfrac{u_3}{0.4,0.9,0.5},\Big), 0.6\Big], \\
& \Big[(\neg e_3, p, 0), \Big(\tfrac{u_1}{0.6,0.7,0.4}, \tfrac{u_2}{0.6,0.9,0.5}, \tfrac{u_3}{0.5,0.8,0.6},\Big), 0.5\Big], \\
& \Big[(\neg e_3, q, 0), \Big(\tfrac{u_1}{0.7,0.8,0.6}, \tfrac{u_2}{0.4,0.9,0.8}, \tfrac{u_3}{0.4,0.7,0.5},\Big), 0.3\Big], \\
& \Big[(\neg e_3, r, 0), \Big(\tfrac{u_1}{0.6,0.6,0.5}, \tfrac{u_2}{0.3,0.6,0.6}, \tfrac{u_3}{0.1,0.8,0.7},\Big), 0.8\Big]\Big\}.
\end{aligned}
$$

Proposition 1. *If (F^u, A) is a generalized neutrosophic soft expert set over U, then*

1. $((F^u, A)^c)^c = (F^u, A)$
2. $((F^u, A)_1)^c = (F^u, A)_0$
3. $((F^u, A)_0)^c = (F^u, A)_1$

Proof. (1) From Definition 16, we have $(F^u, A)^c = (F^{u(c)}, \neg A)$,

where $F^{u(c)}(\propto) = T_{F(\propto)^{(c)}} = F_{F(\propto)}, \mathrm{I}_{F(\propto)^{(c)}} = \bar{1} - \mathrm{I}_{F(\propto)}, F_{F(\propto)^{(c)}} = T_{F(\propto)}$ and $u^c(\propto) = \bar{1} - u(\propto)$ for each $\propto \in E$.

Now $((F^u, A)^c)^c = ((F^{u(c)})^c, A)$ where

$$\begin{aligned}(F^{u(c)})^c(\propto) &= \begin{bmatrix} T_{F(\propto)^{(c)}} = F_{F(\propto)}, \mathrm{I}_{F(\propto)^{(c)}} = \bar{1} - \mathrm{I}_{F(\propto)}, \\ F_{F(\propto)^{(c)}} = T_{F(\propto)}, u^c(\propto) = \bar{1} - u(\propto) \end{bmatrix}^c \\ &= \begin{bmatrix} T_{F(\propto)} = F_{F(\propto)^{(c)}}, \mathrm{I}_{F(\propto)} = \bar{1} - \mathrm{I}_{F(\propto)^{(c)}}, \\ F_{F(\propto)} = T_{F(\propto)^{(c)}}, u(\propto) = \bar{1} - u^c(\propto) \end{bmatrix} \\ &= \bar{1} - \left(\bar{1} - \mathrm{I}_{F(\propto)}\right) = \bar{1} - (\bar{1} - u(\propto)) = \mathrm{I}_{F(\propto)} \\ &= u(\propto).\end{aligned}$$

Thus $((F^u, A)^c)^c = ((F^{u(c)})^c, A) = (F^u, A)$, for all $\propto \in E$.

The proofs of assertions (2) and (3) are obvious. □

Definition 17. *The union of two GNSESs (F^u, A) and (G^η, B) over U, denoted by $(F^u, A) \tilde{\cup} (G^\eta, B)$, is the GNSESs (H^Ω, C), where $C = A \cup B$ and the truth-membership, indeterminacy-membership, and falsity-membership of (H^Ω, C) are as follows:*

$$T_{H^\Omega(e)} = \begin{cases} T_{F^u(e)}(m) & if\ e \in A - B \\ T_{G^\eta(e)}(m) & if\ e \in B - A \\ max\left(T_{F^u(e)}(m), T_{G^\eta(e)}(m)\right) & if\ e \in A \cap B \end{cases}$$

$$I_{H^\Omega(e)} = \begin{cases} I_{F^u(e)}(m) & if\ e \in A - B \\ I_{G^\eta(e)}(m) & if\ e \in B - A \\ min\left(I_{F^u(e)}(m),_{G^\eta(e)}(m)\right) & if\ e \in A \cap B \end{cases}$$

$$F_{H^\Omega(e)} = \begin{cases} F_{F^u(e)}(m) & if\ e \in A - B \\ F_{G^\eta(e)}(m) & if\ e \in B - A \\ min\left(F_{F^u(e)}(m), F_{G^\eta(e)}(m)\right) & if\ e \in A \cap B \end{cases}$$

where $\Omega(m) = max\left(u_{(e)}(m), \eta_{(e)}(m)\right)$.

Example 6. *Suppose that (F^u, A) and (G^η, B) are two GNSESs over U, such that*

$$\begin{aligned}(F^u, A) = \ & \left\{\left[(e_1, p, 1), \left(\tfrac{u_1}{0.4, 0.3, 0.2}, \tfrac{u_2}{0.6, 0.1, 0.8}, \tfrac{u_3}{0.5, 0.7, 0.2}\right), 0.3\right]\right. \\ & \left[(e_2, q, 1), \left(\tfrac{u_1}{0.7, 0.3, 0.6}, \tfrac{u_2}{0.5, 0.1, 0.4}, \tfrac{u_3}{0.7, 0.6, 0.3}\right), 0.2\right], \\ & \left[(e_2, q, 0), \left(\tfrac{u_1}{0.4, 0.3, 0.6}, \tfrac{u_2}{0.7, 0.2, 0.5}, \tfrac{u_3}{0.8, 0.1, 0.4}\right), 0.6\right], \\ & \left.\left[(e_3, r, 1), \left(\tfrac{u_1}{0.8, 0.4, 0.3}, \tfrac{u_2}{0.5, 0.3, 0.6}, \tfrac{u_3}{0.1, 0.4, 0.2}\right), 0.5\right]\right\}. \\ (G^\eta, B) = \ & \left\{\left[(e_1, p, 1), \left(\tfrac{u_1}{0.6, 0.5, 0.1}, \tfrac{u_2}{0.8, 0.2, 0.3}, \tfrac{u_3}{0.9, 0.2, 0.3}\right), 0.1\right],\right. \\ & \left[(e_2, q, 1), \left(\tfrac{u_1}{0.6, 0.7, 0.1}, \tfrac{u_2}{0.8, 0.4, 0.7}, \tfrac{u_3}{0.5, 0.1, 0.7}\right), 0.4\right], \\ & \left.\left[(e_3, r, 1), \left(\tfrac{u_1}{0.4, 0.1, 0.2}, \tfrac{u_2}{0.5, 0.4, 0.2}, \tfrac{u_3}{0.3, 0.6, 0.4}\right), 0.8\right]\right\}.\end{aligned}$$

Then $(F^u, A)\,\widetilde{\cup}\,(G^\eta, B) = \left(H^\Omega, C\right)$ *where*

$$\left(H^\Omega, C\right) = \begin{Bmatrix} \left[(e_1, p, 1), \left(\frac{u_1}{0.6,0.3,0.1}, \frac{u_2}{0.8,0.1,0.3}, \frac{u_3}{0.9,0.2,0.2}\right), 0.3\right], \\ \left[(e_2, q, 1), \left(\frac{u_1}{0.6,0.3,0.1}, \frac{u_2}{0.8,0.2,0.5}, \frac{u_3}{0.7,0.1,0.4}\right), 0.4\right], \\ \left[(e_2, q, 0), \left(\frac{u_1}{0.4,0.3,0.6}, \frac{u_2}{0.7,0.2,0.5}, \frac{u_3}{0.8,0.1,0.4}\right), 0.6\right], \\ \left[(e_3, r, 1), \left(\frac{u_1}{0.8,0.1,0.2}, \frac{u_2}{0.5,0.3,0.2}, \frac{u_3}{0.3,0.4,0.2}\right), 0.8\right] \end{Bmatrix}.$$

Proposition 2. *If* (F^u, A), (G^η, B) *and* $\left(H^\Omega, C\right)$ *are three GNSESs over* U, *then*

1. $\left((F^u, A)\,\widetilde{\cup}\,(G^\eta, B)\right)\widetilde{\cup}\left(H^\Omega, C\right) = (F^u, A)\,\widetilde{\cup}\left((G^\eta, B)\,\widetilde{\cup}\left(H^\Omega, C\right)\right)$.
2. $(F^u, A)\,\widetilde{\cup}\,(F^u, A) \subseteq (F^u, A)$.

Proof. (1) We want to prove that

$$\left((F^u, A)\,\widetilde{\cup}\,(G^\eta, B)\right)\widetilde{\cup}\left(H^\Omega, C\right) = (F^u, A)\,\widetilde{\cup}\left((G^\eta, B)\,\widetilde{\cup}\left(H^\Omega, C\right)\right)$$

By using Definition 17, we consider the case when $e \in A \cap B$, as other cases are trivial. We will have

$$(F^u, A)\,\widetilde{\cup}\,(G^\eta, B) = \left\{\left(u \Big/ \begin{matrix} max\begin{pmatrix} T_{F^u(e)}(m), \\ T_{G^\eta(e)}(m) \end{pmatrix}, \\ min\begin{pmatrix} I_{F^u(e)}(m), \\ I_{G^\eta(e)}(m) \end{pmatrix}, \\ min\begin{pmatrix} F_{F^u(e)}(m), \\ F_{G^\eta(e)}(m) \end{pmatrix} \end{matrix}\right),\ max\begin{pmatrix} u_{(e)}(m), \\ \eta_{(e)}(m) \end{pmatrix}, u \in U\right\}.$$

Also consider the case when $e \in H$, as the other cases are trivial. We will have

$$\begin{aligned} \left((F^u, A)\,\widetilde{\cup}\,(G^\eta, B)\right)\widetilde{\cup}\left(H^\Omega, C\right) &= \left\{\left(u \Big/ \begin{matrix} max\left(T_{F^u(e)}(m), T_{G^\eta(e)}(m)\right), \\ min\left(I_{F^u(e)}(m), I_{G^\eta(e)}(m)\right), \\ min\left(F_{F^u(e)}(m), F_{G^\eta(e)}(m)\right) \end{matrix}\right),\ \begin{matrix} \left(u/T_{H^\Omega(e)}(m), I_{H^\Omega(e)}(m), F_{H^\Omega(e)}(m)\right), \\ max\left(u_{(e)}(m), \eta_{(e)}(m), \Omega(m)\right), u \in U \end{matrix}\right\} \\ &= \left\{\begin{matrix} \left(u/T_{F^\Omega(e)}(m), I_{F^\Omega(e)}(m), F_{F^\Omega(e)}(m)\right), \\ \left(u \Big/ \begin{matrix} max\left(T_{G^u(e)}(m), T_{H^\eta(e)}(m)\right), \\ min\left(I_{G^u(e)}(m), I_{H^\eta(e)}(m)\right), \\ min\left(F_{G^u(e)}(m), F_{H^\eta(e)}(m)\right) \end{matrix}\right) \end{matrix}\ max\left(u_{(e)}(m), \eta_{(e)}(m), \Omega(m)\right), u \in U\right\} \\ &= (F^u, A)\,\widetilde{\cup}\left((G^\eta, B)\,\widetilde{\cup}\left(H^\Omega, C\right)\right). \end{aligned}$$

(2) The proof is straightforward. □

Definition 18. *Let (F^u, A) and (G^η, B) be two GNSESs over a common universe U. Then the intersection of (F^u, A) and (G^η, B) is denoted by $(F^u, A) \tilde{\cap} (G^\eta, B) = \left(K^\delta, C\right)$, where $C = A \cap B$ and the truth-membership, indeterminacy-membership, and falsity-membership of $\left(K^\delta, C\right)$ are as follows:*

$$T_{K^\delta(e)} = \begin{cases} T_{F^u(e)}(m) & if\ e \in A - B \\ T_{G^\eta(e)}\ (m) & if\ e \in B - A \\ min\left(T_{F^u(e)}(m), T_{G^\eta(e)}(m)\right) & if\ e \in A \cap B \end{cases}$$
$$I_{K^\delta(e)} = \begin{cases} \mathrm{I}_{F^u(e)}(m) & if\ e \in A - B \\ \mathrm{I}_{G^\eta(e)}\ (m) & if\ e \in B - A \\ min\left(\mathrm{I}_{F^u(e)}(m), \mathrm{I}_{G^\eta(e)}(m)\right) & if\ e \in A \cap B \end{cases}$$
$$F_{K^\delta(e)} = \begin{cases} \mathrm{F}_{F^u(e)}(m) & if\ e \in A - B \\ \mathrm{F}_{G^\eta(e)}\ (m) & if\ e \in B - A \\ max\left(\mathrm{F}_{F^u(e)}(m), \mathrm{F}_{G^\eta(e)}(m)\right) & if\ e \in A \cap B \end{cases}$$

where $\delta(m) = min\left(u_{(e)}(m), \eta_{(e)}(m)\right)$.

Example 7. *Suppose that (F^u, A) and (G^η, B) are two GNSESs over U, such that*

$$\begin{aligned} (F^u, A) = & \left\{\left[(e_1, p, 1), \left(\frac{u_1}{0.4,0.3,0.2}, \frac{u_2}{0.6,0.1,0.8}, \frac{u_3}{0.5,0.7,0.2},\right), 0.3\right],\right. \\ & \left[(e_2, q, 1), \left(\frac{u_1}{0.7,0.3,0.6}, \frac{u_2}{0.5,0.1,0.4}, \frac{u_3}{0.7,0.6,0.3},\right), 0.2\right], \\ & \left.\left[(e_2, q, 0), \left(\frac{u_1}{0.4,0.3,0.6}, \frac{u_2}{0.7,0.2,0.5}, \frac{u_3}{0.8,0.1,0.4},\right), 0.6\right]\right\}. \\ (G^\eta, B) = & \left\{\left[(e_1, p, 1), \left(\frac{u_1}{0.6,0.5,0.1}, \frac{u_2}{0.8,0.2,0.3}, \frac{u_3}{0.9,0.2,0.3},\right), 0.1\right],\right. \\ & \left.\left[(e_3, r, 1), \left(\frac{u_1}{0.4,0.1,0.2}, \frac{u_2}{0.5,0.4,0.2}, \frac{u_3}{0.3,0.6,0.4},\right), 0.8\right]\right\}. \end{aligned}$$

Then $(F^u, A) \tilde{\cap} (G^\eta, B) = \left(K^\delta, C\right)$ where

$$\left(K^\delta, C\right) = \left\{\left[(e_1, p, 1), \left(\frac{u_1}{0.4, 0.3, 0.2}, \frac{u_2}{0.6, 0.1, 0.8}, \frac{u_3}{0.5, 0.2, 0.3},\right), 0.1\right]\right\}.$$

Proposition 3. *If (F^u, A), (G^η, B) and $\left(H^\Omega, C\right)$ are three GNSESs over U, then*

1. $\left((F^u, A) \tilde{\cap} (G^\eta, B)\right) \tilde{\cap} (K^\delta, C) = (F^u, A) \tilde{\cap} \left((G^\eta, B) \tilde{\cap} (K^\delta, C)\right)$
2. $(F^u, A) \tilde{\cap} (F^u, A) \subseteq (F^u, A)$.

Proof. (1) We want to prove that

$$\left((F^u, A) \tilde{\cap} (G^\eta, B)\right) \tilde{\cap} \left(K^\delta, C\right) = (F^u, A) \tilde{\cap} \left((G^\eta, B) \tilde{\cap} \left(K^\delta, C\right)\right)$$

By using Definition 18, consider the case when $e \in A \cap B$, since other cases are trivial. We have

$$(F^u, A) \tilde{\cap} (G^\eta, B) = \left\{\left(u \middle/ \begin{array}{c} min\left(T_{F^u(e)}(m), T_{G^\eta(e)}(m)\right), \\ min\left(\mathrm{I}_{F^u(e)}(m), \mathrm{I}_{G^\eta(e)}(m)\right), \\ max\left(\mathrm{F}_{F^u(e)}(m), \mathrm{F}_{G^\eta(e)}(m)\right) \end{array}\right), \ min\left(u_{(e)}(m), \eta_{(e)}(m)\right), u \in U\right\}.$$

Also consider the case when $e \in K$, as the other cases are trivial. Then we have

$$\begin{aligned}\left((F^u,A)\,\tilde{\cap}\,(G^\eta,B)\right)\tilde{\cap}\,(K^\delta,C) &= \left\{\left(u\Big/\begin{array}{c} min\left(T_{F^u(e)}(m),T_{G^\eta(e)}(m)\right),\\ min\left(I_{F^u(e)}(m),I_{G^\eta(e)}(m)\right),\\ max\left(F_{F^u(e)}(m),F_{G^\eta(e)}(m)\right)\end{array}\right),\quad \begin{array}{c}\left(u/T_{K^\delta(e)}(m),I_{K^\delta(e)}(m),F_{K^\delta(e)}(m)\right),\\ min\left(u_{(e)}(m),\eta_{(e)}(m),\delta(m)\right),u\in U\end{array}\right\}.\\ &= \left\{\begin{array}{l}\left(u/T_{F^\Omega(e)}(m),I_{F^\Omega(e)}(m),F_{F^\Omega(e)}(m)\right),\\ \left(u\Big/\begin{array}{c} min\left(T_{G^u(e)}(m),T_{K^\delta(e)}(m)\right),\\ min\left(I_{G^u(e)}(m),I_{K^\delta(e)}(m)\right),\\ max\left(F_{G^u(e)}(m),F_{K^\delta(e)}(m)\right)\end{array}\right)\end{array}\right.\quad min\left(u_{(e)}(m),\eta_{(e)}(m),\delta(m)\right),u\in U\Big\}\\ &= (F^u,A)\,\tilde{\cap}\left((G^\eta,B)\,\tilde{\cap}\,(K^\delta,C)\right).\end{aligned}$$

(2) The proof is straightforward. □

Proposition 4. *If* (F^u,A), (G^η,B) *and* (K^δ,C) *are three GNSESs over* U. *Then*

1. $\left((F^u,A)\,\tilde{\cup}\,(G^\eta,B)\right)\tilde{\cap}\,(K^\delta,C) = \left((F^u,A)\,\tilde{\cap}\,(K^\delta,C)\right)\tilde{\cup}\left((G^\eta,B)\,\tilde{\cap}\,(K^\delta,C)\right)$.
2. $\left((F^u,A)\,\tilde{\cap}\,(G^\eta,B)\right)\tilde{\cup}\,(K^\delta,C) = \left((F^u,A)\,\tilde{\cup}\,(K^\delta,C)\right)\tilde{\cap}\left((G^\eta,B)\,\tilde{\cup}\,(K^\delta,C)\right)$.

Proof. The proofs can be easily obtained from Definitions 17 and 18. □

Definition 19. *If* (F^u,A) *and* (G^η,B) *are two GNSESs over* U, *then "*(F^u,A) *AND* (G^η,B)*" denoted by* $(F^u,A)\wedge(G^\eta,B)$, *is defined by*

$$(F^u,A)\wedge(G^\eta,B) = \left(H^\Omega, A\times B\right)$$

such that, $H^\Omega(\alpha,\beta) = F^u(\alpha)\cap G^\eta(\beta)$ *and the truth-membership, indeterminacy-membership, and falsity-membership of* $\left(H^\Omega, A\times B\right)$ *are as follows.*

$$\begin{aligned}T_{H^\Omega(\alpha,\beta)}(m) &= min\left(T_{F^u(\alpha)}(m),T_{G^\eta(\beta)}(m)\right),\\ I_{H^\Omega(\alpha,\beta)}(m) &= min\left(I_{F^u(\alpha)}(m),I_{G^\eta(\beta)}(m)\right),\\ F_{H^\Omega(\alpha,\beta)}(m) &= max\left(F_{F^u(\alpha)}(m),F_{G^\eta(\beta)}(m)\right)\end{aligned}$$

and $\Omega(m) = min\left(u_{(e)}(m),\eta_{(e)}(m)\right)$, $\forall\alpha\in A,\forall\beta\in B$.

Example 8. *Suppose that* (F^u,A) *and* (G^η,B) *are two GNSESs over* U, such that

$$\begin{aligned}(F^u,A) = &\left\{\left[(e_1,p,1),\left(\frac{u_1}{0.2,0.3,0.5},\frac{u_2}{0.4,0.1,0.2},\frac{u_3}{0.6,0.3,0.7}\right),0.4\right],\right.\\ &\left.\left[(e_3,r,0),\left(\frac{u_1}{0.5,0.2,0.1},\frac{u_2}{0.6,0.3,0.7},\frac{u_3}{0.2,0.1,0.8}\right),0.3\right]\right\}\\ (G^\eta,B) = &\left\{\left[(e_1,p,1),\left(\frac{u_1}{0.3,0.2,0.6},\frac{u_2}{0.6,0.3,0.2},\frac{u_3}{0.8,0.1,0.2}\right),0.5\right],\right.\\ &\left.\left[(e_2,q,0),\left(\frac{u_1}{0.1,0.3,0.5},\frac{u_2}{0.7,0.1,0.6},\frac{u_3}{0.4,0.3,0.6}\right),0.6\right]\right\}.\end{aligned}$$

Then $(F^u, A) \wedge (G^\eta, B) = (H^\Omega, A \times B)$ *where*

$$\begin{aligned}(H^\Omega, A \times B) = \Big\{ & \Big[(e_1, p, 1), (e_1, p, 1)\Big(\tfrac{u_1}{0.2,0.2,0.6}, \tfrac{u_2}{0.4,0.1,0.2}, \tfrac{u_3}{0.6,0.1,0.7},\Big), 0.4\Big], \\ & \Big[(e_1, p, 1), (e_2, q, 0), \Big(\tfrac{u_1}{0.1,0.3,0.5}, \tfrac{u_2}{0.4,0.1,0.6}, \tfrac{u_3}{0.4,0.3,0.7},\Big), 0.4\Big], \\ & \Big[(e_3, r, 0), (e_1, p, 1), \Big(\tfrac{u_1}{0.3,0.2,0.6}, \tfrac{u_2}{0.6,0.3,0.7}, \tfrac{u_3}{0.2,0.1,0.8},\Big), 0.3\Big], \\ & \Big[(e_3, r, 0), (e_2, q, 0), \Big(\tfrac{u_1}{0.1,0.2,0.5}, \tfrac{u_2}{0.6,0.1,0.7}, \tfrac{u_3}{0.2,0.1,0.8},\Big), 0.3\Big]\Big\}.\end{aligned}$$

Definition 20. *If* (F^u, A) *and* (G^η, B) *are two GNSESs over* U*, then "*(F^u, A) *OR* (G^η, B)*" denoted by* $(F^u, A) \vee (G^\eta, B)$*, is defined by*

$$(F^u, A) \vee (G^\eta, B) = \left(K^\delta, A \times B\right)$$

such that $K^\delta(\alpha, \beta) = F^u(\alpha) \cup G^\eta(\beta)$ *and the truth-membership, indeterminacy-membership, and falsity-membership of* $(K^\delta, A \times B)$ *are as follows.*

$$\begin{aligned} T_{K^\delta(\alpha,\beta)}(m) &= max\left(T_{F^u(\alpha)}(m), T_{G^\eta(\beta)}(m)\right), \\ I_{K^\delta(\alpha,\beta)}(m) &= min\left(I_{F^u(\alpha)}(m), I_{G^\eta(\beta)}(m)\right), \\ F_{K^\delta(\alpha,\beta)}(m) &= min\left(F_{F^u(\alpha)}(m), F_{G^\eta(\beta)}(m)\right) \end{aligned}$$

and $\delta(m) = max\left(u_{(e)}(m), \eta_{(e)}(m)\right)$, $\forall \alpha \in A, \forall \beta \in B$.

Example 9. *Suppose that* (F^u, A) *and* (G^η, B) *are two GNSESs over* U*, such that*

$$\begin{aligned}(F^u, A) = & \Big\{\Big[(e_1, p, 1), \Big(\tfrac{u_1}{0.2,0.3,0.5}, \tfrac{u_2}{0.4,0.1,0.2}, \tfrac{u_3}{0.6,0.3,0.7},\Big), 0.4\Big], \\ & \Big[(e_3, r, 0), \Big(\tfrac{u_1}{0.5,0.2,0.1}, \tfrac{u_2}{0.6,0.3,0.7}, \tfrac{u_3}{0.2,0.1,0.8},\Big), 0.3\Big]\Big\} \\ (G^\eta, B) = & \Big\{\Big[(e_1, p, 1), \Big(\tfrac{u_1}{0.3,0.2,0.6}, \tfrac{u_2}{0.6,0.3,0.2}, \tfrac{u_3}{0.8,0.1,0.2},\Big), 0.5\Big], \\ & \Big[(e_2, q, 0), \Big(\tfrac{u_1}{0.1,0.3,0.5}, \tfrac{u_2}{0.7,0.1,0.6}, \tfrac{u_3}{0.4,0.3,0.6},\Big), 0.6\Big]\Big\}.\end{aligned}$$

Then $(F^u, A) \vee (G^\eta, B) = (K^\delta, A \times B)$ *where*

$$\begin{aligned}(K^\delta, A \times B) = \Big\{ & \Big[(e_1, p, 1), (e_1, p, 1)\Big(\tfrac{u_1}{0.3,0.2,0.5}, \tfrac{u_2}{0.6,0.1,0.2}, \tfrac{u_3}{0.8,0.1,0.2},\Big), 0.5\Big], \\ & \Big[(e_1, p, 1), (e_2, q, 0), \Big(\tfrac{u_1}{0.2,0.3,0.5}, \tfrac{u_2}{0.7,0.1,0.2}, \tfrac{u_3}{0.6,0.3,0.6},\Big), 0.6\Big], \\ & \Big[(e_3, r, 0), (e_1, p, 1), \Big(\tfrac{u_1}{0.5,0.2,0.1}, \tfrac{u_2}{0.7,0.3,0.6}, \tfrac{u_3}{0.8,0.1,0.2},\Big), 0.5\Big], \\ & \Big[(e_3, r, 0), (e_2, q, 0), \Big(\tfrac{u_1}{0.5,0.2,0.1}, \tfrac{u_2}{0.7,0.1,0.6}, \tfrac{u_3}{0.4,0.1,0.6},\Big), 0.6\Big]\Big\}.\end{aligned}$$

Proposition 5. *Let* (F^u, A) *and* (G^η, B) *be GNSESs over* U*. Then*

1. $((F^u, A) \wedge (G^\eta, B)\ )^c = (F^u, A)^c \vee (G^\eta, B)^c$
2. $((F^u, A) \vee (G^\eta, B))^c = (F^u, A)^c \wedge (G^\eta, B)^c$

Proof. The proofs can be easily obtained from Definitions 16, 19 and 20. □

4. GNSES-Aggregation Operator

In this section, we define a GNSES-aggregation operator of a GNSES to construct a decision method by which approximate functions of a soft expert set are combined to produce a neutrosophic set that can be used to evaluate each alternative.

Definition 21. *Let* $\Upsilon_A \in$ *GNSESs. Then a GNSES-aggregation operator of* Υ_A*, denoted by* Υ_A^{agg}*, is defined by*

$$\Upsilon_A^{agg} = \left\{\langle\left(u, T_A^{agg}(u), I_A^{agg}(u), F_A^{agg}(u)\right)\rangle : u \in U\right\}, \tag{5}$$

which is a GNSES over U*,*

$$T_A^{agg} : U \to [0,1], \quad T_A^{agg}(u) = \frac{1}{|U|}\sum_{\substack{e \in E \\ u \in U}} T_A(u).\mu, \tag{6}$$

$$F_A^{agg} : U \to [0,1],\ F_A^{agg}(u) = \frac{1}{|U|}\sum_{\substack{e \in E \\ u \in U}} F_A(u).\mu, \tag{7}$$

$$I_A^{agg} : U \to [0,1],\ I_A^{agg}(u) = \frac{1}{|U|}\sum_{\substack{e \in E \\ u \in U}} I_A(u).\ \mu, \tag{8}$$

where $|U|$ *is the cardinality of* U *and* μ *is defined below*

$$\mu = \frac{1}{n}.\sum\nolimits_{i=1}^{n} \mu(e_i). \qquad (e_i,\ i = 1,2,3,\ldots,n). \tag{9}$$

Definition 22. *Let* $\Upsilon_A \in$ *GNSESs,* Υ_A^{agg} *be the corresponding GNSES aggregation operator. Then a reduced fuzzy set of* Υ_A^{agg} *is a fuzzy set over* U*, denoted by*

$$\Upsilon_A^{agg} = \left\{\frac{\tau\Upsilon_A^{agg}(u)}{u} : u \in U\right\}, \tag{10}$$

where $\tau\Upsilon_A^{agg}(u) : U \to [0,1]$ *and* $u_i = \left|T_{A_i}^{agg} - F_{A_i}^{agg} - I_{A_i}^{agg}\right|$.

5. An Application of Generalized Neutrosophic Soft Expert Set

In this section, we present an application of generalized neutrosophic soft expert set theory in a decision-making problem. Based on Definitions 21 and 22, we constructed an algorithm for the GNSES decision-making method as follows.

Step 1—Choose a feasible subset of the set of parameters.
Step 2—Construct the GNSES tables for each opinion (agree, disagree) of experts.
Step 3—Compute the aggregation operator GNSES Υ_A^{agg} of Υ_A and the reduced fuzzy set $T_{A_i}^{agg}, F_{A_i}^{agg}, I_{A_i}^{agg}$ of Υ_A^{agg}.
Step 4—$\text{Score}(u_I) = \max agree(u_i) - \min disagree(u_i)$.
Step 5—Choose the element of u_i that has maximum score. This will be the optimal solution.

Example 10. *Suppose a company needs to employ a worker, which is to be decided by a few experts. The employee has to be chosen from five potential workers,* $U = \{u_1, u_2, u_3, u_4, u_5\}$*. Suppose there are four parameters* $E = \{e_1, e_2, e_3, e_4\}$ *where the parameters* e_i $(i = 1,2,3,4)$ *stand for "education," "age," "capability" and* *"experience", respectively. Let $X = \{p, q, r\}$ be a set of experts. After a serious discussion, the experts construct the following generalized neutrosophic soft expert set.*

Step 1—Choose a feasible subset of the set of parameters

$$
\begin{aligned}
(F^u, Z) = \Big\{ & \left[(e_1,p,1),\left(\tfrac{u_1}{0.2,0.3,0.4},\tfrac{u_2}{0.8,0.2,0.6},\tfrac{u_3}{0.6,0.3,0.5},\tfrac{u_4}{0.4,0.2,0.3},\tfrac{u_5}{0.6,0.3,0.1},\right),0.7\right], \\
& \left[(e_1,q,1),\left(\tfrac{u_1}{0.3,0.1,0.4},\tfrac{u_2}{0.2,0.1,0.5},\tfrac{u_3}{0.4,0.2,0.3},\tfrac{u_4}{0.4,0.2,0.3},\tfrac{u_5}{0.7,0.2,0.5},\right),0.6\right], \\
& \left[(e_1,r,1),\left(\tfrac{u_1}{0.3,0.5,0.1},\tfrac{u_2}{0.6,0.2,0.5},\tfrac{u_3}{0.1,0.4,0.2},\tfrac{u_4}{0.5,0.2,0.3},\tfrac{u_5}{0.4,0.3,0.2},\right),0.2\right], \\
& \left[(e_2,p,1),\left(\tfrac{u_1}{0.6,0.2,0.3},\tfrac{u_2}{0.4,0.2,0.5},\tfrac{u_3}{0.3,0.4,0.1},\tfrac{u_4}{0.7,0.3,0.6},\tfrac{u_5}{0.5,0.2,0.4},\right),0.8\right], \\
& \left[(e_2,q,1),\left(\tfrac{u_1}{0.1,0.3,0.6},\tfrac{u_2}{0.7,0.3,0.1},\tfrac{u_3}{0.6,0.2,0.5},\tfrac{u_4}{0.3,0.1,0.6},\tfrac{u_5}{0.4,0.3,0.2},\right),0.4\right], \\
& \left[(e_2,r,1),\left(\tfrac{u_1}{0.6,0.3,0.5},\tfrac{u_2}{0.7,0.3,0.6},\tfrac{u_3}{0.5,0.3,0.4},\tfrac{u_4}{0.2,0.1,0.3},\tfrac{u_5}{0.6,0.2,0.5},\right),0.5\right], \\
& \left[(e_3,p,1),\left(\tfrac{u_1}{0.2,0.4,0.6},\tfrac{u_2}{0.7,0.4,0.2},\tfrac{u_3}{0.4,0.1,0.2},\tfrac{u_4}{0.8,0.4,0.3},\tfrac{u_5}{0.7,0.3,0.4},\right),0.3\right], \\
& \left[(e_3,q,1),\left(\tfrac{u_1}{0.4,0.2,0.6},\tfrac{u_2}{0.5,0.3,0.6},\tfrac{u_3}{0.6,0.2,0.7},\tfrac{u_4}{0.8,0.2,0.4},\tfrac{u_5}{0.6,0.2,0.3},\right),0.4\right], \\
& \left[(e_3,r,1),\left(\tfrac{u_1}{0.3,0.6,0.5},\tfrac{u_2}{0.6,0.2,0.5},\tfrac{u_3}{0.2,0.1,0.4},\tfrac{u_4}{0.5,0.3,0.2},\tfrac{u_5}{0.4,0.1,0.5},\right),0.5\right], \\
& \left[(e_4,p,1),\left(\tfrac{u_1}{0.2,0.3,0.6},\tfrac{u_2}{0.7,0.1,0.5},\tfrac{u_3}{0.4,0.2,0.8},\tfrac{u_4}{0.9,0.2,0.4},\tfrac{u_5}{0.3,0.4,0.6},\right),0.6\right], \\
& \left[(e_4,q,1),\left(\tfrac{u_1}{0.5,0.2,0.1},\tfrac{u_2}{0.2,0.3,0.4},\tfrac{u_3}{0.4,0.1,0.5},\tfrac{u_4}{0.6,0.3,0.2},\tfrac{u_5}{0.7,0.3,0.4},\right),0.6\right], \\
& \left[(e_4,r,1),\left(\tfrac{u_1}{0.5,0.2,0.1},\tfrac{u_2}{0.6,0.3,0.5},\tfrac{u_3}{0.2,0.5,0.3},\tfrac{u_4}{0.5,0.1,0.4},\tfrac{u_5}{0.3,0.2,0.5},\right),0.3\right], \\
& \left[(e_1,p,0),\left(\tfrac{u_1}{0.2,0.3,0.4},\tfrac{u_2}{0.5,0.3,0.1},\tfrac{u_3}{0.6,0.3,0.4},\tfrac{u_4}{0.6,0.2,0.4},\tfrac{u_5}{0.7,0.5,0.6},\right),0.9\right], \\
& \left[(e_1,q,0),\left(\tfrac{u_1}{0.5,0.1,0.7},\tfrac{u_2}{0.4,0.2,0.3},\tfrac{u_3}{0.8,0.5,0.4},\tfrac{u_4}{0.7,0.3,0.6},\tfrac{u_5}{0.5,0.3,0.4},\right),0.7\right], \\
& \left[(e_1,r,0),\left(\tfrac{u_1}{0.3,0.1,0.6},\tfrac{u_2}{0.6,0.3,0.7},\tfrac{u_3}{0.3,0.2,0.4},\tfrac{u_4}{0.8,0.1,0.4},\tfrac{u_5}{0.6,0.4,0.5},\right),0.6\right], \\
& \left[(e_2,p,0),\left(\tfrac{u_1}{0.7,0.3,0.5},\tfrac{u_2}{0.6,0.2,0.4},\tfrac{u_3}{0.4,0.3,0.5},\tfrac{u_4}{0.3,0.2,0.5},\tfrac{u_5}{0.4,0.3,0.5},\right),0.8\right], \\
& \left[(e_2,q,0),\left(\tfrac{u_1}{0.6,0.2,0.4},\tfrac{u_2}{0.5,0.3,0.7},\tfrac{u_3}{0.8,0.1,0.3},\tfrac{u_4}{0.2,0.3,0.6},\tfrac{u_5}{0.6,0.2,0.4},\right),0.4\right], \\
& \left[(e_2,r,0),\left(\tfrac{u_1}{0.6,0.3,0.4},\tfrac{u_2}{0.5,0.2,0.4},\tfrac{u_3}{0.7,0.4,0.5},\tfrac{u_4}{0.5,0.2,0.4},\tfrac{u_5}{0.4,0.3,0.5},\right),0.2\right], \\
& \left[(e_3,p,0),\left(\tfrac{u_1}{0.6,02,0.4},\tfrac{u_2}{0.6,0.1,0.5},\tfrac{u_3}{0.5,0.4,0.6},\tfrac{u_4}{0.8,0.3,0.6},\tfrac{u_5}{0.7,0.2,0.4},\right),0.5\right], \\
& \left[(e_3,q,0),\left(\tfrac{u_1}{0.7,0.1,0.6},\tfrac{u_2}{0.4,0.5,0.8},\tfrac{u_3}{0.4,0.3,0.5},\tfrac{u_4}{0.6,0.2,0.5},\tfrac{u_5}{0.4,0.3,0.5},\right),0.3\right], \\
& \left[(e_3,r,0),\left(\tfrac{u_1}{0.2,0.3,0.6},\tfrac{u_2}{0.7,0.4,0.5},\tfrac{u_3}{0.4,0.2,0.8},\tfrac{u_4}{0.9,0.1,0.4},\tfrac{u_5}{0.6,0.3,0.2},\right),0.3\right], \\
& \left[(e_4,p,0),\left(\tfrac{u_1}{0.4,0.2,0.6},\tfrac{u_2}{0.5,0.2,0.6},\tfrac{u_3}{0.9,0.5,0.1},\tfrac{u_4}{0.3,0.2,0.6},\tfrac{u_5}{0.4,0.3,0.5},\right),0.6\right], \\
& \left[(e_4,q,0),\left(\tfrac{u_1}{0.3,0.2,0.1},\tfrac{u_2}{0.6,0.1,0.5},\tfrac{u_3}{0.6,0.2,0.5},\tfrac{u_4}{0.8,0.3,0.2},\tfrac{u_5}{0.2,0.3,0.4},\right),0.5\right], \\
& \left[(e_4,r,0),\left(\tfrac{u_1}{0.6,0.2,0.5},\tfrac{u_2}{0.7,0.1,0.6},\tfrac{u_3}{0.5,0.3,0.1},\tfrac{u_4}{0.3,0.2,0.6},\tfrac{u_5}{0.4,0.2,0.5},\right),0.1\right]\Big\}.
\end{aligned}
$$

Step 2—Construct the GNSES tables for each opinion (agree, disagree) of experts, as shown in Tables 1 and 2.

Table 1. Agree-GNSES.

U	u_1	u_2	u_3	u_4	u_5	μ
(e_1,p)	0.2, 0.3, 0.4	0.8, 0.2, 0.6	0.6, 0.3, 0.5	0.4, 0.2, 0.3	0.6, 0.3, 0.1	0.7
(e_2,p)	0.6, 0.2, 0.3	0.4, 0.2, 0.5	0.3, 0.4, 0.1	0.7, 0.3, 0.6	0.5, 0.2, 0.4	0.8
(e_3,p)	0.2, 0.4, 0.6	0.7, 0.4, 0.2	0.4, 0.1, 0.2	0.8, 0.4, 0.3	0.7, 0.3, 0.4	0.3
(e_4,p)	0.2, 0.3, 0.6	0.7, 0.1, 0.5	0.4, 0.2, 0.8	0.9, 0.2, 0.4	0.3, 0.4, 0.6	0.6
(e_1,q)	0.3, 0.1, 0.4	0.2, 0.1, 0.5	0.4, 0.2, 0.3	0.4, 0.2, 0.3	0.7, 0.2, 0.5	0.6
(e_2,q)	0.1, 0.3, 0.6	0.7, 0.3, 0.1	0.6, 0.2, 0.5	0.3, 0.1, 0.6	0.4, 0.3, 0.2	0.4
(e_3,q)	0.4, 0.2, 0.6	0.5, 0.3, 0.6	0.6, 0.2, 0.7	0.8, 0.2, 0.4	0.6, 0.2, 0.3	0.4
(e_4,q)	0.5, 0.2, 0.1	0.2, 0.3, 0.4	0.4, 0.1, 0.5	0.6, 0.3, 0.2	0.7, 0.3, 0.4	0.6
(e_1,r)	0.3, 0.5, 0.1	0.6, 0.2, 0.5	0.1, 0.4, 0.2	0.5, 0.2, 0.3	0.4, 0.3, 0.2	0.2
(e_2,r)	0.6, 0.3, 0.5	0.7, 0.3, 0.6	0.5, 0.3, 0.4	0.2, 0.1, 0.3	0.6, 0.2, 0.5	0.5
(e_3,r)	0.3, 0.6, 0.5	0.6, 0.2, 0.5	0.2, 0.1, 0.4	0.5, 0.3, 0.2	0.4, 0.1, 0.5	0.5
(e_4,r)	0.5, 0.2, 0.1	0.6, 0.3, 0.5	0.2, 0.5, 0.3	0.5, 0.1, 0.4	0.3, 0.2, 0.5	0.3

Table 2. Disagree-GNSES.

U	u_1	u_2	u_3	u_4	u_5	μ
(e_1, p)	0.2, 0.3, 0.4	0.5, 0.3, 0.1	0.6, 0.3, 0.4	0.6, 0.2, 0.4	0.7, 0.5, 0.6	0.9
(e_2, p)	0.7, 0.3, 0.5	0.6, 0.2, 0.4	0.4, 0.3, 0.5	0.3, 0.2, 0.5	0.4, 0.3, 0.5	0.8
(e_3, p)	0.6, 02, 0.4	0.6, 0.1, 0.5	0.5, 0.4, 0.6	0.8, 0.3, 0.6	0.7, 0.2, 0.4	0.5
(e_4, p)	0.4, 0.2, 0.6	0.5, 0.2, 0.6	0.9, 0.5, 0.1	0.3, 0.2, 0.6	0.4, 0.3, 0.5	0.6
(e_1, q)	0.5, 0.1, 0.7	0.4, 0.2, 0.3	0.8, 0.5, 0.4	0.7, 0.3, 0.6	0.5, 0.3, 0.4	0.7
(e_2, q)	0.6, 0.2, 0.4	0.5, 0.3, 0.7	0.8, 0.1, 0.3	0.2, 0.3, 0.6	0.6, 0.2, 0.4	0.4
(e_3, q)	0.7, 0.1, 0.6	0.4, 0.5, 0.8	0.4, 0.3, 0.5	0.6, 0.2, 0.5	0.4, 0.3, 0.5	0.3
(e_4, q)	0.3, 0.2, 0.1	0.6, 0.1, 0.5	0.6, 0.2, 0.5	0.8, 0.3, 0.2	0.2, 0.3, 0.4	0.5
(e_1, r)	0.3, 0.1, 0.6	0.6, 0.3, 0.7	0.3, 0.2, 0.4	0.8, 0.1, 0.4	0.6, 0.4, 0.5	0.6
(e_2, r)	0.6, 0.3, 0.4	0.5, 0.2, 0.4	0.7, 0.4, 0.5	0.5, 0.2, 0.4	0.4, 0.3, 0.5	0.2
(e_3, r)	0.2, 0.3, 0.6	0.7, 0.4, 0.5	0.4, 0.2, 0.8	0.9, 0.1, 0.4	0.6, 0.3, 0.2	0.3
(e_4, r)	0.6, 0.2, 0.5	0.7, 0.1, 0.6	0.5, 0.3, 0.1	0.3, 0.2, 0.6	0.4, 0.2, 0.5	0.1

Step 3—Now calculate the scores of agree (u_i) by using the data in Table 1, to obtain values in Table 3.

$$\begin{aligned}
T_A^{agg}(p, u_1) &= \left(\tfrac{T_{A1}+T_{A2}+T_{A3}+T_{A4}}{4}\right).\left(\tfrac{\mu_1+\mu_2+\mu_3+\mu_4}{4}\right). \\
&= \left(\tfrac{0.2+0.6+0.2+0.2}{4}\right).\left(\tfrac{0.7+0.8+0.3+0.6}{4}\right) \\
&= 0.18 \\
I_A^{agg}(q, u_1) &= \left(\tfrac{I_{A1}+I_{A2}+I_{A3}+I_{A4}}{4}\right).\left(\tfrac{\mu_1+\mu_2+\mu_3+\mu_4}{4}\right). \\
&= \left(\tfrac{0.3+0.2+0.4+0.3}{4}\right).\left(\tfrac{0.7+0.8+0.3+0.6}{4}\right) \\
&= 0.18 \\
F_A^{agg}(r, u_1) &= \left(\tfrac{F_{A1}+F_{A2}+F_{A3}+F_{A4}}{4}\right).\left(\tfrac{\mu_1+\mu_2+\mu_3+\mu_4}{4}\right). \\
&= \left(\tfrac{0.4+0.3+0.6+0.6}{4}\right).\left(\tfrac{0.7+0.8+0.3+0.6}{4}\right) \\
&= 0.285
\end{aligned}$$

$$u_1 = \left|T_{A_i}^{agg} - F_{A_i}^{agg} - I_{A_i}^{agg}\right| = |0.18 - 0.18 - 0.285| = 0.285.$$

Table 3. Degree table of agree-GNSES.

U	u_1	u_2	u_3	u_4	u_5
p	0.285	0.015	0.135	0.015	0.09
q	0.18	0.15	0.105	0.12	0.015
r	0.165	0.09	0.24	0.06	0.045

Now calculate the score of disagree (u_i) by using the data in Table 2, to obtain values in Table 4.

$$\begin{aligned}
T_A^{agg}(p, u_1) &= \left(\tfrac{T_{A1}+T_{A2}+T_{A3}+T_{A4}}{4}\right).\left(\tfrac{\mu_1+\mu_2+\mu_3+\mu_4}{4}\right) \\
&= \left(\tfrac{0.2+0.7+0.6+0.4}{4}\right).\left(\tfrac{0.9+0.8+0.5+0.6}{4}\right) \\
&= 0.3325 \\
I_A^{agg}(q, u_1) &= \left(\tfrac{I_{A1}+I_{A2}+I_{A3}+I_{A4}}{4}\right).\left(\tfrac{\mu_1+\mu_2+\mu_3+\mu_4}{4}\right). \\
&= \left(\tfrac{0.3+0.3+0.2+0.2}{4}\right).\left(\tfrac{0.9+0.8+0.5+0.6}{4}\right) \\
&= 0.175 \\
F_A^{agg}(r, u_1) &= \left(\tfrac{F_{A1}+F_{A2}+F_{A3}+F_{A4}}{4}\right).\left(\tfrac{\mu_1+\mu_2+\mu_3+\mu_4}{4}\right). \\
&= \left(\tfrac{0.4+0.5+0.4+0.6}{4}\right).\left(\tfrac{0.9+0.8+0.5+0.6}{4}\right) \\
&= 0.3325
\end{aligned}$$

$$u_1 = \left|T_{A_i}^{agg} - F_{A_i}^{agg} - I_{A_i}^{agg}\right| = |0.3325 - 0.175 - 0.3325| = 0.175.$$

Table 4. Degree table of disagree-GNSES.

U	u_1	u_2	u_3	u_4	u_5
p	0.175	0.035	0.1225	0.175	0.1925
q	0.0525	0.2625	0.035	0.1225	0.0875
r	0.2275	0.1225	0.175	0.0175	0.1575

Step 4—The final score of u_i is computed as follows.

$$\begin{aligned}
\text{Score}(u_1) &= 0.285 - 0.0525 = 0.2325, \\
\text{Score}(u_2) &= 0.15 - 0.035 = 0.115, \\
\text{Score}(u_3) &= 0.24 - 0.035 = 0.205, \\
\text{Score}(u_4) &= 0.12 - 0.0175 = 0.1025, \\
\text{Score}(u_5) &= 0.09 - 0.0875 = 0.0025.
\end{aligned}$$

Step 5—$\text{Score}(u_1) = 0.2325$ is the maximum. Hence, the best decision for the experts is to select worker u_1 as the company's employee.

6. Comparison Analysis

A generalized neutrosophic soft expert model gives more precision, flexibility, and compatibility than the existing neutrosophic models. These are verified by a comparison analysis, using neutrosophic soft expert decision method, with those methods used by Sahin et al. [27], Hassan [44], and Maji [20], as given in Table 5. The comparison is done based on the same example as in Section 5. The ranking order results obtained are consistent with those in [20,27,44].

Table 5. Comparison of neutrosophic soft set to other variants.

Methods	Neutrosophic Soft Set	Neutrosophic Soft Expert Set	*Q*-Neutrosophic Soft Expert Set	Generalized Neutrosophic Soft Expert Set
Authors	Maji [20]	Sahin et al. [27]	Hassan et al. [44]	Proposed Method
Domain	Universe of discourse	Universe of discourse	Universe of discourse	Universe of discourse
Co-domain	$[0,1]^3$	$[0,1]^3$	$[0,1]^3$	$[0,1]^3$
True	Yes	Yes	Yes	Yes
Falsity	Yes	Yes	Yes	Yes
Indeterminacy	Yes	Yes	Yes	Yes
Expert	No	Yes	Yes	Yes
Q	No	No	Yes	No
Ranking	$u_2 > u_3 > u_1 > u_4 > u_5$	$u_2 > u_2 > u_1 > u_4 > u_5$	$u_3 > u_1 > u_2 > u_4 > u_5$	$u_1 > u_3 > u_2 > u_4 > u_5$

7. Conclusions

We have established the concept of generalized neutrosophic soft expert set (GNSES) as a generalization of NSES. The basic operations of GNSES of complement, union, intersection AND, and OR were defined. Subsequently, a definition of GNSES-aggregation operator was proposed to construct an algorithm of a GNSES decision method. Finally, an application of the constructed algorithm, to solve a decision-making, was provided. This new extension provides a significant contribution to current theories for handling indeterminacy, and it spurs the development of further research and pertinent applications.

Author Contributions: All authors contributed equally.

Funding: This research was funded by Universiti Kebangsaan Malaysia grant number GUP-2017-105.

Acknowledgments: We are indebted to Universiti Kebangsaan Malaysia for providing financial support and facilities for this research under the grant GUP-2017-105.

Conflicts of Interest: The authors declare no conflict of interest.

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Neutrosophic Cubic Power Muirhead Mean Operators with Uncertain Data for Multi-Attribute Decision-Making

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Received: 31 August 2018; Accepted: 25 September 2018; Published: 28 September 2018

Abstract: The neutrosophic cubic set (NCS) is a hybrid structure, which consists of interval neutrosophic sets (INS) (associated with the undetermined part of information associated with entropy) and single-valued neutrosophic set (SVNS) (associated with the determined part of information). NCS is a better tool to handle complex decision-making (DM) problems with INS and SVNS. The main purpose of this article is to develop some new aggregation operators for cubic neutrosophic numbers (NCNs), which is a basic member of NCS. Taking the advantages of Muirhead mean (MM) operator and power average (PA) operator, the power Muirhead mean (PMM) operator is developed and is scrutinized under NC information. To manage the problems upstretched, some new NC aggregation operators, such as the NC power Muirhead mean (NCPMM) operator, weighted NC power Muirhead mean (WNCPMM) operator, NC power dual Muirhead mean (NCPMM) operator and weighted NC power dual Muirhead mean (WNCPDMM) operator are proposed and related properties of these proposed aggregation operators are conferred. The important advantage of the developed aggregation operator is that it can remove the effect of awkward data and it considers the interrelationship among aggregated values at the same time. Furthermore, a novel multi-attribute decision-making (MADM) method is established over the proposed new aggregation operators to confer the usefulness of these operators. Finally, a numerical example is given to show the effectiveness of the developed approach.

Keywords: NC power dual MM operator (NCPDMM) operator; NCPMM operator; MADM; MM operator; Neutrosophic cubic sets; PA operator

1. Introduction

One of the drawbacks of real MADM problems is expressing attribute values in fuzzy and indeterminate DM environments. Fuzzy sets (FSs) developed by Zadeh [1] emerged as a tool for describing and communicating uncertainties and vagueness. Since its beginning, FS has gained a significant focus from researchers all over the world who studied its practical and theoretical aspects. Several extensions of FSs have been developed, such as interval-valued FS (IVFS) [2], which explained the truth membership degree (TMD) on a closed interval value in the interval [0, 1], and intuitionistic FS (IFS) [3], which explained the TMD and falsity-membership degree (FMD). Therefore, IFS defines fuzziness and uncertainty more comprehensively than FS. However, neither FS nor IFS are capable to handle indeterminate and inconsistent information. For example, when we take a student opinion about the teaching skills of a professor with about 0.6 being the possibility that the teaching skills of the professor are good, 0.5 being the possibility that the teaching skills of the professor are bad and 0.3 is the possibility that he/she may not be sure about the teaching skills of the professor whether bad or good. To handle such type of information, Smarandache [4] added a new component "indeterminacy membership degree" (IMD) to the TMD and FMD, all being independent elements lying in $]0^-, 1^+[$. The resulting set is now familiarly known as neutrosophic set (NS). To use NS in practical and engineering problems, some scholars developed simplified forms of NS, such as SVNS [5], INS [6,7], simplified neutrosophic sets [8,9], multi-valued NS [10], Q-neutrosophic soft set [11], complex neutrosophic soft expert set [12] and others.

In the real world, sometimes it is difficult to express the TMD in some fuzzy problems completely by an exact value or interval value. Therefore, Jun et al. [13] developed the concept of cubic set (CS) by combining FS and IVFS. CS defined uncertainty and vagueness by an interval value and a fuzzy value concurrently. In recent years, some researchers established some extended forms of CS. Garg et al. [14,15] combined IFS and interval-valued intuitionistic FS (IVIFS) to form cubic IFS (CIFS), while Ali et al. [16] and Jun et al. [17] combined INS and SVNS to develop the cubic NS (CNS), consisting of internal and external NCSs. Jun et al. [18] further investigated P-union and P-intersection of NCS and discussed their related properties. Since then, various studies to solve MADM problems based on NCSs are developed. Zhang et al. [19] and Ye [20] developed some aggregation operators such as weighted averaging operators and weighted geometric operators on NCSs and applied these to MADM. Shi et al. [21], developed some aggregation operator for NCNs based on Dombi T-norm and T-conorm and applied these to MADM. To solve MADM problems under NC information, various similarity measures are developed for NCSs [22,23]. Pramanik et al. [24] introduced the NC-TODIM method to solve multiple-attribute group decision-making (MAGDM) problem.

Aggregation operator (AO) plays a dominant role in DM. Consequently, many scholars proposed different aggregation operators and their generalizations, such as Bonferroni mean (BM) operator [25,26], Heronian mean (HM) operator [27], Muirhead mean (MM) operator [28], Maclaurin symmetric mean (MSM) operator [29,30] and others. Certainly, different AOs have different functions. Some can remove the effect of awkward data given by prejudiced DMs, such as power average (PA) operator [31,32] developed by Yager [31] which can aggregate the input information by giving the weighted vector based on support degree among the input arguments. Some aggregation operators are capable to consider the interrelationship among two or more input arguments such as BM operator, HM operators, MSM operator and MM operator.

Due to the enhanced complexity in real decision-making problems, it is necessary to look over the following questions when selecting the best alternative. Firstly, the values of the attributes provided by the decision makers may be too low or too high, thus giving a negative impact on the final ranking results. The PA operator, however, permits the evaluated values to be mutually supported and enhanced. Therefore, we may use the PA operator to diminish such awful impact by designating distinct weights produced by the support measure. Secondly, the values of attributes are required to be dependent. Hence, the interrelationship among the values of the attributes should be examined. Some advantages of MM operator over BM and HM are discussed by Liu et al. [33,34]. Some existing aggregation operator such as the BM and MSM operators are special cases of the MM operator. The MM operator consists of the parameter vector, which enlarges the flexibility in the aggregation process. Recently, Li et al. [35] developed the concept of power Muirhead mean operator under Pythagorean fuzzy environment. From the existing literature, the PA operator and MM operator have not been yet combined to deal with NC information. To handle the issues raised, a few new aggregation operators will be proposed by incorporating both the PA and MM operators. These new aggregation operators are NC power MM operator (NCPMM), weighted NC power MM operator, NC power dual MM operator (NCPDMM) and weighted NC power dual MM (WNCPDMM) operator. Discussions on some basic properties and related cases with respect to the parameter vector will be dealt at length. The advantages of these proposed aggregation operators are to capture the interrelationship among input arguments by the MM operator, and simultaneously eliminate the effect of awkward data. Finally, a novel approach to solve MADM problems based on these proposed aggregation operators will be developed.

The rest of the article is organized as follows. In Section 2, some basic definitions and properties of NCSs, MM and PA operators are recalled. In Section 3, the PA and MM operators in the construction of new operators, namely NCPMM, WNCPMM, NCPDMM and WNCPDMM operators are incorporated followed by discussions on their related properties. In Section 4, a novel method to MADM is established based on the developed aggregation operators. In Section 5, a numerical example is illustrated to show the effectiveness of the proposed method to solve a MADM problem. In Section 6, a comparison with the existing methods is given followed by the conclusion.

2. Preliminaries

In this part, some basic concepts about SVNSs, INSs, NCSs, PA and MM operators are briefly overviewed.

2.1. The NCSs and Their Operations

Definition 1 ([4]). *Let Γ be a space of points (objects), with a generic element in Γ denoted by n. A neutrosophic set N in Γ is defined as $N = \{\langle n; T_N(n), I_N(n), F_N(n)\rangle n \in \Gamma\}$ where $T_N(n), I_N(n)$ and $F_N(n)$ are the truth membership function, the indeterminacy membership function and the falsity-membership function respectively, such that $T; F; I : \Gamma \to]0^-, 1^+[$ and $0^- \leq T_N(n) + I_N(n) + F_N(n) \leq 3^+$.*

Smarandache [4] developed the concept of NS as a generalization of FS, IFS and IVIFS. To apply NS to real and engineering problems easily, its parameters should be specified. Hence, Wang et al. [5] provided the following definition.

Definition 2 ([5]). *Let Γ be a space of points (objects), with a generic element in Γ denoted by n. A single-valued neutrosophic set S in Γ is defined as:*

$$S = \int_\Gamma \langle T_S(n), I_S(n), F_S(n)\rangle |n, n \in \Gamma \tag{1}$$

when Γ is continuous, and

$$S = \sum_{i=1}^{m} \langle T_S(n_i), I_S(n_i), F_S(n_i)\rangle |n_i, n_i \in \Gamma \tag{2}$$

when Γ is discrete, where $T_S(n), I_S(n)$ and $F_S(n)$ are the truth membership function, the indeterminacy membership function and the falsity-membership function respectively, such that $T; F; I : \Gamma \to [0, 1]$ and $0 \leq T_S(n) + I_S(n) + F_S(n) \leq 3$.

Definition 3 ([6]). *Let Γ be a space of points (objects), with a generic element in Γ denoted by n. An interval neutrosophic set A in Γ is defined as:*

$$A = \int_\Gamma \langle T_A(n), I_A(n), F_A(n)\rangle |n, n \in \Gamma \tag{3}$$

when Γ is continuous, and

$$A = \sum_{i=1}^{m} \langle T_A(n_i), I_A(n_i), F_A(n_i)\rangle |n_i, n_i \in \Gamma \tag{4}$$

when Γ is discrete, where $T_A(n), I_A(n)$ and $F_A(n)$ are the truth membership function, the indeterminacy membership function and the falsity-membership function respectively. For each element n in Γ, we have

$T_A(n) = \left[T_A^L(n), T_A^U(n)\right] \subseteq [0,1], I_A(n) = \left[I_A^L(n), I_A^U(n)\right] \subseteq [0,1]$, and $F_A(n) = \left[F_A^L(n), F_A^U(n)\right] \subseteq [0,1]$ such that

$0 \le \sup T_A^U(n) + \sup I_A^U(n) + \sup F_A^U(n) \le 3.$

Definition 4 ([16,17]). *Let Γ be a non-empty set. A neutrosophic cubic set (NCS) in Γ is a pair $Z = \langle A, \lambda \rangle$, where $A = \{\langle n, T_A(n), I_A(n), F_A(n) \rangle | n \in \Gamma\}$ is an interval neutrosophic set in Γ and $\lambda = \{\langle n, \lambda_T(n), \lambda_I(n), \lambda_F(n) \rangle | n \in \Gamma\}$ is a neutrosophic set in Γ.*

For simplicity, a basic element $\{n, \langle T(n), I(n), F(n) \rangle, \langle \lambda_T(n), \lambda_I(n), \lambda_F(n) \rangle\}$ in a NCS can be expressed by $z = (\langle T, I, F \rangle, \langle \lambda_T, \lambda_I, \lambda_F \rangle)$, which is called neutrosophic cubic number (NCN), where $T, I, F \subseteq [0,1]$ and $\lambda_T, \lambda_I, \lambda_F \in [0,1]$, satisfying $0 \le T^U + I^U + F^U \le 3$ and $0 \le \lambda_T + \lambda_I + \lambda_F \le 3$.

Definition 5 ([20]). *Let $z_1 = (\langle [T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U] \rangle, \langle \lambda_{T_1}, \lambda_{I_1}, \lambda_{F_1} \rangle)$ and $z_2 = (\langle [T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U] \rangle, \langle \lambda_{T_2}, \lambda_{I_2}, \lambda_{F_2} \rangle)$ be any two NCNs and $\xi > 0$. Then the operational laws for NCNs defined by Ye [20] are as follows:*

$$(1)\ z_1 \oplus z_2 = \begin{array}{l} (\langle [T_1^L + T_2^L - T_1^L T_2^L,\ T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U] \rangle, \\ \langle \lambda_{T_1} + \lambda_{T_2} - \lambda_{T_1}\lambda_{T_2}, \lambda_{I_1}\lambda_{I_2}, \lambda_{F_2}\lambda_{F_2} \rangle); \end{array} \tag{5}$$

$$(2)\ z_1 \otimes z_2 = \begin{array}{l} (\langle [T_1^L T_2^L,\ T_1^U T_2^U], [I_1^L + I_2^L - I_1^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U], [F_1^L + F_2^L - F_1^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U] \rangle, \\ \langle \lambda_{T_1}\lambda_{T_2}, \lambda_{I_1} + \lambda_{I_2} - \lambda_{I_1}\lambda_{I_2}, \lambda_{F_2} + \lambda_{F_2} - \lambda_{F_2}\lambda_{F_2} \rangle); \end{array} \tag{6}$$

$$(3)\ \xi z_1 = \left(\left\langle \left[1-(1-(T_1^L))^{\xi}, 1-(1-(T_1^U))^{\xi}\right], \left[(I_1^L)^{\xi}, (I_1^U)^{\xi}\right], \left[(F_1^L)^{\xi}, (F_1^U)^{\xi}\right] \right\rangle, \left\langle 1-(1-\lambda_{T_1})^{\xi}, (\lambda_{I_1})^{\xi}, (\lambda_{F_1})^{\xi} \right\rangle \right); \tag{7}$$

$$(4)\ z_1^{\xi} = \left\langle \left[(T_1^L)^{\xi}, 1-(T_1^U)^{\xi}\right], \left[1-(1-I_1^L)^{\xi}, 1-(1-I_1^U)^{\xi}\right], \left[1-(1-F_1^L)^{\xi}, 1-(1-F_1^U)^{\xi}\right] \right\rangle, \left\langle (\lambda_{T_1})^{\xi}, 1-(1-\lambda_{I_1})^{\xi}, 1-(1-\lambda_{F_1})^{\xi} \right\rangle. \tag{8}$$

Definition 6 ([21]). *Let $z_1 = (\langle [T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U] \rangle, \langle \lambda_{T_1}, \lambda_{I_1}, \lambda_{F_1} \rangle)$ be an NCN. Then, the score, accuracy, and certainty functions of NCN are defined as follows:*

$$\hat{S}(z_1) = \frac{4 + T_1^L - I_1^L - F_1^L + T_1^U - I_1^U - F_1^U + \lambda_{T_1} + 2 - \lambda_{I_1} - \lambda_{F_1}}{9}; \tag{9}$$

$$\hat{A}(z_1) = \frac{T_1^L - I_1^L + T_1^U - I_1^U + \lambda_{T_1} - \lambda_{F_1}}{3} \text{ and } \hat{C}(z_1) = \frac{T_1^L + T_1^U + \lambda_{T_1}}{3}. \tag{10}$$

Theorem 1 ([21]). *Let $z_1 = (\langle [T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U] \rangle, \langle \lambda_{T_1}, \lambda_{I_1}, \lambda_{F_1} \rangle)$ and $z_2 = (\langle [T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U] \rangle, \langle \lambda_{T_2}, \lambda_{I_2}, \lambda_{F_2} \rangle)$. Then the comparison rules for NCNs can be defined as follows:*

(i) *If $\hat{S}(z_1) > \hat{S}(z_2)$, then z_1 is greater than z_2, and is denoted by $z_1 > z_2$;*
(ii) *If $\hat{S}(z_1) = \hat{S}(z_2)$, and $\hat{A}(z_1) > \hat{A}(z_2)$, then z_1 is greater than z_2, and is denoted by $z_1 > z_2$;*
(iii) *If $\hat{S}(z_1) = \hat{S}(z_2)$, $\hat{A}(z_1) = \hat{A}(z_2)$, and $\hat{C}(z_1) > \hat{C}(z_2)$, then z_1 is greater than z_2, and is denoted by $z_1 > z_2$;*
(iv) *If $\hat{S}(z_1) = \hat{S}(z_2)$, $\hat{A}(z_1) = \hat{A}(z_2)$, and $\hat{C}(z_1) = \hat{C}(z_2)$, then z_1 is equal to z_2, and is denoted by $z_1 = z_2$.*

2.2. Power Average (PA) Operator

The PA operator was first introduced by Yager [31] for classical number. The dominant edge of PA operator is its capacity to diminish the inadequate effect of unreasonably too high and too low arguments on the inconclusive results.

Definition 7 ([31]). *Let $\Re_g (g = 1, 2, \ldots, a)$ be a group of classical numbers. The PA operator is then represented as follows:*

$$PA(\Re_1, \Re_2, \ldots, \Re_a) = \sum_{g=1}^{a} \left(\frac{(1 + T(\Re_g))}{\sum_{x=1}^{a} (1 + T(\Re_x))} \Re_g \right) \quad (11)$$

where, $T(\Re_z) = \sum_{\substack{x=1 \\ g \neq x}}^{a} Supp(\Re_g, \Re_x)$ and $Supp(\Re_z, \Re_x)$ is the support degree for $\Re_g$ and $\Re_x$. The support degree must satisfy the following axioms:

(1) $Supp(\Re_g, \Re_x) \in [0, 1]$;
(2) $Supp(\Re_g, \Re_x) = Supp(\Re_x, \Re_g)$;
(3) *If $\overline{\overline{D}}(\Re_g, \Re_x) < \overline{\overline{D}}(\Re_l, \Re_m)$, then $Supp(\Re_g, \Re_x) > Supp(\Re_l, \Re_m)$, where $\overline{\overline{D}}(\Re_g, \Re_x)$ is the distance measure among $\Re_g$ and $\Re_x$.*

2.3. Muirhead Mean (MM) Operator

The MM operator was first introduced by Muirhead [28] for classical numbers. MM operator has the advantage of considering the interrelationship among all aggregated arguments.

Definition 8 ([28]). *Let $\Re_g (g = 1, 2, \ldots, a)$ be a group of classical numbers and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. Then, the MM operator is described as:*

$$MM^Q(\Re_1, \Re_2, \ldots, \Re_a) = \left(\frac{1}{a!} \sum_{\theta \in S_a} \prod_{g=1}^{a} \Re_{\theta(g)}^{q_g} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \quad (12)$$

where, S_a is the group of permutation of $(1, 2, \ldots, a)$ and $\theta(g)$ is any permutation of $(1, 2, \ldots, a)$.

Now we can give some special cases with respect to the parameter vector Q of the MM operator, which are shown as follows:

(1) If $Q = (1, 0, 0, \ldots, 0)$, then the MM operator degenerates to the following form:

$$MM^{(1,0,\ldots,0)}(\Re_1, \Re_2, \ldots, \Re_a) = \frac{1}{a} \sum_{g=1}^{a} \Re_g. \quad (13)$$

That is, the MM operator degenerates into arithmetic averaging operator.

(2) If $Q = \left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a} \right)$, then the MM operator degenerates to the following form:

$$MM^{\left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a}\right)}(\Re_1, \Re_2, \ldots, \Re_a) = \prod_{g=1}^{a} \Re_g^{\frac{1}{a}}. \quad (14)$$

That is, the MM operator degenerates into geometric averaging operator.

(3) If $Q = (1, 1, 0, \ldots, 0)$, then the MM operator degenerates to the following form:

$$MM^{(1,1,0,\ldots,0)}(\Re_1, \Re_2, \ldots, \Re_a) = \left(\frac{1}{a(a+1)} \sum_{\substack{g,x=1 \\ g \neq x}}^{a} \Re_g \Re_x \right)^{\frac{1}{2}}. \quad (15)$$

That is, the MM operator degenerates into BM operator.

(4) If $Q = \left(\overbrace{1,1,\ldots,1}^{c},\overbrace{0,\ldots,0}^{a-c}\right)$, then the MM operator degenerates to the following form:

$$MM^{(\overbrace{1,1,\ldots,1}^{d},\overbrace{0,\ldots,0}^{a-d})}(\Re_1,\Re_2,\ldots,\Re_a) = \left(\frac{\sum\limits_{1\le x_1<x_2<\ldots<x_d\le a}\prod\limits_{y=1}^{d}\Re_{g_y}}{C_a^d}\right)^{\frac{1}{d}}. \tag{16}$$

That is, the MM operator degenerates into MSM operator.

3. Some Power Muirhead Mean Operator for NCNs

In this part, we first give the definitions of PMM operator and propose the concept of power dual Muirhead mean (PDMM) operator. Then, we extended both the aggregation operator to NCN environment.

Definition 9 ([35]). *Let $\Re_g (g = 1,2,\ldots,a)$ be a group of classical numbers and $Q = (q_1,q_2,\ldots,q_a) \in R^a$ be a vector of parameters. Then, the PMM operator is explained as,*

$$PMM^Q(\Re_1,\Re_2,\ldots,\Re_a) = \left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(\frac{a\left(1+T\left(\Re_{\theta(g)}\right)\right)}{\sum\limits_{x=1}^{a}(1+T(\Re_x))}\Re_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{\sum\limits_{g=1}^{a}q_g}} \tag{17}$$

where, $T(\Re_g) = \sum\limits_{x=1,x\neq g}^{a} Supp(\Re_g,\Re_x)$ and $Supp(\Re_g,\Re_x)$ is the support degree for $\Re_g$ and $\Re_x$, satisfying the above conditions.

Definition 10. *Let $\Re_g (g = 1,2,\ldots,a)$ be a group of classical numbers and $Q = (q_1,q_2,\ldots,q_a) \in R^a$ be a vector of parameters. Then, the PDMM operator is described as,*

$$PDMM^Q(\Re_1,\Re_2,\ldots,\Re_a) = \frac{1}{\sum\limits_{g=1}^{a}q_g}\left(\sum_{\theta\in S_a}\prod_{g=1}^{a}q_g\Re_{\theta(g)}^{\frac{a(1+T(\Re_{\theta(g)}))}{\sum\limits_{x=1}^{a}(1+T(\Re_x))}}\right)^{\frac{1}{a!}} \tag{18}$$

where, $T(\Re_g) = \sum\limits_{x=1x\neq 1}^{a} Supp(\Re_g,\Re_x)$ and $Supp(\Re_g,\Re_x)$ is the support degree for $\Re_g$ and $\Re_x$, satisfying the above conditions.

3.1. The Neutrosophic Cubic Power Muirhead Mean (NCPMM) Operator

In this subsection, we extend the PMM operator to neutrosophic cubic environment and discuss some basic properties, and special cases of these developed aggregation operators with respect to the parameter Q.

Definition 11. *Let $z_g(g = 1, 2, \ldots, a)$ be a group of NCNs and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. If,*

$$NCPMM^Q(z_1, z_2, \ldots, z_a) = \left(\frac{1}{a!} \sum_{\theta \in S_a} \prod_{g=1}^{a} \left(\frac{a\left(1 + T\left(z_{\theta(g)}\right)\right)}{\sum\limits_{x=1}^{a} (1 + T(z_x))} z_{\theta(g)} \right)^{q_g} \right)^{\frac{1}{\sum\limits_{g=1}^{a} q_g}} \tag{19}$$

then, we call $NCPMM^Q$ the neutrosophic cubic power Muirhead mean operator, where S_a is the group of all permutation, $\theta(g)$ is any permutation of $(1, 2, \ldots, a)$ and $T(z_x) = \sum\limits_{x=1, x \neq g}^{a} Supp(z_g, z_x)$, $Supp(z_g, z_x)$ is the support degree for z_g and z_x, satisfying the following axioms:

(1) $Supp(z_g, z_x) \in [0, 1]$;

(2) $Supp(z_g, z_x) = Supp(z_x, z_z)$;

(3) *If $\overline{\overline{D}}(z_g, z_x) < \overline{\overline{D}}(z_u, z_v)$, then $Supp(z_g, z_x) > Supp(z_u, z_v)$, where $\overline{\overline{D}}(z_g, z_x)$ is the distance among z_g and z_x.*

To write Equation (20) in a simple form, we can specify it as:

$$\Theta_g = \frac{(1 + T(z_g))}{\sum\limits_{x=1}^{a} (1 + T(z_x))}. \tag{20}$$

For suitability, we can call $(\Theta_1, \Theta_2, \ldots, \Theta_a)^T$ the power weight vector (PMV), such that $\Theta_g \in [0, 1]$ and $\sum\limits_{g=1}^{a} \Theta_g = 1$. From the use of Equation (20), Equation (19) can be expressed as:

$$NCPMM^Q(z_1, z_2, \ldots, z_a) = \left(\frac{1}{a!} \sum_{\theta \in S_a} \prod_{g=1}^{a} \left(a\Theta_g z_{\theta(g)} \right)^{q_g} \right)^{\frac{1}{\sum\limits_{g=1}^{a} q_g}}. \tag{21}$$

Based on the operational rules given in Definition 3 for NCNs, and Definition 11, we can have the following Theorem 2.

Theorem 2. *Let $z_g(g = 1, 2, \ldots, a)$ be a group of NCNs and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. Then, the aggregated value obtained by using Equation (21) is still an NCN and,*

$$\begin{gathered} NCPMM^Q(z_1, z_2, \ldots, z_a) = \\ \left(\left\langle \left[\left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (T^L)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (T^U)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right], \left[1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (I^L)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \right. \right. \\ \left. 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (I^U)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right], \left[1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (F^L)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (F^U)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right] \Big\rangle \\ \left. \left\langle \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (\lambda_T)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (\lambda_I)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (\lambda_F)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right\rangle \right). \end{gathered} \tag{22}$$

Proof. According to the operational laws for NCNs, we have

$$a\Theta_g z_{\theta(g)} = \left(\left\langle \left[1 - \left(1 - (T^L)_{\theta(g)} \right)^{a\Theta_g}, 1 - \left(1 - (T^U)_{\theta(g)} \right)^{a\Theta_g} \right], \left[(I^L)_{\theta(g)}^{a\Theta_g}, (I^U)_{\theta(g)}^{a\Theta_g} \right], \left[(F^L)_{\theta(g)}^{a\Theta_g}, (F^U)_{\theta(g)}^{a\Theta_g} \right] \right\rangle, \left\langle 1 - \left(1 - (\lambda_T)_{\theta(g)} \right)^{a\Theta_g}, (\lambda_I)_{\theta(g)}^{a\Theta_g}, (\lambda_F)_{\theta(g)}^{a\Theta_g} \right\rangle \right).$$

Therefore,

$$\begin{aligned} \left(a\Theta_g z_{\theta(g)} \right)^{q_g} = \ & \left(\left\langle \left[\left(1 - \left(1 - (T^L)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g}, \left(1 - \left(1 - (T^U)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right], \left[, 1 - \left(1 - (I^L)_{\theta(g)}^{a\Theta_g} \right)^{q_g} 1 - \left(1 - (I)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right], \left[1 - \left(1 - (F^L)_{\theta(g)}^{a\Theta_g} \right)^{q_g}, 1 - \left(1 - (F^U)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right] \right\rangle, \right. \\ & \left. \left\langle \left(1 - \left(1 - (\lambda_T)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g}, 1 - \left(1 - (\lambda_I)_{\theta(g)}^{a\Theta_g} \right)^{q_g}, 1 - \left(1 - (\lambda_F)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right\rangle \right). \end{aligned}$$

Therefore,

$$\prod_{g=1}^{a}\left(a\Theta_g z_{\theta(g)}\right)^{q_g}=\left(\left\langle\left[\prod_{g=1}^{a}\left(1-\left(1-\left(T^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g},\prod_{g=1}^{a}\left(1-\left(1-\left(T^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right],\left[1-\prod_{g=1}^{a}\left(1-\left(I^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g},1-\prod_{g=1}^{a}\left(1-\left(I^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right],\left[1-\prod_{g=1}^{a}\left(1-\left(F^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g},\right.\right.\right.$$
$$\left.\left.\left.1-\prod_{g=1}^{a}\left(1-\left(F^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right]\right\rangle,\left\langle\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g},1-\prod_{g=1}^{a}\left(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\right)^{q_g},1-\prod_{g=1}^{a}\left(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right\rangle\right),$$

and

$$\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(a\Theta_g z_{\theta(g)}\right)^{q_g}=\left\langle\left[1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right),1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right],\left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right]\right),$$
$$\left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right]\right\rangle,\left\langle1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\lambda_f\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right\rangle\right).$$

Furthermore,

$$\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(a\Theta_g z_{\theta(g)}\right)^{q_g}=\left(\left\langle\left[1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right],\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\right.\right.\right.$$
$$\left\langle\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right],\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right]\right\rangle,\left\langle1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right.$$
$$\left.\left.,\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right\rangle\right).$$

Hence,

$$\left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(a\Theta_g\Gamma_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}=\left(\left\langle\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\right.\right.$$
$$\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\right.$$
$$\left.\left.1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle,\left\langle\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right.$$
$$\left.\left.,1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right),$$

$$NCPMM^{Q}(z_1,z_2,\ldots,z_a)=$$
$$\left(\left\langle\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\right.\right.\right.$$
$$\left.1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle,$$
$$\left\langle\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right).$$

This is the required proof of Theorem 2. □

In the above equations, we calculate the PWV Θ, after calculating the support degree $Supp(z_g,z_x)$. First, we determined the $Supp(z_g,z_x)$ using

$$Supp(z_g,z_x)=1-\overline{\overline{D}}(z_g,z_x), \tag{23}$$

where,

$$\overline{\overline{D}}(z_g,z_x)=\sqrt{\begin{array}{l}\frac{1}{9}\left(\left(T_g^L-T_x^L\right)^2+\left(T_g^U-T_x^U\right)^2+\left(I_g^L-I_x^L\right)^2+\left(I_g^U-I_x^U\right)^2+\left(F_g^L-F_x^L\right)^2+\left(F_g^U-F_x^U\right)^2\right.\\ \left.\left(\lambda_{T_g}-\lambda_{T_x}\right)^2+\left(\lambda_{I_g}-\lambda_{I_x}\right)^2+\left(\lambda_{F_g}-\lambda_{F_x}\right)^2\right)\end{array}}. \tag{24}$$

Therefore, we use the equation

$$T(z_g)=\sum_{g=1,g\neq x}^{a}Supp(z_g,z_x) \tag{25}$$

to obtain the values of $T(z_g)(g=1,2,\ldots,a)$. Then using Equation (20) we can get the PWV.

Theorem 3. *(Idempotency) Let $z_g (g = 1, 2, \ldots, a)$ be a group of NCNs, and $z_g = z$, for all $g = 1, 2, \ldots, a$. Then,*

$$NCPMM^{Q}(z_1, z_2, \ldots, z_a) = CN. \tag{26}$$

Proof. As $z_g = z$ for all $g = 1, 2, \ldots, a$, we have $Supp(z_g, z_x) = 1$ for all $g, x = 1, 2, \ldots, a$. Therefore, we can get $\Theta_g = \frac{1}{a}$ for all g. Moreover,

$$\begin{aligned}
& CNPMM^{Q}(z_1, z_2, \ldots, z_a) = CNPMM^{Q}(z, z, \ldots, z) \\
&= \Bigg(\Big\langle\Big[\Big(1-\Big(\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-(1-T^{L})^{a\frac{1}{a}}\Big)^{q_g}\Big)\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \Big(1-\Big(\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-(1-T^{L})^{a\frac{1}{a}}\Big)^{q_g}\Big)\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big], \Big[1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-I^{La\frac{1}{a}}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \\
&\quad 1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-I^{Ua\frac{1}{a}}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big], \Big[1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-F^{La\frac{1}{a}}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{z=1}^{a} q_g}}, 1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-F^{Ua\frac{1}{a}}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big]\Big\rangle, \\
&\quad \Big\langle\Big(1-\Big(\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-(1-\lambda_T)^{a\frac{1}{a}}\Big)^{q_g}\Big)\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\lambda_I^{a\frac{1}{a}}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\lambda_F^{a\frac{1}{a}}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big\rangle\Bigg), \\
&= \Bigg(\Big\langle\Big[\Big(1-\Big(\Big(1-(1-(1-T^{L}))^{\sum_{g=1}^{a} q_g}\Big)^{a!}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \Big(1-\Big(\Big(1-(1-(1-T^{U}))^{\sum_{g=1}^{a} q_g}\Big)^{a!}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big], \Big[1-\Big(1-\Big(1-(1-I^{L})^{\sum_{g=1}^{a} q_g}\Big)^{a!\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \\
&\quad 1-\Big(1-\Big(1-(1-I^{U})^{\sum_{g=1}^{a} q_g}\Big)^{a!\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big], \Big[1-\Big(1-\Big(1-(1-F^{L})^{\sum_{g=1}^{a} q_g}\Big)^{a!\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\Big(1-\Big(1-(1-F^{U})^{\sum_{g=1}^{a} q_g}\Big)^{a!\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big]\Big\rangle, \\
&\quad \Big\langle\Big(1-\Big(\Big(1-(1-(1-\lambda_T))^{\sum_{g=1}^{a} q_g}\Big)^{a!}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\Big(1-\Big(1-(1-\lambda_I)^{\sum_{g=1}^{a} q_g}\Big)^{a!\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\Big(1-\Big(1-(1-\lambda_F)^{\sum_{g=1}^{a} q_g}\Big)^{a!\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big\rangle\Bigg), \\
&= (\langle[T^{L}, T^{U}], [I^{L}, I^{U}], [F^{L}, F^{U}]\rangle, \langle\lambda_T, \lambda_I, \lambda_F\rangle) = z.
\end{aligned}$$

This is the required proof of Theorem 3. □

Theorem 4. *(Boundedness) Let $z_g (g = 1, 2, \ldots, a)$ be a group of NCNs. Where*

$$\overset{-}{z} = \min(z_1, z_2, \ldots, z_a) = (\langle[T^{-L}, T^{-U}], [I^{+L}, I^{+U}], [F^{+L}, F^{+U}]\rangle, \langle\lambda_T{}^{-}, \lambda_I{}^{+}, \lambda_F{}^{+}\rangle), \text{ and}$$
$$\overset{+}{z} = \max(z_1, z_2, \ldots, z_a) = (\langle[T^{+L}, T^{+U}], [I^{-L}, I^{-U}], [F^{-L}, F^{-U}]\rangle, \langle\lambda_T{}^{+}, \lambda_I{}^{-}, \lambda_F{}^{-}\rangle).$$

Then,

$$m \leq NCPMM^{Q}(z_1, z_2, \ldots, z_a) \leq n \tag{27}$$

where,

$$\begin{aligned}
m &= \Bigg(\Big\langle\Big[\Big(1-\Big(\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\Big(1-\overset{-}{T}{}^{L}_{\theta(g)}\Big)^{a\Theta_g}\Big)^{q_g}\Big)\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \Big(1-\Big(\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\Big(1-\overset{-}{T}{}^{U}_{\theta(g)}\Big)^{a\Theta_g}\Big)^{q_g}\Big)\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big], \Big[1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\overset{+}{I}{}^{La\Theta_g}_{\theta(g)}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \\
&\quad 1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\overset{+}{I}{}^{Ua\Theta_z}_{\theta(g)}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big], \Big[1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\overset{+}{F}{}^{La\Theta_z}_{\theta(g)}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\overset{+}{F}{}^{Ua\Theta_g}_{\theta(g)}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big]\Big\rangle, \\
&\quad \Big\langle\Big(1-\Big(\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\Big(1-\big(\overset{-}{\lambda}_T\big)_{\theta(g)}\Big)^{a\Theta_g}\Big)^{q_g}\Big)\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{z=1}^{a} q_g}}, 1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\big(\overset{+}{\lambda}_I\big)^{a\Theta_g}_{\theta(g)}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\big(\overset{+}{\lambda}_F\big)^{a\Theta_g}_{\theta(g)}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big\rangle\Bigg).
\end{aligned}$$

and

$$\begin{aligned}
n &= \Bigg(\Big\langle\Big[\Big(1-\Big(\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\Big(1-\overset{+}{T}{}^{L}_{\theta(g)}\Big)^{a\Theta_g}\Big)^{q_g}\Big)\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \Big(1-\Big(\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\Big(1-\overset{+}{T}{}^{U}_{\theta(g)}\Big)^{a\Theta_g}\Big)^{q_g}\Big)\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big], \Big[1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\overset{-}{I}{}^{La\Theta_g}_{\theta(g)}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \\
&\quad 1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\overset{-}{I}{}^{Ua\Theta_z}_{\theta(g)}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big], \Big[1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\overset{-}{F}{}^{La\Theta_z}_{\theta(g)}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\overset{-}{F}{}^{Ua\Theta_g}_{\theta(g)}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big]\Big\rangle, \\
&\quad \Big\langle\Big(1-\Big(\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\Big(1-\big(\overset{+}{\lambda}_T\big)_{\theta(g)}\Big)^{a\Theta_g}\Big)^{q_g}\Big)\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{z=1}^{a} q_g}}, 1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\big(\overset{-}{\lambda}_I\big)^{a\Theta_g}_{\theta(g)}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1-\Big(1-\prod_{\theta\in S_a}\Big(1-\prod_{g=1}^{a}\Big(1-\big(\overset{-}{\lambda}_F\big)^{a\Theta_g}_{\theta(g)}\Big)^{q_g}\Big)^{\frac{1}{a!}}\Big)^{\frac{1}{\sum_{g=1}^{a} q_g}}\Big\rangle\Bigg).
\end{aligned}$$

Proof.

$$\begin{aligned}a\Theta_g z_{\theta(g)} &= \left(\left\langle\left[1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}, 1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right], \left[(I^L)_{\theta(g)}^{a\Theta_g}, (I^U)_{\theta(g)}^{a\Theta_g}\right], \left[(F^L)_{\theta(g)}^{a\Theta_g}, (F^U)_{\theta(g)}^{a\Theta_g}\right]\right\rangle, \left\langle 1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}, (\lambda_I)_{\theta(g)}^{a\Theta_g}, (\lambda_F)_{\theta(g)}^{a\Theta_g}\right\rangle\right) \\ &\geq \left(\left\langle\left[1-\left(1-\left(\overset{-}{T}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}, 1-\left(1-\left(\overset{-}{T}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right], \left[\left(\overset{+}{I}^{L}\right)_{\theta(g)}^{a\Theta_g}, \left(\overset{+}{I}^{U}\right)_{\theta(g)}^{a\Theta_g}\right], \left[\left(\overset{+}{F}^{L}\right)_{\theta(g)}^{a\Theta_g}, \left(\overset{+}{F}^{U}\right)_{\theta(g)}^{a\Theta_g}\right]\right\rangle, \left\langle 1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}, \left(\overset{+}{\lambda}_I\right)_{\theta(g)}^{a\Theta_g}, \left(\overset{+}{\lambda}_F\right)_{\theta(g)}^{a\Theta_g}\right\rangle\right).\end{aligned}$$

and

$$\begin{aligned}\left(a\Theta_g z_{\theta(g)}\right)^{q_g} &= \left(\left\langle\left[\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, \left(1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right], \left[,1-\left(1-(I^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g} 1-\left(1-(I^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right],\right.\right. \\ &\left.\left[1-\left(1-(F^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g} 1-\left(1-(F^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right]\right\rangle, \left.\left\langle\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, 1-\left(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\left(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right\rangle\right) \\ &\geq \left(\left\langle\left[\left(1-\left(1-\left(\overset{-}{T}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, \left(1-\left(1-\left(\overset{-}{T}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right], \left[,1-\left(1-\left(\overset{+}{I}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g} 1-\left(1-\left(\overset{+}{I}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right],\right.\right. \\ &\left.\left[1-\left(1-\left(\overset{+}{F}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g} 1-\left(1-\left(\overset{+}{F}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right]\right\rangle, \left.\left\langle\left(1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, 1-\left(1-\left(\overset{+}{\lambda}_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\left(1-\left(\overset{+}{\lambda}_F\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right\rangle\right).\end{aligned}$$

Thus,

$$\begin{aligned}&\prod_{g=1}^{a}\left(a\Theta_g z_{\theta(g)}\right)^{q_g} = \left(\left\langle\left[\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, \prod_{g=1}^{a}\left(1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right], \left[1-\prod_{g=1}^{a}\left(1-(I^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-(I^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right],\right.\right. \\ &\left.\left[1-\prod_{g=1}^{a}\left(1-(F^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-(F^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right]\right\rangle, \left.\left\langle\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right\rangle\right) \geq \\ &\left(\left\langle\left[\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, \prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right], \left[1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right],\right.\right. \\ &\left.\left[1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right]\right\rangle, \left.\left\langle\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_F\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right\rangle\right).\end{aligned}$$

and

$$\begin{aligned}&\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(a\Theta_g z_{\theta(g)}\right)^{q_g} = \left(\left\langle\left[1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right), 1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right], \left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right],\right.\right. \\ &\left.\left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right]\right\rangle, \left.\left\langle 1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right\rangle\right) \\ &\geq \left(\left\langle\left[1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right), 1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right], \left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right],\right.\right. \\ &\left.\left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right]\right\rangle, \left.\left\langle 1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right), \prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_F\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right\rangle\right).\end{aligned}$$

Furthermore,

$$\begin{aligned}&\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(a\Theta_g z_{\theta(g)}\right)^{q_g} = \left(\left\langle\left[1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, 1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right],\right.\right. \\ &\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right], \left.\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right]\right\rangle, \\ &\left.\left\langle 1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-(\lambda_F)_{\theta(z)}^{a\Theta_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}\right\rangle\right) \\ &\geq \left(\left\langle\left[1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, 1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right],\right.\right. \\ &\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right], \left.\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right]\right\rangle, \\ &\left.\left\langle 1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(\overset{+}{\lambda}_F\right)_{\theta(z)}^{a\Theta_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}\right\rangle\right).\end{aligned}$$

Hence,

$$\begin{aligned}&\left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(a\Theta_z\Gamma_{\theta(z)}\right)^{q_z}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}} = \left(\left\langle\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\right.\right. \\ &\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right], \left.\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle, \\ &\left.\left\langle\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right) \\ &\geq \left(\left\langle\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{T}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right], \left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\right.\right.\right. \\ &\left.1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{I}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right], \left.\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{F}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle, \\ &\left.\left\langle\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_F\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right).\end{aligned}$$

This implies that $m \leq NCPMM^{Q}(z_1, z_2, \ldots, z_a)$. $\square$

In a similar way we can show that $NCPMM^{Q}(z_1, z_2, \ldots, z_a) \leq n$. Hence, $m \leq NCPMM^{Q}(z_1, z_2, \ldots, z_a) \leq n$.

The NCPMM operator does not have the property of monotonicity.

One of the leading advantages of NCPMM is its capacity to represent the interrelationship among NCNs. Furthermore, the NCPMM operator is more flexible in aggregation process due to parameter vector. Now, we discuss some special cases of NCPMM operators by assigning different values to the parameter vector.

Case 1. If $Q = (1, 0, \ldots, 0)$, then the NCPMM operator degenerates into the following form:

$$NCPMM^{(1,0,\ldots,0)}(z_1, z_2, \ldots, z_a) = \left(\sum_{g=1}^{a} \frac{(1 + T(z_g))}{\sum\limits_{x=1}^{a} (1 + T(z_x))} z_g \right). \tag{28}$$

This is the NC power averaging operator.

Case 2. If $Q = \left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a} \right)$, then the NCPMM operator degenerates into the following form:

$$NCPMM^{\left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a}\right)}(z_1, z_2, \ldots, z_a) = \prod_{g=1}^{a} z_g^{\frac{(1+T(z_{\theta(g)}))}{\sum_{x=1}^{a}(1+T(z_x))}}. \tag{29}$$

This is the NC power geometric operator.

Case 3. If $Q = (1, 1, \ldots, 0)$, then the NCPMM operator degenerates into the following form:

$$\begin{aligned}
&NCPMM^{(1,1,0,\ldots,0)} = (z_1, z_2, \ldots, z_a) = \\
&\left(\left\langle \left[\left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - \left(1 - T_g^L \right)^{\Theta_g} \right) \left(1 - \left(1 - T_x^L \right)^{\Theta_x} \right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}}, \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - \left(1 - T_g^U \right)^{\Theta_g} \right) \left(1 - \left(1 - T_x^U \right)^{\Theta_x} \right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}} \right], \left[1 - \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - I_g^{L\Theta_g} \right) \left(1 - I_x^{L\Theta_x} \right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}}, \right. \right. \\
&\left. 1 - \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - I_g^{U\Theta_g} \right) \left(1 - I_x^{U\Theta_x} \right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}} \right], \left[1 - \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - F_g^{L\Theta_g} \right) \left(1 - F_x^{L\Theta_x} \right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}}, 1 - \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - F_g^{U\Theta_g} \right) \left(1 - F_x^{U\Theta_x} \right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}} \right], \\
&\left. \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - \left(1 - (\lambda_T)_g \right)^{\Theta_z} \right) \left(1 - \left(1 - T_x \right)^{\Theta_x} \right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}}, 1 - \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - (\lambda_I)_g^{\Theta_g} \right) \left(1 - (\lambda_T)_x^{\Theta_x} \right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}}, 1 - \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - (\lambda_F)_z^{\Theta_z} \right) \left(1 - (\lambda_F)_x^{\Theta_x} \right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}} \right\rangle \right).
\end{aligned} \tag{30}$$

This is the NC power Bonferroni mean operator $(p = q = 1)$.

Case 4. If $Q = \left(\overbrace{1,1,\ldots,1}^{i},\overbrace{0,0,\ldots,0}^{z-i}\right)$, then the NCPMM operator degenerates into the following form:

$$
\begin{aligned}
&NCPMM^{(\overbrace{1,1,..,1}^{i},\overbrace{0,0,\ldots,0}^{z-i})} = (z_1,z_2,\ldots,z_a) = \\
&\left\langle \left[\left(1-\prod_{1\le y_1<y_2<\ldots.<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-T^L_{g_x}\right)^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}, \left(1-\prod_{1\le y_1<y_2<\ldots.<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-T^U_{g_x}\right)^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}} \right], \left[,1-\left(1-\prod_{1\le y_1<y_2<\ldots.<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-I^{L\Theta_{g_x}}_{g_x}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}, \right.\\
&\left. 1-\left(1-\prod_{1\le y_1<y_2<\ldots.<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-I^{U\Theta_{g_x}}_{g_x}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\right], \left[1-\left(1-\prod_{1\le y_1<y_2<\ldots.<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-F^{L\Theta_{g_x}}_{g_x}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}, 1-\left(1-\prod_{1\le y_1<y_2<\ldots.<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-F^{U\Theta_{g_x}}_{g_x}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\right]\right\rangle, \\
&\left\langle \left(1-\prod_{1\le y_1<y_2<\ldots.<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(\lambda_T)_{g_x}\right)^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}, 1-\left(1-\prod_{1\le y_1<y_2<\ldots.<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-(\lambda_I)^{\Theta_{g_x}}_{g_x}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}, 1-\left(1-\prod_{1\le y_1<y_2<\ldots.<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-(\lambda_F)^{\Theta_{g_x}}_{g_x}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}} \right\rangle.
\end{aligned}
\tag{31}
$$

This is the NC power Maclaurin symmetric mean operator.

3.2. Weighted Neutrosophic Cubic Power Muirhead Mean (WNCPMM) Operator

The NCPMM operator does not consider the weight of the aggregated NCNs. In this subsection, we develop the WNCPMM operator, which has the capacity of taking the weights of NCNs.

Definition 12. *Let $z_g(g = 1,2,\ldots,a)$ be a group of NCNs and $Q = (q_1,q_2,\ldots,q_a) \in R^a$ be a vector of parameters. If,*

$$
WNCPMM^Q(z_1,z_2,\ldots.,z_a) = \left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(\frac{a\Xi_{\theta(g)}\Theta_{\theta(g)}}{\sum\limits_{x=1}^{a}\Xi_x\Theta_x}z_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{\sum\limits_{g=1}^{a}q_g}}
\tag{32}
$$

then, we $WNCPMM^Q$ the weighted neutrosophic cubic power Muirhead mean operator, where $\Xi = (\Xi_1,\Xi_2,\ldots,\Xi_a)^T$ is the weight vector of $z_g(g = 1,2,\ldots,a)$ such that $\Xi_z \in [0,1]$, $\sum\limits_{z=1}^{a}\Xi_z = 1$, S_a is the group of all permutation, $\theta(z)$ is any permutation of $(1,2,\ldots,a)$ and Θ_g is power weight vector (PWV) satisfying $\Theta_g = \frac{(1+T(z_g))}{\sum\limits_{g=1}^{a}(1+T(z_g))}$, $\sum\limits_{g=1}^{a}\Theta_g = 1$, $T(z_x) = \sum\limits_{x=1,x\neq g}^{a} Supp(z_g,z_x)$, $Supp(z_g,z_x)$ is the support degree for z_g and z_x, satisfying the following axioms:

(1) $Supp(z_g,z_x) \in [0,1]$;
(2) $Supp(z_g,z_x) = Supp(z_x,z_g)$;
(3) *If $\overline{\overline{D}}(z_g,z_x) < \overline{\overline{D}}(z_u,z_v)$, then $Supp(z_g,z_x) > Supp(z_u,z_v)$, where $\overline{\overline{D}}(z_g,z_x)$ is distance among z_g and z_x.*

From Definition 12, we have the following Theorem 5.

Theorem 5. *Let $z_g(g = 1,2,\ldots,a)$ be a group of NCNs and $Q = (q_1,q_2,\ldots,q_a) \in R^a$ be a vector of parameters. Then, the aggregated value obtained by using Equation (32) is still an NCN and*

$$
\begin{aligned}
&WNCPMM^Q(z_1,z_2,\ldots,z_a) = \\
&\left(\left\langle \left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^L)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(T^U)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right], \left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^L)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \right.\right.\\
&\left. 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(I^U)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right], \left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^L)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(F^U)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle, \\
&\left\langle \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_F)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right).
\end{aligned}
\tag{33}
$$

Proof. Proof of Theorem 5 is same as Theorem 2. □

3.3. The Neutrosophic Cubic Power Dual Muirhead Mean (NCPDMM) Operator

In this subsection, we develop the NCPDMM operator and discuss some related properties.

Definition 13. *Let $z_g (g = 1, 2, \ldots, a)$ be a group of NCNs and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. If,*

$$NCPDMM^Q(z_1, z_2, \ldots, z_a) = \frac{1}{\sum_{g=1}^{a} q_g} \left(\prod_{\theta \in S_a} \sum_{g=1}^{a} \left(q_g z_{\theta(g)}^{\frac{a(1+T(z_{\theta(g)}))}{\sum_{x=1}^{a}(1+T(z_x))}} \right) \right)^{\frac{1}{a!}} \tag{34}$$

then, we call $NCPDMM^Q$ the neutrosophic cubic power dual Muirhead mean operator, where S_a is the group of all permutation, $\theta(g)$ is any permutation of $(1, 2, \ldots, a)$ and $T(z_x) = \sum_{x=1, x \neq g}^{a} Supp(z_g, z_x)$, $Supp(z_g, z_x)$ is the support degree for z_g and z_x, satisfying the following axioms:

(1) $Supp(z_g, z_x) \in [0, 1]$;

(2) $Supp(z_g, z_x) = Supp(z_x, z_g)$;

(3) *If $\overline{\overline{D}}(z_g, z_x) < \overline{\overline{D}}(z_u, z_v)$, then $Supp(z_g, z_x) > Supp(z_u, z_v)$, where $\overline{\overline{D}}(z_g, z_x)$ is distance among z_g and z_x.*

To write Equation (34) in a simple form, we can specify it as:

$$\Theta_g = \frac{(1 + T(z_g))}{\sum_{x=1}^{a}(1 + T(z_x))}. \tag{35}$$

For suitability, we can call $(\Theta_1, \Theta_2, \ldots, \Theta_a)^T$ the power weight vector (PMV), such that $\Theta_g \in [0, 1]$ and $\sum_{g=1}^{a} \Theta_g = 1$. From, the use of Equation (35), Equation (34) can be expressed as,

$$NCPDMM^Q(z_1, z_2, \ldots, z_a) = \frac{1}{\sum_{g=1}^{a} q_g} \left(\prod_{\theta \in S_a} \sum_{g=1}^{a} \left(q_g z_{\theta(g)}^{a\Theta_{\theta(g)}} \right) \right)^{\frac{1}{a!}}. \tag{36}$$

Theorem 6. *Let $z_g (g = 1, 2, \ldots, a)$ be a group of SVNNs and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. Then, the aggregated value obtained by using Equation (36) is still an NCN and,*

$$\begin{gathered} NCPDMM^Q(z_1, z_2, \ldots, z_a) = \\ \left(\left\langle \left[1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (T^L)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (T^U)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right], \left[\left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (I^L)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \right. \right. \right. \\ \left. \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (I^U)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right], \left. \left[\left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (F^L)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (F^U)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right] \right\rangle \\ \left. \left\langle 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - (\lambda_T)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{z=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (\lambda_I)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - (\lambda_F)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right\rangle \right). \end{gathered} \tag{37}$$

Proof. Proof of Theorem 6 is similar to that of Theorem 2. □

Theorem 7 (Idempotency)**.** *Let $z_g (g = 1, 2, \ldots, a)$ be a group of NCNs, and $z_g = z$, for all $g = 1, 2, \ldots, a$. Then,*

$$NCPDMM^Q(z_1, z_2, \ldots, z_a) = z. \tag{38}$$

Theorem 8 (Boundedness)**.** *Let $z_g(g = 1,2,\ldots,a)$ be a group of NCNs, $\overset{-}{z} = \min(z_1,z_2,\ldots,z_a) = \left(\left\langle\left[\overset{-}{T}^{L},\overset{-}{T}^{U}\right],\left[\overset{+}{I}^{L},\overset{+}{I}^{U}\right],\left[\overset{+}{F}^{L},\overset{+}{F}^{U}\right]\right\rangle,\left\langle{\lambda_T}^{-},{\lambda_I}^{+},{\lambda_F}^{+}\right\rangle\right)$, and $z^{+} = \max(z_1,z_2,\ldots,z_a) = \left(\left\langle\left[\overset{+}{T}^{L},\overset{+}{T}^{U}\right],\left[\overset{-}{I}^{L},\overset{-}{I}^{U}\right],\left[\overset{-}{F}^{L},\overset{-}{F}^{U}\right]\right\rangle,\left\langle{\lambda_T}^{+},{\lambda_I}^{-},{\lambda_F}^{-}\right\rangle\right)$.*

Then,

$$m \leq NCPDMM^{Q}(z_1,z_2,\ldots,z_a) \leq n. \tag{39}$$

where,

$$m = \left(\left\langle\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{-}{T}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{-}{T}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right], \left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{I}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\right.\right.\right.$$
$$\left.\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{I}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right], \left.\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{F}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{F}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle,$$
$$\left.\left\langle 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(z)}^{a\Theta_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{\lambda}_I\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{\lambda}_F\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right),$$

and

$$n = \left(\left\langle\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{T}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{T}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right], \left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{I}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\right.\right.\right.$$
$$\left.\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{I}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right], \left.\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{F}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{F}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle,$$
$$\left.\left\langle 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(\overset{+}{\lambda}_T\right)_{\theta(z)}^{a\Theta_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_I\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_F\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right).$$

Now we will discuss some special cases of NCPDMM operator with respect to the parameter vector Q.

Case 1. If $Q = (1,0,\ldots,0)$, then NCPDMM operators degenerate into the following form:

$$NCPDMM^{(1,0,\ldots,0)}(z_1,z_2,\ldots,z_a) = \left(\prod_{g=1}^{a} z_g^{\frac{(1+T(z_g))}{\sum_{x=1}^{a}(1+T(z_x))}}\right) \tag{40}$$

This is the NC power geometric averaging operator.

Case 2. If $Q = \left(\frac{1}{a},\frac{1}{a},\ldots,\frac{1}{a}\right)$, then NCPMM operators degenerate into the following form:

$$NCPDMM^{\left(\frac{1}{a},\frac{1}{a},\ldots,\frac{1}{a}\right)}(z_1,z_2,\ldots,z_a) = \sum_{g=1}^{a}\frac{(1+T(z_g))}{\sum_{x=1}^{a}(1+T(z_x))}z_g \tag{41}$$

This is NC power arithmetic averaging operator.

Case 3. If $Q = (1, 1, 0, \ldots, 0)$, then NCPDMM operators degenerate into the following form:

$$
\begin{aligned}
& NCPDMM^{(1,1,0,\ldots,0)}(z_1, z_2, \ldots, z_a) = \left(\left\langle \left[1 - \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - (T^L)_g^{\Theta_g}\right)\left(1 - (T^L)_x^{\Theta_x}\right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}}, 1 - \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - (T^U)_g^{\Theta_g}\right)\left(1 - (T^U)_x^{\Theta_x}\right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}} \right], \right. \right. \\
& \left[\left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - (1 - (I^L)_g)^{\Theta_g}\right)\left(1 - (1 - (I^L)_x)^{\Theta_x}\right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}}, \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - (1 - (I^U)_g)^{\Theta_g}\right)\left(1 - (1 - (I^U)_x)^{\Theta_x}\right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}} \right], \\
& \left. \left[\left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - (1 - (F^L)_g)^{\Theta_g}\right)\left(1 - (1 - (F^L)_x)^{\Theta_x}\right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}}, \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - (1 - (F^U)_g)^{\Theta_g}\right)\left(1 - (1 - (F^U)_x)^{\Theta_x}\right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}} \right] \right\rangle, \\
& \left\langle 1 - \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - (\lambda_T)_g^{\Theta_g}\right)\left(1 - (\lambda_T)_x^{\Theta_x}\right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}}, \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - (1 - (\lambda_I)_g)^{\Theta_g}\right)\left(1 - (1 - (\lambda_I)_x)^{\Theta_x}\right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}}, \right. \\
& \left. \left. \left(1 - \left(\prod_{\substack{g,x=1 \\ g \neq x}}^{a} \left(1 - \left(1 - (1 - (\lambda_F)_g)^{\Theta_g}\right)\left(1 - (1 - (\lambda_F)_x)^{\Theta_x}\right) \right) \right)^{\frac{1}{a^2-a}} \right)^{\frac{1}{2}} \right\rangle \right).
\end{aligned}
\tag{42}
$$

This is the NC power geometric Bonferroni mean operator ($p = q = 1$).

Case 4. If $Q = \left(\overbrace{1, 1, \ldots, 1}^{i}, \overbrace{0, 0, \ldots, 0}^{z-i} \right)$, then the NCPDMM operator degenerates into the following form:

$$
\begin{aligned}
& NCPDMM^{(\overbrace{1,1,..,1}^{i},\overbrace{0,0,\ldots,0}^{z-i})}(z_1, z_2, \ldots, z_a) = \\
& \left(\left\langle \left[1 - \left(1 - \prod_{1 \leq y_1 < y_2 < \ldots < y_i \leq a} \left(1 - \prod_{x=1}^{i} \left(1 - (T^L)_{g_x}^{\Theta_{z_g}} \right) \right)^{\frac{1}{C_a^i}} \right)^{\frac{1}{k}}, 1 - \left(1 - \prod_{1 \leq y_1 < y_2 < \ldots < y_i \leq a} \left(1 - \prod_{x=1}^{i} \left(1 - (T^U)_{g_x}^{\Theta_{z_g}} \right) \right)^{\frac{1}{C_a^i}} \right)^{\frac{1}{k}} \right], \left[\left(1 - \prod_{1 \leq y_1 < y_2 < \ldots < y_i \leq a} \left(1 - \prod_{x=1}^{i} \left(1 - \left(1 - (I^L)_{g_x} \right)^{\Theta_{gx}} \right) \right)^{\frac{1}{C_a^i}} \right)^{\frac{1}{k}}, \right. \right. \right. \\
& \left. \left(1 - \prod_{1 \leq y_1 < y_2 < \ldots < y_i \leq a} \left(1 - \prod_{x=1}^{i} \left(1 - \left(1 - (I^U)_{g_x} \right)^{\Theta_{gx}} \right) \right)^{\frac{1}{C_a^i}} \right)^{\frac{1}{k}} \right], \left. \left[\left(1 - \prod_{1 \leq y_1 < y_2 < \ldots < y_i \leq a} \left(1 - \prod_{x=1}^{i} \left(1 - \left(1 - (F^L)_{g_x} \right)^{\Theta_{gx}} \right) \right)^{\frac{1}{C_a^i}} \right)^{\frac{1}{k}}, \left(1 - \prod_{1 \leq y_1 < y_2 < \ldots < y_i \leq a} \left(1 - \prod_{x=1}^{i} \left(1 - \left(1 - (F^U)_{g_x} \right)^{\Theta_{gx}} \right) \right)^{\frac{1}{C_a^i}} \right)^{\frac{1}{k}} \right] \right\rangle \\
& \left. \left\langle 1 - \left(1 - \prod_{1 \leq y_1 < y_2 < \ldots < y_i \leq a} \left(1 - \prod_{x=1}^{i} \left(1 - (\lambda_T)_{g_x}^{\Theta_{gx}} \right) \right)^{\frac{1}{C_a^i}} \right)^{\frac{1}{k}}, \left(1 - \prod_{1 \leq y_1 < y_2 < \ldots < y_i \leq a} \left(1 - \prod_{x=1}^{i} \left(1 - \left(1 - (\lambda_I)_{g_x} \right)^{\Theta_{gx}} \right) \right)^{\frac{1}{C_a^i}} \right)^{\frac{1}{k}}, \left(1 - \prod_{1 \leq y_1 < y_2 < \ldots < y_i \leq a} \left(1 - \prod_{x=1}^{i} \left(1 - \left(1 - (\lambda_F)_{g_x} \right)^{\Theta_{gx}} \right) \right)^{\frac{1}{C_a^i}} \right)^{\frac{1}{k}} \right\rangle \right).
\end{aligned}
\tag{43}
$$

This is the NC power dual Maclaurin symmetric mean operator.

3.4. Weighted Neutrosophic Cubic Power Dual Muirhead Mean (WNCPDMM) Operator

The NCPDMM operator does not consider the weight of the aggregated NCNs. In this subsection, we develop the WNCPDMM operator, which has the capacity of taking the weights of NCNs.

Definition 14. *Let $z_g (g = 1, 2, \ldots, a)$ be a group of NCNs and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. If,*

$$
WNCPDMM^{Q}(z_1, z_2, \ldots, z_a) = \frac{1}{\sum\limits_{g=1}^{a} q_g} \left(\prod_{\theta \in S_a} \sum_{g=1}^{a} \left(q_g z_{\theta(g)}^{\frac{a \Xi_{\theta(g)} \Theta_{\theta(g)}}{\sum_{x=1}^{a} \Xi_x \Theta_x}} \right) \right)^{\frac{1}{a!}} \tag{44}
$$

then, we call $WNCPDMM^{Q}$ the weighted neutrosophic cubic power dual Muirhead mean operator, where $\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_a)^T$ is the weight vector of $z_g (g = 1, 2, \ldots, a)$ such that $\Xi_g \in [0, 1]$, $\sum_{g=1}^{a} \Xi_g = 1$, S_a is the group of all permutation, $\theta(g)$ is any permutation of $(1, 2, \ldots, a)$ and Θ_g is PVW satisfying $\Theta_g = \frac{(1+T(z_g))}{\sum_{g=1}^{a}(1+T(z_g))}$, $\sum_{g=1}^{a} \Theta_g = 1$, $T(z_x) = \sum_{x=1, x \neq g}^{a} Supp(z_g, z_x)$, and $Supp(z_g, z_x)$ is the support degree for z_g and z_x, satisfying the following axioms:

(1) $Supp(z_g, z_x) \in [0, 1]$;

(2) $Supp(z_g, z_x) = Supp(z_x, z_g)$;

(3) *If $\overline{\overline{D}}(z_g, z_x) < \overline{\overline{D}}(z_u, z_v)$, then $Supp(z_g, z_x) > Supp(z_u, z_v)$, where $\overline{\overline{D}}(z_g, z_x)$ is distance among z_g and z_x.*

From Definition 14, we have the following Theorem 9.

Theorem 9. *Let $z_g (g = 1, 2, \ldots, a)$ be a group of NCNs and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. Then, the aggregated value obtained by using Equation (44) is still an NCN and*

$$\begin{aligned} WNCPDMM^{Q}(z_1, z_2, \ldots, z_a) = \Bigg\langle & \Bigg[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\binom{L}{T}_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\binom{U}{T}_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg], \\ & \Bigg[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\binom{L}{I}_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\binom{U}{I}_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg], \\ & \Bigg[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\binom{L}{F}_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\binom{U}{F}_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg]\Bigg\rangle, \\ & 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_T)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\lambda_I)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}} \\ & , \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\lambda_F)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Xi_{\theta(g)}}{\sum_{x=1}^{a}\Theta_x\Xi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg\rangle\Bigg). \end{aligned} \tag{45}$$

Proof. Proof of Theorem 9 is similar to that of Theorem 2. □

4. The MADM Approach Based on WNCPMM Operator and WNCPDMM Operator

In this section, we give a novel method to MADM with NCNs, in which the attributes values gain the form of NCNs. For a MADM problem, let the series of alternatives is represented by $\hbar = \{\hbar_1, \hbar_2, \ldots, \hbar_a\}$, and the series of attributes is represented by $ƛ = \{ƛ_1, ƛ_2, \ldots, ƛ_b\}$. The weight vector of the attributes is denoted by $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_b)^T$ such that $\varpi_p \in [0, 1]$, $\sum_{p=1}^{b} \varpi_p = 1$. Assume that $z_{gh} = \left(\left\langle \left[T_{gh}^{L}, T_{gh}^{U}\right], \left[I_{gh}^{L}, I_{gh}^{U}\right], \left[F_{gh}^{L}, F_{gh}^{U}\right]\right\rangle, \left\langle \lambda_{T_{gh}}, \lambda_{I_{gh}}, \lambda_{F_{gh}}\right\rangle\right)$ is the assessment values of the alternatives $\hbar_g$ on the attribute l_h, which is expressed by the form of NCN. Then, the main aim is to rank the alternatives. The following decision steps are to be followed.

Step 1. Standardize the decision matrix. Generally, there are two types of attributes, one is of cost type and the other is of benefit type. We need to convert the cost type of attributes into benefit types of attributes by using the following formula:

$$\begin{aligned} z_{gh} &= \left(\left\langle \left[T_{gh}^{L}, T_{gh}^{U}\right], \left[I_{gh}^{L}, I_{gh}^{U}\right], \left[F_{gh}^{L}, F_{gh}^{U}\right]\right\rangle, \left\langle \lambda_{T_{gh}}, \lambda_{I_{gh}}, \lambda_{F_{gh}}\right\rangle\right), \\ &= \begin{cases} \left(\left\langle \left[T_{gh}^{L}, T_{gh}^{U}\right], \left[I_{gh}^{L}, I_{gh}^{U}\right], \left[F_{gh}^{L}, F_{gh}^{U}\right]\right\rangle, \lambda_{T_{gh}}, \lambda_{I_{gh}}, \lambda_{F_{gh}}\right\rangle), & \text{for benefit attribute } \Gamma_h, \\ \left(\left\langle \left[F_{gh}^{L}, F_{gh}^{U}\right], \left[1-I_{gh}^{U}, 1-I_{gh}^{L}\right], \left[T_{gh}^{L}, T_{gh}^{U}\right]\right\rangle, \lambda_{T_{gh}}, 1-\lambda_{I_{gh}}, \lambda_{F_{gh}}\right\rangle), & \text{for cost attribute } \Gamma_h. \end{cases} \end{aligned} \tag{46}$$

Hence, the decision matrix $M = \left[z_{gh}\right]_{a\times b}$ can be transformed into normalized decision matrix $N = \left[\delta_{gh}\right]_{a\times b}$.

Step 2. Determine the supports $Supp\left(\delta_{gh}, \delta_{gl}\right)(g = 1, 2, \ldots, a; h, l = 1, 2, \ldots, b)$ by,

$$Supp\left(\delta_{gh}, \delta_{gl}\right) = 1 - \overline{\overline{D}}\left(\delta_{gh}, \delta_{gh}\right) \tag{47}$$

where, $\overline{\overline{D}}\left(\delta_{gh}, \delta_{gh}\right)$ is the distance measure among two NCNs δ_{gh} and δ_{gl} defined in Equation (25).

Step 3. Determine $T\left(\delta_{gh}\right)$ by,

$$T\left(\delta_{gh}\right) = \sum_{\substack{l=1\\ l\neq h}}^{b} Supp\left(\delta_{gh}, \delta_{gl}\right)(g = 1, 2, \ldots, a; h, l = 1, 2, \ldots, b) \tag{48}$$

Step 4. Determine the weights related with the NCN $\delta_{gh}(g = 1, 2, \ldots, a; h = 1, 2, \ldots, b)$ with the formula

$$\Psi_{gh} = \frac{b\varpi_h\left(1 + T\left(\delta_{gh}\right)\right)}{\sum_{d=1}^{b} \varpi_d\left(1 + T\left(\delta_{gh}\right)\right)}(g = 1, 2, \ldots, a; h, d = 1, 2, \ldots, b), \tag{49}$$

where, $T\left(\delta_{gh}\right) = \sum_{\substack{l=1\\ l\neq h}}^{b} Supp\left(\delta_{gh}, \delta_{gl}\right)(g = 1, 2, \ldots, a; h, l = 1, 2, \ldots, b)$ is weighted support of NCN δ_{gh} by the other NCN $\delta_{gl}(g = 1, 2, \ldots, a; h, l = 1, 2, \ldots, b)$.

Step 5. Use the WNCPMM or WNCPDMM operators

$$\delta_g = \left\langle \left[T_g^L, T_g^U\right], \left[I_g^L, I_g^U\right], \left[F_g^L, F_g^U\right], \lambda_{Tg}, \lambda_{Ig}, \lambda_{Fg} \right\rangle = WNCPMM^Q\left(\delta_{g1}, \delta_{g2}, \ldots, \delta_{gb}\right) \tag{50}$$

or

$$\delta_g = \left\langle \left[T_g^L, T_g^U\right], \left[I_g^L, I_g^U\right], \left[F_g^L, F_g^U\right], \lambda_{Tg}, \lambda_{Ig}, \lambda_{Fg} \right\rangle = WNCPDMM^Q\left(\delta_{g1}, \delta_{g2}, \ldots, \delta_{gb}\right) \tag{51}$$

to calculate the overall NCNs, $\delta_g(g = 1, 2, \ldots, a)$.

Step 6. Determine the score values of the collective NCNs $\delta_g(g = 1, 2, \ldots, a)$, using Definition 6.

Step 7. Rank all the alternatives according to their score values, and the select the best one using Theorem 1.

5. An Illustrative Example

To show the application of the developed MADM method, an illustrative example is embraced from [19,21] with NC information.

Example 1. *A passenger wants to travel and select the best vans (alternatives) $\hbar_g(g = 1, 2, 3, 4)$ among the possible four vans. The customer takes the following four attributes into account to evaluate the possible four alternatives: (1) the facility λ_1; (2) saving rent λ_2; (3) comfort λ_3; (4) safety λ_4. The importance degree of the attributes is expressed by $\varpi = (0.5, 0.25, 0.125, 0.125)^T$. Therefore, the following decision matrix $M = \left[z_{gh}\right]_{4\times 4}$ can be obtained in the form of NCNs shown in Table 1.*

Table 1. The decision matrix $M = \left[CN_{gh}\right]_{4\times 4}$.

	$\lambdabar_1$	$\lambdabar_2$	$\lambdabar_3$	$\lambdabar_4$
$\hbar_1$	$(\langle[0.2,0.5],[0.3,0.7],[0.1,\ 0.2]\rangle,\langle 0.9,0.7,0.2\rangle)$	$(\langle[0.2,0.4],[0.4,0.5],[0.2,0.5]\rangle,\langle 0.7,0.4,0.5\rangle)$	$(\langle[0.2,0.7],[0.4,0.9],[0.5,0.7]\rangle,\langle 0.7,0.7,0.5\rangle)$	$(\langle[0.1,0.6],[0.3,0.4],[0.5,0.8]\rangle,\langle 0.5,0.5,0.7\rangle)$
$\hbar_2$	$(\langle[0.3,0.9],[0.2,0.7],[0.3,0.5]\rangle,\langle 0.5,0.7,0.5\rangle)$	$(\langle[0.3,0.7],[0.6,0.8],[0.2,0.4]\rangle,\langle 0.7,0.6,0.8\rangle)$	$(\langle[0.3,0.9],[0.4,0.6],[0.6,0.8]\rangle,\langle 0.9,\ 0.4,0.6\rangle)$	$(\langle[0.2,0.5],[0.4,0.9],[0.5,0.8]\rangle,\langle 0.5,0.2,0.7\rangle)$
$\hbar_3$	$(\langle[0.3,0.4],[0.4,0.8],[0.2,0.6]\rangle,\langle 0.1,0.4,0.2\rangle)$	$(\langle[0.2,0.4],[0.2,0.3],[0.2,0.5]\rangle,\langle 0.2,0.2,0.2\rangle)$	$(\langle[0.4,0.7],[0.1,0.2],[0.4,0.5]\rangle,\langle 0.9,0.5,0.5\rangle)$	$(\langle[0.6,0.7],[0.3,0.6],[0.3,0.7]\rangle,\langle 0.7,0.5,0.3\rangle)$
$\hbar_4$	$(\langle[0.5,0.9],[0.1,0.8],[0.2,0.6]\rangle,\langle 0.4,0.6,0.2\rangle)$	$(\langle[0.4,0.6],[0.5,0.7],[0.1,0.2]\rangle,\langle 0.5,0.3,0.2\rangle)$	$(\langle[0.5,0.6],[0.2,0.4],[0.3,0.5]\rangle,\langle 0.5,0.4,0.5\rangle)$	$(\langle[0.3,0.7],[0.7,0.8],[0.6,0.7]\rangle,\langle 0.4,0.2,0.8\rangle)$

Then, we apply the WNCPMM operator or WNCPDMM operator to solve the MADM problem. Now, we use the WNCPMM operator for this decision-making problem as follows:

Step 1. Since all the attributes are the same, hence there is no need for conversion.

Step 2. Use Equation (47), to calculate the support degrees $Supp\left(z_{gh}, z_{gl}\right)(1,2,\ldots,4; h,l = 1,2,\ldots,4)$. We denote $Supp\left(z_{gh}, z_{gl}\right)$ by $Supp_{gh,gl}$.

$$Supp_{11,12} = Supp_{12,11} = 0.79452, Supp_{11,13} = Supp_{13,11} = 0.735425, Supp_{11,14} = Supp_{14,11} = 0.65359,$$
$$Supp_{12,13} = Supp_{13,12} = 0.771478, Supp_{12,14} = Supp_{14,12} = 0.805635, Supp_{13,14} = Supp_{14,13} = 0.786563;$$
$$Supp_{21,22} = Supp_{22,21} = 0.7972, Supp_{21,23} = Supp_{23,21} = 0.7667, Supp_{21,24} = Supp_{24,21} = 0.727155,$$
$$Supp_{22,23} = Supp_{23,22} = 0.750556, Supp_{22,24} = Supp_{24,22} = 0.750556, Supp_{23,24} = Supp_{23,24} = 0.76906,$$
$$Supp_{31,32} = Supp_{32,31} = 0.8, Supp_{31,33} = Supp_{33,31} = 0.614139, Supp_{31,34} = Supp_{34,31} = 0.735425,$$
$$Supp_{32,33} = Supp_{33,32} = 0.690879, Supp_{32,34} = Supp_{34,32} = 0.711325, Supp_{33,34} = Supp_{33,34} = 0.797241,$$
$$Supp_{41,42} = Supp_{42,41} = 0.7551, Supp_{41,43} = Supp_{43,41} = 0.783975, Supp_{41,44} = Supp_{44,41} = 0.645662,$$
$$Supp_{42,43} = Supp_{43,42} = 0.783975, Supp_{42,44} = Supp_{44,42} = 0.675107, Supp_{43,44} = Supp_{44,43} = 0.7152.$$

Step 3. Use Equation (48), to get $T\left(\delta_{gh}\right)(g,h = 1\ to\ 4)$. We denote $T\left(\delta_{gh}\right)$ by T_{gh}.

$$\mathrm{T}_{11} = 2.183534, \mathrm{T}_{12} = 2.371633, \mathrm{T}_{13} = 2.293466, \mathrm{T}_{14} = 2.245787;$$
$$\mathrm{T}_{21} = 2.291063, \mathrm{T}_{22} = 2.298354, \mathrm{T}_{23} = 2.286283, \mathrm{T}_{24} = 2.246771$$
$$\mathrm{T}_{31} = 2.149564, \mathrm{T}_{32} = 2.202204, \mathrm{T}_{33} = 2.102259, \mathrm{T}_{34} = 2.243991,$$
$$\mathrm{T}_{41} = 2.184688, \mathrm{T}_{42} = 2.214133, \mathrm{T}_{43} = 2.28315, \mathrm{T}_{44} = 2.035969.$$

Step 4. Use Equation (49), to obtain $\Psi_{gh}(g,h = 1,2,3,4)$.

$$\Psi_{11} = 1.957844, \Psi_{12} = 1.036761, \Psi_{13} = 0.506363, \Psi_{14} = 0.499032,$$
$$\Psi_{21} = 2.002623, \Psi_{22} = 1.00353, \Psi_{23} = 0.499929, \Psi_{24} = 0.493918,$$
$$\Psi_{31} = 1.987975, \Psi_{32} = 1.010601, \Psi_{33} = 0.489529, \Psi_{34} = 0.511894,$$

Step 5. Use the WNCPMM given in Equation (50),

$$z_g = \left(\left\langle \left[T_g^L, T_g^U\right], \left[I_g^L, I_g^U\right], \left[F_g^L, F_g^U\right]\right\rangle, \left\langle \lambda_{Tg}, \lambda_{Ig}, \lambda_{Fg}\right\rangle\right) = WCNPMM^Q(z_{g1}, z_{g2}, \ldots, z_{g4})(g = 1,2,\ldots,4).$$

To get the overall NCNs $z_g(g = 1,2,\ldots,4)$. Assume that $Q = (1,1,1,1)$.

$$z_1 = (\langle[0.1399, 0.4650], [0.4421, 0.7027], [0.4691, 0.6847]\rangle, \langle 0.5483, 0.6368, 0.6029\rangle);$$
$$z_2 = (\langle[0.2238, 0.6021], [0.5236, 0.8162], [0.5122, 0.715]\rangle, \langle 0.5617, 0.5505, 0.7294\rangle);$$
$$z_3 = \langle[0.3002, 0.4736], [0.3232, 0.5782], [0.3881, 0.6445]\rangle, \langle 0.3255, 0.4952, 0.415668\rangle;$$
$$z_4 = \langle[0.3413, 0.5540], [0.5437, 0.7485], [0.4487, 0.5965]\rangle, \langle 0.3762, 0.4451, 0.5976\rangle.$$

Step 6. Using Definition 6, we calculate the score values of the collective NCNs $z_g(g = 1, 2, \ldots, a)$.

$$\widetilde{SC}(z_1) = 0.4022, \widetilde{SC}(z_2) = 0.393352, \widetilde{SC}(z_3) = 0.472717, \widetilde{SC}(z_4) = 0.4324.$$

Step 7. According to the score values, ranking order of the alternative is $\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2$.

Hence using Theorem 1, the best alternative is $\hbar_3$ and the worst is $\hbar_2$.

Similarly, by using WNCPDMM operator for this decision-making problem, we will have, the Steps 1 to 4 are similar to that of weighted neutrosophic cubic power Muirhead mean operator.

Step 5. Use the WNCPDMM given in Equation (51),

$$z_g = \left\langle \left[T_g^L, T_g^U\right], \left[I_g^L, I_g^U\right], \left[F_g^L, F_g^U\right] \right\rangle, \left\langle \lambda_{Tg}, \lambda_{Ig}, \lambda_{Fg} \right\rangle = WNCPDMM^Q(z_{g1}, z_{g2}, \ldots, z_{g4})(g = 1, 2, \ldots, 4).$$

To get the overall NCNs $z_g(g = 1, 2, \ldots, 4)$. Assume that, $Q = (1, 1, 1, 1)$.

$$\begin{aligned}
z_1 &= \langle [0.2569, 0.6239], [0.2929, 0.5112], [0.2375, 0.4571] \rangle, \langle 0.7682, 0.4666, 0.3905 \rangle; \\
z_2 &= \langle [0.3642, 0.8179], [0.3110, 0.6479], [0.3194, 0.5430] \rangle, \langle 0.7416, 0.3336, 0.5561 \rangle; \\
z_3 &= \langle [0.4935, 0.6438], [0.1794, 0.3224], [0.2248, 0.4812] \rangle, \langle 0.6502, 0.3206, 0.2330 \rangle; \\
z_4 &= \langle [0.4995, 0.7691], [0.2570, 0.5332], [0.2130, 0.3815] \rangle, \langle 0.5355, 0.2744, 0.3248 \rangle.
\end{aligned}$$

Step 6. Using Definition 6, we calculate the score values of the collective NCNs $z_g(g = 1, 2, \ldots, a)$.

$$\widetilde{SC}(z_1) = 0.5881, \widetilde{SC}(z_2) = 0.5782, \widetilde{SC}(z_3) = 0.6688, \widetilde{SC}(z_4) = 0.6467.$$

Step 7. According to the score values, ranking order of the alternative is $\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2$.

Hence using Theorem 1, the best alternatives is $\hbar_3$, while the worst is $\hbar_2$.

From the above obtained results, we can see that by using WNCPMM operator or WNCPDMM operator, the best alternative obtained is $\hbar_3$, while the worst is $\hbar_2$.

Effect of the Parameter Q on the Decision Result

In this subsection, different values to the parameter vector and the results obtained from these values are shown in Tables 2 and 3. From Tables 2 and 3, it can be seen that, when the value of the parameter vector Q is $(1, 0, 0, 0)$, that is, when the interrelationship among the attributes is not considered, then according to the score values the best alternative is $\hbar_4$ while the worst is $\hbar_2$. Similarly, when the value of the parameter vector Q is $(1, 1, 0, 0)$, that is, when WCNPMM operator and WNCPDMM operator degenerate into neutrosophic cubic power Bonferroni mean operator and neutrosophic cubic power geometric Bonferroni mean operator respectively, the best alternative is $\hbar_3$ and $\hbar_4$ while the worst for both cases is $\hbar_2$. When the value of the parameter vector Q is $(1, 1, 1, 0)$, the best alternative is $\hbar_3$ and the worst is $\hbar_2$. When the value of the parameter vector Q is $(1, 1, 1, 1)$, the best alternative is $\hbar_3$ and the worst is $\hbar_2$. Similarly, for other values of the parameter vector the score values and ranking order vary. Thus, one can select the value of the parameter vector according to the needs of the situations.

Table 2. Score values and ranking orders for different parameter values in WCNPMM operator.

Parameter Vector Q	Score Values	Ranking Orders
$Q(1,0,0,0)$	$\widetilde{SC}(CN_1) = 0.5671, \widetilde{SC}(CN_2) = 0.5230,$ $\widetilde{SC}(CN_3) = 0.5593, \widetilde{SC}(CN_4) = 0.6031.$	$\hbar_4 > \hbar_1 > \hbar_3 > \hbar_2.$
$Q(1,1,0,0)$	$\widetilde{SC}(CN_1) = 0.4579, \widetilde{SC}(CN_2) = 0.4468,$ $\widetilde{SC}(CN_3) = 0.5092, \widetilde{SC}(CN_4) = 0.5027.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(1,1,1,0)$	$\widetilde{SC}(CN_1) = 0.4227, \widetilde{SC}(CN_2) = 0.4133,$ $\widetilde{SC}(CN_3) = 0.4866, \widetilde{SC}(CN_4) = 0.4607.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(1,1,1,1)$	$\widetilde{SC}(CN_1) = 0.5881, \widetilde{SC}(CN_2) = 0.5782,$ $\widetilde{SC}(CN_3) = 0.6688, \widetilde{SC}(CN_4) = 0.6467.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(0.5,0.5,0.5,0.5)$	$\widetilde{SC}(CN_1) = 0.3988, \widetilde{SC}(CN_2) = 0.3910,$ $\widetilde{SC}(CN_3) = 0.4708, \widetilde{SC}(CN_4) = 0.4306.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(5,0,0,0)$	$\widetilde{SC}(CN_1) = 0.6608, \widetilde{SC}(CN_2) = 0.6235,$ $\widetilde{SC}(CN_3) = 0.6313, \widetilde{SC}(CN_4) = 0.6854.$	$\hbar_4 > \hbar_1 > \hbar_3 > \hbar_2.$

Table 3. Score values and ranking orders for different parameter values in weighted neutrosophic cubic power dual Muirhead mean operator.

Parameter Vector Q	Score Values	Ranking Orders
$Q(1,0,0,0)$	$\widetilde{SC}(CN_1) = 0.5588, \widetilde{SC}(CN_2) = 0.5346,$ $\widetilde{SC}(CN_3) = 0.6040, \widetilde{SC}(CN_4) = 0.6081.$	$\hbar_4 > \hbar_1 > \hbar_3 > \hbar_2.$
$Q(1,1,0,0)$	$\widetilde{SC}(CN_1) = 0.5881, \widetilde{SC}(CN_2) = 0.5782,$ $\widetilde{SC}(CN_3) = 0.6688, \widetilde{SC}(CN_4) = 0.6467.$	$\hbar_4 > \hbar_3 > \hbar_1 > \hbar_2.$
$Q(1,1,1,0)$	$\widetilde{SC}(CN_1) = 0.5760, \widetilde{SC}(CN_2) = 0.5582,$ $\widetilde{SC}(CN_3) = 0.6478, \widetilde{SC}(CN_4) = 0.6276.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(1,1,1,1)$	$\widetilde{SC}(CN_1) = 0.5881, \widetilde{SC}(CN_2) = 0.5782,$ $\widetilde{SC}(CN_3) = 0.6688, \widetilde{SC}(CN_4) = 0.6467.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(0.5,0.5,0.5,0.5)$	$\widetilde{SC}(CN_1) = 0.5909, \widetilde{SC}(CN_2) = 0.5817,$ $\widetilde{SC}(CN_3) = 0.6741, \widetilde{SC}(CN_4) = 0.6488.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
$Q(5,0,0,0)$	$\widetilde{SC}(CN_1) = 0.4671, \widetilde{SC}(CN_2) = 0.4073,$ $\widetilde{SC}(CN_3) = 0.4022, \widetilde{SC}(CN_4) = 0.4559.$	$\hbar_1 > \hbar_4 > \hbar_2 > \hbar_3.$

6. Comparison with Existing Methods

To show the efficiency and advantages of the proposed method, we give a comparative analysis. We exploit some existing methods to solve the same example and examine the final results. We compare our method in this paper with the methods developed by Qin et al. [30] based on weighted IFMSM operator, and the one developed by Liu et al. [32]-based generalized INPWA operator. We extend the IFMSM operator method [30] for intuitionistic fuzzy information to neutrosophic cubic Maclaurin symmetric mean operator. We also extend the GINPWA operator [32] for interval neutrosophic information to generalized neutrosophic cubic power average operator.

The method developed by Qin et al. [30], is based on MSM operator, which can consider the interrelationship among the attribute values, but unable to remove the effect of awkward data. The MSM operator is a special case of the proposed aggregation operator. Also, the ranking result obtained using the method of Qin et al. [30], is different from the one obtained using the proposed method.

Similarly, the method developed by Liu et al. [32], is based on power weighted averaging operator, which can remove the effect of awkward data but cannot consider the interrelationship among the attributes values. From Table 4, it can be seen that the ranking result obtained using Liu et al. [32] is the same as the ranking order obtained from the proposed method, when $Q(1,0,0,0)$. That is, when the interrelationship between NCNs are not considered. This shows the validity of the proposed approach. The ranking order is different when $Q(1,1,1,1)$. That is, when the interrelationship among four attributes are considered, then the ranking order is different. The main reason behind the different ranking results is due to the existing aggregation operators, can only consider a single characteristic at a time while aggregating the NCNs, meaning that they can only either consider interrelationship among attributes or remove the effect of awkward data. Our proposed aggregation operator, however, can consider two characteristics at a time. It can consider the interrelationship among the attributes and remove the effect of awkward data. In fact, these existing aggregation operators can be regarded as special cases to our proposed aggregation operator. Hence, our proposed aggregation operator is more practical and flexible to be used in decision-making problems.

Table 4. Score values and ranking orders for different parameter values in WCNPDMM operator.

Aggregation Operator	Score Values	Ranking Orders
NCMSM operator [30]	$\widetilde{SC}(CN_1) = 0.6263, \widetilde{SC}(CN_2) = 0.6153,$ $\widetilde{SC}(CN_3) = 0.6355, \widetilde{SC}(CN_4) = 0.6373.$	$\hbar_4 > \hbar_3 > \hbar_1 > \hbar_2.$
GNCPWA operator [32]	$\widetilde{SC}(CN_1) = 0.5694, \widetilde{SC}(CN_2) = 0.5266,$ $\widetilde{SC}(CN_3) = 0.5646, \widetilde{SC}(CN_4) = 0.6054.$	$\hbar_4 > \hbar_1 > \hbar_3 > \hbar_2.$
Proposed WNCPMM operator $Q(1,0,0,0)$	$\widetilde{SC}(CN_1) = 0.5671, \widetilde{SC}(CN_2) = 0.5230,$ $\widetilde{SC}(CN_3) = 0.5593, \widetilde{SC}(CN_4) = 0.6031.$	$\hbar_4 > \hbar_1 > \hbar_3 > \hbar_2.$
Proposed WNCPDMM operator $Q(1,0,0,0)$	$\widetilde{SC}(CN_1) = 0.5588, \widetilde{SC}(CN_2) = 0.5346,$ $\widetilde{SC}(CN_3) = 0.6040, \widetilde{SC}(CN_4) = 0.6081.$	$\hbar_4 > \hbar_1 > \hbar_3 > \hbar_2.$
Proposed WNCPMM operator $Q(1,1,1,1)$	$\widetilde{SC}(CN_1) = 0.5881, \widetilde{SC}(CN_2) = 0.5782,$ $\widetilde{SC}(CN_3) = 0.6688, \widetilde{SC}(CN_4) = 0.6467.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$
Proposed WNCPDMM operator $Q(1,1,1,1)$	$\widetilde{SC}(CN_1) = 0.5881, \widetilde{SC}(CN_2) = 0.5782,$ $\widetilde{SC}(CN_3) = 0.6688, \widetilde{SC}(CN_4) = 0.6467.$	$\hbar_3 > \hbar_4 > \hbar_1 > \hbar_2.$

7. Conclusions

In this article, we incorporate both the PA operator and MM operator to form a few new aggregation operators to aggregate CNNs, such as the cubic neutrosophic power Muirhead mean (CNPMM) operator, WCNPMM operator, CNPDMM operator and WCNPDMM operator. We discussed several basic results and properties, along with a few special cases of the proposed aggregation operators. In other words, the developed aggregation operators do not only consider the interrelationship among the NCNs, but also remove the influence of too high or too low arguments in the final results. Based on these aggregation operators, a novel approach to MADM problem is developed. Finally, a numerical example is illustrated to show the effectiveness and practicality of the proposed approach.

Our main contribution is enhancing the neutrosophic cubic aggregation operator and its MADM method under neutrosophic cubic environment. In future, we will incorporate the PA operator with the MM operator under the intuitionistic fuzzy environment [3], interval neutrosophic environment [6] and multi-valued neutrosophic environment [10], to develop new operators such as IFPMM, IFPDMM, INPMM, INPDMM, multi-valued neutrosophic power Muirhead mean (NPMM) and multi-valued neutrosophic power dual Muirhead mean (NPDMM) operators along with their weighted forms. We will apply these to MAGDM, data mining, decision support, recommender system and pattern recognition.

Author Contributions: Q.K. drafted the initial manuscript, T.M. supervised, N.H. corrected the draft and sourced the funding. All authors approved the publication work.

Funding: The APC is funded by Universiti Kebangsaan Malaysia under the grant GUP-2017-105.

Acknowledgments: We, the authors of this project are deeply grateful for the valuable suggestions and comments provided by the respected reviewers to improve the quality and worth of this research. We are also indebted to Universiti Kebangsaan Malaysia for providing financial support and facilities for this research under the grant GUP-2017-105.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

FS	Fuzzy set
IFS	Intuitionistic fuzzy set
INS	Interval neutrosophic set
INN	Interval neutrosophic number
MADM	Multiple-attribute decision-making
MAGDM	Multiple-attribute group decision-making
MM	Muirhead Mean
NS	Neutrosophic set
NC	Neutrosophic cubic
NCN	Neutrosophic cubic number
NCPMM	Neutrosophic cubic power Muirhead mean operator
NCPDMM	Neutrosophic cubic power dual Muirhead mean operator
PA	Power average operator
PWV	Power weight vector
SVNS	Single-valued neutrosophic set
SVNN	Single-valued neutrosophic number
WNCPMM	Weighted neutrosophic cubic power Muirhead mean
WNCPDMM	Weighted neutrosophic cubic power dual Muirhead mean operator

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Some Interval Neutrosophic Dombi Power Bonferroni Mean Operators and Their Application in Multi–Attribute Decision–Making

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Received: 16 August 2018; Accepted: 28 September 2018; Published: 2 October 2018

Abstract: The power Bonferroni mean (PBM) operator is a hybrid structure and can take the advantage of a power average (PA) operator, which can reduce the impact of inappropriate data given by the prejudiced decision makers (DMs) and Bonferroni mean (BM) operator, which can take into account the correlation between two attributes. In recent years, many researchers have extended the PBM operator to handle fuzzy information. The Dombi operations of T-conorm (TCN) and T-norm (TN), proposed by Dombi, have the supremacy of outstanding flexibility with general parameters. However, in the existing literature, PBM and the Dombi operations have not been combined for the above advantages for interval-neutrosophic sets (INSs). In this article, we first define some operational laws for interval neutrosophic numbers (INNs) based on Dombi TN and TCN and discuss several desirable properties of these operational rules. Secondly, we extend the PBM operator based on Dombi operations to develop an interval-neutrosophic Dombi PBM (INDPBM) operator, an interval-neutrosophic weighted Dombi PBM (INWDPBM) operator, an interval-neutrosophic Dombi power geometric Bonferroni mean (INDPGBM) operator and an interval-neutrosophic weighted Dombi power geometric Bonferroni mean (INWDPGBM) operator, and discuss several properties of these aggregation operators. Then we develop a multi-attribute decision-making (MADM) method, based on these proposed aggregation operators, to deal with interval neutrosophic (IN) information. Lastly, an illustrative example is provided to show the usefulness and realism of the proposed MADM method. The developed aggregation operators are very practical for solving MADM problems, as it considers the interaction among two input arguments and removes the influence of awkward data in the decision-making process at the same time. The other advantage of the proposed aggregation operators is that they are flexible due to general parameter.

Keywords: interval neutrosophic sets; Bonferroni mean; power operator; multi-attribute decision making (MADM)

1. Introduction

While dealing with any real world problems, a decision maker (DM) often feels discomfort when expressing his\her evaluation information by utilizing a single real number in multi-attribute decision making (MADM) or multi-attribute group decision making (MAGDM) problems due to the intellectual fuzziness of DMs. For this cause, Zadeh [1] developed fuzzy sets (FSs), which are assigned by a truth-membership degree (TMD) in $[0, 1]$ and are a better tool to present fuzzy information for handling MADM or MAGDM problems. After the introduction of FSs, different fuzzy modelling approaches were developed to deal with uncertainty in various fields [2–4]. However, in some situations, it is difficult to express truth-membership degree with an exact number. In order to overcome this defect and to express TMD in a more appropriate way, Turksen [5] developed interval valued FSs (IVFSs), in which TMD is represented by interval numbers instead of exact numbers. Since only TMD was considered in FSs or IVFSs and the falsity-membership degree (FMD) came automatically by subtracting TMD from one, it is hard to explain some complicated fuzzy information, for example, for the selection of Dean of a faculty, if the results received from five professors are in favor, two are against and three are neither in favor nor against. Then, this type of information cannot be expressed by FSs. So, in order to handle such types of information, Atanassov [6] developed intuitionistic fuzzy sets (IFSs), which were assigned by TMD and FMD. Atanassov et al. [7] further enlarged IFSs and developed the interval valued IFS (IVIFSs). However, the shortcoming of FSs, IVFSs, IFSs and IVIFSs are that they cannot deal with unreliable or indefinite information. To solve such problems, Smarandache [8,9] developed neutrosophic sets (NSs). In neutrosophic set, every member of the domain set has TMD, an indeterminacy-membership degree (IMD) and FMD, which capture values in $]0^-, 1^+[$. Due to the containment of subsets of $]0^-, 1^+[$ in NS, it is hard to utilize NS in real world and engineering problems. To make NSs helpful in these cases, some authors developed subclasses of NSs, such as single valued neutrosophic sets (SVNSs) [10], interval neutrosophic sets (INSs) [11,12], simplified neutrosophic sets (SNSs) [13,14] and so forth. In recent years, INSs have gained much attention from the researchers and a great number of achievement have been made, such as distance measures [15–17], entropies of INS [18–20], correlation coefficient [21–23]. The theory of NSs has been extensively utilized to handle MADM and MAGDM problems.

For the last many years, information aggregation operators [24–27] have stimulated much awareness of authors and have become very dominant research topic of MADM and MAGDM problems. The conventional aggregation operators (AGOs) proposed by Xu, Xu and Yager [28,29] can only aggregate a group of real numbers into a single real number. Now these conventional AGOs were further extended by many authors, for example, Sun et al. [30] proposed the interval neutrosophic number Choquet integral operator for MADM and Liu et al. [31] developed prioritized ordered weighted AGOs for INSs and applied them to MADM. In addition, some decision-making methods were also developed for MADM problems, for example, Mukhametzyanov et al. [32] developed a statistically based model for sensitivity analysis in MADM problems. Petrovic et al. [33] developed a model for the selection of aircrafts based on decision making trial and evaluation laboratory and analytic hierarchy process (DEMATEL-AHP). Roy et al. [34] proposed a rough relational DEMATEL model to analyze the key success factor of hospital quality. Sarkar et al. [35] developed an optimization technique for national income determination model with stability analysis of differential equation in discrete and continuous process under uncertain environment. These methods can only give a ranking result, however, AGOs can not only give the ranking result, but also give the comprehensive value of each alternative by aggregating its attribute values.

It is obvious that, different aggregation operators have distinct functions, a few of them can reduce the impact of some awkward data produced by predispose DMs, such as power average (PA) operator proposed by Yager [36]. The PA operator can aggregate the input data by designating the weight vector based on the support degree among the input arguments, and can attain this function. Now the PA operator was further extended by many researchers into different environments. Liu et al. [37] proposed some generalized PA operator for INNs, and applied them to MADM. Consequently, some aggregation operators can include the interrelationship between the aggregating parameters, such as the Bonferroni mean (BM) operators developed by Bonferroni [38], the Heronian mean (HM) operator introduced by Sykora [39], Muirhead Mean (MM) operator [40], Maclaurin symmetric mean [41] operators. In addition, these aggregation operators have also been extended by many authors to deal with fuzzy information [42–46].

For aggregating INNs, some AGOs are developed by utilizing different T-norms (TNs) and T-conorms (TCNs), such as algebraic, Einstein and Hamacher. Usually, the Archimedean TN and TCN are the generalizations of various TNs and TCNs such as algebraic, Einstein, Hamacher, Frank, and Dombi [47] TNs and TCNs. Dombi TN and TCN have the characteristics of general TN and TCN by a general parameter, and this can make the aggregation process more flexible. Recently, several authors defined some operational laws for IFSs [48], SVNSs [49], hesitant fuzzy sets (HFSs) [50,51] based on Dombi TN and TCN. In practical decision making, we generally need to consider interrelationship among attributes and eliminated the influence of awkward data. For this purpose, some researchers combined BM and PA operators to propose some PBM operators and extended them to various fields [52–55]. The PBM operators have two characteristics. Firstly, it can consider the interaction among two input arguments by BM operator, and secondly, it can remove the effect of awkward data by PA operator. The Dombi TN and TCN have a general parameter, which makes the decision-making process more flexible. From the existing literatures, we know that PBM operators are combined with algebraic operations to aggregate IFNs, or IVIFNs, and there is no research on combining PBM operator with Dombi operations to aggregate INNs.

In a word, by considering the following advantages. (1) Since INSs are the more précised class by which one can handle the vague information in a more accurate way when compared with FSs and all other extensions like IVFSs, IFSs, IVIFSs and so forth, they are more suitable to describe the attributes of MADM problems, so in this study, we will select the INSs as information expression; (2) Dombi TN and TCN are more flexible in the decision making process due to general parameter which is regarded as decision makers' risk attitude; (3) The PBM operators have the properties of considering interaction between two input arguments and vanishes the effect of awkward data at the same time. Hence, the purpose and motivation are that we try to combine these three concepts to take the above defined advantages and proposed some new powerful tools to aggregate INNs. (1) we define some Dombi operational laws for INNs; (2) we propose some new PBM aggregation operators based on these new operational laws; (3) we develop a novel MADM based on these developed aggregation operators.

The following sections of this article are shown as follows. In Section 2, we review some basic concepts of INSs, PA operators, BM operators, and GBM operators. In Section 3, we review basic concept of Dombi TN and TCN. After that, we propose some Dombi operations for INNs, and discuss some properties. In Section 4, we define INDPBM operator, INWDPBM operator, INDPGBM operator and INWDPGBM operator and discuss their properties. In Section 5, we propose a MADM method based on the proposed aggregation operators with INNs. In Section 6, we use an illustrative example to show the effectiveness of the proposed MADM method. The conclusion is discussed in Section 7.

2. Preliminaries

In this part, some basic definitions, properties about INSs, BM operators and PA operators are discussed.

2.1. The INSs and Their Operational Laws

Definition 1. *Let Ω be the domain set [8,9], with a non-specific member in Ω expressed by $\overline{\overline{v}}$. A NS $\overline{\overline{NS}}$ in Ω is expressed by*

$$\overline{\overline{NS}} = \left\{ \left\langle \overline{\overline{v}}, t_{\overline{\overline{NS}}}(\overline{\overline{v}}), i_{\overline{\overline{NS}}}(\overline{\overline{v}}), f_{\overline{\overline{NS}}}(\overline{\overline{v}}) \right\rangle \middle| \overline{\overline{v}} \in \Omega \right\}, \tag{1}$$

where, $t_{\overline{\overline{NS}}}(\overline{\overline{v}}), i_{\overline{\overline{NS}}}(\overline{\overline{v}})$ and $f_{\overline{\overline{NS}}}(\overline{\overline{v}})$ respectively express the TMD, IMD and FMD of the element $\overline{\overline{v}} \in \widetilde{U}$ to the set $\overline{\overline{NS}}$. For each point $\overline{\overline{v}} \in \widetilde{U}$, we have, $t_{\overline{\overline{NS}}}(\overline{\overline{v}}), i_{\overline{\overline{NS}}}(\overline{\overline{v}}) f_{\overline{\overline{NS}}}(\overline{\overline{v}}) \in]0^-, 1^+[$ and $0^- \leq t_{\overline{\overline{NS}}}(\overline{\overline{v}}) + i_{\overline{\overline{NS}}}(\overline{\overline{v}}) + f_{\overline{\overline{NS}}}(\overline{\overline{v}}) \leq 3^+$.

The NS was predominantly developed from philosophical perspective, and it is hard to be applied to engineering problems due to the containment of subsets of $]0^-, 1^+[$. So, in order to use it more easily in real life or engineering problem, Wang et al. [8] presented a subclass of NS by changing $]0^-, 1^+[$ to $[0,1]$ and was named SVNS, and is defined as follow:

Definition 2. *Let Ω be the domain set [10], with a non-specific member in Ω expressed by $\overline{\overline{v}}$. A SVNS $\overline{\overline{SV}}$ in Ω* is expressed by

$$\overline{\overline{SV}} = \left\{ \left\langle \overline{\overline{u}}, t_{\overline{\overline{SV}}}(\overline{\overline{v}}), i_{\overline{\overline{SV}}}(\overline{\overline{v}}), f_{\overline{\overline{SV}}}(\overline{\overline{v}}) \right\rangle \middle| \overline{\overline{v}} \in \Omega \right\}, \tag{2}$$

where $t_{\overline{\overline{SV}}}(\overline{\overline{v}}), i_{\overline{\overline{SV}}}(\overline{\overline{v}})$ and $f_{\overline{\overline{SV}}}(\overline{\overline{v}})$ express the TMD, IMD and FMD of the element $\overline{\overline{v}} \in \Omega$ to the set $\overline{\overline{SV}}$ respectively. For each point $\overline{\overline{v}} \in \Omega$, we have, $t_{\overline{\overline{SV}}}(\overline{\overline{v}}), i_{\overline{\overline{SV}}}(\overline{\overline{v}}), f_{\overline{\overline{SV}}}(\overline{\overline{v}}), \in [0,1]$ and $0 \le t_{\overline{\overline{SV}}}(\overline{\overline{v}}) + i_{\overline{\overline{SV}}}(\overline{\overline{v}}) + f_{\overline{\overline{SV}}}(\overline{\overline{v}}) \le 3$.

In order to define more complex information, Wang et al. [9] further developed INS which is define as follows:

Definition 3. *Let Ω be the domain set and $\overline{\overline{v}} \in \Omega$ [11]. Then an INS $\overline{\overline{IN}}$ in Ω* is expressed by

$$\overline{\overline{IN}} = \left\{ \left\langle \overline{\overline{v}}, \overline{\overline{TR}}_{\hat{I}N}(\overline{\overline{v}}), \overline{\overline{ID}}_{\hat{I}N}(\overline{\overline{v}}), \overline{\overline{FL}}_{\hat{I}N}(\overline{\overline{v}}) \right\rangle \middle| \overline{\overline{v}} \in \Omega \right\}, \tag{3}$$

where, $\overline{\overline{TR}}_{\hat{I}N}(\overline{\overline{v}}), \overline{\overline{ID}}_{\hat{I}N}(\overline{\overline{v}})$ and $\overline{\overline{FL}}_{\hat{I}N}(\overline{\overline{v}})$ respectively, express the TMD, IMD and FMD of the element $\overline{\overline{v}} \in \Omega$ to the set $\overline{\overline{IN}}$. For each point $\overline{\overline{v}} \in \tilde{U}$, we have, $\overline{\overline{TR}}_{IN}(\overline{\overline{v}}), \overline{\overline{ID}}_{IN}(\overline{\overline{v}}), \overline{\overline{FL}}_{IN}(\overline{\overline{v}}) \subseteq [0,1]$ and $0 \le \max\overline{\overline{ID}}_{IN}(\overline{\overline{v}}) + \max\overline{\overline{ID}}_{IN}(\overline{\overline{v}}) + \max\overline{\overline{FL}}_{IN}(\overline{\overline{v}}) \le 3$.

For computational simplicity, we can use $\overline{\overline{in}} = \left\langle \left[\overline{\overline{TR}}^L, \overline{\overline{TR}}^U\right], \left[\overline{\overline{ID}}^L, \overline{\overline{ID}}^U\right], \left[\overline{\overline{FL}}^L, \overline{\overline{FL}}^U\right] \right\rangle$ to express an element $\overline{\overline{in}}$ in an INS, and the element $\overline{\overline{in}}$ is called an interval neutrosophic number (INN). Where $\left[\overline{\overline{TR}}^L, \overline{\overline{TR}}^U\right] \subseteq [0,1]$, $\left[\overline{\overline{ID}}^L, \overline{\overline{ID}}^U\right] \subseteq [0,1]$, $\left[\overline{\overline{FL}}^L, \overline{\overline{FL}}^U\right] \subseteq [0,1]$ and $0 \le \overline{\overline{TR}}^U + \overline{\overline{ID}}^U + \overline{\overline{FL}}^U \le 3$.

Definition 4. *Let $\overline{\overline{in}}_1 = \left\langle \left[\overline{\overline{TR}}_1^L, \overline{\overline{TR}}_1^U\right], \left[\overline{\overline{ID}}_1^L, \overline{\overline{ID}}_1^U\right], \left[\overline{\overline{FL}}_1^L, \overline{\overline{FL}}_1^U\right] \right\rangle$ and $\overline{\overline{in}}_2 = \left\langle \left[\overline{\overline{TR}}_2^L, \overline{\overline{TR}}_2^U\right], \left[\overline{\overline{ID}}_2^L, \overline{\overline{ID}}_2^U\right], \left[\overline{\overline{FL}}_2^L, \overline{\overline{FL}}_2^U\right] \right\rangle$ be any two INNs [12], and $\zeta > 0$. Then the operational laws of INNs can be defined as follows:*

$$(1)\ \overline{\overline{in}}_1 \oplus \overline{\overline{in}}_2 = \left\langle \left[\overline{\overline{TR}}_1^L + \overline{\overline{TR}}_2^L - \overline{\overline{TR}}_1^L\overline{\overline{TR}}_2^L, \overline{\overline{TR}}_1^U + \overline{\overline{TR}}_2^U - \overline{\overline{TR}}_1^U\overline{\overline{TR}}_2^U\right], \left[\overline{\overline{ID}}_1^L\overline{\overline{ID}}_2^L, \overline{\overline{ID}}_1^U\overline{\overline{ID}}_2^U\right], \left[\overline{\overline{FL}}_1^L\overline{\overline{FL}}_2^L, \overline{\overline{FL}}_1^U\overline{\overline{FL}}_2^U\right] \right\rangle; \tag{4}$$

$$(2)\ \overline{\overline{in}}_1 \otimes \overline{\overline{in}}_2 = \left\langle \left[\overline{\overline{TR}}_1^L\overline{\overline{TR}}_2^L, \overline{\overline{TR}}_1^U\overline{\overline{TR}}_2^U\right], \left[\overline{\overline{ID}}_1^L + \overline{\overline{ID}}_2^L - \overline{\overline{ID}}_1^L\overline{\overline{ID}}_2^L, \overline{\overline{ID}}_1^U + \overline{\overline{ID}}_2^U - \overline{\overline{ID}}_1^U\overline{\overline{ID}}_2^U\right], \left[\overline{\overline{FL}}_1^L + \overline{\overline{FL}}_2^L - \overline{\overline{FL}}_1^L\overline{\overline{FL}}_2^L, \overline{\overline{FL}}_1^U + \overline{\overline{FL}}_2^U - \overline{\overline{FL}}_1^U\overline{\overline{FL}}_2^U\right] \right\rangle; \tag{5}$$

$$(3)\ \overline{\overline{in}}_1^{\zeta} = \left\langle \left[\left(\overline{\overline{TR}}_1^L\right)^{\zeta}, \left(\overline{\overline{TR}}_1^U\right)^{\zeta}\right], \left[1 - \left(1 - \overline{\overline{ID}}_1^L\right)^{\zeta}, 1 - \left(1 - \overline{\overline{ID}}_1^U\right)^{\zeta}\right], \left[1 - \left(1 - \overline{\overline{FL}}_1^L\right)^{\zeta}, 1 - \left(1 - \overline{\overline{FL}}_1^U\right)^{\zeta}\right] \right\rangle; \tag{6}$$

$$(4)\ \zeta\overline{\overline{in}}_1 = \left\langle \left[1 - \left(1 - \overline{\overline{TR}}_1^L\right)^{\zeta}, 1 - \left(1 - \overline{\overline{TR}}_1^U\right)^{\zeta}\right], \left[\left(\overline{\overline{ID}}_1^L\right)^{\zeta}, \left(\overline{\overline{ID}}_1^U\right)^{\zeta}\right], \left[\left(\overline{\overline{FL}}_1^L\right)^{\zeta}, \left(\overline{\overline{FL}}_1^U\right)^{\zeta}\right] \right\rangle. \tag{7}$$

Definition 5. *Let $\overline{\overline{in}} = \left\langle \left[\overline{\overline{TR}}^L, \overline{\overline{TR}}^U\right], \left[\overline{\overline{ID}}^L, \overline{\overline{ID}}^U\right], \left[\overline{\overline{FL}}^L, \overline{\overline{FL}}^U\right] \right\rangle$ [42], be an INN. Then the score function $S\left(\overline{\overline{in}}\right)$ and accuracy function $A\left(\overline{\overline{in}}\right)$ can be defined as follows:*

$$(i)\ S\left(\overline{\overline{in}}\right) = \frac{\overline{\overline{TR}}^L + \overline{\overline{TR}}^U}{2} + 1 - \frac{\overline{\overline{ID}}^L + \overline{\overline{ID}}^U}{2} + 1 - \frac{\overline{\overline{FL}}^L + \overline{\overline{FL}}^U}{2}; \tag{8}$$

$$(ii)\ A\left(\overline{\overline{in}}\right) = \frac{\overline{\overline{TR}}^L + \overline{\overline{TR}}^U}{2} + 1 - \frac{\overline{\overline{ID}}^L + \overline{\overline{ID}}^U}{2} + \frac{\overline{\overline{FL}}^L + \overline{\overline{FL}}^U}{2}. \tag{9}$$

In order to compare two INNs, the comparison rules were defined by Liu et al. [36], which can be stated as follows.

Definition 6. *Let* $\overline{\overline{in}}_1 = \left\langle \left[\overline{\overline{TR}}_1^L, \overline{\overline{TR}}_1^U\right], \left[\overline{\overline{ID}}_1^L, \overline{\overline{ID}}_1^U\right], \left[\overline{\overline{FL}}_1^L, \overline{\overline{FL}}_1^U\right] \right\rangle$ *and* $\overline{\overline{in}}_2 = \left\langle \left[\overline{\overline{TR}}_2^L, \overline{\overline{TR}}_2^U\right], \left[\overline{\overline{ID}}_2^L, \overline{\overline{ID}}_2^U\right], \left[\overline{\overline{FL}}_2^L, \overline{\overline{FL}}_2^U\right] \right\rangle$ *be any two INNs [42]. Then we have:*

(1) *If* $S\left(\overline{\overline{in}}_1\right) > S\left(\overline{\overline{in}}_2\right)$, *then* $\overline{\overline{in}}_1$ *is better than* $\overline{\overline{in}}_2$, *and denoted by* $\overline{\overline{in}}_1 > \overline{\overline{in}}_2$;
(2) *If* $S\left(\overline{\overline{in}}_1\right) = S\left(\overline{\overline{in}}_2\right)$, *and* $A\left(\overline{\overline{in}}_1\right) > A\left(\overline{\overline{in}}_2\right)$, *then* $\overline{\overline{in}}_1$ *is better than* $\overline{\overline{in}}_2$, *and denoted by* $\overline{\overline{in}}_1 > \overline{\overline{in}}_2$;
(3) *If* $S\left(\overline{\overline{in}}_1\right) = S\left(\overline{\overline{in}}_2\right)$, *and* $A\left(\overline{\overline{in}}_1\right) = A\left(\overline{\overline{in}}_2\right)$, *then* $\overline{\overline{in}}_1$ *is equal to* $\overline{\overline{in}}_2$, *and denoted by* $\overline{\overline{in}}_1 = \overline{\overline{in}}_2$.

Definition 7. *Let* $\overline{\overline{in}}_1 = \left\langle \left[\overline{\overline{TR}}_1^L, \overline{\overline{TR}}_1^U\right], \left[\overline{\overline{ID}}_1^L, \overline{\overline{ID}}_1^U\right], \left[\overline{\overline{FL}}_1^L, \overline{\overline{FL}}_1^U\right] \right\rangle$ *and* $\overline{\overline{in}}_2 = \left\langle \left[\overline{\overline{TR}}_2^L, \overline{\overline{TR}}_2^U\right], \left[\overline{\overline{ID}}_2^L, \overline{\overline{ID}}_2^U\right], \left[\overline{\overline{FL}}_2^L, \overline{\overline{FL}}_2^U\right] \right\rangle$ *be any two INNs [15]. Then the normalized Hamming distance between* n_1 *and* n_2 *is described as follows.*

$$D\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2\right) = \frac{1}{6}\left(\left|\overline{\overline{TR}}_1^L - \overline{\overline{TR}}_2^L\right| + \left|\overline{\overline{TR}}_1^U - \overline{\overline{TR}}_2^U\right| + \left|\overline{\overline{ID}}_1^L - \overline{\overline{ID}}_2^L\right| + \left|\overline{\overline{ID}}_1^U - \overline{\overline{ID}}_2^U\right| + \left|\overline{\overline{FL}}_1^L - \overline{\overline{FL}}_2^L\right| + \left|\overline{\overline{FL}}_1^U - \overline{\overline{FL}}_2^U\right| \right) \tag{10}$$

2.2. The PA Operator

The PA operator was first presented by Yager [36] and it is described as follows.

Definition 8. *For positive real numbers* $\wp_h (h = 1, 2, \ldots, l)$ *[36], the PA operator is described as*

$$PA(\wp_1, \wp_2, \ldots, \wp_l) = \frac{\sum_{h=1}^{l} (1 + T(\wp_h))\wp_h}{\sum_{h=1}^{l} (1 + T(\wp_h))}, \tag{11}$$

where, $T(\wp_h) = \sum_{y=1, h \neq y}^{l} sup(\wp_h, \wp_y)$, *and* $sup(\wp_h, \wp_y)$ *is the degree to which* $\wp_h$ *supports* $\wp_y$. *The support degree (SPD) satisfies the following properties.*

(1) $sup(\wp_h, \wp_y) = sup(\wp_y, \wp_h)$;
(2) $sup(\wp_h, \wp_y) \in [0, 1]$;
(3) $sup(\wp_h, \wp_y) \geq sup(\wp_c, \wp_d)$, *if* $|\wp_h - \wp_y| \leq |\wp_c - \wp_d|$.

2.3. The BM Operator

The BM operator was initially presented by Bonferroni [38], and it was explained as follows:

Definition 9. *For non-negative real numbers* $\wp_h (h = 1, 2, \ldots, l)$, *and* $x, y \geq 0$ *[38], the BM operator is described as*

$$BM^{x,y}(\wp_1, \wp_2, \ldots, \wp_l) = \left(\frac{1}{l^2 - l} \sum_{h=1}^{l} \sum_{s=1, h \neq s}^{l} \wp_h^x \wp_s^y \right)^{\frac{1}{x+y}}. \tag{12}$$

The BM operator ignores the importance degree of each input argument, which can be given by decision makers according to their interest. To overcome this shortcoming of BM operator, He et al. [52] defined the weighted Bonferroni mean (WBM) operators which can be explained as follows:

Definition 10. *For positive real numbers $\wp_h (h = 1, 2, \ldots, l)$ and $x, y \geq 0$ [52], then the weighted BM operator (WBM) is described as*

$$WBM^{x,y}(\wp_1, \wp_2, \ldots, \wp_l) = \left(\frac{1}{l^2 - l} \sum_{h=1}^{l} \sum_{s=1, h \neq s}^{l} \frac{\widetilde{\kappa}_h \widetilde{\kappa}_s}{1 - \widetilde{\kappa}_h} \wp_h^x \wp_s^y \right)^{\frac{1}{x+y}}, \tag{13}$$

where $\widetilde{\kappa} = (\widetilde{\kappa}_1, \widetilde{\kappa}_2, \ldots, \widetilde{\kappa}_l)^T$ is the importance degree of every $\wp_h (h = 1, 2, \ldots, l)$.

The WBM operator has the following characteristics:

Theorem 1. *(Reducibility) If the weight vector is $\widetilde{\kappa} = \left(\frac{1}{l}, \frac{1}{l}, \ldots, \frac{1}{l} \right)^T$, then*

$$\begin{aligned} WBM^{x,y}(\wp_1, \wp_2, \ldots, \wp_l) &= \left(\frac{1}{l^2 - l} \sum_{h=1}^{l} \sum_{s=1, z \neq s}^{l} \wp_h^x \wp_s^y \right)^{\frac{1}{x+y}} \\ &= BM^{x,y}(\wp_1, \wp_2, \ldots, \wp_m). \end{aligned} \tag{14}$$

Theorem 2. *(Idempotency) Let $\wp_h = \wp, (h = 1, 2, \ldots, l)$. Then $BM^{x,y}(\wp_1, \wp_2, \ldots, \wp_l) = \wp$.*

Theorem 3. *(Permutation) Let $(\wp_1, \wp_2, \ldots, \wp_l)$ be any permutation of $\left(Z_1', Z_2', \ldots, Z_l' \right)$. Then*

$$WMB^{x,y}\left(Z_1', Z_2', \ldots, Z_l' \right) = WBM(\wp_1, \wp_2, \ldots, \wp_l). \tag{15}$$

Theorem 4. *(Monotonicity) Let $\wp_h \geq K_h' (h = 1, 2, \ldots, l)$. Then*

$$WBM^{x,y}(\wp_1, \wp_2, \ldots, \wp_l) \geq WBM^{x,y}\left(K_1', K_2', \ldots, K_l' \right). \tag{16}$$

Theorem 5. *(Boundedness) The $WBM^{x,y}$ lies in the min and max operators, that is,*

$$\min(\wp_1, \wp_2, \ldots, \wp_l) \leq WBM^{x,y}(\wp_1, \wp_2, \ldots, \wp_l) \leq \max(\wp_1, \wp_2, \ldots, \wp_l). \tag{17}$$

Similar to BM operator, the geometric BM operator also considers the correlation among the input arguments. It can be explained as follows:

Definition 11. *For positive real numbers $\wp_h (h = 1, 2, \ldots, l)$ and $x, y \geq 0$ [53], the geometric BM operator (GBM) is described as*

$$GBM^{x,y}(\wp_1, \wp_2, \ldots, \wp_l) = \frac{1}{x + y} \prod_{h=1}^{l} \prod_{s=1, h \neq s}^{l} (x \wp_h + y \wp_s)^{\frac{1}{l^2 - l}}. \tag{18}$$

The GBM operator ignores the importance degree of each input argument, which can be given by decision makers according to their interest. In a similar way to WBM, the weighted geometric

BM (WGBM) operator was also presented. The extension process is same as that of WBM, so it is omitted here.

The definition of power Bonferroni mean (PBM) and power geometric Bonferroni mean (PGBM) operators are given in Appendix A.

3. Some Operations of INSs Based on Dombi TN and TCN

Dombi TN and TCN

Dombi operations consist of the Dombi sum and Dombi product.

Definition 12. *Let* $\Im$ *and* $\aleph$ *be any two real numbers [47]. Then the Dombi TN and TCN among* $\Im$ *and* $\aleph$ *are explained as follows:*

$$T_D(\Im,\aleph) = \frac{1}{1+\left\{\left(\frac{1-\Im}{\Im}\right)^l+\left(\frac{1-\aleph}{\aleph}\right)^l\right\}^{\frac{1}{l}}}; \tag{19}$$

$$T_D{}^*(\Im,\aleph) = 1-\frac{1}{1+\left\{\left(\frac{\Im}{1-\Im}\right)^l+\left(\frac{\aleph}{1-\aleph}\right)^l\right\}^{\frac{1}{l}}}, \tag{20}$$

where, $l \geq 1$, *and* $(\Im,\aleph) \in [0,1]\times[0,1]$.

According to the Dombi TN and TCN, we develop a few operational rules for INNs.

Definition 13. *Let* $\overline{\overline{in}} = \left\langle \left[\overline{\overline{TR}}^L, \overline{\overline{TR}}^U\right], \left[\overline{\overline{ID}}^L, \overline{\overline{ID}}^U\right], \left[\overline{\overline{FL}}^L, \overline{\overline{FL}}^U\right] \right\rangle$, $\overline{\overline{in}}_1 = \left\langle \left[\overline{\overline{TR}}_1^L, \overline{\overline{TR}}_1^U\right], \left[\overline{\overline{ID}}_1^L, \overline{\overline{ID}}_1^U\right], \left[\overline{\overline{FL}}_1^L, \overline{\overline{FL}}_1^U\right] \right\rangle$ *and* $\overline{\overline{in}}_2 = \left\langle \left[\overline{\overline{TR}}_2^L, \overline{\overline{TR}}_2^U\right], \left[\overline{\overline{ID}}_2^L, \overline{\overline{ID}}_2^U\right], \left[\overline{\overline{FL}}_2^L, \overline{\overline{FL}}_2^U\right] \right\rangle$ *be any three INNs and* $\Phi > 0$. *Then, based on Dombi TN and TCN, the following operational laws are developed for INNs.*

(1)
$$\overline{\overline{in}}_1 \oplus \overline{\overline{in}}_2 = \left\langle \begin{array}{l} \left[1-\frac{1}{1+\left(\left(\frac{\overline{\overline{TR}}_1^L}{1-\overline{\overline{TR}}_1^L}\right)^\gamma+\left(\frac{\overline{\overline{TR}}_2^L}{1-\overline{\overline{TR}}_2^L}\right)^\gamma\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\left(\frac{\overline{\overline{TR}}_1^U}{1-\overline{\overline{TR}}_1^U}\right)^\gamma+\left(\frac{\overline{\overline{TR}}_2^U}{1-\overline{\overline{TR}}_2^U}\right)^\gamma\right)^{\frac{1}{\gamma}}}\right], \left[\frac{1}{1+\left(\left(\frac{1-\overline{\overline{ID}}_1^L}{\overline{\overline{ID}}_1^L}\right)^\gamma+\left(\frac{1-\overline{\overline{ID}}_2^L}{\overline{\overline{ID1}}_2^L}\right)^\gamma\right)^{\frac{1}{\gamma}}},\right. \\ \left.\frac{1}{1+\left(\left(\frac{1-\overline{\overline{ID}}_1^U}{\overline{\overline{ID}}_1^U}\right)^\gamma+\left(\frac{1-\overline{\overline{ID}}_2^U}{\overline{\overline{ID}}_2^U}\right)^\gamma\right)^{\frac{1}{\gamma}}}\right], \left[\frac{1}{1+\left(\left(\frac{1-\overline{\overline{FL}}_1^L}{\overline{\overline{FL}}_1^L}\right)^\gamma+\left(\frac{1-\overline{\overline{FL}}_2^L}{\overline{\overline{FL}}_2^L}\right)^\gamma\right)^{\frac{1}{v}}}, \frac{1}{1+\left(\left(\frac{1-\overline{\overline{FL}}_1^U}{\overline{\overline{FL}}_1^U}\right)^\gamma+\left(\frac{1-\overline{\overline{FL}}_2^U}{\overline{\overline{FL}}_2^U}\right)^\gamma\right)^{\frac{1}{\gamma}}}\right] \end{array} \right\rangle; \tag{21}$$

(2)
$$\overline{\overline{in}}_1 \otimes \overline{\overline{in}}_2 = \left\langle \begin{array}{l} \left[\frac{1}{1+\left(\left(\frac{1-\overline{\overline{TR}}_1^L}{\overline{\overline{TR}}_1^L}\right)^\gamma+\left(\frac{1-\overline{\overline{TR}}_2^L}{\overline{\overline{TR}}_2^L}\right)^\gamma\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\left(\frac{1-\overline{\overline{TR}}_1^U}{\overline{\overline{TR}}_1^U}\right)^\gamma+\left(\frac{1-\overline{\overline{TR}}_2^U}{\overline{\overline{TR}}_2^U}\right)^\gamma\right)^{\frac{1}{\gamma}}}\right], \left[1-\frac{1}{1+\left(\left(\frac{\overline{\overline{ID}}_1^L}{1-\overline{\overline{ID}}_1^L}\right)^\gamma+\left(\frac{\overline{\overline{ID}}_2^L}{1-\overline{\overline{ID}}_2^L}\right)^\gamma\right)^{\frac{1}{\gamma}}},\right. \\ \left.1-\frac{1}{1+\left(\left(\frac{\overline{\overline{ID}}_1^U}{1-\overline{\overline{ID}}_1^U}\right)^\gamma+\left(\frac{\overline{\overline{ID}}_2^U}{1-\overline{\overline{ID}}_2^U}\right)^\gamma\right)^{\frac{1}{\gamma}}}\right], \left[1-\frac{1}{1+\left(\left(\frac{\overline{\overline{FL}}_1^L}{1-\overline{\overline{FL}}_1^L}\right)^\gamma+\left(\frac{\overline{\overline{FL}}_2^L}{1-\overline{\overline{FL}}_2^L}\right)^\gamma\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\left(\frac{\overline{\overline{FL}}_1^U}{1-\overline{\overline{FL}}_1^U}\right)^\gamma+\left(\frac{\overline{\overline{FL}}_2^U}{1-\overline{\overline{FL}}_2^U}\right)^\gamma\right)^{\frac{1}{\gamma}}}\right] \end{array} \right\rangle; \tag{22}$$

$$(3)\quad \Phi\overline{\overline{in}} = \left\langle \left[1-\frac{1}{1+\left(\Phi\left(\frac{\overline{\overline{TR}}^{L}}{1-\overline{\overline{TR}}^{L}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\Phi\left(\frac{\overline{\overline{TR}}^{U}}{1-\overline{\overline{TR}}^{U}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right], \left[\frac{1}{1+\left(\Phi\left(\frac{1-\overline{\overline{ID}}^{L}}{\overline{\overline{ID}}^{L}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\Phi\left(\frac{1-\overline{\overline{ID}}^{U}}{\overline{\overline{ID}}^{U}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right], \left[\frac{1}{1+\left(\Phi\left(\frac{1-\overline{\overline{FL}}^{L}}{\overline{\overline{FL}}^{L}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\Phi\left(\frac{1-\overline{\overline{FL}}^{U}}{\overline{\overline{FL}}^{U}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right] \right\rangle; \tag{23}$$

$$(4)\quad \overline{\overline{in}}^{\Phi} = \left\langle \left[\frac{1}{1+\left(\Phi\left(\frac{1-\overline{\overline{TR}}^{L}}{\overline{\overline{TR}}^{L}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\Phi\left(\frac{1-\overline{\overline{TR}}^{U}}{\overline{\overline{TR}}^{U}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right], \left[1-\frac{1}{1+\left(\Phi\left(\frac{\overline{\overline{ID}}^{L}}{1-\overline{\overline{ID}}^{L}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\Phi\left(\frac{\overline{\overline{ID}}^{U}}{1-\overline{\overline{ID}}^{U}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right], \left[1-\frac{1}{1+\left(\Phi\left(\frac{\overline{\overline{FL}}^{L}}{1-\overline{\overline{FL}}^{L}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\Phi\left(\frac{\overline{\overline{FL}}^{U}}{1-\overline{\overline{FL}}^{U}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right] \right\rangle \tag{24}$$

Now, based on these new operational laws for INNs, we develop some aggregation operators to aggregate IN information in the preceding sections.

4. The INPBM Operator Based on Dombi TN and Dombi TCN

In this part, based on the Dombi operational laws for INNs, we combine PA operator and BM to introduce interval neutrosophic Dombi power Bonferroni mean (INDPBM), interval neutrosophic weighted Dombi power Bonferroni mean, interval neutrosophic Dombi power geometric Bonferroni mean (INDPGBM) and interval neutrosophic weighted Dombi power Bonferroni mean (INWDPGBM) operators and discuss some related properties.

4.1. The INDPBM Operator and INWDPBM Operator

Definition 14. *Let* $\overline{\overline{in}}_i = \left\langle \left[\overline{\overline{TR}}_i^{L}, \overline{\overline{TR}}_i^{U}\right], \left[\overline{\overline{ID}}_i^{L}, \overline{\overline{ID}}_i^{U}\right], \left[\overline{\overline{FL}}_i^{L}, \overline{\overline{FL}}_i^{L}\right] \right\rangle, (i = 1, 2, \ldots, l)$, *be a group of INNs, and* $x, y \geq 0$. *If*

$$INDPBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right) = \left(\frac{1}{l^2-l}\left(\underset{\substack{i,j=1\\ i\neq j}}{\overset{l}{\oplus}}\left(\left(\frac{l\left(1+T\left(\overline{\overline{in}}_i\right)\right)}{\overset{l}{\underset{u=1}{\oplus}}\left(1+T\left(\overline{\overline{in}}_u\right)\right)}\overline{\overline{in}}_i\right)^{x} \otimes_D \left(\frac{l\left(1+T\left(\overline{\overline{in}}_j\right)\right)}{\overset{l}{\underset{u=1}{\oplus}}\left(1+T\left(\overline{\overline{in}}_u\right)\right)}\overline{\overline{in}}_j\right)^{y}\right)\right)\right)^{\frac{1}{x+y}}. \tag{25}$$

then $INDPBM^{x,y}$ *is said to be IN Dombi power Bonferroni mean (INDPBM) operator, where* $T\left(\overline{\overline{in}}_z\right) = \overset{l}{\underset{s=1,s\neq z}{\oplus}} Sup\left(\overline{\overline{in}}_z, \overline{\overline{in}}_s\right)$. $Sup\left(\overline{\overline{in}}_z, \overline{\overline{in}}_s\right)$ *is the support degree for* $\overline{\overline{in}}_z$ *from* $\overline{\overline{in}}_s$, *which satisfies the following axioms: (1)* $Sup\left(\overline{\overline{in}}_z, \overline{\overline{in}}_s\right) \in [0,1]$; *(2)* $Sup\left(\overline{\overline{in}}_z, \overline{\overline{in}}_s\right) = Sup\left(\overline{\overline{in}}_s, \overline{\overline{in}}_z\right)$; *(3)* $Sup\left(\overline{\overline{in}}_z, \overline{\overline{in}}_s\right) \geq Sup\left(\overline{\overline{in}}_a, \overline{\overline{in}}_b\right)$, *if* $D\left(\overline{\overline{in}}_z, \overline{\overline{in}}_s\right) < D\left(\overline{\overline{in}}_a, \overline{\overline{in}}_b\right)$, *in which* $D\left(\overline{\overline{in}}_a, \overline{\overline{in}}_b\right)$ *is the distance measure between INNs* $\overline{\overline{in}}_a$ and $\overline{\overline{in}}_b$ *defined in Definition 7.*

In order to simplify Equation (25), we can give

$$\Lambda_z = \frac{\left(1+T\left(\overline{\overline{in}}_z\right)\right)}{\overset{l}{\underset{z=1}{\oplus}}\left(1+T\left(\overline{\overline{in}}_z\right)\right)} \tag{26}$$

and call $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_l)^T$ is the power weight vector (PWV), such that $\Lambda_z \geq 0$, $\overset{l}{\underset{z=1}{\oplus}} \Lambda_z = 1$. It turns Equation (25) into the following form

$$INDPBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right) = \left(\frac{1}{l^2 - l} \underset{\substack{i,j=1 \\ i \neq j}}{\overset{l}{\oplus}} \left(l\Lambda_i \overline{\overline{in}}_i\right)^x \otimes_D \left(l\Lambda_j \overline{\overline{in}}_j\right)^y \right)^{\frac{1}{x+y}}. \tag{27}$$

Theorem 6. *Let* $\overline{\overline{in}}_i = \left\langle \left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L\right] \right\rangle, (i = 1, 2, \ldots, l)$ *be a group of INNs. Then the value obtained by utilizing Equation (25) is expressed as*

$$INDPBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_e\right)$$

$$= \left\langle \left[1 \middle/ \left(1 + \frac{l^2 - l}{x+y} \times 1 \middle/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{e} 1 \middle/ \left(\frac{x}{l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^L}{1 - \overline{\overline{TR}}_i^L}\right)^\gamma} + \frac{y}{l\Lambda_j \left(\frac{\overline{\overline{TR}}_j^L}{1 - \overline{\overline{TR}}_j^L}\right)^\gamma} \right) \right)^{\frac{1}{\gamma}} \right), 1 \middle/ \left(1 + \frac{l^2 - l}{x+y} \times 1 \middle/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \middle/ \left(\frac{x}{l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^U}{1 - \overline{\overline{TR}}_i^U}\right)^\gamma} + \frac{y}{l\Lambda_j \left(\frac{\overline{\overline{TR}}_j^U}{1 - \overline{\overline{TR}}_j^U}\right)^\gamma} \right) \right)^{\frac{1}{\gamma}} \right) \right], \right.$$

$$\left[1 - 1 \middle/ \left(1 + \frac{l^2 - l}{x+y} \times 1 \middle/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \middle/ \left(\frac{x}{l\Lambda_i \left(\frac{1 - \overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L}\right)^\gamma} + \frac{y}{l\Lambda_j \left(\frac{1 - \overline{\overline{ID}}_j^L}{\overline{\overline{ID}}_j^L}\right)^\gamma} \right) \right)^{\frac{1}{\gamma}} \right), 1 - 1 \middle/ \left(1 + \frac{l^2 - l}{x+y} \times 1 \middle/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \middle/ \left(\frac{x}{l\Lambda_i \left(\frac{1 - \overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U}\right)^\gamma} + \frac{y}{l\Lambda_j \left(\frac{1 - \overline{\overline{ID}}_j^U}{\overline{\overline{ID}}_j^U}\right)^\gamma} \right) \right)^{\frac{1}{\gamma}} \right) \right], \tag{28}$$

$$\left. \left[1 - 1 \middle/ \left(1 + \frac{l^2 - l}{x+y} \times 1 \middle/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \middle/ \left(\frac{x}{l\Lambda_i \left(\frac{1 - \overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L}\right)^\gamma} + \frac{y}{l\Lambda_j \left(\frac{1 - \overline{\overline{FL}}_j^L}{\overline{\overline{FL}}_j^L}\right)^\gamma} \right) \right)^{\frac{1}{\gamma}} \right), 1 - 1 \middle/ \left(1 + \frac{l^2 - l}{x+y} \times 1 \middle/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \middle/ \left(\frac{x}{l\Lambda_i \left(\frac{1 - \overline{\overline{FL}}_i^U}{\overline{\overline{FL}}_i^U}\right)^\gamma} + \frac{y}{l\Lambda_j \left(\frac{1 - \overline{\overline{FL}}_j^U}{\overline{\overline{FL}}_j^U}\right)^\gamma} \right) \right)^{\frac{1}{\gamma}} \right) \right] \right\rangle.$$

Proof. Proof of Theorem 6 is given in Appendix B. □

In order to determine the PWV Λ, we firstly need to determine the support degree among INNs. In general, the similarity measure among INNs can replace the support degree among INNs. That is,

$$Sup\left(\overline{\overline{in}}_i, \overline{\overline{in}}_m\right) = 1 - D\left(\overline{\overline{in}}_i, \overline{\overline{in}}_m\right)\ (i, m = 1, 2, \ldots, l). \tag{29}$$

Example 1. *Let* $\overline{\overline{in}}_1 = \langle [0.3, 0.7], [0.2, 0.4], [0.3, 0.5] \rangle, \overline{\overline{in}}_2 = \langle [0.4, 0.6], [0.1, 0.3], [0.2, 0.4] \rangle$ *and* $\overline{\overline{in}}_3 = \langle [0.1, 0.3], [0.4, 0.6], [0.2, 0.4] \rangle$ *be any three INNs,* $x = 1, y = 1, \gamma = 3$. *Then by Theorem 6 in Equation (28), we can aggregate these three INNs and generate the comprehensive value* $\overline{\overline{in}} = \left\langle \left[\overline{\overline{TR}}^L, \overline{\overline{TR}}^U\right], \left[\overline{\overline{ID}}^L, \overline{\overline{ID}}^U\right], \left[\overline{\overline{FL}}^L, \overline{\overline{FL}}^U\right] \right\rangle$ *which is calculated as follows:*

Step 1. *Determine the supports* $Sup\left(\overline{\overline{in}}_i, \overline{\overline{in}}_j\right), i, j = 1, 2, 3$ *by using Equation (29), and then we get* $Sup\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2\right) = Sup\left(\overline{\overline{in}}_2, \overline{\overline{in}}_1\right) = 0.9, Sup\left(\overline{\overline{in}}_1, \overline{\overline{in}}_3\right) = Sup\left(\overline{\overline{in}}_3, \overline{\overline{in}}_1\right) = 0.933, Sup\left(\overline{\overline{in}}_2, \overline{\overline{in}}_3\right) = Sup\left(\overline{\overline{in}}_3, \overline{\overline{in}}_2\right) = 1.$

Step 2. *Determine the PWV Because* $T\left(\overline{\overline{in}}_z\right) = \sum_{s=1, s\neq z}^{3} Sup\left(\overline{\overline{in}}_z, \overline{\overline{in}}_s\right)$*, and*

$$T\left(\overline{\overline{in}}_1\right) = Sup\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2\right) + Sup\left(\overline{\overline{in}}_1, \overline{\overline{in}}_3\right) = 1.833, T\left(\overline{\overline{in}}_2\right) = Sup\left(\overline{\overline{in}}_2, \overline{\overline{in}}_1\right) + Sup\left(\overline{\overline{in}}_2, \overline{\overline{in}}_3\right) = 1.9, T\left(\overline{\overline{in}}_3\right) = Sup\left(\overline{\overline{in}}_3, \overline{\overline{in}}_1\right) + Sup\left(\overline{\overline{in}}_3, \overline{\overline{in}}_2\right) = 1.933,$$

then

$$\Lambda_1 = \frac{\left(T\left(\overline{\overline{in}}_1\right) + 1\right)}{\left(T\left(\overline{\overline{in}}_1\right) + 1\right) + \left(T\left(\overline{\overline{in}}_2\right) + 1\right) + \left(T\left(\overline{\overline{in}}_3\right) + 1\right)} = 0.3269,$$

$$\Lambda_2 = \frac{\left(T\left(\overline{\overline{in}}_2\right) + 1\right)}{\left(T\left(\overline{\overline{in}}_1\right) + 1\right) + \left(T\left(\overline{\overline{in}}_2\right) + 1\right) + \left(T\left(\overline{\overline{in}}_3\right) + 1\right)} = 0.3346,$$

$$\Lambda_3 = \frac{\left(T\left(\overline{\overline{in}}_3\right) + 1\right)}{\left(T\left(\overline{\overline{in}}_1\right) + 1\right) + \left(T\left(\overline{\overline{in}}_2\right) + 1\right) + \left(T\left(\overline{\overline{in}}_3\right) + 1\right)} = 0.3385.$$

Step 3. *Determine the comprehensive value* $\overline{\overline{in}} = \left\langle \left[\overline{\overline{TR}}^L, \overline{\overline{TR}}^U\right], \left[\overline{\overline{ID}}^L, \overline{\overline{ID}}^U\right], \left[\overline{\overline{FL}}^L, \overline{\overline{FL}}^U\right] \right\rangle$ *by using Equation (28), we have*

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3} 1\Bigg/\left(\frac{x}{3\Lambda_i\left(\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L}\right)^3} + \frac{y}{3\Lambda_j\left(\frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L}\right)^3}\right) = 1\Bigg/\left(\frac{1}{3\Lambda_1\left(\frac{\overline{\overline{TR}}_1^L}{1-\overline{\overline{TR}}_1^L}\right)^3} + \frac{2}{3\Lambda_2\left(\frac{\overline{\overline{TR}}_2^L}{1-\overline{\overline{TR}}_2^L}\right)^3}\right) + 1\Bigg/\left(\frac{1}{3\Lambda_1\left(\frac{\overline{\overline{TR}}_1^L}{1-\overline{\overline{TR}}_1^L}\right)^3} + \frac{2}{3\Lambda_3\left(\frac{\overline{\overline{TR}}_3^L}{1-\overline{\overline{TR}}_3^L}\right)^3}\right) + 1\Bigg/\left(\frac{1}{3\Lambda_2\left(\frac{\overline{\overline{TR}}_2^L}{1-\overline{\overline{TR}}_2^L}\right)^3} + \frac{2}{3\Lambda_1\left(\frac{\overline{\overline{TR}}_1^L}{1-\overline{\overline{TR}}_1^L}\right)^3}\right)$$

$$+ 1\Bigg/\left(\frac{1}{3\Lambda_2\left(\frac{\overline{\overline{TR}}_2^L}{1-\overline{\overline{TR}}_2^L}\right)^3} + \frac{2}{3\Lambda_3\left(\frac{\overline{\overline{TR}}_3^L}{1-\overline{\overline{TR}}_3^L}\right)^3}\right) + 1\Bigg/\left(\frac{1}{3\Lambda_3\left(\frac{\overline{\overline{TR}}_3^L}{1-\overline{\overline{TR}}_3^L}\right)^3} + \frac{2}{3\Lambda_1\left(\frac{\overline{\overline{TR}}_1^L}{1-\overline{\overline{TR}}_1^L}\right)^3}\right) + 1\Bigg/\left(\frac{1}{3\Lambda_3\left(\frac{\overline{\overline{TR}}_3^L}{1-\overline{\overline{TR}}_3^L}\right)^3} + \frac{2}{3\Lambda_2\left(\frac{\overline{\overline{TR}}_2^L}{1-\overline{\overline{TR}}_2^L}\right)^3}\right) = 0.1281$$

$$1\Bigg/\left(1 + \left(\frac{3^2-3}{1+1} \times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{3} 1\Bigg/\left(\frac{1}{3\Lambda_i\left(\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L}\right)^\gamma} + \frac{1}{3\Lambda_j\left(\frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L}\right)^\gamma}\right)\right)\right)^{\frac{1}{3}}\right) = 0.2590$$

Similarly, we can get $n = \langle[0.2590, 0.5525], [0.2221, 0.4373], [0.2334, 0.4365]\rangle$.

Theorem 7. *(Idempotency) Let* $\overline{\overline{in}}_i = \left\langle \left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right] \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L\right] \right\rangle, (i = 1, 2, \ldots, l)$*, be a group of INNs, if all* $\overline{\overline{in}}_i (i = 1, 2, \ldots, l)$ *are equal, that is* $\overline{\overline{in}}_i = \overline{\overline{in}} = \left\langle \left[\overline{\overline{TR}}^L, \overline{\overline{TR}}^U\right], \left[\overline{\overline{ID}}^L, \overline{\overline{ID}}^U\right], \left[\overline{\overline{FL}}^L, \overline{\overline{FL}}^U\right] \right\rangle, (i = 1, 2, \ldots, l)$*,. Then*

$$INDPBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right) = \overline{\overline{in}}. \tag{30}$$

Proof. Proof of Theorem 7 is given in Appendix C. □

Theorem 8. *(Commutativity) Assume that* $\overline{\overline{in'}}_u$ *is any permutation of* $\overline{\overline{in}}_u (u = 1, 2, \ldots, l)$, *then*

$$INDPBM^{x,y}\left(\overline{\overline{in'}}_1, \overline{\overline{in'}}_2, \ldots, \overline{\overline{in'}}_l\right) = INDPBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right). \tag{31}$$

Proof. From Definition 14, we have

$$INDPBM^{x,y}\left(\overline{\overline{in'}}_1, \overline{\overline{in'}}_2, \ldots, \overline{\overline{in'}}_e\right) = \left(\frac{1}{l^2 - l} \sum_{\substack{i,j=1 \\ i \neq j}}^{l} \left(l\Lambda'_i \overline{\overline{in'}}_i\right)^x \otimes_D \left(l\Lambda'_j \overline{\overline{in'}}_j\right)^y\right)^{\frac{1}{x+y}},$$

and

$$INDPBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right) = \left(\frac{1}{l^2 - l} \sum_{\substack{i,j=1 \\ i \neq j}}^{l} \left(l\Lambda_i \overline{\overline{in}}_i\right)^x \otimes_D \left(l\Lambda_j \overline{\overline{in}}_j\right)^y\right)^{\frac{1}{x+y}}.$$

Because,

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{l} \left(l\Lambda'_i \overline{\overline{in'}}_i\right)^x \otimes_D \left(l\Lambda'_j \overline{\overline{in'}}_j\right)^y = \sum_{\substack{i,j=1 \\ i \neq j}}^{l} \left(l\Lambda_i \overline{\overline{in}}_i\right)^x \otimes_D \left(l\Lambda_j \overline{\overline{in}}_j\right)^y,$$

Hence, $INDPBM^{x,y}\left(\overline{\overline{in'}}_1, \overline{\overline{in'}}_2, \ldots, \overline{\overline{in'}}_l\right) = INDPBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right)$. □

Theorem 9. *(Boundedness) Let* $\overline{\overline{in}}_i = \left\langle \left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L\right] \right\rangle, (i = 1, 2, \ldots, l)$ *be a group of INNs, and* $\overline{\overline{in}}^+ = \left\langle \max_{i=1}^{l}\left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \min_{i=1}^{l}\left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \min_{i=1}^{l}\left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^U\right] \right\rangle, \overline{\overline{in}}^- = \left\langle \min_{i=1}^{l}\left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \max_{i=1}^{l}\left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \max_{i=1}^{l}\left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^U\right] \right\rangle$,. *Then*

$$\overline{\overline{in}}^- \leq INDPBM\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right) \leq \overline{\overline{in}}^+. \tag{32}$$

Proof. Proof of Theorem 9 is given in Appendix D. □

Now, we shall study a few special cases of the $INDPBM^{x,y}$ with respect to x and y. (1) When $y \to 0, \gamma > 0,$ then we can get

$$INDPBM^{x,0}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right)$$

$$= \lim_{y \to 0} \left\langle \begin{array}{l} \left[\left(1 \Big/ \left(1 + \left(\frac{l^2 - l}{x} \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^L}{1 - \overline{\overline{TR}}_i^L} \right)^{\gamma}} \right) \right) \right) \right) \right)^{\frac{1}{\gamma}}, \left(1 \Big/ \left(1 + \left(\frac{l^2 - l}{x} \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^U}{1 - \overline{\overline{TR}}_i^U} \right)^{\gamma}} \right) \right) \right) \right) \right)^{\frac{1}{\gamma}} \right], \\ \left[\left(1 - 1 \Big/ \left(1 + \left(\frac{l^2 - l}{x} \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1 - \overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L} \right)^{\gamma}} \right) \right) \right) \right) \right)^{\frac{1}{\gamma}}, \left(1 \Big/ \left(1 + \left(\frac{l^2 - l}{x} \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1 - \overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U} \right)^{\gamma}} \right) \right) \right) \right) \right)^{\frac{1}{\gamma}} \right], \\ \left[\left(1 - 1 \Big/ \left(1 + \left(\frac{l^2 - l}{x} \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1 - \overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L} \right)^{\gamma}} \right) \right) \right) \right) \right)^{\frac{1}{\gamma}}, \left(1 \Big/ \left(1 + \left(\frac{l^2 - l}{x} \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1 - \overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U} \right)^{\gamma}} \right) \right) \right) \right) \right)^{\frac{1}{\gamma}} \right]. \end{array} \right. \tag{33}$$

(2) When $x = 1, y \to 0, \gamma > 0,$ then we can get

$$INDPBM^{1,0}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right)$$

$$= \lim_{y \to 0} \left\langle \begin{array}{l} \left[\left(1 \Big/ \left(1 + \left(l^2 - l \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{1}{l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^L}{1 - \overline{\overline{TR}}_i^L} \right)^{\gamma}} \right) \right) \right) \right) \right)^{\frac{1}{\gamma}}, \left(1 \Big/ \left(1 + \left(l^2 - l \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{1}{l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^U}{1 - \overline{\overline{TR}}_i^U} \right)^{\gamma}} \right) \right) \right) \right) \right)^{\frac{1}{\gamma}} \right], \\ \left[\left(1 - 1 \Big/ \left(1 + \left(l^2 - l \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{1}{l\Lambda_i \left(\frac{1 - \overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L} \right)^{\gamma}} \right) \right) \right) \right) \right)^{\frac{1}{\gamma}}, \left(1 \Big/ \left(1 + \left(l^2 - l \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{1}{l\Lambda_i \left(\frac{1 - \overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U} \right)^{\gamma}} \right) \right) \right) \right) \right)^{\frac{1}{\gamma}} \right], \\ \left[\left(1 - 1 \Big/ \left(1 + \left(l^2 - l \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{1}{l\Lambda_i \left(\frac{1 - \overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L} \right)^{\gamma}} \right) \right) \right) \right) \right)^{\frac{1}{\gamma}}, \left(1 \Big/ \left(1 + \left(l^2 - l \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{1}{l\Lambda_i \left(\frac{1 - \overline{\overline{FL}}_i^U}{\overline{\overline{FL}}_i^U} \right)^{\gamma}} \right) \right) \right) \right) \right)^{\frac{1}{\gamma}} \right]. \end{array} \right. \tag{34}$$

(3) When $x = y = 1, \gamma > 0$, then we can get

$$INDPBM^{x,0}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right)$$

$$=\left\langle\left[1\Bigg/\left(1+\frac{l^2-l}{2}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}},\ 1\Bigg/\left(1+\frac{l^2-l}{2}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^U}{1-\overline{\overline{TR}}_i^U}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^U}{1-\overline{\overline{TR}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right],\right.$$

$$\left[1-1\Bigg/\left(1+\frac{l^2-l}{2}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^L}{\overline{\overline{ID}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}},1-1\Bigg/\left(1+\frac{l^2-l}{2}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^U}{\overline{\overline{ID}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right], \tag{35}$$

$$\left.\left[1-1\Bigg/\left(1+\frac{l^2-l}{2}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^L}{\overline{\overline{FL}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}},1-1\Bigg/\left(1+\frac{l^2-l}{2}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^U}{\Upsilon_i^U}\right)^{\gamma}}+\frac{y}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_j^U}{\overline{\overline{FL}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right]\right\rangle.$$

In the INDPBM operator, we can only take the correlation among the input arguments and cannot consider the importance degree of input arguments. In what follows, the INWPDBM operator shall be proposed to overcome the shortcoming of the INDPBM operator.

Definition 15. *Let* $\overline{\overline{in}}_i = \left\langle \left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L\right] \right\rangle, (i = 1, 2, \ldots, l)$*, be a group of INNs, then the INWDPBM operator is defined as*

$$INWDPBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,\ldots,\overline{\overline{in}}_l\right)=\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\ i\neq j}}^{l}\left(\left(\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\overline{\overline{in}}_i\right)^{x}\otimes_D\left(\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\overline{\overline{in}}_j\right)^{y}\right)\right)^{\frac{1}{x+y}}, \tag{36}$$

where, $T\left(\overline{\overline{in}}_i\right) = \sum_{j=1, i\neq j}^{l} Sup\left(\overline{\overline{in}}_i, \overline{\overline{in}}_j\right), x, y > 0, w = (w_1, w_2, \ldots, w_l)^T$ *is the importance degree of the INNs, such that* $0 \le w_z \le 1 (z = 1, 2, \ldots, l)$ *and* $\sum_{k=1}^{l} w_k = 1$.

Theorem 10. *Let* $\overline{\overline{in}}_i = \left\langle \left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L\right] \right\rangle, (i = 1, 2, \ldots, l)$ *be a group of INNs. Then the value obtained using Definition 15, is represented by*

$$INDPBM^{x,0}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,\ldots.,\overline{\overline{in}}_l\right)$$

$$
= \left\langle \left[\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum\limits_{\substack{i,j=1\\i\neq j}}^{l}\frac{1}{\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum\limits_{\substack{i,j=1\\i\neq j}}^{l}\frac{1}{\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{\overline{\overline{TR}}_i^U}{1-\overline{\overline{TR}}_i^U}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{\overline{\overline{TR}}_j^U}{1-\overline{\overline{TR}}_j^U}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}} \right],
$$

$$
\left[1-\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum\limits_{\substack{i,j=1\\i\neq j}}^{l}\frac{1}{\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{ID}}_j^L}{\overline{\overline{ID}}_j^L}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum\limits_{\substack{i,j=1\\i\neq j}}^{l}\frac{1}{\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{ID}}_j^U}{\overline{\overline{ID}}_j^U}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}} \right], \quad (37)
$$

$$
\left[1-\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum\limits_{\substack{i,j=1\\i\neq j}}^{l}\frac{1}{\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{FL}}_j^L}{\overline{\overline{FL}}_j^L}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum\limits_{\substack{i,j=1\\i\neq j}}^{l}\frac{1}{\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{FL}}_i^U}{\overline{\overline{FL}}_i^U}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{FL}}_j^U}{\overline{\overline{FL}}_j^U}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}} \right] \right\rangle.
$$

Proof. Proof of Theorem 10 is similar to Theorem 6. □

Similar to the INDPBM operator, the INWDPBM operator has the properties of boundedness, idempotency and commutativity.

4.2. The INDPGBM Operator and INWDPGBM Operator

In this subpart, we develop INDPGBM and INWDPGBM operators.

Definition 16. *Let* $\overline{\overline{in}}_i = \left\langle \left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L\right] \right\rangle, (i = 1, 2, \ldots, l)$ *be a group of INNs. Then the INDPGBM operator is defined as*

$$
INDPGBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right) = \frac{1}{x+y}\left(\prod_{\substack{i,j=1\\i\neq j}}^{l}\left(x\overline{\overline{in}}_i^{\frac{l(T(\overline{\overline{in}}_i)+1)}{\sum_{z=1}^{l}(T(\overline{\overline{in}}_z)+1)}} + y\overline{\overline{in}}_j^{\frac{l(T(\overline{\overline{in}}_j)+1)}{\sum_{z=1}^{l}(T(\overline{\overline{in}}_z)+1)}}\right)\right)^{\frac{1}{l^2-l}}. \quad (38)
$$

Then, $INDPGBM^{x,y}$ *is said to be an interval neutrosophic Dombi power geometric Bonferroni mean (INDPGBM) operator. Where* $T\left(\overline{\overline{in}}_z\right) = \sum_{s=1, s\neq z}^{l} Sup\left(\overline{\overline{in}}_z, \overline{\overline{in}}_s\right)$, $Sup\left(\overline{\overline{in}}_z, \overline{\overline{in}}_s\right)$ *is the support degree for* n_z *from* n_s*, which satisfies the following axioms: (1)* $Sup\left(\overline{\overline{in}}_z, \overline{\overline{in}}_s\right) \in [0, 1]$*; (2)* $Sup\left(\overline{\overline{in}}_z, \overline{\overline{in}}_s\right) = Sup\left(\overline{\overline{in}}_s, \overline{\overline{in}}_z\right)$*; (3)* $Sup\left(\overline{\overline{in}}_z, \overline{\overline{in}}_s\right) \geq Sup\left(\overline{\overline{in}}_a, \overline{\overline{in}}_b\right)$*, if* $D\left(\overline{\overline{in}}_z, \overline{\overline{in}}_s\right) < D\left(\overline{\overline{in}}_a, \overline{\overline{in}}_b\right)$*, in which* $D(n_a, n_b)$ *is the distance measure between INNs* $\overline{\overline{in}}_a$ *and* $\overline{\overline{in}}_b$ *defined in Definition 7.*

In order to simplify Equation (38), we can describe

$$
\Lambda_z = \frac{\left(1 + T\left(\overline{\overline{in}}_z\right)\right)}{\sum_{z=1}^{l}\left(1 + T\left(\overline{\overline{in}}_z\right)\right)}, \quad (39)
$$

and call $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_l)^T$ is the power weight vector (PWV), such that $\Lambda_z \geq 0$, $\sum_{z=1}^{l} \Lambda_z = 1$. Then Equation (38) can be written as follows:

$$INDPGBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right) = \frac{1}{x+y}\left(\prod_{\substack{i,j=1\\ i\neq j}}^{l}\left(x\overline{\overline{in}}_i^{\,l\Lambda_i} + y\overline{\overline{in}}_j^{\,l\Lambda_j}\right)\right)^{\frac{1}{l^2-l}}. \quad (40)$$

Theorem 11. *Let* $\overline{\overline{in}}_i = \left\langle\left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L\right]\right\rangle, (i = 1, 2, \ldots, l)$ *be a group of INNs. Then the result obtained from Equation (38) is expressed as*

$$INDPGBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right)$$

$$= \left\langle \begin{array}{l} \left[\left(1 - 1\Bigg/\left(1 + \left(\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{TR}}_i^L}{\overline{\overline{TR}}_i^L}\right)^{\gamma}} + \frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{TR}}_j^L}{\overline{\overline{TR}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1\Bigg/\left(1 + \left(\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{TR}}_i^U}{\overline{\overline{TR}}_i^U}\right)^{\gamma}} + \frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{TR}}_j^U}{\overline{\overline{TR}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right], \\ \left[\left(1\Bigg/\left(1 + \left(\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{ID}}_i^L}{1-\overline{\overline{ID}}_i^L}\right)^{\gamma}} + \frac{y}{l\Lambda_j\left(\frac{\overline{\overline{ID}}_j^L}{1-\overline{\overline{ID}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1\Bigg/\left(1 + \left(\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{ID}}_i^U}{1-\overline{\overline{ID}}_i^U}\right)^{\gamma}} + \frac{y}{l\Lambda_j\left(\frac{\overline{\overline{ID}}_j^U}{1-\overline{\overline{ID}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right], \\ \left[\left(1\Bigg/\left(1 + \left(\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{FL}}_i^L}{1-\overline{\overline{FL}}_i^L}\right)^{\gamma}} + \frac{y}{l\Lambda_j\left(\frac{\overline{\overline{FL}}_j^L}{1-\overline{\overline{FL}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1\Bigg/\left(1 + \left(\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{FL}}_i^U}{1-\overline{\overline{FL}}_i^U}\right)^{\gamma}} + \frac{y}{l\Lambda_i\left(\frac{\overline{\overline{FL}}_j^U}{1-\overline{\overline{FL}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right]. \end{array} \right. \quad (41)$$

Theorem 12. *(Idempotency) Let* $\overline{\overline{in}}_i = \left\langle\left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L\right]\right\rangle, (i = 1, 2, \ldots, l)$ *be a group of INNs, if all* $\overline{\overline{in}}_i (i = 1, 2, \ldots, l)$ *are equal, that is* $\overline{\overline{in}}_i = \overline{\overline{in}} = \left\langle\left[\overline{\overline{TR}}^L, \overline{\overline{TR}}^U\right], \left[\overline{\overline{ID}}^L, \overline{\overline{ID}}^U\right], \left[\overline{\overline{FL}}^L, \overline{\overline{FL}}^U\right]\right\rangle, (i = 1, 2, \ldots, l)$*, then*

$$INDPGBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right) = \overline{\overline{in}}. \quad (42)$$

Theorem 13. *(Commutativity) Assume that* $\overline{\overline{in'}}_u$ *is any permutation of* $\overline{\overline{in}}_u (u = 1, 2, \ldots, l)$*, then*

$$INDPGBM^{x,y}\left(\overline{\overline{in'}}_1, \overline{\overline{in'}}_2, \ldots, \overline{\overline{in'}}_l\right) = INDPGBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right). \quad (43)$$

Theorem 14. *(Boundedness) Let* $\overline{\overline{in}}_i = \left\langle \left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L\right] \right\rangle, (i = 1, 2, \ldots, l)$ *be a group of INNs, and* $\overline{\overline{in}}^+ = \left\langle \max_{i=1}^{l}\left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \min_{i=1}^{l}\left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \min_{i=1}^{l}\left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^U\right] \right\rangle, \overline{\overline{in}}^- = \left\langle \min_{i=1}^{l}\left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \max_{i=1}^{l}\left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \max_{i=1}^{l}\left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^U\right] \right\rangle$, *then*

$$\overline{\overline{in}}^- \leq INDPGBM\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right) \leq \overline{\overline{in}}^+. \tag{44}$$

Definition 17. *Let* $\overline{\overline{in}}_i = \left\langle \left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L\right] \right\rangle, (i = 1, 2, \ldots, l)$ *be a group of INNs, then the INWDPGBM operator is defined as*

$$INWDPGBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right) = \frac{1}{x+y}\left(\prod_{\substack{i,j=1\\ i\neq j}}^{l}\left(x\overline{\overline{in}}_i^{\frac{lw_i(T(\overline{\overline{in}}_i)+1)}{\sum_{z=1}^{l} w_z(T(\overline{\overline{in}}_z)+1)}} + y\overline{\overline{in}}_j^{\frac{lw_j(T(\overline{\overline{in}}_j)+1)}{\sum_{z=1}^{l} w_z(T(\overline{\overline{in}}_z)+1)}}\right)\right)^{\frac{1}{l^2-l}}. \tag{45}$$

Theorem 15. *Let* $\overline{\overline{in}}_i = \left\langle \left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L\right] \right\rangle, (i = 1, 2, \ldots, l)$ *be a group of INNs.*

Then the aggregated result from Equation (45) is expressed as

$$INWDPGBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right)$$

$$= \left\langle \left[\left(1 - 1\Big/\left(1 + \left(\frac{l^2-l}{x+y} \times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l} w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{1-\overline{\overline{TR}}_i^L}{\overline{\overline{TR}}_i^L}\right)^{\gamma}} + \frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_j)+1\right)}{\sum_{z=1}^{l} w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{1-\overline{\overline{TR}}_j^L}{\overline{\overline{TR}}_j^L}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right), \left(1\Big/\left(1 + \left(\frac{l^2-l}{x+y} \times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l} w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{1-\overline{\overline{TR}}_i^U}{\overline{\overline{TR}}_i^U}\right)^{\gamma}} + \frac{y}{\frac{lwj\left(T(\overline{\overline{in}}_j)+1\right)}{\sum_{z=1}^{l} w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{1-\overline{\overline{TR}}_j^U}{\overline{\overline{TR}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right],$$

$$\left[\left(1\Big/\left(1 + \left(\frac{l^2-l}{x+y} \times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l} w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{ID}}_i^L}{1-\overline{\overline{ID}}_i^L}\right)^{\gamma}} + \frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_j)+1\right)}{\sum_{z=1}^{l} w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{ID}}_j^L}{1-\overline{\overline{ID}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1\Big/\left(1 + \left(\frac{l^2-l}{x+y} \times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l} w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{ID}}_i^U}{1-\overline{\overline{ID}}_i^U}\right)^{\gamma}} + \frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_j)+1\right)}{\sum_{z=1}^{l} w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{ID}}_j^U}{1-\overline{\overline{ID}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right], \tag{46}$$

$$\left[\left(1\Big/\left(1 + \left(\frac{l^2-l}{x+y} \times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l} w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{FL}}_i^L}{1-\overline{\overline{FL}}_i^L}\right)^{\gamma}} + \frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_j)+1\right)}{\sum_{z=1}^{\overline{\overline{in}}} w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{FL}}_j^L}{1-\overline{\overline{FL}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1\Big/\left(1 + \left(\frac{l^2-l}{x+y} \times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l} w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{FL}}_i^U}{1-\overline{\overline{FL}}_i^U}\right)^{\gamma}} + \frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_j)+1\right)}{\sum_{z=1}^{l} w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{FL}}_j^U}{1-\overline{\overline{FL}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right].$$

5. MADM Approach Based on the Developed Aggregation Operator

In this section, based upon the developed INWDPBM and INWDPGBM operators, we will propose a novel MADM method, which is defined as follows.

Assume that in a MADM problem, we need to evaluate u alternatives $\widetilde{M} = \left\{\widetilde{M}_1, \widetilde{M}_2, \ldots, \widetilde{M}_u\right\}$ with respect to v attributes $\widetilde{C} = \left\{\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_v\right\}$, and the importance degree of the attributes is represented by $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_v)^T$, satisfying the condition $\varpi_h \in [0,1]$, $\sum_{h=1}^{v} \varpi_h = 1$. The decision matrix for this decision problem is denoted by $\widetilde{D} = \left[\widetilde{d}_{gh}\right]_{m \times n}$, where $\widetilde{d}_{gh} = \left\langle \left[\overline{\overline{TR}}^L_{gh}, \overline{\overline{TR}}^U_{gh}\right], \left[\overline{\overline{ID}}^L_{gh}, \overline{\overline{ID}}^U_{gh}\right], \left[\overline{\overline{FL}}^L_{gh} \overline{\overline{FL}}^U_{gh}\right] \right\rangle$ is an INN for the alternative $\widetilde{M}_g$ with respect to the attribute $\widetilde{C}_h$, $(g = 1,2,\ldots,u; h = 1,2,\ldots,v)$. Then the main purpose is to rank the alternative and select the best alternative.

In the following, we will use the proposed INWDPBM and INWDPGBM operators to solve this MADM problem, and the detailed decision steps are shown as follows:

Step 1. Standardize the attribute values. Normally, in real problems, the attributes are of two types, (1) cost type, (2) benefit type. To get right result, it is necessary to change cost type of attribute values to benefit type using the following formula:

$$\widetilde{d}_{gh} = \left\langle \left[\overline{\overline{FL}}^L_{gh}, \overline{\overline{FL}}^U_{gh}\right], \left[1 - \overline{\overline{ID}}^U_{gh}, 1 - \overline{\overline{ID}}^L_{gh}\right], \left[\overline{\overline{TR}}^L_{gh}, \overline{\overline{TR}}^U_{gh}\right] \right\rangle. \tag{47}$$

Step 2. Calculate the supports

$$Supp\left(\widetilde{d}_{gh}, \widetilde{d}_{gl}\right) = 1 - D\left(\widetilde{d}_{gh}, \widetilde{d}_{gl}\right),\ (g = 1,2,\ldots,u; h,l = 1,2,\ldots,v), \tag{48}$$

where, $D\left(\widetilde{d}_{gh}, \widetilde{d}_{gl}\right)$ is the distance measure defined in Equation (10).

Step 3. Calculate $T\left(\widetilde{d}_{gh}\right)$

$$T\left(\widetilde{d}_{gh}\right) = \sum_{\substack{l=1 \\ l \neq h}}^{u} Supp\left(\widetilde{d}_{gh}, \widetilde{d}_{gl}\right), (g = 1,2,\ldots,u; h,l = 1,2,\ldots,v). \tag{49}$$

Step 4. Aggregate all the attribute values $\widetilde{d}_{gh}$ $(h = 1,2,\ldots,v)$ to the comprehensive value R_g by using INWDPBM or INWDPGBM operators shown as follows.

$$R_g = INWDPBM\left(\widetilde{d}_{g1}, \widetilde{d}_{g2}, \ldots, \widetilde{d}_{gv}\right); \tag{50}$$

or

$$R_g = INWDPGBM\left(\widetilde{d}_{g1}, \widetilde{d}_{g2}, \ldots, \widetilde{d}_{gv}\right). \tag{51}$$

Step 5. Determine the score values, accuracy values of $R_g (g = 1,2,\ldots,u)$, using Definition 5.

Step 6. Rank all the alternatives according to their score and accuracy values, and select the best alternative using Definition 6.

Step 7. End.

This decision steps are also described in Figure 1.

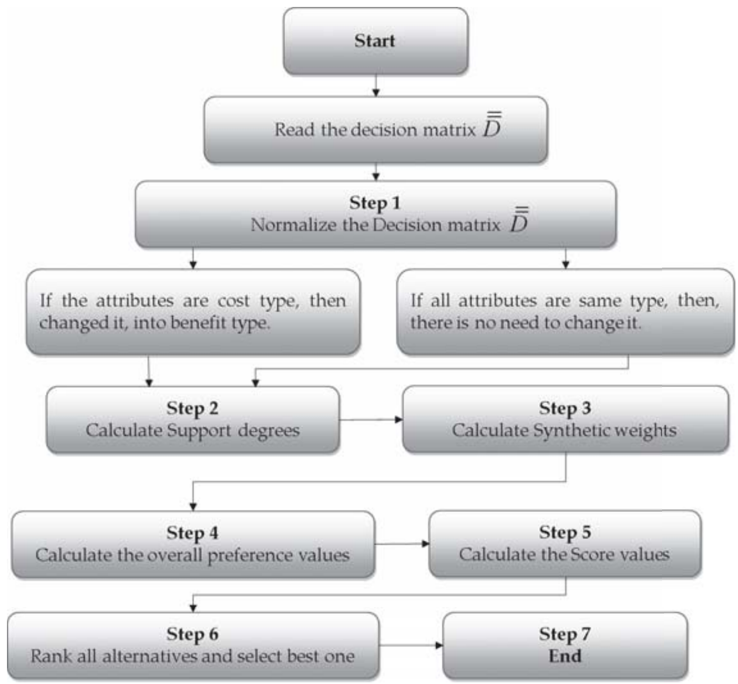

Figure 1. Flow chart for developed approach.

6. Illustrative Example

In this part, an example adapted from [42] is used to illustrate the application and effectiveness of the developed method in MADM problem.

An investment company wants to invest a sum of money in the best option. The company must invest a sum of money in the following four possible companies (alternatives): (1) car company $\widetilde{M}_1$; (2) food company $\widetilde{M}_2$; (3) Computer company $\widetilde{M}_3$; (4) An arm company $\widetilde{M}_4$, and the attributes under consideration are (1) risk analysis $\widetilde{C}_1$; (2) growth analysis $\widetilde{C}_2$; (3) environmental impact analysis $\widetilde{C}_3$. The importance degree of the attributes is $\varpi = (0.35, 0.4, 0.25)^T$. The four possible alternatives $\widetilde{M}_g(g = 1, 2, 3, 4)$ are evaluated with respect to the above attributes $\widetilde{C}_h(h = 1, 2, 3)$ by the form of INN, and the IN decision matrix $\widetilde{D}$ is listed in Table 1. The purpose of this decision-making problem is to rank the alternatives.

Table 1. The IN decision matrix $\overline{\overline{D}}$.

Alternatives/Attributes	$\widetilde{C}_1$	$\widetilde{C}_2$	$\widetilde{C}_3$
$\widetilde{M}_1$	$\langle[0.4, 0.5], [0.2, 0.3], [0.3, 0.4]\rangle$	$\langle[0.4, 0.6], [0.1, 0.3], [0.2, 0.4]\rangle$	$\langle[0.7, 0.9], [0.7, 0.8], [0.4, 0.5]\rangle$
$\widetilde{M}_2$	$\langle[0.6, 0.8], [0.1, 0.2], [0.1, 0.2]\rangle$	$\langle[0.6, 0.7], [0.15, 0.25], [0.2, 0.3]\rangle$	$\langle[0.3, 0.6], [0.2, 0.3], [0.8, 0.9]\rangle$
$\widetilde{M}_3$	$\langle[0.3, 0.6], [0.2, 0.3], [0.3, 0.4]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3], [0.3, 0.4]\rangle$	$\langle[0.4, 0.5], [0.2, 0.4], [0.7, 0.9]\rangle$
$\widetilde{M}_4$	$\langle[0.7, 0.8], [0.01, 0.1], [0.2, 0.3]\rangle$	$\langle[0.6, 0.7], [0.1, 0.2], [0.3, 0.4]\rangle$	$\langle[0.4, 0.6], [0.5, 0.6], [0.8, 0.9]\rangle$

6.1. The Decision-Making Steps

Step 1. Since $\widetilde{C}_1, \widetilde{C}_2$ are of benefit type, and $\widetilde{C}_3$ is of cost type. So, $\widetilde{C}_3$ will be changed into benefit type using Equation (47). So, the normalize decision matrix $\overline{\overline{D}}$ is given in Table 2.

Table 2. The Normalize IN decision matrix $\overline{\overline{D}}$.

Alternatives/Attributes	$\widetilde{C}_1$	$\widetilde{C}_2$	$\widetilde{C}_3$
$\widetilde{M}_1$	$\langle[0.4, 0.5], [0.2, 0.3], [0.3, 0.4]\rangle$	$\langle[0.4, 0.6], [0.1, 0.3], [0.2, 0.4]\rangle$	$\langle[0.4, 0.5], [0.2, 0.3], [0.7, 0.9]\rangle$
$\widetilde{M}_2$	$\langle[0.6, 0.8], [0.1, 0.2], [0.1, 0.2]\rangle$	$\langle[0.6, 0.7], [0.15, 0.25], [0.2, 0.3]\rangle$	$\langle[0.8, 0.9], [0.6, 0.7], [0.3, 0.6]\rangle$
$\widetilde{M}_3$	$\langle[0.3, 0.6], [0.2, 0.3], [0.3, 0.4]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3], [0.3, 0.4]\rangle$	$\langle[0.7, 0.9], [0.6, 0.8], [0.4, 0.5]\rangle$
$\widetilde{M}_4$	$\langle[0.7, 0.8], [0.01, 0.1], [0.2, 0.3]\rangle$	$\langle[0.6, 0.7], [0.1, 0.2], [0.3, 0.4]\rangle$	$\langle[0.8, 0.9], [0.4, 0.5], [0.4, 0.6]\rangle$

Step 2. Determine the supports $Supp\left(\widetilde{d}_{gh}, \widetilde{d}_{gl}\right)$, $(g = 1, 2, 3, 4; h, l = 1, 2, 3)$ by Equation (48) (for simplicity we denote $Supp\left(\widetilde{d}_{gh}, \widetilde{d}_{gl}\right)$ with $S^g_{gh,gl}$), we have

$$S^1_{11,12} = S^1_{12,11} = 0.950; S^1_{12,13} = S^1_{13,12} = 0.800; S^1_{11,13} = S^1_{13,11} = 0.85; S^2_{11,12} = S^2_{12,11} = 0.933; S^2_{12,13} = S^2_{13,12} = 0.717; S^2_{11,13} = S^2_{13,11} = 0.683;$$
$$S^3_{11,12} = S^3_{12,11} = 0.967; S^3_{12,13} = S^3_{13,12} = 0.733; S^3_{11,13} = S^3_{13,11} = 0.700; S^4_{11,12} = S^4_{12,11} = 0.902; S^4_{12,13} = S^4_{13,12} = 0.783; S^4_{11,13} = S^4_{13,11} = 0.752;$$

Step 3. Determine $T\left(\widetilde{d}_{gh}\right); (g = 1, 2, 3, 4; h = 1, 2, 3)$ by Equation (49), and we get

$$T^1_{11} = 1.800, T^1_{12} = 1.750, T^1_{13} = 1.650, T^2_{11} = 1.617, T^2_{12} = 1.650, T^2_{13} = 1.400,$$
$$T^3_{11} = 1.667, T^3_{12} = 1.700, T^3_{13} = 1.433, T^4_{11} = 1.653, T^4_{12} = 1.685, T^4_{13} = 1.535.$$

Step 4. **(a)** Determine the comprehensive value of every alternative using the INWDPBM operator, that is, Equation (50) $(Assume\ that\ x = y = 1; \gamma = 3)$, we have

$$R_1 = \langle[0.3974, 0.5195], [0.1823, 0.3023], [0.3353, 0.4796]\rangle;$$
$$R_2 = \langle[0.6457, 0.7954], [0.1700, 0.2885], [0.2044, 0.3265]\rangle;$$
$$R_3 = \langle[0.4846, 0.6503], [0.2556, 0.3711], [0.3376, 0.4394]\rangle;$$
$$R_4 = \langle[0.6938, 0.7953], [0.1062, 0.2154], [0.3069, 0.4278]\rangle.$$

(b) Determine the comprehensive value of every alternative using the INWDPGBM operator, that is Equation (51), $(Assume\ that\ x = y = 1; \gamma = 3)$, we have

$$R_1 = \langle[0.4026, 0.5381], [0.1570, 0.2977], [0.2998, 0.4520]\rangle;$$
$$R_2 = \langle[0.6654, 0.8193], [0.1558, 0.2686], [0.1836, 0.3035]\rangle;$$
$$R_3 = \langle[0.5159, 0.6732], [0.2366, 0.3473], [0.3265, 0.4279]\rangle;$$
$$R_4 = \langle[0.5159, 0.8193], [0.0938, 0.1952], [0.2862, 0.4037]\rangle.$$

Step 5. **(a)** Determine the score values of $R_g (g = 1, 2, 3, 4)$ by Definition 5, we have

$$S(R_1) = 1.8087,\ S(R_2) = 2.2259, S(R_3) = 1.8656, S(R_4) = 2.2164;$$

(b) Determine the score values of $R_g (g = 1, 2, 3, 4)$ by Definition 5, we have

$$S(R_1) = 1.8671,\ S(R_2) = 2.2866, S(R_3) = 1.9254, S(R_4) = 2.1781;$$

Step 6. **(a)** According to their score and accuracy values, by using Definition 6, the ranking order is $\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1$. So the best alternative is $\widetilde{M}_2$, while the worst alternative is $\widetilde{M}_1$.

(b) According to their score and accuracy values, by using Definition 6, the ranking order is $\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1$. So the best alternative is $\widetilde{M}_2$, while the worst alternative is $\widetilde{M}_1$.

So, by using INWDPBM or INWDPGBM operators, the best alternative is $\widetilde{M}_2$, while the worst alternative is $\widetilde{M}_1$.

6.2. Effect of Parameters γ, x and y on Ranking Result of this Example

In order to show the effect of the parameters x and y on the ranking result of this example, we set different parameter values for x and y, and $\gamma = 3$ is fixed, to show the ranking results of this example. The ranking results are given in Table 3.

As we know from Tables 3 and 4, the score values and ranking order are different for different values of the parameters x and y, when we use INWDPBM operator and INWDPGBM operator. We can see from Tables 3 and 4, when the parameter values x = 1 or 0 and y = 0 or 1, the best choice is $\widetilde{M}_4$ and the worst one is $\widetilde{M}_1$. In simple words, when the interrelationship among attributes are not considered, the best choice is $\widetilde{M}_4$ and the worst one is $\widetilde{M}_1$. On the other hand, when different values for the parameters x and y are utilized, for INWPBM and INWDPGBM operators, the ranking result is changed. That is, from Table 4, we can see that when the parameter values $x = 1, y = 1$, the ranking results are changed as the one obtained for x = 1 or 0 and y = 0 or 1. In this case the best alternative is $\widetilde{M}_2$ while the worst alternative remains the same.

Table 3. Ranking orders of decision result using different values for x and y for INWDPBM.

Parameter Values	INWDPBM Operator	Ranking Orders
$x = 1, y = 0, \gamma = 3$	$S(R_1) = 1.9319,\ S(R_2) = 2.4172,$ $S(R_3) = 2.0936, S(R_4) = 2.4222;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$
$x = 1, y = 5, \gamma = 3$	$S(R_1) = 1.8338,\ S(R_2) = 2.2684,$ $S(R_3) = 1.9049, S(R_4) = 2.2666;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
$x = 3, y = 7, \gamma = 3$	$S(R_1) = 1.8169,\ S(R_2) = 2.2398,$ $S(R_3) = 1.8777, S(R_4) = 2.2327;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
$x = 5, y = 10, \gamma = 3$	$S(R_1) = 1.8143,\ S(R_2) = 2.2354,$ $S(R_3) = 1.8738, S(R_4) = 2.2275;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$
$x = 1, y = 10, \gamma = 3$	$S(R_1) = 1.8501,\ S(R_2) = 2.2966,$ $S(R_3) = 1.9355, S(R_4) = 2.3012;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$
$x = 10, y = 4, \gamma = 3$	$S(R_1) = 1.8182,\ S(R_2) = 2.2419,$ $S(R_3) = 1.8796, S(R_4) = 2.2352;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$
$x = 3, y = 12, \gamma = 3$	$S(R_1) = 1.8285,\ S(R_2) = 2.2592,$ $S(R_3) = 1.8958, S(R_4) = 2.2557;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$

Table 4. Ranking orders of decision result using different values for x and y for INWDPGBM.

Parameter Values	INWDPGBM Operator	Ranking Orders
$x = 1, y = 0, \gamma = 3$	$S(R_1) = 1.5032,\ S(R_2) = 1.7934,$ $S(R_3) = 1.5136, S(R_4) = 1.8037;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$
$x = 1, y = 5, \gamma = 3$	$S(R_1) = 1.8220,\ S(R_2) = 2.2256,$ $S(R_3) = 1.8717, S(R_4) = 2.1140;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
$x = 3, y = 7, \gamma = 3$	$S(R_1) = 1.8539,\ S(R_2) = 2.2686,$ $S(R_3) = 1.9094, S(R_4) = 2.1584;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
$x = 5, y = 10, \gamma = 3$	$S(R_1) = 1.8583,\ S(R_2) = 2.2745,$ $S(R_3) = 1.9146, S(R_4) = 2.1647;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
$x = 1, y = 10, \gamma = 3$	$S(R_1) = 1.7814,\ S(R_2) = 2.1710,$ $S(R_3) = 1.8248, S(R_4) = 2.0632;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
$x = 10, y = 4, \gamma = 3$	$S(R_1) = 1.8087,\ S(R_2) = 2.2259,$ $S(R_3) = 1.8656, S(R_4) = 2.2164;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
$x = 3, y = 12, \gamma = 3$	$S(R_1) = 1.8671,\ S(R_2) = 2.2866,$ $S(R_3) = 1.9254, S(R_4) = 2.1781;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$

From Tables 3 and 4, we can observe that when the values of the parameter increase, the score values obtained using INWDPBM decrease. While using the INWDPGBM operator, the score values increase but the best choice is M_2 for $x = y \geq 1$.

From Table 5, we can see that different ranking orders are obtained for different values of γ. When $\gamma = 0.5$ and $\gamma = 2$, the best choice is M_4 by the INWPBM operator; when we use the INWPGBM operator, it is M_2. Similarly, for other values of $\gamma > 2$, the best choice is M_2 while the worst is M_1.

Table 5. Ranking orders of decision result using different values for γ.

Parameter Values	INWDPBM Operator	INWDPGBM Operator	Ranking Orders
$x=1, y=1, \gamma=0.5$	$S(R_1)=1.6662,\ S(R_2)=2.1025,$ $S(R_3)=1.7606, S(R_4)=2.1972;$	$S(R_1)=1.7870,\ S(R_2)=2.2347,$ $S(R_3)=1.9103, S(R_4)=2.1812;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$ $\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
$x=1, y=1, \gamma=2$	$S(R_1)=1.7783,\ S(R_2)=2.2015,$ $S(R_3)=1.8408, S(R_4)=2.2091;$	$S(R_1)=1.8491,\ S(R_2)=2.2786,$ $S(R_3)=1.9213, S(R_4)=2.1799;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$ $\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
$x=1, y=1, \gamma=4$	$S(R_1)=1.8229,\ S(R_2)=2.2363,$ $S(R_3)=1.8803, S(R_4)=2.2219;$	$S(R_1)=1.8740,\ S(R_2)=2.2856,$ $S(R_3)=1.9275, S(R_4)=2.1751;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$ $\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
$x=1, y=1, \gamma=7$	$S(R_1)=1.8375,\ S(R_2)=2.2455,$ $S(R_3)=1.9037, S(R_4)=2.2315;$	$S(R_1)=1.8747,\ S(R_2)=2.2763,$ $S(R_3)=1.9331, S(R_4)=2.1669;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$ $\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
$x=1, y=1, \gamma=10$	$S(R_1)=1.8418,\ S(R_2)=2.2477,$ $S(R_3)=1.9160, S(R_4)=2.2365;$	$S(R_1)=1.8701,\ S(R_2)=2.2698,$ $S(R_3)=1.9373, S(R_4)=2.1622;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$ $\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
$x=1, y=1, \gamma=15$	$S(R_1)=1.8447,\ S(R_2)=2.2488,$ $S(R_3)=1.9270, S(R_4)=2.2409;$	$S(R_1)=1.8642,\ S(R_2)=2.2637,$ $S(R_3)=1.9414, S(R_4)=2.1582;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$ $\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
$x=1, y=1, \gamma=20$	$S(R_1)=1.8460,\ S(R_2)=2.2492,$ $S(R_3)=1.9328, S(R_4)=2.2432;$	$S(R_1)=1.8608,\ S(R_2)=2.2604,$ $S(R_3)=1.9435, S(R_4)=2.1562;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$ $\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$

6.3. Comparing with the Other Methods

To illustrate the advantages and effectiveness of the developed method in this article, we solve the above example by four existing MADM methods, including IN weighted averaging operator, IN weighted geometric operator [12], the similarity measure defined by Ye [15], Muirhead mean operators developed by Liu et al. [42], IN power aggregation operator developed by Liu et al. [37].

From Table 6, we can see that the ranking orders are the same as the ones produced by the existing aggregation operators when the parameter values $x = 1, y = 0, \gamma = 3$, but the ranking orders are different when the interrelationship among attributes are considered. That is why the developed method based on the proposed aggregation operators is more flexible due the parameter and practical as it can consider the interrelationship among input arguments.

Table 6. Ranking order of the alternatives using different aggregation operators.

Aggregation Operator	Parameter	Score Values	Ranking Order
INWA operator [12]	No	$S(R_1)=1.8430,\ S(R_2)=2.2497,$ $S(R_3)=1.9151, S(R_4)=2.2788;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$
INWGA operator [12]	No	$S(R_1)=1.7286,\ S(R_2)=2.0991,$ $S(R_3)=1.7751, S(R_4)=2.1608;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$
Similarity measure Hamming distance [15]	No	$D_1(R^*, R_1)=0.7948,\ D_1(R^*, R_2)=0.9581,$ $D_1(R^*, R_3)=0.8805, D_1(R^*, R_4)=0.9725;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$
Generalized power Aggregation operator [37]	Yes $\lambda=1$	$S(R_1)=1.8460,\ S(R_2)=2.2543,$ $S(R_3)=1.9163, S(R_4)=2.2799;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$
INWMM operator [42]	Yes $P(1,1,1)$	$S(R_1)=1.8054,\ S(R_2)=2.2321,$ $S(R_3)=1.9172, S(R_4)=2.2773;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$
INWDMM operator [42]	Yes $P(1,1,1)$	$S(R_1)=1.6260,\ S(R_2)=1.9202,$ $S(R_3)=1.7061, S(R_4)=2.0798;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$
Proposed INWDPBM $x=1, y=0, \gamma=3$	Yes	$S(R_1)=1.9319,\ S(R_2)=2.4172,$ $S(R_3)=2.0936, S(R_4)=2.4222;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$
Proposed INWDPGBM $x=1, y=0, \gamma=3$	Yes	$S(R_1)=1.5032,\ S(R_2)=1.7934,$ $S(R_3)=1.5136, S(R_4)=1.8037;$	$\widetilde{M}_4 > \widetilde{M}_2 > \widetilde{M}_3 > \widetilde{M}_1.$
INWDPBM operator in this article	Yes $x=y=1, \gamma=3$	$S(R_1)=1.8087,\ S(R_2)=2.2259,$ $S(R_3)=1.8656, S(R_4)=2.2164;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$
INWDPBM operator in this article	Yes $x=y=1, \gamma=3$	$S(R_1)=1.8671,\ S(R_2)=2.2866,$ $S(R_3)=1.9254, S(R_4)=2.1781;$	$\widetilde{M}_2 > \widetilde{M}_4 > \widetilde{M}_3 > \widetilde{M}_1.$

From the above comparative analysis, we can know the proposed method has the following advantages, that is, it can consider the interrelationship among the input arguments and can relieve the effect of the awkward data by PWV at the same time, and it can permit more precise ranking order than the existing methods. The proposed method can take the advantages of PA operator and BM operator concurrently, these factors makes it a little complex in calculations.

The score values and ranking orders by these methods are shown in Table 6.

7. Conclusions

The PBM operator can take the advantage of PA operator, which can eliminate the impact of awkward data given by the predisposed DMs, and BM operator, which can consider the correlation between two attributes. The Dombi operations of TN and TCN proposed by Dombi have the edge of good flexibility with general parameter. In this article, we combined PBM with Dombi operation and proposed some aggregation operators to aggregate INNs. Firstly, we defined some operational laws for INSs based on Dombi TN and TCN and discussed some properties of these operations. Secondly, we extended PBM operator based on Dombi operations to introduce INDPBM operator, INWDPBM operator, INDPGBM operator, INWDPGBM operator and discussed some properties of these aggregation operators. The developed aggregation operators have the edge that they can take the correlation among the attributes by BM operator, and can also remove the effect of awkward data by PA operator at the same and due to general parameter, so they are more flexible in the aggregation process. Further, we developed a novel MADM method based on developed aggregation operators to deal with interval neutrosophic information. Finally, an illustrative example is used to show the effectiveness and practicality of the proposed MADM method and comparison were made with the existing methods. The proposed aggregation operators are very useful to solve MADM problems.

In future research, we shall define some distinct aggregation operators for SVHFSs, INHFSs, double valued neutrosophic sets and so on based on Dombi operations and apply them to MAGDM and MADM.

Author Contributions: Q.K. and P.L. visualized and work together to achieve this work. Q.K. and P.L. wrote the paper. T.M., F.S. and K.U. offered plenty of advice enhancing the readability of work. All authors approved the publication work.

Funding: National Natural Science Foundation of China: 71771140, 71471172, Special Funds of Taishan Scholars Project of Shandong Province: ts201511045.

Acknowledgments: We, the authors of this project are deeply grateful for the valuable suggestions and comments provided by the respected reviewers to improve the quality and value of this research.

Conflicts of Interest: No potential conflict of interest was reported by authors.

Appendix A. Basic Concept of PBM Operator

Definition A1. *For positive real numbers $\wp_h (h = 1, 2, \ldots, l)$ and $x, y > 0$ the aggregation mapping [54]*

$$PBM^{x,y}(\wp_1, \wp_2, \ldots, \wp_l) = \left(\frac{1}{l^2 - l} \sum_{\substack{i=1, j=1 \\ i \neq j}}^{l} \left(\left(\frac{l(T(\wp_i)+1)}{\sum_{o=1}^{l}(T(\wp_o)+1)} \wp_i \right)^x \times \left(\frac{l(T(\wp_i)+1)}{\sum_{o=1}^{l}(T(\wp_o)+1)} \wp_j \right)^y \right) \right)^{\frac{1}{x+y}} \quad \text{(A1)}$$

is said to be power Bonferroni mean (PBM) mean operator.

Definition A2. *For positive real numbers $\wp_h (h = 1, 2, \ldots, l)$ and $x, y > 0$ the aggregation mapping [54]*

$$PBM^{x,y}(\wp_1, \wp_2, \ldots, \wp_l) = \frac{1}{x+y}\left(\prod_{\substack{i=1,j=1\\ i\neq j}}^{l}\left(x\wp_i^{\frac{l(T(\wp_i)+1)}{\sum_{o=1}^{l}(T(\wp_o)+1)}} + y\wp_j^{\frac{l(T(\wp_j)+1)}{\sum_{o=1}^{l}(T(\wp_o)+1)}}\right)\right)^{\frac{1}{l^2-l}} \tag{A2}$$

is said to be power geometric Bonferroni mean (PGBM) mean operator.

In Definitions A1 and A2, $T(\wp_i) = \sum_{j=1, j\neq i}^{l} \text{supp}(\wp_i, \wp_j)$, and $\text{supp}(\wp_i, \wp_j)$ is the SPD for $\wp_i$ from $\wp_j$ satisfying the axioms as;

(1) $sup(\wp_i, \wp_j) = 1 - D(\wp_i, \wp_j)$, so $sup(\wp_i, \wp_j) \in [0, 1]$;
(2) $sup(\wp_i, \wp_j) = sup(\wp_j, \wp_i)$;
(3) $sup(\wp_i, \wp_j) \geq sup(\wp_c, \wp_d)$, if $|\wp_i - \wp_y| \leq |\wp_c - \wp_d|$.

where $D(\wp_i, \wp_j)$ is the distance measure among $\wp_i$ and $\wp_j$.

Appendix B. Proof of Theorem 6

Proof. Since

$$l\Lambda_i \overline{\overline{in}}_i = \left\langle \left[1-\frac{1}{1+\left(l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^U}{1-\overline{\overline{TR}}_i^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right], \left[\frac{1}{1+\left(l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right], \left[\frac{1}{1+\left(l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^U}{\overline{\overline{FL}}_i^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right] \right\rangle;$$

and

$$l\Lambda_j \overline{\overline{in}}_j = \left\langle \left[1-\frac{1}{1+\left(l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^U}{1-\overline{\overline{TR}}_j^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right], \left[\frac{1}{1+\left(l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^L}{\overline{\overline{ID}}_j^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^U}{\overline{\overline{ID}}_j^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right], \left[\frac{1}{1+\left(l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^L}{\overline{\overline{FL}}_j^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^U}{\overline{\overline{FL}}_j^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right] \right\rangle.$$

Let

$$a_i = \frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L}, b_i = \frac{\overline{\overline{TR}}_i^U}{1-\overline{\overline{TR}}_i^U}, c_i = \frac{1-\overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L}, d_i = \frac{1-\overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U}, g_i = \frac{1-\overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L}, h_i = \frac{1-\overline{\overline{FL}}_i^U}{\overline{\overline{FL}}_i^U}, a_j = \frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L}, b_j = \frac{\overline{\overline{TR}}_j^U}{1-\overline{\overline{TR}}_j^U},$$
$$c_j = \frac{1-\overline{\overline{ID}}_j^L}{\overline{\overline{ID}}_j^L}, d_j = \frac{1-\overline{\overline{ID}}_j^U}{\overline{\overline{ID}}_j^U}, g_j = \frac{1-\overline{\overline{FL}}_j^L}{\overline{\overline{FL}}_j^L}, h_j = \frac{1-\overline{\overline{FL}}_j^U}{\overline{\overline{FL}}_j^U}.$$

then, we have

$$l\Lambda_i \overline{\overline{in}}_i = \left\langle \left[1-\frac{1}{1+\left(l\Lambda_i\right)^{\frac{1}{\gamma}} a_i}, 1-\frac{1}{1+\left(l\Lambda_i\right)^{\frac{1}{\gamma}} b_i}\right], \left[\frac{1}{1+\left(l\Lambda_i\right)^{\frac{1}{\gamma}} c_i}, \frac{1}{1+\left(l\Lambda_i\right)^{\frac{1}{\gamma}} d_i}\right], \left[\frac{1}{1+\left(l\Lambda_i\right)^{\frac{1}{\gamma}} g_i}, \frac{1}{1+\left(l\Lambda_i\right)^{\frac{1}{\gamma}} h_i}\right] \right\rangle;$$

$$l\Lambda_j \overline{\overline{in}}_j = \left\langle \left[1-\frac{1}{1+\left(l\Lambda_j\right)^{\frac{1}{\gamma}} a_j}, 1-\frac{1}{1+\left(l\Lambda_j\right)^{\frac{1}{\gamma}} b_j}\right], \left[\frac{1}{1+\left(l\Lambda_j\right)^{\frac{1}{\gamma}} c_j}, \frac{1}{1+\left(l\Lambda_j\right)^{\frac{1}{\gamma}} d_j}\right], \left[\frac{1}{1+\left(l\Lambda_j\right)^{\frac{1}{\gamma}} g_j}, \frac{1}{1+\left(l\Lambda_j\right)^{\frac{1}{\gamma}} h_j}\right] \right\rangle.$$

and

$$\left(l\Lambda_i \overline{\overline{in}}_i\right)^x = \left\langle \left[\frac{1}{1+x^{\frac{1}{\gamma}}\Big/\left(l\Lambda_i\right)^{\frac{1}{\gamma}} a_i}, \frac{1}{1+x^{\frac{1}{\gamma}}\Big/\left(l\Lambda_i\right)^{\frac{1}{\gamma}} b_i}\right], \left[1-\frac{1}{1+x^{\frac{1}{\gamma}}\Big/\left(l\Lambda_i\right)^{\frac{1}{\gamma}} c_i}, 1-\frac{1}{1+x^{\frac{1}{\gamma}}\Big/\left(l\Lambda_i\right)^{\frac{1}{\gamma}} d_i}\right], \left[1-\frac{1}{1+x^{\frac{1}{\gamma}}\Big/\left(l\Lambda_i\right)^{\frac{1}{\gamma}} g_i}, 1-\frac{1}{1+x^{\frac{1}{\gamma}}\Big/\left(l\Lambda_i\right)^{\frac{1}{\gamma}} h_i}\right] \right\rangle;$$

$$\left(l\omega_j n_j\right)^y = \left\langle \left[\frac{1}{1+y^{\frac{1}{\gamma}}\Big/\left(l\omega_j\right)^{\frac{1}{\gamma}} a_j}, \frac{1}{1+y^{\frac{1}{\gamma}}\Big/\left(l\omega_j\right)^{\frac{1}{\gamma}} b_j}\right], \left[1-\frac{1}{1+y^{\frac{1}{\gamma}}\Big/\left(l\omega_j\right)^{\frac{1}{\gamma}} c_j}, 1-\frac{1}{1+y^{\frac{1}{\gamma}}\Big/\left(l\omega_j\right)^{\frac{1}{\gamma}} d_j}\right], \left[1-\frac{1}{1+y^{\frac{1}{\gamma}}\Big/\left(l\omega_j\right)^{\frac{1}{\gamma}} g_j}, 1-\frac{1}{1+y^{\frac{1}{\gamma}}\Big/\left(l\omega_j\right)^{\frac{1}{\gamma}} h_j}\right] \right\rangle.$$

Moreover, we have

$$\left(l\Lambda_i \overline{\overline{in}}_i\right)^x \otimes_D \left(l\Lambda_j \overline{\overline{in}}_j\right)^y = \left\langle \left[\frac{1}{1+\left(x/l\Lambda_i a_i^\gamma + y/l\Lambda_j a_j^\gamma\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(x/l\Lambda_i b_i^\gamma + y/l\Lambda_j b_j^\gamma\right)^{\frac{1}{\gamma}}}\right], \left[1-\frac{1}{1+\left(x/l\Lambda_i c_i^\gamma + y/l\Lambda_j c_j^\gamma\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(x/l\Lambda_i d_i^\gamma + y/l\Lambda_j d_j^\gamma\right)^{\frac{1}{\gamma}}}\right], \left[1-\frac{1}{1+\left(x/l\Lambda_i g_i^\gamma + y/l\Lambda_j g_j^\gamma\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(x/l\Lambda_i h_i^\gamma + y/l\Lambda_j h_j^\gamma\right)^{\frac{1}{\gamma}}}\right] \right\rangle,$$

and

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{l} \left(l\Lambda_i \overline{\overline{in}}_i\right)^x \otimes_D \left(l\Lambda_j \overline{\overline{in}}_j\right)^y$$

$$=\left\langle\left[1-1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(\frac{1}{1+\left(x/l\Lambda_i a_i^{\gamma}+y/l\omega_j a_j^{\gamma}\right)^{\frac{1}{\gamma}}}\Bigg/\left(1-\frac{1}{1+\left(x/l\Lambda_i a_i^{\gamma}+y/l\Lambda_j a_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}},1-1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(\frac{1}{1+\left(x/l\Lambda_i b_i^{\gamma}+y/l\Lambda_j b_j^{\gamma}\right)^{\frac{1}{\gamma}}}\Bigg/\left(1-\frac{1}{1+\left(x/l\Lambda_i b_i^{\gamma}+y/l\Lambda_j b_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right]\right.$$

$$\left[1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(\left(1-1+\frac{1}{1+\left(x/l\Lambda_i c_i^{\gamma}+y/l\Lambda_j c_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\Bigg/\left(1-\frac{1}{1+\left(x/l\Lambda_i c_i^{\gamma}+y/l\Lambda_j c_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}},1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(\left(1-1+\frac{1}{1+\left(x/l\Lambda_i d_i^{\gamma}+y/l\omega_j d_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\Bigg/\left(1-\frac{1}{1+\left(x/l\Lambda_i d_i^{\gamma}+y/l\Lambda_j d_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right]$$

$$\left.\left[1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(\left(1-1+\frac{1}{1+\left(x/l\Lambda_i g_i^{\gamma}+y/e\Lambda_j g_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\Bigg/\left(1-\frac{1}{1+\left(x/l\Lambda_i g_i^{\gamma}+y/l\Lambda_j g_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}},1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(\left(1-1+\frac{1}{1+\left(x/l\Lambda_i h_i^{\gamma}+y/l\Lambda_j h_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\Bigg/\left(1-\frac{1}{1+\left(x/l\Lambda_i h_i^{\gamma}+y/lv_j h_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right]\right\rangle$$

$$=\left\langle\left[1-1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i a_i^{\gamma}}+\frac{y}{l\Lambda_j a_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right),1-1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i b_i^{\gamma}}+\frac{y}{l\Lambda_j b_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right],\left[1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i c_i^{\gamma}}+\frac{y}{l\Lambda_j c_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right),\right.\right.$$

$$\left.\left.1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i d_i^{\gamma}}+\frac{y}{l\Lambda_j d_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right],\left[1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i g_i^{\gamma}}+\frac{y}{l\Lambda_j g_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right),1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i h_i^{\gamma}}+\frac{y}{l\Lambda_j h_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right]\right\rangle.$$

So, we can have

$$\frac{1}{l^2-l}\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(l\Lambda_i\overline{\overline{in}}_i\right)^{x}\otimes_D\left(l\Lambda_j\overline{\overline{in}}_j\right)^{y}$$

$$=\left\langle\left[\left(1-1\Bigg/\left(1+\left(\frac{1}{l^2-l}\left(\left(1-1\Big/1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i a_i^{\gamma}}+\frac{y}{l\Lambda_j a_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Bigg/\left(1-1+1\Big/1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{e\Lambda_i a_i^{\gamma}}+\frac{y}{e\Lambda_j a_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right),\right.\right.$$

$$\left.\left(1-1\Bigg/\left(1+\left(\frac{1}{l^2-l}\left(\left(1-1\Big/1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i b_i^{\gamma}}+\frac{y}{l\Lambda_j b_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Bigg/\left(1-1+1\Big/1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i b_i^{\gamma}}+\frac{y}{l\Lambda_j b_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)\right],$$

$$\left[\left(1\Bigg/\left(1+\left(\frac{1}{l^2-l}\left(\left(1-1\Big/1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i c_i^{\gamma}}+\frac{y}{l\Lambda_j c_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Bigg/\left(1\Big/1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i c_i^{\gamma}}+\frac{y}{l\Lambda_j c_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right),\right.$$

$$\left.\left(1\Bigg/\left(1+\left(\frac{1}{l^2-l}\left(\left(1-1\Big/1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i d_i^{\gamma}}+\frac{y}{l\Lambda_j d_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Bigg/\left(1\Big/1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i d_i^{\gamma}}+\frac{y}{l\Lambda_j d_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)\right]$$

$$\left[\left(1\Bigg/\left(1+\left(\frac{1}{l^2-l}\left(\left(1-1\Bigg/1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i e_i^{\gamma}}+\frac{y}{l\Lambda_j e_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Bigg/\left(1\Bigg/1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i e_i^{\gamma}}+\frac{y}{l\Lambda_j e_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)\right)^{\frac{1}{\gamma}}\right)\right],$$

$$\left(1\Bigg/\left(1+\left(\frac{1}{l^2-l}\left(\left(1-1\Bigg/1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i h_i^{\gamma}}+\frac{y}{l\Lambda_j h_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Bigg/\left(1\Bigg/1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i h_i^{\gamma}}+\frac{y}{l\Lambda_j h_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)\right)^{\frac{1}{\gamma}}\right)\Bigg].$$

$$=\Bigg\langle\left[\left(1-1\Bigg/\left(1+\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i a_i^{\gamma}}+\frac{y}{l\Lambda_j a_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1-1\Bigg/\left(1+\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i b_i^{\gamma}}+\frac{y}{l\Lambda_j b_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right],\left[\left(1\Bigg/\left(1+\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i c_i^{\gamma}}+\frac{y}{l\Lambda_j c_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right),\right.$$

$$\left.\left(1-1\Bigg/\left(1+\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i d_i^{\gamma}}+\frac{y}{l\Lambda_j d_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right],\left[\left(1\Bigg/\left(1+\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i g_i^{\gamma}}+\frac{y}{l\Lambda_j g_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1\Bigg/\left(1+\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i h_i^{\gamma}}+\frac{y}{l\Lambda_j h_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right]\Bigg\rangle$$

Then,

$$\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(l\Lambda_i\overline{\overline{in}}_i\right)^{x}\otimes_D\left(l\Lambda_j\overline{\overline{in}}_j\right)^{y}\right)^{\frac{1}{x+y}}$$

$$=\Bigg\langle\left[\left(1\Bigg/1+\left(\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i a_i^{\gamma}}+\frac{y}{l\Lambda_j a_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right),\left(1\Bigg/1+\left(\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i b_i^{\gamma}}+\frac{y}{l\Lambda_j b_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],\left[\left(1-1\Bigg/1+\left(\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i c_i^{\gamma}}+\frac{y}{l\Lambda_j c_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right),\right. \quad \text{(A3)}$$

$$\left.\left(1-1\Bigg/1+\left(\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i d_i^{\gamma}}+\frac{y}{l\Lambda_j d_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],\left[\left(1-1\Bigg/1+\left(\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i g_i^{\gamma}}+\frac{y}{l\Lambda_j g_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\left(1-1\Bigg/1+\left(\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i h_i^{\gamma}}+\frac{y}{l\Lambda_j h_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right]\Bigg\rangle.$$

Now, put

$$a_i=\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L},b_i=\frac{\overline{\overline{TR}}_i^U}{1-\overline{\overline{TR}}_i^U},c_i=\frac{1-\overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L},d_i=\frac{1-\overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U},g_i=\frac{1-\overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L},h_i=\frac{1-\overline{\overline{FL}}_i^U}{\overline{\overline{FL}}_i^U},a_j=\frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L},b_j=\frac{\overline{\overline{TR}}_j^U}{1-\overline{\overline{TR}}_j^U},$$
$$c_j=\frac{1-\overline{\overline{ID}}_j^L}{\overline{\overline{ID}}_j^L},d_j=\frac{1-\overline{\overline{ID}}_j^U}{\overline{\overline{ID}}_j^U},g_j=\frac{1-\overline{\overline{FL}}_j^L}{\overline{\overline{FL}}_j^L},h_j=\frac{1-\overline{\overline{FL}}_j^U}{\overline{\overline{FL}}_j^U}.$$

in Equation (A3), we can get

$$\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(l\Lambda_i\overline{\overline{in}}_i\right)^{x}\otimes_D\left(l\Lambda_j\overline{\overline{in}}_j\right)^{y}\right)^{\frac{1}{x+y}}$$

$$=\left\langle\left[1\Big/\left(1+\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^{L}}{1-\overline{\overline{TR}}_i^{L}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^{L}}{1-\overline{\overline{TR}}_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}},\ 1\Big/\left(1+\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^{U}}{1-\overline{\overline{TR}}_i^{U}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^{U}}{1-\overline{\overline{TR}}_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right],\right.$$

$$\left[1-1\Big/\left(1+\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^{L}}{\overline{\overline{ID}}_i^{L}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^{L}}{\overline{\overline{ID}}_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}},1-1\Big/\left(1+\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^{U}}{\overline{\overline{ID}}_i^{U}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^{U}}{\overline{\overline{ID}}_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right],$$

$$\left.\left[1-1\Big/\left(1+\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^{L}}{\overline{\overline{FL}}_i^{L}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^{L}}{\overline{\overline{FL}}_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}},1-1\Big/\left(1+\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^{U}}{\overline{\overline{FL}}_i^{U}}\right)^{\gamma}}+\frac{y}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_j^{U}}{\overline{\overline{FL}}_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right]\right\rangle.$$

This is the required proof of the Theorem 6. □

Appendix C. Proof of Theorem 7

Proof. Since all $\overline{\overline{in}}_i = \overline{\overline{in}} = \left\langle\left[\overline{\overline{TR}}^{L},\overline{\overline{TR}}^{U}\right],\left[\overline{\overline{ID}}^{L},\overline{\overline{ID}}^{U}\right],\left[\overline{\overline{FL}}^{L},\overline{\overline{FL}}^{U}\right]\right\rangle, (i=1,2,\ldots,l)$, so we have $Supp\left(\overline{\overline{in}}_p,\overline{\overline{in}}_q\right)=1$, for all $p,q=1,2,\ldots,l$, so $\Lambda_p=\frac{1}{l}$, for all $p=1,2,\ldots,l$. Then

$$INDPBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,\ldots,\overline{\overline{in}}_l\right)=INDPBM^{x,y}\left(\overline{\overline{in}},\overline{\overline{in}},\ldots,\overline{\overline{in}}\right)$$

$$=\left\langle\left[\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\frac{1}{l}\left(\frac{\overline{\overline{TR}}^{L}}{1-\overline{\overline{TR}}^{L}}\right)^{\gamma}}+\frac{x}{l\frac{1}{l}\left(\frac{\overline{\overline{TR}}^{L}}{1-\overline{\overline{TR}}^{L}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right),\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\frac{1}{l}\left(\frac{\overline{\overline{TR}}^{U}}{1-\overline{\overline{TR}}^{U}}\right)^{\gamma}}+\frac{x}{l\frac{1}{l}\left(\frac{\overline{\overline{TR}}^{U}}{1-\overline{\overline{TR}}^{U}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],\right.$$

$$\left[\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{ID}}^{L}}{\overline{\overline{ID}}^{L}}\right)^{\gamma}}+\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{ID}}^{L}}{\overline{\overline{ID}}^{L}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right),\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{ID}}^{U}}{\overline{\overline{ID}}^{U}}\right)^{\gamma}}+\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{ID}}^{U}}{\overline{\overline{ID}}^{U}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],$$

$$\left[\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{FL}}^{L}}{\overline{\overline{FL}}^{L}}\right)^{\gamma}}+\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{FL}}^{L}}{\overline{\overline{FL}}^{L}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right),\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{FL}}^{U}}{\overline{\overline{FL}}^{U}}\right)^{\gamma}}+\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{FL}}^{U}}{\overline{\overline{FL}}^{U}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right].,$$

$$
=\left\langle\left[\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{l}\frac{1}{\frac{x+y}{\left(\frac{\overline{\overline{TR}}^{L}}{1-\overline{\overline{TR}}^{L}}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}},\ \frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{l}\frac{1}{\frac{x+y}{\left(\frac{\overline{\overline{TR}}^{U}}{1-\overline{\overline{TR}}^{U}}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right],\right.
$$

$$
\left[1-\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{l}\frac{1}{\frac{x+y}{\left(\frac{1-\overline{\overline{ID}}^{L}}{\overline{\overline{ID}}^{L}}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}},\ \frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{l}\frac{1}{\frac{x+y}{\left(\frac{1-\overline{\overline{ID}}^{U}}{\overline{\overline{ID}}^{U}}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right],
$$

$$
\left.\left[1-\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{l}\frac{1}{\frac{x+y}{\left(\frac{1-\overline{\overline{FL}}^{L}}{\overline{\overline{FL}}^{L}}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}},\ \frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum\limits_{\substack{i,j=1\\ i\neq j}}^{l}\frac{1}{\frac{x+y}{\left(\frac{1-\overline{\overline{FL}}^{U}}{\overline{\overline{FL}}^{U}}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right]\right\rangle,
$$

$$
=\left\langle\left[\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\frac{1}{\frac{x+y}{\left(\frac{\overline{\overline{TR}}^{L}}{1-\overline{\overline{TR}}^{L}}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}},\ \frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\frac{1}{\frac{x+y}{\left(\frac{\overline{\overline{TR}}^{U}}{1-\overline{\overline{TR}}^{U}}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right],\right.
$$

$$
\left[1-\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\frac{1}{\frac{x+y}{\left(\frac{1-\overline{\overline{ID}}^{L}}{\overline{\overline{ID}}^{L}}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}},\ \frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\frac{1}{\frac{x+y}{\left(\frac{1-\overline{\overline{ID}}^{U}}{\overline{\overline{ID}}^{U}}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right],
$$

$$
\left.\left[1-\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\frac{1}{\frac{x+y}{\left(\frac{1-\overline{\overline{FL}}^{L}}{\overline{\overline{FL}}^{L}}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}},\ \frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\frac{1}{\frac{x+y}{\left(\frac{1-\overline{\overline{FL}}^{U}}{\overline{\overline{FL}}^{U}}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right]\right\rangle,
$$

$$
\left\langle\left[\overline{\overline{TR}}^{L},\overline{\overline{TR}}^{U}\right],\left[\overline{\overline{ID}}^{L},\overline{\overline{ID}}^{U}\right],\left[\overline{\overline{FL}}^{L},\overline{\overline{FL}}^{U}\right]\right\rangle=\overline{\overline{in}}.
$$

□

Appendix D. Proof of Theorem 9

Proof. Since

$$
\overline{\overline{in}}^{+}=\left\langle\max_{i=1}^{l}\left[\overline{\overline{TR}}_i^{L},\overline{\overline{TR}}_i^{U}\right],\min_{i=1}^{l}\left[\overline{\overline{ID}}_i^{L},\overline{\overline{ID}}_i^{U}\right],\min_{i=1}^{l}\left[\overline{\overline{FL}}_i^{L},\overline{\overline{FL}}_i^{U}\right]\right\rangle,\overline{\overline{in}}^{-}=\left\langle\min_{i=1}^{l}\left[\overline{\overline{TR}}_i^{L},\overline{\overline{TR}}_i^{U}\right],\max_{i=1}^{l}\left[\overline{\overline{ID}}_i^{L},\overline{\overline{ID}}_i^{U}\right],\max_{i=1}^{l}\left[\overline{\overline{FL}}_i^{L},\overline{\overline{FL}}_i^{U}\right]\right\rangle,
$$

then, there are

$$\overline{\overline{TR}}^{L-} \leq \overline{\overline{TR}}^{L}(n_i) \leq \overline{\overline{TR}}^{L+}, \overline{\overline{TR}}^{U-} \leq \overline{\overline{TR}}^{U}(n_i) \leq \overline{\overline{TR}}^{U+}, \overline{\overline{ID}}^{L-} \leq \overline{\overline{ID}}^{L}(n_i) \leq \overline{\overline{ID}}^{L+}, \overline{\overline{ID}}^{U-} \leq \overline{\overline{ID}}^{U}(n_i) \leq \overline{\overline{ID}}^{U+}, \overline{\overline{FL}}^{L-} \leq \overline{\overline{FL}}^{L}(n_i) \leq \overline{\overline{FL}}^{L+},$$
$$\overline{\overline{FL}}^{U-} \leq \overline{\overline{FL}}^{U}(n_i) \leq \overline{\overline{FL}}^{U+},$$

for all $i = 1, 2, \ldots, l$. We have

$$\overline{\overline{TR}}^{L}(\overline{\overline{in}}_i) = \left(1 \Big/ \left(1 + \left(\frac{l^2-l}{x+y} \times 1 \Big/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^{L}}{1-\overline{\overline{TR}}_i^{L}} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{\overline{\overline{TR}}_j^{L}}{1-\overline{\overline{TR}}_j^{L}} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) \geq \left(1 \Big/ \left(1 + \left(\frac{l^2-l}{x+y} \times 1 \Big/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^{L-}}{1-\overline{\overline{TR}}_i^{L-}} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{\overline{\overline{TR}}_j^{L-}}{1-\overline{\overline{TR}}_j^{L-}} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) = \overline{\overline{TR}}^{L-},$$

$$\overline{\overline{TR}}^{U}(\overline{\overline{in}}_i) = \left(1 \Big/ \left(1 + \left(\frac{l^2-l}{x+y} \times 1 \Big/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^{U}}{1-\overline{\overline{TR}}_i^{U}} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{\overline{\overline{TR}}_j^{U}}{1-\overline{\overline{TR}}_j^{U}} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) \geq \left(1 \Big/ \left(1 + \left(\frac{l^2-l}{x+y} \times 1 \Big/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^{U-}}{1-\overline{\overline{TR}}_i^{U-}} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{\overline{\overline{TR}}_j^{U-}}{1-\overline{\overline{TR}}_j^{U-}} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) = \overline{\overline{TR}}^{U-},$$

$$\overline{\overline{ID}}^{L}\left(\overline{\overline{in}}_i\right) = \left(1 - 1 \Big/ \left(1 + \left(\frac{l^2-l}{x+y} \times 1 \Big/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1-\overline{\overline{ID}}_i^{L}}{\overline{\overline{ID}}_i^{L}} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{1-\overline{\overline{ID}}_j^{L}}{\overline{\overline{ID}}_j^{L}} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) \leq \left(1 - 1 \Big/ \left(1 + \left(\frac{l^2-l}{x+y} \times 1 \Big/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1-\overline{\overline{ID}}_i^{L+}}{\overline{\overline{ID}}_i^{L+}} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{1-\overline{\overline{ID}}_j^{L+}}{\overline{\overline{ID}}_j^{L+}} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) = \overline{\overline{ID}}^{L+}$$

$$\overline{\overline{ID}}^{U}\left(\overline{\overline{in}}_i\right) = \left(1 - 1 \Big/ \left(1 + \left(\frac{l^2-l}{x+y} \times 1 \Big/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1-\overline{\overline{ID}}_i^{U}}{\overline{\overline{ID}}_i^{U}} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{1-\overline{\overline{ID}}_j^{U}}{\overline{\overline{ID}}_j^{U}} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) \leq \left(1 - 1 \Big/ \left(1 + \left(\frac{l^2-l}{x+y} \times 1 \Big/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1-\overline{\overline{ID}}_i^{U+}}{\Psi_i^{U+}} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{1-\overline{\overline{ID}}_j^{U+}}{\overline{\overline{ID}}_j^{U+}} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) = \overline{\overline{ID}}^{U+}$$

$$\overline{\overline{FL}}^{L}\left(\overline{\overline{in}}_i\right) = \left(1 - 1 \Big/ \left(1 + \left(\frac{l^2-l}{x+y} \times 1 \Big/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1-\overline{\overline{FL}}_i^{L}}{\overline{\overline{FL}}_i^{L}} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{1-\overline{\overline{FL}}_j^{L}}{\overline{\overline{FL}}_j^{L}} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) \leq \left(1 - 1 \Big/ \left(1 + \left(\frac{l^2-l}{x+y} \times 1 \Big/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1-\overline{\overline{FL}}_i^{L+}}{\overline{\overline{FL}}_i^{L+}} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{1-\overline{\overline{FL}}_j^{L+}}{\overline{\overline{FL}}_j^{L+}} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) = \overline{\overline{FL}}^{L+}$$

$$\Upsilon^{U}\left(\overline{\overline{in}}_i\right) = \left(1 - 1 \Big/ \left(1 + \left(\frac{l^2-l}{x+y} \times 1 \Big/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1-\Upsilon_i^{U}}{\Upsilon_i^{U}} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{1-\Upsilon_j^{U}}{\Upsilon_j^{U}} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) \leq \left(1 - 1 \Big/ \left(1 + \left(\frac{l^2-l}{x+y} \times 1 \Big/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1-\Upsilon_i^{U+}}{\Upsilon_i^{U+}} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{1-\Upsilon_j^{U+}}{\Upsilon_j^{U+}} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) = \Upsilon^{U+}$$

Then there are the following scores

$$\frac{\overline{\overline{TR}}^{L}+\overline{\overline{TR}}^{U}}{2}+1-\frac{\overline{\overline{ID}}^{L}+\overline{\overline{ID}}^{U}}{2}+1-\frac{\overline{\overline{FL}}^{L}+\overline{\overline{FL}}^{U}}{2}=$$

$$\left(\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^{L}}{1-\overline{\overline{TR}}_i^{L}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^{L}}{1-\overline{\overline{TR}}_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)+1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^{U}}{1-\overline{\overline{TR}}_i^{U}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^{U}}{1-\overline{\overline{TR}}_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\Big/2\right.$$

$$+1-\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^{L}}{\overline{\overline{ID}}_i^{L}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^{L}}{\overline{\overline{ID}}_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)+1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^{U}}{\overline{\overline{ID}}_i^{U}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^{U}}{\overline{\overline{ID}}_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\Big/2$$

$$\left.+1-\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^{L}}{\overline{\overline{FL}}_i^{L}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^{L}}{\overline{\overline{FL}}_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)+1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^{U}}{\overline{\overline{FL}}_i^{U}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^{U}}{\overline{\overline{FL}}_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\Big/2\right)$$

$$\geq\frac{\overline{\overline{TR}}^{L-}+\overline{\overline{TR}}^{U-}}{2}+1-\frac{\overline{\overline{ID}}^{L+}+\overline{\overline{ID}}^{U+}}{2}+1-\frac{\overline{\overline{FL}}^{L+}+\overline{\overline{FL}}^{U+}}{2}.$$

Therefore according to the Definition 6, we have

$$\overline{\overline{in}}^{-}\leq INDPBM\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,\ldots,\overline{\overline{in}}_l\right).$$

In a similar way, the other part can be proved. That is $\overline{\overline{in}}^{-}\leq INDPBM\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,\ldots,\overline{\overline{in}}_m\right)\leq\overline{\overline{in}}^{+}$. Hence

$$\overline{\overline{in}}^{-}\leq INDPBM\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,\ldots,\overline{\overline{in}}_m\right)\leq\overline{\overline{in}}^{+}.$$

□

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TODIM Method for Multiple Attribute Group Decision Making under 2-Tuple Linguistic Neutrosophic Environment

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Received: 24 September 2018; Accepted: 8 October 2018; Published: 11 October 2018

Abstract: In this article, we extend the original TODIM (Portuguese acronym for Interactive Multi-Criteria Decision Making) method to the 2-tuple linguistic neutrosophic fuzzy environment to propose the 2TLNNs TODIM method. In the extended method, we use 2-tuple linguistic neutrosophic numbers (2TLNNs) to present the criteria values in multiple attribute group decision making (MAGDM) problems. Firstly, we briefly introduce the definition, operational laws, some aggregation operators and the distance calculating method of 2TLNNs. Then, the calculation steps of the original TODIM model are presented in simplified form. Thereafter, we extend the original TODIM model to the 2TLNNs environment to build the 2TLNNs TODIM model, our proposed method, which is more reasonable and scientific in considering the subjectivity of DM's behaviors and the dominance of each alternative over others. Finally, a numerical example for the safety assessment of a construction project is proposed to illustrate the new method, and some comparisons are also conducted to further illustrate the advantages of the new method.

Keywords: multiple attribute group decision making (MAGDM); 2-tuple linguistic neutrosophic sets (2TLNSs); TODIM model; 2TLNNs TODIM method; construction project

1. Introduction

The Interactive Multi-Criteria Decision Making (TODIM) model, first defined by Gomes and Lima [1], is a useful tool to investigate multiple attribute group decision making (MAGDM) problems and has been widely used in industrial, commercial economy, and management science areas. Some traditional MAGDM models have been investigated in the previous literature, such as: the ELimination Et Choix Traduisant la Realité (ELECTRE) model [2]; the Preference Ranking Organization Method for Enrichment of Evaluations (PROMETHEE) model [3]; the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) model [4,5]; the grey relational analysis (GRA) model [6–8]; the multi-objective optimization by ratio analysis plus the full multiplicative form (MULTIMOORA) model [9,10]; and, the VIseKriterijumska Optimizacija I KOmpromisno Resenje (VIKOR) model [11–13]. Compared with these existing methods, the TODIM model, which is based on prospect theory (PT), [14] has the advantages of considering the subjectivity of decision maker's (DM's) behaviors and providing the dominance of each alternative over others with particular operation formulas, and can be more reasonable and scientific in the application of MAGDM problems.

In practical decision problems, it is difficult to present the criteria values with real values for the complexity and fuzziness of the alternatives, and so it can be more useful and effective to express the criteria values with fuzzy numbers. Fuzzy set theory, which was initially introduced by Zadeh, [15] has been proved as a feasible means in the application of MAGDM [16,17]. Smarandache [18,19] provided the neutrosophic set (NS). Then, Wang et al. [20,21] investigated theories about single-valued neutrosophic sets (SVNSs) and provided the definition of interval neutrosophic sets (INSs). Ye [22] studied multiple attribute decision making (MADM) problems under the hesitant linguistic neutrosophic (HLN) environment. Wang et al. [23] studied the dual generalized Bonferroni mean (DGBM) aggregation operators under the SVNNs environment. Liu and You [24] proposed some linguistic neutrosophic Hamy mean (LNHM) aggregation operators. Wu et al. [25] gave the definition of SVN 2-tuple linguistic sets (SVN2TLSs) and proposed some new Hamacher aggregation operators. Ju et al. [26] extended the SVN2TLSs to the interval-valued environment and presented some single-valued neutrosophic interval 2-tuple linguistic Maclaurin symmetric mean (SVN-ITLMSM) operators. Wu et al. [27] studied SVNNs with Hamy operators under the 2-tuple linguistic variable environment. Wang et al. [28] provided the definition of the 2-tuple linguistic neutrosophic number (2TLNN) in which the degree of truth-membership, indeterminacy-membership and falsity-membership are depicted by 2TLNNs. Thereafter, the SVNS theory has been widely used to study MAGDM problems.

Gomes and Lima [1] used the TODIM model to investigate MADM problems taking the DM's confidence level into account to obtain more rational selection under risk. Wei et al. [29] extended the TODIM method to the hesitant fuzzy environment. Ren et al. [30] studied the TODIM model under the Pythagorean fuzzy environment. Fan et al. [31] established an extended TODIM model to solve MADM problems. Wang and Liu [32] developed an extended TODIM model based on intuitionistic linguistic information. Krohling et al. [33] extended the original TODIM method to the intuitionistic fuzzy numbers environment to propose the IF-TODIM method, and Lourenzutti and Krohling [34] built an intuitionistic fuzzy TODIM model based on the random environment. Wang et al. [35] combined the TODIM method with multi-hesitant fuzzy linguistic information to propose a likelihood-based TODIM method. Liu and Teng [36] provided an extension of the TODIM method under the 2-dimension uncertain linguistic variable. Sang and Liu [37] extended the TODIM method to interval type-2 fuzzy environments. Pramanik et al. [38] provide the NC-TODIM method under the neutrosophic cubic sets. Xu et al. [39] considered both the traditional TODIM model and SVNSs to build the SVN TODIM and IN TODIM models. Hu et al. [40] proposed a three-way decision TODIM model. Huang & Wei [41] proposed the TODIM method for Pythagorean 2-tuple linguistic multiple attribute decision making. However, there has been no study about the TIDOM model for MAGDM problems with 2TLNNs and there is a need to take the 2TLNNs TIDOM model into account. The goal of our article is to combine the original TIDOM model with 2TLNNs to study MAGDM problems. The structure of our paper is as follows. Section 2 introduces the concepts, operation formulas, distance calculating method, some aggregation operators of 2TLNNs and the calculation steps of the original TODIM model. Section 3 extends the original TIDOM model to the 2TLNNs environment and introduces the calculation steps of the 2TLNNs TIDOM method. Section 4 provides a numerical example and introduces the comparison between our proposed methods and the existing method. Section 5 provides some conclusions from our article.

2. Preliminaries

2.1. 2-Tuple Linguistic Neutrosophic Sets

Based on the concepts of 2-tuple linguistic fuzzy set (2TLS) and the fundamental theories of the single valued neutrosophic set (SVNS), the 2-tuple linguistic neutrosophic sets (2TLNSs) first defined by Wang et al. [28] can be depicted as follows.

Definition 1 ([28]). *Let $\eta_1, \eta_2, \ldots, \eta_k$ be a linguistic term set. Any label η_i shows a possible linguistic variable, and $\eta = \{\eta_0 = extremely\ poor, \eta_1 = very\ poor, \eta_2 = poor, \eta_3 = medium,\ \eta_4 = good, \eta_5 = very\ good, \eta_6 = extremely\ good.\}$, the 2TLNSs η can be depicted as:*

$$\eta = \{(s_\alpha, \phi), (s_\beta, \varphi), (s_\chi, \gamma)\} \tag{1}$$

where $s_\alpha, s_\beta, s_\chi \in \eta, \phi, \varphi, \gamma \in [-0{,}5,\ 0.5)$, $(s_\alpha, \phi), (s_\beta, \varphi)$ *and* (s_χ, γ) *represent the degree of the truth membership, the indeterminacy membership and the falsity membership which are expressed by 2TLNNs and satisfies the condition* $\Delta^{-1}(s_\alpha, \phi), \Delta^{-1}(s_\beta, \varphi)$ *and* $\Delta^{-1}(s_\chi, \gamma) \in [0, k]$, $0 \le \Delta^{-1}(s_\alpha, \phi) + \Delta^{-1}(s_\beta, \varphi) + \Delta^{-1}(s_\chi, \gamma) \le 3k$.

Definition 2 ([28]). *Assume there are three 2TLNNs* $\eta_1 = \{(s_{\alpha_1}, \phi_1), (s_{\beta_1}, \varphi_1), (s_{\chi_1}, \gamma_1)\}$, $\eta_2 = \{(s_{\alpha_2}, \phi_2), (s_{\beta_2}, \varphi_2), (s_{\chi_2}, \gamma_2)\}$ *and* $\eta = \{(s_\alpha, \phi), (s_\beta, \varphi), (s_\chi, \gamma)\}$, *the operation laws of them can be defined:*

$$\eta_1 \oplus \eta_2 = \left\{ \begin{array}{l} \Delta\left(k\left(\frac{\Delta^{-1}(s_{\alpha_1},\phi_1)}{k} + \frac{\Delta^{-1}(s_{\alpha_2},\phi_2)}{k} - \frac{\Delta^{-1}(s_{\alpha_1},\phi_1)}{k} \cdot \frac{\Delta^{-1}(s_{\alpha_2},\phi_2)}{k}\right)\right), \\ \Delta\left(k\left(\frac{\Delta^{-1}(s_{\beta_1},\varphi_1)}{k} \cdot \frac{\Delta^{-1}(s_{\beta_2},\varphi_2)}{k}\right)\right), \Delta\left(k\left(\frac{\Delta^{-1}(s_{\chi_1},\gamma_1)}{k} \cdot \frac{\Delta^{-1}(s_{\gamma_2},\gamma_2)}{k}\right)\right) \end{array} \right\};$$

$$\eta_1 \otimes \eta_2 = \left\{ \begin{array}{l} \Delta\left(k\left(\frac{\Delta^{-1}(s_{\alpha_1},\phi_1)}{k} \cdot \frac{\Delta^{-1}(s_{\alpha_2},\phi_2)}{k}\right)\right), \\ \Delta\left(k\left(\frac{\Delta^{-1}(s_{\beta_1},\varphi_1)}{k} + \frac{\Delta^{-1}(s_{\beta_2},\varphi_2)}{k} - \frac{\Delta^{-1}(s_{\beta_1},\varphi_1)}{k} \cdot \frac{\Delta^{-1}(s_{\beta_2},\varphi_2)}{k}\right)\right), \\ \Delta\left(k\left(\frac{\Delta^{-1}(s_{\chi_1},\gamma_1)}{k} + \frac{\Delta^{-1}(s_{\chi_2},\gamma_2)}{k} - \frac{\Delta^{-1}(s_{\chi_1},\gamma_1)}{k} \cdot \frac{\Delta^{-1}(s_{\chi_2},\gamma_2)}{k}\right)\right) \end{array} \right\};$$

$$\lambda\eta = \left\{ \Delta\left(k\left(1 - \left(1 - \frac{\Delta^{-1}(s_\alpha, \phi)}{k}\right)^\lambda\right)\right), \Delta\left(k\left(\frac{\Delta^{-1}(s_\beta, \varphi)}{k}\right)^\lambda\right), \Delta\left(k\left(\frac{\Delta^{-1}(s_\chi, \gamma)}{k}\right)^\lambda\right) \right\}, \lambda > 0;$$

$$\eta^\lambda = \left\{ \Delta\left(k\left(\frac{\Delta^{-1}(s_\alpha, \phi)}{k}\right)^\lambda\right), \Delta\left(k\left(1 - \left(1 - \frac{\Delta^{-1}(s_\beta, \varphi)}{k}\right)^\lambda\right)\right), \Delta\left(k\left(1 - \left(1 - \frac{\Delta^{-1}(s_\chi, \gamma)}{k}\right)^\lambda\right)\right) \right\}, \lambda > 0.$$

According to Definition 2, it is clear that the operation laws have the following properties:

$$\eta_1 \oplus \eta_2 = \eta_2 \oplus \eta_1, \eta_1 \otimes \eta_2 = \eta_2 \otimes \eta_1, \left((\eta_1)^{\lambda_1}\right)^{\lambda_2} = (\eta_1)^{\lambda_1 \lambda_2}; \tag{2}$$

$$\lambda(\eta_1 \oplus \eta_2) = \lambda\eta_1 \oplus \lambda\eta_2,\ (\eta_1 \otimes \eta_2)^\lambda = (\eta_1)^\lambda \otimes (\eta_2)^\lambda; \tag{3}$$

$$\lambda_1\eta_1 \oplus \lambda_2\eta_1 = (\lambda_1 + \lambda_2)\eta_1,\ (\eta_1)^{\lambda_1} \otimes (\eta_1)^{\lambda_2} = (\eta_1)^{(\lambda_1 + \lambda_2)}. \tag{4}$$

Definition 3 ([28]). *Let* $\eta = \{(s_\alpha, \phi), (s_\beta, \varphi), (s_\chi, \gamma)\}$ *be a 2TLNN, the score and accuracy functions of* η *can be expressed:*

$$s(\eta) = \frac{(2k + \Delta^{-1}(s_\alpha, \phi) - \Delta^{-1}(s_\beta, \varphi) - \Delta^{-1}(s_\chi, \gamma))}{3k}, s(\eta) \in [0, 1] \tag{5}$$

$$h(\eta) = \Delta^{-1}(s_\alpha, \phi) - \Delta^{-1}(s_\chi, \gamma), h(\eta) \in [-k, k] \tag{6}$$

For two 2TLNNs η_1 and η_2, based on Definition 3, then

(1) *if* $s(\eta_1) \prec s(\eta_2)$, *then* $\eta_1 \prec \eta_2$;
(2) *if* $s(\eta_1) \succ s(\eta_2)$, *then* $\eta_1 \succ \eta_2$;
(3) *if* $s(\eta_1) = s(\eta_2), h(\eta_1) \prec h(\eta_2)$, *then* $\eta_1 \prec \eta_2$;
(4) *if* $s(\eta_1) = s(\eta_2),\ h(\eta_1) \succ h(\eta_2)$, *then* $\eta_1 \succ \eta_2$;
(5) *if* $s(\eta_1) = s(\eta_2), h(\eta_1) = h(\eta_2)$, *then* $\eta_1 = \eta_2$.

2.2. The Normalized Hamming Distance

Definition 4. *Let* $\eta_1 = \{(s_{\alpha_1}, \phi_1), (s_{\beta_1}, \varphi_1), (s_{\chi_1}, \gamma_1)\}$ *and* $\eta_2 = \{(s_{\alpha_2}, \phi_2), (s_{\beta_2}, \varphi_2), (s_{\chi_2}, \gamma_2)\}$ *be two 2TLNNs, then we can get the normalized Hamming distance:*

$$d(\eta_1, \eta_2) = \frac{1}{3k}\left(\begin{array}{l} \left|\Delta^{-1}(s_{\alpha_1}, \phi_1) - \Delta^{-1}(s_{\alpha_2}, \phi_2)\right| + \left|\Delta^{-1}(s_{\beta_1}, \varphi_1) - \Delta^{-1}(s_{\beta_2}, \varphi_2)\right| \\ +\left|\Delta^{-1}(s_{\chi_1}, \gamma_1) - \Delta^{-1}(s_{\chi_2}, \gamma_2)\right| \end{array}\right) \tag{7}$$

Theorem 1. *Assume there are three 2TLNNs* $\eta_1 = \{(s_{\alpha_1}, \phi_1), (s_{\beta_1}, \varphi_1), (s_{\chi_1}, \gamma_1)\}$, $\eta_2 = \{(s_{\alpha_2}, \phi_2), (s_{\beta_2}, \varphi_2), (s_{\chi_2}, \gamma_2)\}$ *and* $\eta_3 = \{(s_{\alpha_3}, \phi_3), (s_{\beta_3}, \varphi_3), (s_{\chi_3}, \gamma_3)\}$, *the Hamming distance d has the following properties:*

$$\begin{array}{ll} \text{(P1) } 0 \le d(\eta_1, \eta_2) \le 1; & \text{(P2) } if\ d(\eta_1, \eta_2) = 0,\ then\ \eta_1 = \eta_2; \\ \text{(P3) } d(\eta_1, \eta_2) = d(\eta_2, \eta_1); & \text{(P4) } d(\eta_1, \eta_2) + d(\eta_2, \eta_3) \ge d(\eta_1, \eta_3). \end{array}$$

Proof. (P1) $0 \le d(\eta_1, \eta_2) \le 1$

Since $\Delta^{-1}(s_{\alpha_1}, \phi_1), \Delta^{-1}(s_{\alpha_2}, \phi_2) \in [0, k]$, then $0 \le \left|\Delta^{-1}(s_{\alpha_1}, \phi_1) - \Delta^{-1}(s_{\alpha_2}, \phi_2)\right| \le k$, similarly we can get $0 \le \left|\Delta^{-1}(s_{\beta_1}, \varphi_1) - \Delta^{-1}(s_{\beta_2}, \varphi_2)\right| \le k, 0 \le \left|\Delta^{-1}(s_{\chi_1}, \gamma_1) - \Delta^{-1}(s_{\chi_2}, \gamma_2)\right| \le k$, then $0 \le \left|\Delta^{-1}(s_{\alpha_1}, \phi_1) - \Delta^{-1}(s_{\alpha_2}, \phi_2)\right| + \left|\Delta^{-1}(s_{\beta_1}, \varphi_1) - \Delta^{-1}(s_{\beta_2}, \varphi_2)\right| + \left|\Delta^{-1}(s_{\chi_1}, \gamma_1) - \Delta^{-1}(s_{\chi_2}, \gamma_2)\right| \le 3k$,

So $0 \le \left(\left|\Delta^{-1}(s_{\alpha_1}, \phi_1) - \Delta^{-1}(s_{\alpha_2}, \phi_2)\right| + \left|\Delta^{-1}\left(s_{\beta_1}, \varphi_1\right) - \Delta^{-1}\left(s_{\beta_2}, \varphi_2\right)\right| + \left|\Delta^{-1}(s_{\chi_1}, \gamma_1) - \Delta^{-1}(s_{\chi_2}, \gamma_2)\right|\right) \le 3k$.

Therefore $0 \le d(\eta_1, \eta_2) \le 1$, the proof is completed.

(P2) $if\ d(\eta_1, \eta_2) = 0,\ then\ \eta_1 = \eta_2$

$$\begin{array}{l} d(\eta_1, \eta_2) = \frac{1}{3k}\left(\left|\Delta^{-1}(s_{\alpha_1}, \phi_1) - \Delta^{-1}(s_{\alpha_2}, \phi_2)\right| + \left|\Delta^{-1}(s_{\beta_1}, \varphi_1) - \Delta^{-1}(s_{\beta_2}, \varphi_2)\right| + \left|\Delta^{-1}(s_{\chi_1}, \gamma_1) - \Delta^{-1}(s_{\chi_2}, \gamma_2)\right|\right) = 0 \\ \Rightarrow \left(\left|\Delta^{-1}(s_{\alpha_1}, \phi_1) - \Delta^{-1}(s_{\alpha_2}, \phi_2)\right| = 0, \left|\Delta^{-1}(s_{\beta_1}, \varphi_1) - \Delta^{-1}(s_{\beta_2}, \varphi_2)\right| = 0, \left|\Delta^{-1}(s_{\chi_1}, \gamma_1) - \Delta^{-1}(s_{\chi_2}, \gamma_2)\right| = 0\right) \\ \Rightarrow \left(\Delta^{-1}(s_{\alpha_1}, \phi_1) = \Delta^{-1}(s_{\alpha_2}, \phi_2), \Delta^{-1}(s_{\beta_1}, \varphi_1) = \Delta^{-1}(s_{\beta_2}, \varphi_2), \Delta^{-1}(s_{\chi_1}, \gamma_1) = \Delta^{-1}(s_{\chi_2}, \gamma_2)\right) \end{array}$$

That means $\eta_1 = \eta_2$, so (P2) $if\ d(\eta_1, \eta_2) = 0,\ then\ \eta_1 = \eta_2$ is right.

(P3) $d(\eta_1, \eta_2) = d(\eta_2, \eta_1)$

$$\begin{array}{l} d(\eta_1, \eta_2) = \frac{1}{3k}\left(\left|\Delta^{-1}(s_{\alpha_1}, \phi_1) - \Delta^{-1}(s_{\alpha_2}, \phi_2)\right| + \left|\Delta^{-1}(s_{\beta_1}, \varphi_1) - \Delta^{-1}(s_{\beta_2}, \varphi_2)\right| + \left|\Delta^{-1}(s_{\chi_1}, \gamma_1) - \Delta^{-1}(s_{\chi_2}, \gamma_2)\right|\right) \\ = \frac{1}{3k}\left(\left|\Delta^{-1}(s_{\alpha_2}, \phi_2) - \Delta^{-1}(s_{\alpha_1}, \phi_1)\right| + \left|\Delta^{-1}(s_{\beta_2}, \varphi_2) - \Delta^{-1}(s_{\beta_1}, \varphi_1)\right| + \left|\Delta^{-1}(s_{\chi_2}, \gamma_2) - \Delta^{-1}(s_{\chi_1}, \gamma_1)\right|\right) = d(\eta_2, \eta_1) \end{array}$$

So we complete the proof. (P3) $d(\eta_1, \eta_2) = d(\eta_2, \eta_1)$ holds.

(P4) $d(\eta_1, \eta_2) + d(\eta_2, \eta_3) \ge d(\eta_1, \eta_3)$

$$\begin{array}{l} d(\eta_1, \eta_2) = \frac{1}{3k}\left(\begin{array}{l} \left|\Delta^{-1}(s_{\alpha_1}, \phi_1) - \Delta^{-1}(s_{\alpha_3}, \phi_3)\right| + \left|\Delta^{-1}(s_{\beta_1}, \varphi_1) - \Delta^{-1}(s_{\beta_3}, \varphi_3)\right| \\ +\left|\Delta^{-1}(s_{\chi_1}, \gamma_1) - \Delta^{-1}(s_{\chi_3}, \gamma_3)\right| \end{array}\right) \\ = \frac{1}{3k}\left(\begin{array}{l} \left|\Delta^{-1}(s_{\alpha_1}, \phi_1) - \Delta^{-1}(s_{\alpha_2}, \phi_2) + \Delta^{-1}(s_{\alpha_2}, \phi_2) - \Delta^{-1}(s_{\alpha_3}, \phi_3)\right| \\ +\left|\Delta^{-1}(s_{\beta_1}, \varphi_1) - \Delta^{-1}(s_{\beta_2}, \varphi_2) + \Delta^{-1}(s_{\beta_2}, \varphi_2) - \Delta^{-1}(s_{\beta_3}, \varphi_3)\right| \\ +\left|\Delta^{-1}(s_{\chi_1}, \gamma_1) - \Delta^{-1}(s_{\chi_2}, \gamma_2) + \Delta^{-1}(s_{\chi_2}, \gamma_2) - \Delta^{-1}(s_{\chi_3}, \gamma_3)\right| \end{array}\right) \\ \le \frac{1}{3k}\left(\begin{array}{l} \left|\Delta^{-1}(s_{\alpha_1}, \phi_1) - \Delta^{-1}(s_{\alpha_2}, \phi_2)\right| + \left|\Delta^{-1}(s_{\alpha_2}, \phi_2) - \Delta^{-1}(s_{\alpha_3}, \phi_3)\right| \\ +\left|\Delta^{-1}(s_{\beta_1}, \varphi_1) - \Delta^{-1}(s_{\beta_2}, \varphi_2)\right| + \left|\Delta^{-1}(s_{\beta_2}, \varphi_2) - \Delta^{-1}(s_{\beta_3}, \varphi_3)\right| \\ +\left|\Delta^{-1}(s_{\chi_1}, \gamma_1) - \Delta^{-1}(s_{\chi_2}, \gamma_2)\right| + \left|\Delta^{-1}(s_{\chi_2}, \gamma_2) - \Delta^{-1}(s_{\chi_3}, \gamma_3)\right| \end{array}\right) \\ = d(\eta_1, \eta_2) + d(\eta_2, \eta_3) \end{array}$$

□

2.3. The Aggregation Operators of 2TLNNs

Definition 5 ([28]). *Let* $\eta_j = \left\{ \left(s_{\alpha_j}, \phi_j\right), \left(s_{\beta_j}, \varphi_j\right), \left(s_{\chi_j}, \gamma_j\right) \right\} (j = 1, 2, \ldots, n)$ *be a group of 2TLNNs, then the 2TLNNWA and 2TLNNWG operators proposed by Wang* et al. *[25] are defined as follows.*

$$\text{2TLNNWA}(\eta_1, \eta_2, \ldots, \eta_n) = \omega_1\eta_1 \oplus \omega_2\eta_2 \ldots \oplus \omega_n\eta_n = \mathop{\oplus}_{j=1}^{n} \omega_j\eta_j \quad (8)$$

and

$$\text{2TLNNWG}(\eta_1, \eta_2, \ldots, \eta_n) = (\eta_1)^{\omega_1} \otimes (\eta_2)^{\omega_2} \ldots \otimes (\eta_n)^{\omega_n} = \mathop{\otimes}_{j=1}^{n} (\eta_j)^{\omega_j} \quad (9)$$

where ω_j *is weighting vector of* $\eta_j, j = 1, 2, \ldots, n$*. which satisfies* $0 \leq \omega_j \leq 1, \sum_{j=1}^{n} \omega_j = 1$.

Theorem 2 ([28]). *Let* $\eta_j = \left\{ \left(s_{\alpha_j}, \phi_j\right), \left(s_{\beta_j}, \varphi_j\right), \left(s_{\chi_j}, \gamma_j\right) \right\} (j = 1, 2, \ldots, n)$ *be a group of 2TLNNs, then the operation results by 2TLNNWA and 2TLNNWG operators are also a 2TLNN where*

$$\begin{aligned} &\text{2TLNNWA}(\eta_1, \eta_2, \ldots, \eta_n) = \mathop{\oplus}_{j=1}^{n} \omega_j\eta_j \\ &= \left\langle \begin{array}{l} \Delta\left(k\left(1 - \prod_{j=1}^{n}\left(1 - \frac{\Delta^{-1}\left(s_{\alpha_j}, \phi_j\right)}{k}\right)^{w_j}\right)\right), \Delta\left(k\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\beta_j}, \varphi_j\right)}{k}\right)^{w_j}\right), \\ \Delta\left(k\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\chi_j}, \gamma_j\right)}{k}\right)^{w_j}\right). \end{array} \right\rangle \end{aligned} \quad (10)$$

and

$$\begin{aligned} &\text{2TLNNWG}(\eta_1, \eta_2, \ldots, \eta_n) = \mathop{\otimes}_{j=1}^{n} (\eta_j)^{\omega_j} \\ &= \left\langle \begin{array}{l} \Delta\left(k\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\alpha_j}, \phi_j\right)}{k}\right)^{w_j}\right), \Delta\left(k\left(1 - \prod_{j=1}^{n}\left(1 - \frac{\Delta^{-1}\left(s_{\beta_j}, \varphi_j\right)}{k}\right)^{w_j}\right)\right), \\ \Delta\left(k\left(1 - \prod_{j=1}^{n}\left(1 - \frac{\Delta^{-1}\left(s_{\chi_j}, \gamma_j\right)}{k}\right)^{w_j}\right)\right). \end{array} \right\rangle \end{aligned} \quad (11)$$

2.4. The Original TODIM Method

The TODIM method, which is based on prospect theory (PT), considers the subjectivity of DM's behaviors and can provide the dominance of each alternative over others with particular operation formulas, and is more reasonable and scientific in the application of MAGDM problems.

Assume that $\{\eta_1, \eta_2, \ldots \eta_m\}$ be a group of alternatives, $\{c_1, c_2, \ldots c_n\}$ be a list of criteria with weighting vector be $\{w_1, w_2, \ldots w_n\}$, thereby satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Construct a decision matrix $\eta = [d_{ij}]_{m \times n}$ where d_{ij} means the estimate results of the alternative $\eta_i (i = 1, 2, \ldots, m)$ based on the criterion $c_j (j = 1, 2, \ldots, n)$. Suppose that $w_{jk} = w_j / w_k$ be relative weight of c_j *to* c_t where $w_k = \max(w_j)$ $k, j = 1, 2, \ldots, n$. The traditional TODIM method decision making steps can be summarized as follows:

Step 1. Normalize $\eta = [d_{ij}]_{m \times n}$ into $\eta' = \left[d'_{ij}\right]_{m \times n}$.

Step 2. Calculate the dominance degree of η_i over each alternative η_t based on c_j. Let ρ be the attenuation factor of the losses. Then

$$\delta(\eta_i, \eta_t) = \sum_{j=1}^{n} \vartheta_j(\eta_i, \eta_t) (i, t = 1, 2, \ldots, m) \quad (12)$$

$$\vartheta_j(\eta_i,\eta_t)=\begin{cases}\sqrt{w_{jk}\left(d_{ij}-d_{tj}\right)/\sum_{j=1}^{n}w_{jk}} & if\ d_{ij}-d_{tj}>0\\ 0 & if\ d_{ij}-d_{tj}=0\\ -\frac{1}{\rho}\sqrt{\left(\sum_{j=1}^{n}w_{jk}\right)\left(d_{ij}-d_{tj}\right)/w_{jk}} & if\ d_{ij}-d_{tj}<0\end{cases} \tag{13}$$

where $\vartheta_j(\eta_i,\eta_t)\left(d_{ij}-d_{tj}>0\right)$ means gain and $\vartheta_j(\eta_i,\eta_t)\left(d_{ij}-d_{tj}<0\right)$ indicates loss.

Step 3. Compute the overall value of $\delta(\eta_i)$ with formula (14):

$$\delta(\eta_i)=\frac{\sum\limits_{t=1}^{m}\delta(\eta_i,\eta_t)-\min\limits_{i}\left\{\sum\limits_{t=1}^{m}\delta(\eta_i,\eta_t)\right\}}{\max\limits_{i}\left\{\sum\limits_{t=1}^{m}\delta(\eta_i,\eta_t)\right\}-\min\limits_{i}\left\{\sum\limits_{t=1}^{m}\delta(\eta_i,\eta_t)\right\}} \tag{14}$$

Step 4. To choose the best alternative by rank the values of $\delta(\eta_i)$, the alternative with maximum value is the best choice.

3. The TODIM Method with 2TLNNs

Assume that $\{\eta_1,\eta_2,\ldots\eta_m\}$ be a group of alternatives, $\{d_1,d_2,\ldots d_\lambda\}$ be a list of experts with weighting vector be $\{v_1,v_2,\ldots v_t\}$, and $\{c_1,c_2,\ldots c_n\}$ be a list of criteria with weighting vector be $\{w_1,w_2,\ldots w_n\}$, thereby satisfying $w_i\in[0,1],v_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1,\sum_{i=1}^{t}v_i=1$. Construct a decision matrix $\eta^\lambda=\left[r_{ij}^\lambda\right]_{m\times n}$ where $\eta_{ij}^\lambda=\left\{\left(s_{\alpha_{ij}},\phi_{ij}\right)^\lambda,\left(s_{\beta_{ij}},\varphi_{ij}\right)^\lambda,\left(s_{\chi_{ij}},\gamma_{ij}\right)^\lambda\right\}$ means the estimate results of the alternative $\eta_i(i=1,2,\ldots,m)$ based on the criterion $c_j(j=1,2,\ldots,n)$ by expert d^λ. $\left(s_{\alpha_{ij}},\phi_{ij}\right)^\lambda$ denotes the degree of truth-membership (TMD), $\left(s_{\beta_{ij}},\varphi_{ij}\right)^\lambda$ denotes the degree of indeterminacy-membership (IMD) and $\left(s_{\chi_{ij}},\gamma_{ij}\right)^\lambda$ denotes the degree of falsity-membership (FMD), $0\le\Delta^{-1}\left(s_{\alpha_{ij}},\phi_{ij}\right)^\lambda+\Delta^{-1}\left(s_{\beta_{ij}},\varphi_{ij}\right)^\lambda+\Delta^{-1}\left(s_{\chi_{ij}},\gamma_{ij}\right)^\lambda\le 3k$ $(i=1,2,\ldots,m,j=1,2,\ldots,n)$. let $w_{jk}=w_j/w_k\left(0\le w_{jk}\le 1\right)$ be relative weight of c_j *to* c_t where $w_k=\max\left(w_j\right)$ $(k,j=1,2,\ldots,n)$.

Consider both the 2TLNNs theories and traditional TODIM method which based on prospect theory (PT), we try to propose a 2TLNNs TODIM method to solve MAGDM problems effectively. The model can be depicted as follows:

Step 1. Calculate the value of $w_{jk}=w_j/w_k\left(0\le w_{jk}\le 1\right)$, $w_k=\max\left(w_j\right)$ $(k,j=1,2,\ldots,n)$.

Step 2. According to the computing results of relative weight w_{jk}, we can calculate the dominance degree of η_i^λ over each alternative η_t^λ based on c_j by expert d_λ. let ρ be the attenuation factor of the losses. Then

$$\vartheta_j^\lambda(\eta_i,\eta_t)=\begin{cases}\sqrt{w_{jk}d\left(r_{ij}^\lambda-r_{tj}^\lambda\right)/\sum_{j=1}^{n}w_{jk}} & if\ r_{ij}^\lambda-r_{tj}^\lambda>0\\ 0 & if\ r_{ij}^\lambda-r_{tj}^\lambda=0\\ -\frac{1}{\rho}\sqrt{\left(\sum_{j=1}^{n}w_{jk}\right)d\left(r_{ij}^\lambda-r_{tj}^\lambda\right)/w_{jk}} & if\ r_{ij}^\lambda-r_{tj}^\lambda<0\end{cases} \tag{15}$$

$$d\left(r_{ij}^\lambda-r_{tj}^\lambda\right)=\frac{1}{3k}\left(\begin{array}{l}\left|\Delta^{-1}\left(s_{\alpha_{ij}},\phi_{tj}\right)^\lambda-\Delta^{-1}\left(s_{\alpha_{ij}},\phi_{tj}\right)^\lambda\right|+\left|\Delta^{-1}\left(s_{\beta_{ij}},\varphi_{ij}\right)^\lambda-\Delta^{-1}\left(s_{\beta_{tj}},\varphi_{tj}\right)^\lambda\right|\\+\left|\Delta^{-1}\left(s_{\chi_{ij}},\gamma_{ij}\right)^\lambda-\Delta^{-1}\left(s_{\chi_{tj}},\gamma_{tj}\right)^\lambda\right|\end{array}\right) \tag{16}$$

where $\vartheta_j^\lambda(\eta_i,\eta_t)(r_{ij}^\lambda-r_{tj}^\lambda>0)$ means gain and $\vartheta_j^\lambda(\eta_i,\eta_t)(r_{ij}^\lambda-r_{tj}^\lambda<0)$ indicates loss, and based on Definition 4, $d(r_{ij}^\lambda-r_{tj}^\lambda)$ means the normalized Hamming distance between r_{ij}^λ and r_{tj}^λ.

Next we construct a matrix model of dominance degree $\vartheta_j^\lambda = \left[\vartheta_j^\lambda(\eta_i, \eta_t)\right]_{m\times m}$ under criteria c_j by expert d_λ to express Equation (15) more clearly.

$$\vartheta_j^\lambda(\eta_i,\eta_t) = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{array} \begin{array}{c} \begin{array}{cccc} \eta_1 & \eta_2 & \cdots & \eta_m \end{array} \\ \begin{bmatrix} 0 & \vartheta_j^\lambda(\eta_1,\eta_2) & \cdots & \vartheta_j^\lambda(\eta_1,\eta_m) \\ \vartheta_j^\lambda(\eta_2,\eta_1) & 0 & \cdots & \vartheta_j^\lambda(\eta_2,\eta_m) \\ \vdots & \vdots & \cdots & \vdots \\ \vartheta_j^\lambda(\eta_m,\eta_1) & \vartheta_j^\lambda(\eta_i,\eta_t) & \cdots & 0 \end{bmatrix} \end{array}, j = 1,2,\ldots,n \tag{17}$$

Step 3. Compute overall dominance degree $\vartheta_j^\lambda = \left[\vartheta_j^\lambda(\eta_i, \eta_t)\right]_{m\times m}$ to get the matrix model $\vartheta^\lambda = \left[\vartheta^\lambda(\eta_i, \eta_t)\right]_{m\times m}$.

$$\vartheta^\lambda(\eta_i,\eta_t) = \sum_{j=1}^{n} \vartheta_j^\lambda(\eta_i,\eta_t)(i,t = 1,2,\ldots,m) \tag{18}$$

$$\vartheta^\lambda(\eta_i,\eta_t) = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{array} \begin{array}{c} \begin{array}{cccc} \eta_1 & \eta_2 & \cdots & \eta_m \end{array} \\ \begin{bmatrix} 0 & \vartheta^\lambda(\eta_1,\eta_2) & \cdots & \vartheta^\lambda(\eta_1,\eta_m) \\ \vartheta^\lambda(\eta_2,\eta_1) & 0 & \cdots & \vartheta^\lambda(\eta_2,\eta_m) \\ \vdots & \vdots & \cdots & \vdots \\ \vartheta^\lambda(\eta_m,\eta_1) & \vartheta^\lambda(\eta_i,\eta_t) & \cdots & 0 \end{bmatrix} \end{array} \tag{19}$$

Step 4. Calculate the overall dominance $\delta(\eta_i, \eta_t)$ based on the expert weighting vector $\{v_1, v_2, \ldots v_t\}$ and the results of Equation (19).

$$\delta(\eta_i,\eta_t) = \sum_{j=1}^{\lambda} v_\lambda \vartheta^\lambda(\eta_i,\eta_t)(i,t = 1,2,\ldots,m) \tag{20}$$

The overall dominance $\delta(\eta_i, \eta_t)$ matrix can be constructed by Formula (21) as follows:

$$\delta_j(\eta_i,\eta_t) = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{array} \begin{array}{c} \begin{array}{cccc} \eta_1 & \eta_2 & \cdots & \eta_m \end{array} \\ \begin{bmatrix} 0 & \delta_j(\eta_1,\eta_2) & \cdots & \delta_j(\eta_1,\eta_m) \\ \delta_j(\eta_2,\eta_1) & 0 & \cdots & \delta_j(\eta_2,\eta_m) \\ \vdots & \vdots & \cdots & \vdots \\ \delta_j(\eta_m,\eta_1) & \delta_j(\eta_i,\eta_t) & \cdots & 0 \end{bmatrix} \end{array}, j = 1,2,\ldots,n \tag{21}$$

Step 5. Compute the overall value of $\delta(\eta_i)$ with Formula (22):

$$\delta(\eta_i) = \frac{\sum_{t=1}^{m} \delta(\eta_i,\eta_t) - \min_i\left\{\sum_{t=1}^{m} \delta(\eta_i,\eta_t)\right\}}{\max_i\left\{\sum_{t=1}^{m} \delta(\eta_i,\eta_t)\right\} - \min_i\left\{\sum_{t=1}^{m} \delta(\eta_i,\eta_t)\right\}} \tag{22}$$

Step 6. To choose the best alternative by rank the values of $\delta(\eta_i)$, the alternative with maximum value is the best choice.

4. The Numerical Example

4.1. Calculation Steps Based on MAGDM Problems

Construction engineering projects have the following characteristics: large investment, many participants, complex project environment, and a wide range of risk factors on the basis of the engineering procurement construction (EPC) mode. Therefore, it is necessary to analyze and assess risks during the life cycle of a construction engineering project, with a risk assessment being beneficial for implementing projects and completing project goals. Construction engineering projects face a range of political, economic, social natural and other types of risks during the implementation process. These risks have a great influence on construction companies, and produce many high probability factors which are difficult to estimate and quantify. Thus, we provide a numerical example for construction engineering project risk assessment (adapted from Reference [27]), using the TODIM method with 2TLNNs, in order to illustrate the method proposed in this paper. Assuming that there are five possible construction projects $\eta_i(i = 1, 2, 3, 4, 5)$ to select from and four criteria to assess these construction projects: ① G_1 is the construction work environment; ② G_2 is the construction site safety protection measures; ③ G_3 is the safety management ability of the engineering project management; and ④ G_4 is the safety production responsibility system. The five possible construction projects $\eta_i(i = 1, 2, 3, 4, 5)$ are to be evaluated with 2TLNNs with the four criteria by three experts d^k (criteria weight $w = (0.14, 0.33, 0.29, 0.24)$, experts weight $v = (0.45, 0.15, 0.40)$., listed in Tables 1–3.

Table 1. 2-tuple linguistic neutrosophic numbers (2TLNNs) evaluation matrix by d^1.

	G_1	G_2	G_3	G_4
η_1	{(s_4,0), (s_2,0), (s_1,0)}	{(s_5,0), (s_3,0), (s_2,0)}	{(s_4,0), (s_1,0), (s_1,0)}	{(s_3,0), (s_2,0), (s_2,0)}
η_2	{(s_5,0), (s_4,0), (s_4,0)}	{(s_3,0), (s_4,0), (s_2,0)}	{(s_2,0), (s_1,0), (s_3,0)}	{(s_4,0), (s_1,0), (s_2,0)}
η_3	{(s_5,0), (s_4,0), (s_2,0)}	{(s_2,0), (s_4,0), (s_5,0)}	{(s_3,0), (s_2,0), (s_4,0)}	{(s_2,0), (s_1,0), (s_4,0)}
η_4	{(s_3,0), (s_2,0), (s_3,0)}	{(s_4,0), (s_3,0), (s_2,0)}	{(s_3,0), (s_3,0), (s_4,0)}	{(s_2,0), (s_1,0), (s_1,0)}
η_5	{(s_1,0), (s_4,0), (s_5,0)}	{(s_2,0), (s_3,0), (s_1,0)}	{(s_3,0), (s_4,0), (s_5,0)}	{(s_2,0), (s_4,0), (s_3,0)}

Table 2. 2TLNNs evaluation matrix by d^2.

	G_1	G_2	G_3	G_4
η_1	{(s_5,0), (s_1,0), (s_2,0)}	{(s_4,0), (s_3,0), (s_1,0)}	{(s_4,0), (s_2,0), (s_1,0)}	{(s_5,0), (s_1,0), (s_2,0)}
η_2	{(s_4,0), (s_3,0), (s_3,0)}	{(s_3,0), (s_1,0), (s_4,0)}	{(s_2,0), (s_1,0), (s_3,0)}	{(s_5,0), (s_4,0), (s_1,0)}
η_3	{(s_3,0), (s_4,0), (s_3,0)}	{(s_2,0), (s_4,0), (s_5,0)}	{(s_5,0), (s_1,0), (s_2,0)}	{(s_2,0), (s_1,0), (s_2,0)}
η_4	{(s_4,0), (s_5,0), (s_4,0)}	{(s_2,0), (s_3,0), (s_4,0)}	{(s_3,0), (s_3,0), (s_4,0)}	{(s_4,0), (s_4,0), (s_5,0)}
η_5	{(s_2,0), (s_4,0), (s_5,0)}	{(s_3,0), (s_1,0), (s_5,0)}	{(s_2,0), (s_3,0), (s_4,0)}	{(s_2,0), (s_1,0), (s_3,0)}

Table 3. 2TLNNs evaluation matrix by d^3.

	G_1	G_2	G_3	G_4
η_1	{(s_5,0), (s_1,0), (s_1,0)}	{(s_5,0), (s_1,0), (s_2,0)}	{(s_3,0), (s_3,0), (s_1,0)}	{(s_4,0), (s_2,0), (s_1,0)}
η_2	{(s_5,0), (s_4,0), (s_5,0)}	{(s_3,0), (s_2,0), (s_1,0)}	{(s_2,0), (s_1,0), (s_4,0)}	{(s_4,0), (s_5,0), (s_3,0)}
η_3	{(s_2,0), (s_1,0), (s_4,0)}	{(s_5,0), (s_4,0), (s_3,0)}	{(s_4,0), (s_3,0), (s_3,0)}	{(s_5,0), (s_2,0), (s_3,0)}
η_4	{(s_2,0), (s_1,0), (s_3,0)}	{(s_4,0), (s_1,0), (s_2,0)}	{(s_5,0), (s_3,0), (s_2,0)}	{(s_1,0), (s_4,0), (s_5,0)}
η_5	{(s_1,0), (s_4,0), (s_5,0)}	{(s_2,0), (s_4,0), (s_4,0)}	{(s_3,0), (s_4,0), (s_3,0)}	{(s_2,0), (s_4,0), (s_4,0)}

Step 1. Calculate the value of $w_{jk} = w_j / w_k \left(0 \leq w_{jk} \leq 1\right)$, $w_k = \max\left(w_j\right)$ $(k, j = 1, 2, \ldots, n)$.

$$\begin{aligned} w_k &= \max(0.14, 0.33, 0.29, 0.24) = 0.33 \\ w_{jk} &= w_j / w_k = (0.4242, 1.0000, 0.8788, 0.7273)^T \end{aligned}$$

Step 2. According to the computing results of relative weight w_{jk}, we can calculate the dominance degree of η_i^λ over each alternative η_t based on c_j by λth experts. The operation results are listed as follows. $(\rho = 2.4)$

For expert d_1, the dominance degree η_i^1 can be calculated:

$$\vartheta_1^1 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left[\begin{array}{ccccc} 0.0000 & 0.2160 & 0.1764 & 0.1528 & 0.2646 \\ -0.6429 & 0.0000 & -0.3712 & -0.5869 & 0.1972 \\ -0.5250 & -0.3712 & 0.0000 & -0.5869 & 0.2333 \\ -0.4546 & 0.1972 & -0.5869 & 0.0000 & 0.2160 \\ -0.7874 & -0.5869 & -0.6944 & -0.6429 & 0.0000 \end{array}\right] \end{array}$$

$$\vartheta_2^1 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left[\begin{array}{ccccc} 0.0000 & 0.2345 & 0.3582 & 0.1354 & 0.2708 \\ -0.2961 & 0.0000 & 0.2708 & -0.2418 & -0.2961 \\ -0.4523 & -0.3419 & 0.0000 & -0.4188 & -0.3823 \\ -0.1710 & 0.1915 & 0.3317 & 0.0000 & 0.2345 \\ -0.3419 & 0.2345 & 0.3028 & -0.2961 & 0.0000 \end{array}\right] \end{array}$$

$$\vartheta_3^1 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left[\begin{array}{ccccc} 0.0000 & 0.2539 & 0.2838 & 0.3109 & 0.3590 \\ -0.3647 & 0.0000 & 0.2198 & 0.2539 & 0.3109 \\ -0.4078 & -0.3159 & 0.0000 & 0.1269 & 0.2198 \\ -0.4467 & -0.3647 & -0.1824 & 0.0000 & 0.1795 \\ -0.5158 & -0.4467 & -0.3159 & -0.2579 & 0.0000 \end{array}\right] \end{array}$$

$$\vartheta_4^1 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left[\begin{array}{ccccc} 0.0000 & -0.2835 & 0.2309 & -0.3472 & 0.2309 \\ 0.1633 & 0.0000 & 0.2309 & 0.2000 & 0.2828 \\ -0.4009 & -0.4009 & 0.0000 & -0.3472 & 0.2309 \\ 0.2000 & -0.3472 & 0.2000 & 0.0000 & 0.2582 \\ -0.4009 & -0.4910 & -0.4009 & -0.4483 & 0.0000 \end{array}\right] \end{array}$$

For expert d_2, the dominance degree η_i^2 can be calculated:

$$\vartheta_1^2 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left[\begin{array}{ccccc} 0.0000 & 0.1764 & 0.2160 & 0.2333 & 0.2646 \\ -0.5250 & 0.0000 & 0.1247 & 0.1528 & 0.1972 \\ -0.6429 & -0.3712 & 0.0000 & 0.1528 & 0.1528 \\ -0.6944 & -0.4546 & -0.4546 & 0.0000 & 0.1764 \\ -0.7874 & -0.5869 & -0.4546 & -0.5250 & 0.0000 \end{array}\right] \end{array}$$

$$\vartheta_2^2 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left[\begin{array}{ccccc} 0.0000 & 0.3317 & 0.3582 & 0.3028 & 0.3582 \\ -0.4188 & 0.0000 & 0.3028 & 0.2345 & 0.1354 \\ -0.4523 & -0.3823 & 0.0000 & -0.2418 & -0.3419 \\ -0.3823 & -0.2961 & 0.1915 & 0.0000 & 0.2708 \\ -0.4523 & -0.1710 & 0.2708 & 0.2708 & 0.0000 \end{array}\right] \end{array}$$

$$\vartheta_3^2 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left[\begin{array}{ccccc} 0.0000 & 0.2838 & -0.3159 & 0.2838 & 0.3109 \\ -0.4078 & 0.0000 & -0.3647 & 0.2539 & 0.2198 \\ 0.2198 & 0.2539 & 0.0000 & 0.3109 & 0.3358 \\ -0.4078 & -0.3647 & -0.4467 & 0.0000 & 0.1269 \\ -0.4467 & -0.3159 & -0.4825 & -0.1824 & 0.0000 \end{array}\right] \end{array}$$

$$\vartheta_4^2 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left[\begin{array}{ccccc} 0.0000 & 0.2309 & 0.2000 & 0.3055 & 0.2309 \\ -0.4009 & 0.0000 & 0.3055 & 0.2582 & 0.3266 \\ -0.3472 & -0.5304 & 0.0000 & 0.3266 & 0.1155 \\ -0.5304 & -0.4483 & -0.5670 & 0.0000 & -0.5304 \\ -0.4009 & -0.5670 & -0.2005 & 0.3055 & 0.0000 \end{array}\right] \end{array}$$

For expert d_3, the dominance degree η_i^3 can be calculated:

$$\vartheta_1^3 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left[\begin{array}{ccccc} 0.0000 & 0.2333 & 0.2160 & 0.1972 & 0.2925 \\ -0.6944 & 0.0000 & -0.6944 & -0.7424 & 0.1764 \\ -0.6429 & 0.2333 & 0.0000 & -0.2625 & 0.1972 \\ -0.5869 & 0.2494 & 0.0882 & 0.0000 & 0.2160 \\ -0.8705 & -0.5250 & -0.5869 & -0.6429 & 0.0000 \end{array}\right] \end{array}$$

$$\vartheta_2^3 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left[\begin{array}{ccccc} 0.0000 & 0.2708 & 0.2708 & 0.1354 & 0.3830 \\ -0.3419 & 0.0000 & 0.3317 & -0.2961 & 0.3317 \\ -0.3419 & -0.4188 & 0.0000 & -0.3823 & 0.2708 \\ -0.1710 & 0.2345 & 0.3028 & 0.0000 & 0.3582 \\ -0.4835 & -0.4188 & -0.3419 & -0.4523 & 0.0000 \end{array}\right] \end{array}$$

$$\vartheta_3^3 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left[\begin{array}{ccccc} 0.0000 & 0.3109 & 0.2198 & -0.3159 & 0.2198 \\ -0.4467 & 0.0000 & -0.4078 & -0.4825 & 0.2838 \\ -0.3159 & 0.2838 & 0.0000 & -0.2579 & 0.1795 \\ 0.2198 & 0.3358 & 0.1795 & 0.0000 & 0.2539 \\ -0.3159 & -0.4078 & -0.2579 & -0.3647 & 0.0000 \end{array}\right] \end{array}$$

$$\vartheta_4^3 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \left[\begin{array}{ccccc} 0.0000 & 0.2582 & 0.2000 & 0.3464 & 0.3055 \\ -0.4483 & 0.0000 & -0.4009 & 0.2828 & 0.2309 \\ -0.3472 & 0.2309 & 0.0000 & 0.3266 & 0.2828 \\ -0.6014 & -0.4910 & -0.5670 & 0.0000 & -0.2835 \\ -0.5304 & -0.4009 & -0.4910 & 0.1633 & 0.0000 \end{array}\right] \end{array}$$

Step 3. Compute overall dominance degree $\vartheta_j^\lambda = \left[\vartheta_j^\lambda(\eta_i, \eta_t)\right]_{m\times m}$ to get the matrix $\phi^\lambda = \left[\phi^\lambda(\varphi_i, \varphi_t)\right]_{m\times m}$.

$$\vartheta^1 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \begin{bmatrix} 0.0000 & 0.4209 & 1.0494 & 0.2518 & 1.1253 \\ -1.1405 & 0.0000 & 0.3504 & -0.3748 & 0.4948 \\ -1.7860 & -1.4299 & 0.0000 & -1.2260 & 0.3018 \\ -0.8723 & -0.3233 & -0.2376 & 0.0000 & 0.8882 \\ -2.0461 & -1.2902 & -1.1085 & -1.6452 & 0.0000 \end{bmatrix} \end{array}$$

$$\vartheta^2 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \begin{bmatrix} 0.0000 & 1.0228 & 0.4584 & 1.1254 & 1.1647 \\ -1.7524 & 0.0000 & 0.3683 & 0.8993 & 0.8791 \\ -1.2226 & -1.0300 & 0.0000 & 0.5485 & 0.2621 \\ -2.0149 & -1.5637 & -1.2769 & 0.0000 & 0.0437 \\ -2.0874 & -1.6408 & -0.8668 & -0.1310 & 0.0000 \end{bmatrix} \end{array}$$

$$\vartheta^3 = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \begin{bmatrix} 0.0000 & 1.0732 & 0.9067 & 0.3631 & 1.2008 \\ -1.9313 & 0.0000 & -1.1715 & -1.2382 & 1.0228 \\ -1.6479 & 0.3293 & 0.0000 & -0.5761 & 0.9304 \\ -1.1394 & 0.3287 & 0.0035 & 0.0000 & 0.5446 \\ -2.2003 & -1.7524 & -1.6778 & -1.2967 & 0.0000 \end{bmatrix} \end{array}$$

Step 4. Calculate the overall dominance $\delta(\eta_i, \eta_t)$ based on the expert weighting vector $(0.45, 0.15, 0.40)$ and the results of $\vartheta^\lambda = \left[\vartheta^\lambda(\eta_i, \eta_t)\right]_{m\times m}$.

$$\delta(\eta_i, \eta_t) = \begin{array}{c} \\ \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \end{array} \\ \begin{bmatrix} 0.0000 & 0.7721 & 0.9037 & 0.4274 & 1.1614 \\ -1.5486 & 0.0000 & -0.2557 & -0.5290 & 0.7637 \\ -1.6463 & -0.6662 & 0.0000 & -0.6998 & 0.5473 \\ -1.1505 & -0.2485 & -0.2971 & 0.0000 & 0.6241 \\ -2.1140 & -1.5277 & -1.3000 & -1.2787 & 0.0000 \end{bmatrix} \end{array}$$

Step 5. Compute the overall value of $\delta(\eta_i)$ with the Formula (22):

$$\delta(\eta_1) = 1.0000\ , \delta(\eta_2) = 0.4903, \delta(\eta_3) = 0.3959, \delta(\eta_4) = 0.5428, \delta(\eta_5) = 0.0000.$$

Step 6. To choose the best alternative by rank the values of $\delta(\eta_i)$, the alternative with maximum value is the best choice. According to step 5, the ranking of η_i is $\eta_1 > \eta_4 > \eta_2 > \eta_3 > \eta_5$, and it is clear that the best choice is η_1.

4.2. The Affection Analysis of the Parameter ρ

By altering parameters ρ in the computing process of the 2TLNNs TODIM method, we can depict the effects on ordering. The calculation results follow.

From the calculation results of Table 4, we can easily ascertain that the best alternative is η_1 by altering the values of ρ. Next we will compare our proposed 2TLNNs TODIM method with the existing method using 2TLNNWA and 2TLNNWG operators.

Table 4. Ordering of η_i by altering parameters ρ.

ρ	$\delta(\eta_1)$	$\delta(\eta_2)$	$\delta(\eta_3)$	$\delta(\eta_4)$	$\delta(\eta_5)$	Ordering
1.0	1.0000	0.4947	0.4182	0.5601	0.0000	$\eta_1 > \eta_4 > \eta_2 > \eta_3 > \eta_5$
1.1	1.0000	0.4943	0.4162	0.5586	0.0000	$\eta_1 > \eta_4 > \eta_2 > \eta_3 > \eta_5$
1.2	1.0000	0.4939	0.4143	0.5571	0.0000	$\eta_1 > \eta_4 > \eta_2 > \eta_3 > \eta_5$
1.5	1.0000	0.4929	0.4090	0.5530	0.0000	$\eta_1 > \eta_4 > \eta_2 > \eta_3 > \eta_5$
1.7	1.0000	0.4922	0.4058	0.5505	0.0000	$\eta_1 > \eta_4 > \eta_2 > \eta_3 > \eta_5$
2.0	1.0000	0.4914	0.4013	0.5470	0.0000	$\eta_1 > \eta_4 > \eta_2 > \eta_3 > \eta_5$
2.3	1.0000	0.4906	0.3972	0.5438	0.0000	$\eta_1 > \eta_4 > \eta_2 > \eta_3 > \eta_5$
2.5	1.0000	0.4901	0.3947	0.5418	0.0000	$\eta_1 > \eta_4 > \eta_2 > \eta_3 > \eta_5$
3.0	1.0000	0.4890	0.3889	0.5373	0.0000	$\eta_1 > \eta_4 > \eta_2 > \eta_3 > \eta_5$
4.0	1.0000	0.4871	0.3794	0.5299	0.0000	$\eta_1 > \eta_4 > \eta_2 > \eta_3 > \eta_5$

4.3. Comparative Analyses

In this section, we compare our proposed 2TLNNs TIDOM model with the 2TLNNWA and 2TLNNWG operators defined by Wang et al. [28]. Based on the values of Tables 1–3 and expert weighting vector $(0.45, 0.15, 0.40)^T$, we can utilize overall r_{ij}^{λ} to r_{ij} by 2TLNNWA operator.

Based on the values of Tables 5 and 6 and attributes weighting vector $w = (0.14, 0.33, 0.29, 0.24)^T$, we can utilize overall r_{ij} to r_i by 2TLNNWA and 2TLNNWG operators.

Table 5. Utilizing results r_{ij} with 2TLNNWA operator.

	G_1	G_2
η_1	$\{(s_5, -0.1892), (s_1, 0.1892), (s_1, 0.2746)\}$	$\{(s_4, 0.3182), (s_2, -0.0668), (s_2, -0.4308)\}$
η_2	$\{(s_5, -0.2746), (s_4, -0.3831), (s_4, -0.0455)\}$	$\{(s_3, 0.0000), (s_2, 0.3784), (s_2, -0.0681)\}$
η_3	$\{(s_3, 0.4425), (s_2, 0.2974), (s_3, 0.0414)\}$	$\{(s_4, -0.2974), (s_2, 0.2974), (s_4, 0.0760)\}$
η_4	$\{(s_3, 0.0794), (s_3, -0.2438), (s_3, 0.3178)\}$	$\{(s_3, 0.4509), (s_2, -0.0668), (s_2, 0.0000)\}$
η_5	$\{(s_1, 0.3756), (s_4, 0.0000), (s_5, 0.0000)\}$	$\{(s_2, 0.3831), (s_2, 0.2914), (s_3, 0.0582)\}$
	G_3	G_4
η_1	$\{(s_4, -0.3522), (s_2, -0.0221), (s_1, 0.0000)\}$	$\{(s_4, 0.2634), (s_2, -0.4308), (s_2, 0.0000)\}$
η_2	$\{(s_2, 0.0000), (s_1, 0.0000), (s_3, 0.3659)\}$	$\{(s_4, 0.4308), (s_3, 0.0925), (s_2, 0.3522)\}$
η_3	$\{(s_4, 0.2634), (s_2, -0.1545), (s_3, 0.1383)\}$	$\{(s_4, -0.2974), (s_1, 0.3195), (s_3, -0.2028)\}$
η_4	$\{(s_4, 0.0668), (s_3, 0.0000), (s_3, 0.0314)\}$	$\{(s_3, -0.4313), (s_3, -0.1716), (s_3, 0.3437)\}$
η_5	$\{(s_3, -0.3178), (s_4, -0.3831), (s_4, -0.2303)\}$	$\{(s_2, 0.0000), (s_2, 0.4623), (s_3, 0.3659)\}$

Table 6. Utilizing results r_i with 2TLNNWA and 2TLNNWG operators.

	2TLNNWA Operator	2TLNNWG Operator
η_1	$\{(s_4, 0.2205), (s_2, -0.2707), (s_1, 0.4176)\}$	$\{(s_4, 0.1619), (s_2, -0.2365), (s_1, 0.4819)\}$
η_2	$\{(s_4, -0.4760), (s_2, 0.0894), (s_3, -0.3699)\}$	$\{(s_3, 0.1212), (s_2, 0.4422), (s_3, -0.1731)\}$
η_3	$\{(s_4, -0.1504), (s_2, -0.1127), (s_3, 0.3133)\}$	$\{(s_4, -0.1820), (s_2, -0.0499), (s_3, 0.4089)\}$
η_4	$\{(s_3, 0.4248), (s_3, -0.4716), (s_3, -0.2600)\}$	$\{(s_3, 0.3183), (s_3, -0.3984), (s_3, -0.1444)\}$
η_5	$\{(s_2, 0.2598), (s_3, -0.1229), (s_4, -0.4380)\}$	$\{(s_2, 0.1896), (s_3, 0.0416), (s_4, -0.2731)\}$

Calculating the alternative scores $s(r_i)$ by score functions of 2TLNNs as listed in Table 7.

Table 7. Alternative scores $s(r_i)$ with 2TLNNWA and 2TLNNWG operators.

2TLNNWA Operator	2TLNNWG Operator
$s(\varphi_1) = 0.7263, s(\varphi_2) = 0.6002,$ $s(\varphi_3) = 0.5916, s(\varphi_4) = 0.5642,$ $s(\varphi_5) = 0.4345.$	$s(\varphi_1) = 0.7176, s(\varphi_2) = 0.5473,$ $s(\varphi_3) = 0.5811, s(\varphi_4) = 0.5478,$ $s(\varphi_5) = 0.4123.$

Then we can obtain the ranking of alternatives with 2TLNNWA and 2TLNNWG operators. The calculating result is listed in Table 8.

Table 8. Ranking of alternatives with 2TLNNWA and 2TLNNWG operators.

	Order
2TLNNWA	$\eta_1 > \eta_2 > \eta_3 > \eta_4 > \eta_5$
2TLNNWG	$\eta_1 > \eta_3 > \eta_4 > \eta_2 > \eta_5$
2TLNNs TODIM	$\eta_1 > \eta_4 > \eta_2 > \eta_3 > \eta_5$

Comparing the results between our proposed 2TLNNs TODIM method and 2TLNNWA and 2TLNNWG operators, they have the same best choice η_1 and differ slightly in the ranking of alternatives. However, the 2TLNNs TODIM method considers the subjectivity of DM's behaviors and provides the dominance of each alternative over others with particular operation formulas, and can be more reasonable and scientific in the application of MAGDM problems.

5. Conclusions

In our article, we proposed the 2TLNNs TODIM method based on the fundamental theories of 2TLNNs and the original TODIM model. Firstly, we briefly introduced the definition, operation laws, aggregation operators and the distance calculating method of 2TLNNs. Then, the calculation steps of the original TODIM model were presented in simplified form. Thereafter, we extended the original TODIM model to the 2TLNNs environment to build the 2TLNNs TIDOM model, our proposed method which is more reasonable and scientific in considering the subjectivity of DM's behaviors and the dominance of each alternative over others. Finally, a numerical example for the safety assessment of construction projects was proposed to illustrate the new method and some comparisons were also conducted to further illustrate the advantages of the new method. In the future, the application of the proposed models and methods of 2TLNNs can be investigated in MAGDM problems [42–53], risk analysis and many other uncertain and fuzzy environments [54–65].

Author Contributions: J.W., G.W.W. and M.L. conceived and worked together to achieve this work, J.W. compiled the computing program by Matlab and analyzed the data, J.W. and G.W.W. wrote the paper. Finally, all the authors have read and approved the final manuscript.

Funding: The work was supported by the National Natural Science Foundation of China under Grant No. 71571128 and the Humanities and Social Sciences Foundation of Ministry of Education of the People's Republic of China (16XJA630005) and the Construction Plan of Scientific Research Innovation Team for Colleges and Universities in Sichuan Province (15TD0004).

Conflicts of Interest: The authors declare no conflict of interest.

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An Extended VIKOR Method for Multiple Criteria Group Decision Making with Triangular Fuzzy Neutrosophic Numbers

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Received: 1 August 2018; Accepted: 8 October 2018; Published: 15 October 2018

Abstract: In this article, we combine the original VIKOR model with a triangular fuzzy neutrosophic set to propose the triangular fuzzy neutrosophic VIKOR method. In the extended method, we use the triangular fuzzy neutrosophic numbers (TFNNs) to present the criteria values in multiple criteria group decision making (MCGDM) problems. Firstly, we summarily introduce the fundamental concepts, operation formulas and distance calculating method of TFNNs. Then we review some aggregation operators of TFNNs. Thereafter, we extend the original VIKOR model to the triangular fuzzy neutrosophic environment and introduce the calculating steps of the TFNNs VIKOR method, our proposed method which is more reasonable and scientific for considering the conflicting criteria. Furthermore, a numerical example for potential evaluation of emerging technology commercialization is presented to illustrate the new method, and some comparisons are also conducted to further illustrate advantages of the new method.

Keywords: MCGDM problems; triangular fuzzy neutrosophic sets (TFNSs); VIKOR model; TFNNs VIKOR method; potential evaluation; emerging technology commercialization

1. Introduction

The VIKOR (VIseKriterijumska Optimizacija I KOmpromisno Resenje) method [1] has been used to investigate multiple criteria group decision making (MCGDM) problems and has been widely used in many domains. In the existing literature, more and more traditional MCGDM models have been studied, such as: the grey relational analysis model [2–4]; the multi-objective optimization by ratio analysis plus the full multiplicative form (MULTIMOORA) model [5,6]; the Preference Ranking Organization Method for Enrichment of Evaluations (PROMETHEE) model [7]; the ELimination Et Choix Traduisant la REalité (ELECTRE) model [8]; and the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) model [9,10].

In many real MCGDM problems, it is not easy to describe the criteria values with accurate values due to the fuzziness and complexity of the alternatives, and so it can be more effective and useful to describe the criteria values with fuzzy information. Fuzzy set theory [11] has been used as a feasible tool for MCGDM [12,13] problems. Smarandache [14,15] proposed the neutrosophic set (NS). Then, Wang et al. [16,17] defined the single-valued neutrosophic sets (SVNSs) and interval neutrosophic sets (INSs). Wang et al. [18,19] explored some aggregation operators of SVNNs and extended the SVNS to a 2-tuple linguistic neutrosophic number environment. Wu et al. [20] studied SVNNs with Hamy operators under 2-tuple linguistic neutrosophic numbers. Biswas et al. [21] provided the definition of a triangular fuzzy neutrosophic number (TFNN) in which the degree of truth-membership (MD), indeterminacy-membership (IMD) and falsity-membership (FMD) are depicted by TFNNs. Sahin et al. [22] studied multiple attribute decision making (MADM) problems with centroid single valued triangular neutrosophic numbers. Samah et al. [23] studied two ranking means based on information systems quality (ISQ) theory and the TFNNs environment. Ye [24] provided the definition of trapezoidal neutrosophic sets. Biswas et al. [25] studied some applications under the trapezoidal fuzzy neutrosophic environment. Tan and Zhang [26] defined some trapezoidal fuzzy neutrosophic aggregation operators.

Opricovic [1] used the VIKOR model to investigate some MCGDM problems with conflicting criteria [27,28]. Bausys and Zavadskas [29] established the INS VIKOR model. Liu and Park et al. [30] studied the VIKOR model under interval-valued intuitionistic fuzzy sets (IVIFSs). Selvakumari et al. [31] proposed the extended VIKOR model by constructing an octagonal neutrosophic soft matrix. Wan et al. [32] proposed the VIKOR model with triangular intuitionistic fuzzy numbers (TIFN), Liu et al. [33] provided the linguistic VIKOR model, and Qin et al. [34] developed the interval type-2 fuzzy VIKOR model. Chen [35] proposed the remoteness index-based Pythagorean fuzzy VIKOR methods with a generalized distance measure for multiple criteria decision analysis. Liao et al. [36] explored the VIKOR method with the hesitant fuzzy linguistic information. Ren et al. [37] provided the dual hesitant fuzzy VIKOR model. Li et al. [38] provided the VIKOR model with linguistic intuitionistic fuzzy numbers. Pouresmaeil et al. [39] established the SVNNs VIKOR model. Huang et al. [40] extended the VIKOR method to INSs. Zhang and Wei [41] extended the VIKOR method to a hesitant fuzzy environment.

However, there has been no study about the VIKOR model for MCGDM problems with TFNNs, so taking the TFNNs VIKOR model into account is of necessity. The goal of our article is to combine the original VIKOR model with TFNNs to study MCGDM problems. The structure of our paper is as follows. Section 1 introduces the concepts, operation formulas and the distance calculating method of TFNNs. Section 2 reviews some aggregation operators of TFNNs. Section 3 extends the original VIKOR model to a TFN environment and introduces the required calculating steps of TFNNs VIKOR method. Section 4 provides a numerical example for potential evaluation of emerging technology commercialization and introduces a comparison between our proposed methods and the existing method. Section 5 summarises our conclusions.

2. Preliminaries

2.1. Triangular Fuzzy Neutrosophic Sets

Based on the concepts of a traditional triangular fuzzy set and the fundamental theory of a single valued neutrosophic set (SVNS), the triangular fuzzy neutrosophic sets (TFNSs), which were first defined by Biswas et al., [21] can be depicted as follows:

Definition 1 [21]**.** *Let X be a fixed set. The TFNSs η can be depicted as:*

$$\eta = \left\{ \left(x, \phi_\eta(x), \varphi_\eta(x), \gamma_\eta(x)\right) \middle| x \in X \right\} \tag{1}$$

where $\phi_\eta(x), \varphi_\eta(x)$ and $\gamma_\eta(x) \in [0,1]$ represent the degree of the truth membership, the indeterminacy membership and the falsity membership, respectively, which can be expressed by triangular fuzzy numbers as follows.

$$\phi_\eta(x) = \left(\phi_\eta^L(x), \phi_\eta^M(x), \phi_\eta^U(x)\right), 0 \le \phi_\eta^L(x) \le \phi_\eta^M(x) \le \phi_\eta^U(x) \le 1 \tag{2}$$

$$\varphi_\eta(x) = \left(\varphi_\eta^L(x), \varphi_\eta^M(x), \varphi_\eta^U(x)\right), 0 \le \varphi_\eta^L(x) \le \varphi_\eta^M(x) \le \varphi_\eta^U(x) \le 1 \tag{3}$$

$$\gamma_\eta(x) = \left(\gamma_\eta^L(x), \gamma_\eta^M(x), \gamma_\eta^U(x)\right), 0 \le \gamma_\eta^L(x) \le \gamma_\eta^M(x) \le \gamma_\eta^U(x) \le 1 \tag{4}$$

and the truth membership function can be defined:

$$\phi_\eta(x) = \begin{cases} \frac{x-\phi_\eta^L(x)}{\phi_\eta^M(x)-\phi_\eta^L(x)}, & \phi_\eta^L(x) \le x \le \phi_\eta^M(x), \\ \frac{x-\phi_\eta^U(x)}{\phi_\eta^M(x)-\phi_\eta^U(x)}, & \phi_\eta^M(x) \le x \le \phi_\eta^U(x), \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

For convenience, we let $\eta = \{(\phi^L, \phi^M, \phi^U), (\varphi^L, \varphi^M, \varphi^U), (\gamma^L, \gamma^M, \gamma^U)\}$ *be a TFNN which satisfies the condition* $0 \le \phi^U + \varphi^U + \gamma^U \le 3$.

Definition 2 [21]. *Assume there are three TFNNs* $\eta_1 = \{(\phi_1^L, \phi_1^M, \phi_1^U), (\varphi_1^L, \varphi_1^M, \varphi_1^U), (\gamma_1^L, \gamma_1^M, \gamma_1^U)\}$, $\eta_2 = \{(\phi_2^L, \phi_2^M, \phi_2^U), (\varphi_2^L, \varphi_2^M, \varphi_2^U), (\gamma_2^L, \gamma_2^M, \gamma_2^U)\}$ *and* $\eta = \{(\phi^L, \phi^M, \phi^U), (\varphi^L, \varphi^M, \varphi^U), (\gamma^L, \gamma^M, \gamma^U)\}$, *the operation laws of them can be defined:*

$$(1)\ \eta_1 \oplus \eta_2 = \left\{ \begin{array}{l} (\phi_1^L + \phi_2^L - \phi_1^L\phi_2^L, \phi_1^M + \phi_2^M - \phi_1^M\phi_2^M, \phi_1^U + \phi_2^U - \phi_1^U\phi_2^U), (\varphi_1^L\varphi_2^L, \varphi_1^M\varphi_2^M, \varphi_1^U\varphi_2^U), \\ (\gamma_1^L\gamma_2^L, \gamma_1^M\gamma_2^M, \gamma_1^U\gamma_2^U) \end{array} \right\};$$

$$(2)\ \eta_1 \otimes \eta_2 = \left\{ \begin{array}{l} (\phi_1^L\phi_2^L, \phi_1^M\phi_2^M, \phi_1^U\phi_2^U), (\varphi_1^L + \varphi_2^L - \varphi_1^L\varphi_2^L, \varphi_1^M + \varphi_2^M - \varphi_1^M\varphi_2^M, \varphi_1^U + \varphi_2^U - \varphi_1^U\varphi_2^U), \\ (\gamma_1^L + \gamma_2^L - \gamma_1^L\gamma_2^L, \gamma_1^M + \gamma_2^M - \gamma_1^M\gamma_2^M, \gamma_1^U + \gamma_2^U - \gamma_1^U\gamma_2^U) \end{array} \right\};$$

$$(3)\ \lambda\eta = \left\{ \begin{array}{l} \left(1-(1-\phi^L)^\lambda, 1-(1-\phi^M)^\lambda, 1-(1-\phi^U)^\lambda\right), \left((\varphi^L)^\lambda, (\varphi^M)^\lambda, (\varphi^U)^\lambda\right), \\ \left((\gamma^L)^\lambda, (\gamma^M)^\lambda, (\gamma^U)^\lambda\right) \end{array} \right\}, \lambda > 0;$$

$$(4)\ \eta^\lambda = \left\{ \begin{array}{l} \left((\phi^L)^\lambda, (\phi^M)^\lambda, (\phi^U)^\lambda\right), \left(1-(1-\varphi^L)^\lambda, 1-(1-\varphi^M)^\lambda, 1-(1-\varphi^U)^\lambda\right), \\ \left(1-(1-\gamma^L)^\lambda, 1-(1-\gamma^M)^\lambda, 1-(1-\gamma^U)^\lambda\right) \end{array} \right\}, \lambda > 0.$$

According to Definition 2, it is clear that the operation laws have the following properties:

$$\eta_1 \oplus \eta_2 = \eta_2 \oplus \eta_1,\ \ \eta_1 \otimes \eta_2 = \eta_2 \otimes \eta_1,\ \ \left((\eta_1)^{\lambda_1}\right)^{\lambda_2} = (\eta_1)^{\lambda_1\lambda_2}; \tag{6}$$

$$\lambda(\eta_1 \oplus \eta_2) = \lambda\eta_1 \oplus \lambda\eta_2,\ \ (\eta_1 \otimes \eta_2)^\lambda = (\eta_1)^\lambda \otimes (\eta_2)^\lambda; \tag{7}$$

$$\lambda_1\eta_1 \oplus \lambda_2\eta_1 = (\lambda_1 + \lambda_2)\eta_1,\ \ (\eta_1)^{\lambda_1} \otimes (\eta_1)^{\lambda_2} = (\eta_1)^{(\lambda_1+\lambda_2)}. \tag{8}$$

Definition 3 [21]. *Let* $\eta = \{(\phi^L, \phi^M, \phi^U), (\varphi^L, \varphi^M, \varphi^U), (\gamma^L, \gamma^M, \gamma^U)\}$ *be a TFNN, the score and accuracy functions of* η *can be expressed:*

$$s(\eta) = \frac{1}{12}\left[\begin{array}{l} 8 + (\phi^L + 2\phi^M + \phi^U) - (\varphi^L + 2\varphi^M + \varphi^U) \\ -(\gamma^L + 2\gamma^M + \gamma^U) \end{array} \right],\ s(\eta) \in [0,1] \tag{9}$$

$$h(\eta) = \frac{1}{4}\left[\left(\phi^L + 2\phi^M + \phi^U\right) - \left(\gamma^L + 2\gamma^M + \gamma^U\right)\right],\ h(\eta) \in [-1,1] \tag{10}$$

Let η_1 and η_2 be two TFNNs. Then, based on Definition 3, the following assertion holds true.

(1) *if* $s(\eta_1) < s(\eta_2)$, *then* $\eta_1 < \eta_2$;
(2) *if* $s(\eta_1) > s(\eta_2)$, *then* $\eta_1 > \eta_2$;
(3) *if* $s(\eta_1) = s(\eta_2), h(\eta_1) < h(\eta_2)$, *then* $\eta_1 < \eta_2$;
(4) *if* $s(\eta_1) = s(\eta_2)$, $h(\eta_1) > h(\eta_2)$, *then* $\eta_1 > \eta_2$;
(5) *if* $s(\eta_1) = s(\eta_2), h(\eta_1) = h(\eta_2)$, *then* $\eta_1 = \eta_2$.

2.2. The Normalized Hamming Distance between TFNNs

Definition 4 [32]. *Let* $\eta_1 = \{(\phi_1^L, \phi_1^M, \phi_1^U), (\varphi_1^L, \varphi_1^M, \varphi_1^U), (\gamma_1^L, \gamma_1^M, \gamma_1^U)\}$ *and* $\eta_2 = \{(\phi_2^L, \phi_2^M, \phi_2^U), (\varphi_2^L, \varphi_2^M, \varphi_2^U), (\gamma_2^L, \gamma_2^M, \gamma_2^U)\}$ *be two TFNNs. Then the normalized Hamming distance is defined by:*

$$d(\eta_1, \eta_2) = \frac{1}{9}\begin{pmatrix} |\phi_1^L - \phi_2^L| + |\phi_1^M - \phi_2^M| + |\phi_1^U - \phi_2^U| \\ +|\varphi_1^L - \varphi_2^L| + |\varphi_1^M - \varphi_2^M| + |\varphi_1^U - \varphi_2^U| \\ +|\gamma_1^L - \gamma_2^L| + |\gamma_1^M - \gamma_2^M| + |\gamma_1^U - \gamma_2^U| \end{pmatrix} \tag{11}$$

Theorem 1. *Assume that there are three TFNNs* $\eta_1 = \{(\phi_1^L, \phi_1^M, \phi_1^U), (\varphi_1^L, \varphi_1^M, \varphi_1^U), (\gamma_1^L, \gamma_1^M, \gamma_1^U)\}$, $\eta_2 = \{(\phi_2^L, \phi_2^M, \phi_2^U), (\varphi_2^L, \varphi_2^M, \varphi_2^U), (\gamma_2^L, \gamma_2^M, \gamma_2^U)\}$ *and* $\eta = \{(\phi^L, \phi^M, \phi^U), (\varphi^L, \varphi^M, \varphi^U), (\gamma^L, \gamma^M, \gamma^U)\}$, *the Hamming distance* $d(\eta_1, \eta_2)$ *has the following properties:*

(P1) $0 \le d(\eta_1, \eta_2) \le 1$; (P2) *if* $d(\eta_1, \eta_2) = 0$, *then* $\eta_1 = \eta_2$;
(P3) $d(\eta_1, \eta_2) = d(\eta_2, \eta_1)$; (P4) $d(\eta_1, \eta_2) + d(\eta_2, \eta_3) \ge d(\eta_1, \eta_3)$.

Proof. (P1) $0 \le d(\eta_1, \eta_2) \le 1$

Since $0 \le \phi^L \le 1$, then $0 \le |\phi_1^L - \phi_2^L| \le 1$, similarly we see $0 \le |\phi_1^M - \phi_2^M| \le 1, 0 \le |\phi_1^U - \phi_2^U| \le 1$, $0 \le |\varphi_1^L - \varphi_2^L| \le 1, 0 \le |\varphi_1^M - \varphi_2^M| \le 1, 0 \le |\varphi_1^U - \varphi_2^U| \le 1, 0 \le |\gamma_1^L - \gamma_2^L| \le 1, 0 \le |\gamma_1^M - \gamma_2^M| \le 1, 0 \le |\gamma_1^U - \gamma_2^U| \le 1$. So $0 \le |\phi_1^L - \phi_2^L| + |\phi_1^M - \phi_2^M| + |\phi_1^U - \phi_2^U| + |\varphi_1^L - \varphi_2^L| + |\varphi_1^M - \varphi_2^M| + |\varphi_1^U - \varphi_2^U| + |\gamma_1^L - \gamma_2^L| + |\gamma_2^M - \gamma_1^M| + |\gamma_1^U - \gamma_2^U| \le 9$.

Therefore $0 \le d(\eta_1, \eta_2) \le 1$, which completes the proof.

(P2) *if* $d(\eta_1, \eta_2) = 0$, *then* $\eta_1 = \eta_2$

$$d(\eta_1, \eta_2) = \tfrac{1}{9}\begin{pmatrix} |\phi_1^L - \phi_2^L| + |\phi_1^M - \phi_2^M| + |\phi_1^U - \phi_2^U| + |\varphi_1^L - \varphi_2^L| + |\varphi_1^M - \varphi_2^M| + |\varphi_1^U - \varphi_2^U| + \\ |\gamma_1^L - \gamma_2^L| + |\gamma_1^M - \gamma_2^M| + |\gamma_1^U - \gamma_2^U| \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} |\phi_1^L - \phi_2^L| = 0, |\phi_1^M - \phi_2^M| = 0, |\phi_1^U - \phi_2^U| = 0, |\varphi_1^L - \varphi_2^L| = 0, |\varphi_1^M - \varphi_2^M| = 0, |\varphi_1^U - \varphi_2^U| = 0, \\ |\gamma_1^L - \gamma_2^L| = 0, |\gamma_1^M - \gamma_2^M| = 0, |\gamma_1^U - \gamma_2^U| = 0 \end{pmatrix}$$

$$\Rightarrow (\phi_1^L = \phi_2^L, \phi_1^M = \phi_2^M, \phi_1^U = \phi_2^U, \varphi_1^L = \varphi_2^L, \varphi_1^M = \varphi_2^M, \varphi_1^U = \varphi_2^U, \gamma_1^L = \gamma_2^L, \gamma_1^M = \gamma_2^M, \gamma_1^U = \gamma_2^U)$$

That means $\eta_1 = \eta_2$, and so (P2) *if* $d(\eta_1, \eta_2) = 0$, *then* $\eta_1 = \eta_2$ is correct.

(P3) $d(\eta_1, \eta_2) = d(\eta_2, \eta_1)$

$$d(\eta_1, \eta_2) = \tfrac{1}{9}\begin{pmatrix} |\phi_1^L - \phi_2^L| + |\phi_1^M - \phi_2^M| + |\phi_1^U - \phi_2^U| + |\varphi_1^L - \varphi_2^L| + |\varphi_1^M - \varphi_2^M| + |\varphi_1^U - \varphi_2^U| \\ +|\gamma_1^L - \gamma_2^L| + |\gamma_1^M - \gamma_2^M| + |\gamma_1^U - \gamma_2^U| \end{pmatrix}$$
$$= \tfrac{1}{9}\begin{pmatrix} |\phi_2^L - \phi_1^L| + |\phi_2^M - \phi_1^M| + |\phi_2^U - \phi_1^U| + |\varphi_2^L - \varphi_1^L| + |\varphi_2^M - \varphi_1^M| + |\varphi_2^U - \varphi_1^U| \\ +|\gamma_2^L - \gamma_1^L| + |\gamma_2^M - \gamma_1^M| + |\gamma_2^U - \gamma_1^U| \end{pmatrix} = d(\eta_2, \eta_1)$$

So we complete the proof of (P3), which asserts that equality $d(\eta_1, \eta_2) = d(\eta_2, \eta_1)$ holds.

(P4) $d(\eta_1, \eta_2) + d(\eta_2, \eta_3) \ge d(\eta_1, \eta_3)$

$$d(\eta_1, \eta_3) = \tfrac{1}{9}\begin{pmatrix} |\phi_1^L - \phi_3^L| + |\phi_1^M - \phi_3^M| + |\phi_1^U - \phi_3^U| + |\varphi_1^L - \varphi_3^L| + |\varphi_1^M - \varphi_3^M| \\ +|\varphi_1^U - \varphi_3^U| + |\gamma_1^L - \gamma_3^L| + |\gamma_1^M - \gamma_3^M| + |\gamma_1^U - \gamma_3^U| \end{pmatrix}$$
$$= \tfrac{1}{9}\begin{pmatrix} |\phi_1^L - \phi_2^L + \phi_2^L - \phi_3^L| + |\phi_1^M - \phi_2^M + \phi_2^M - \phi_3^M| + |\phi_1^U - \phi_2^U + \phi_2^U - \phi_3^U| \\ +|\varphi_1^L - \varphi_2^L + \varphi_2^L - \varphi_3^L| + |\varphi_1^M - \varphi_2^M + \varphi_2^M - \varphi_3^M| + |\varphi_1^U - \varphi_2^U + \varphi_2^U - \varphi_3^U| \\ +|\gamma_1^L - \gamma_2^L + \gamma_2^L - \gamma_3^L| + |\gamma_1^M - \gamma_2^M + \gamma_2^M - \gamma_3^M| + |\gamma_1^U - \gamma_2^U + \gamma_2^U - \gamma_3^U| \end{pmatrix}$$
$$\le \tfrac{1}{9}\begin{pmatrix} |\phi_1^L - \phi_2^L| + |\phi_2^L - \phi_3^L| + |\phi_1^M - \phi_2^M| + |\phi_2^M - \phi_3^M| + |\phi_1^U - \phi_2^U| + |\phi_2^U - \phi_3^U| + \\ |\varphi_1^L - \varphi_2^L| + |\varphi_2^L - \varphi_3^L| + |\varphi_1^M - \varphi_2^M| + |\varphi_2^M - \varphi_3^M| + |\varphi_1^U - \varphi_2^U| + |\varphi_2^U - \varphi_3^U| \\ +|\gamma_1^L - \gamma_2^L| + |\gamma_2^L - \gamma_3^L| + |\gamma_1^M - \gamma_2^M| + |\gamma_2^M - \gamma_3^M| + |\gamma_1^U - \gamma_2^U| + |\gamma_2^U - \varphi_3^U| \end{pmatrix}$$
$$= d(\eta_1, \eta_2) + d(\eta_2, \eta_3)$$

Then, triangular fuzzy neutrosophic number weighted averaging (TFNNWA) and triangular fuzzy neutrosophic number weighted geometric (TFNNWG) operators are introduced as follows:

Definition 5 [21]. *Let* $\eta_j = \left\{\left(\phi_j^L, \phi_j^M, \phi_j^U\right), \left(\varphi_j^L, \varphi_j^M, \varphi_j^U\right), \left(\gamma_j^L, \gamma_j^M, \gamma_j^U\right)\right\}(j = 1, 2, \ldots, n)$ *be a group of TFNNs, then the TFNNWA and TFNNWG operators proposed by Biswas et al.* [21] *are defined as follows.*

$$\text{TFNNWA}(\eta_1, \eta_2, \ldots, \eta_n) = \omega_1\eta_1 \oplus \omega_2\eta_2 \ldots \oplus \omega_n\eta_n = \overset{n}{\underset{j=1}{\oplus}} \omega_j\eta_j \tag{12}$$

and

$$\text{TFNNWG}(\eta_1, \eta_2, \ldots, \eta_n) = (\eta_1)^{\omega_1} \otimes (\eta_2)^{\omega_2} \ldots \otimes (\eta_n)^{\omega_n} = \overset{n}{\underset{j=1}{\otimes}} (\eta_j)^{\omega_j} \tag{13}$$

where ω_j *is weight vector of* $\eta_j, j = 1, 2, \ldots, n$*, which satisfies* $0 \le \omega_j \le 1, \sum_{j=1}^{n} \omega_j = 1$.

Theorem 2 [21]. *Let* $\eta_j = \left\{\left(\phi_j^L, \phi_j^M, \phi_j^U\right), \left(\varphi_j^L, \varphi_j^M, \varphi_j^U\right), \left(\gamma_j^L, \gamma_j^M, \gamma_j^U\right)\right\}(j = 1, 2, \ldots, n)$ *be a group of TFNNs, then the operation results by TFNNWA and TFNNWG operators are also a TFNN where*

$$\begin{gathered} \text{TFNNWA}(\eta_1, \eta_2, \ldots, \eta_n) = \overset{n}{\underset{j=1}{\oplus}} \omega_j\eta_j \\ = \left\{\begin{array}{l} \left(1 - \prod_{j=1}^{n}\left(1 - \phi_j^L\right)^{\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - \phi_j^M\right)^{\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - \phi_j^U\right)^{\omega_j}\right), \\ \left(\prod_{j=1}^{n}\left(\varphi_j^L\right)^{\omega_j}, \prod_{j=1}^{n}\left(\varphi_j^M\right)^{\omega_j}, \prod_{j=1}^{n}\left(\varphi_j^U\right)^{\omega_j}\right), \left(\prod_{j=1}^{n}\left(\gamma_j^L\right)^{\omega_j}, \prod_{j=1}^{n}\left(\gamma_j^M\right)^{\omega_j}, \prod_{j=1}^{n}\left(\gamma_j^U\right)^{\omega_j}\right) \end{array}\right\} \end{gathered} \tag{14}$$

and

$$\begin{gathered} \text{TFNNWG}(\eta_1, \eta_2, \ldots, \eta_n) = \overset{n}{\underset{j=1}{\otimes}} (\eta_j)^{\omega_j} \\ = \left\{\begin{array}{l} \left(1 - \prod_{j=1}^{n}\left(1 - \varphi_j^L\right)^{\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - \varphi_j^M\right)^{\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - \varphi_j^U\right)^{\omega_j}\right), \\ \left(1 - \prod_{j=1}^{n}\left(1 - \gamma_j^L\right)^{\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - \gamma_j^M\right)^{\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - \gamma_j^U\right)^{\omega_j}\right), \\ \left(\prod_{j=1}^{n}\left(\phi_j^L\right)^{\omega_j}, \prod_{j=1}^{n}\left(\phi_j^M\right)^{\omega_j}, \prod_{j=1}^{n}\left(\phi_j^U\right)^{\omega_j}\right) \end{array}\right\} \end{gathered} \tag{15}$$

2.3. VIKOR Method

Denote n alternatives under consideration as $O_1, O_2, \cdots, O_n$, the evaluation attribute as $C_1, C_2, \cdots, C_n$, and the rating of each alternative $O_j(j = 1, \cdots, n)$ with respect to attribute $C_j(j = 1, \cdots, m)$ as f_{ij}. Then the compromise ranking algorithm of the VIKOR method [42–45] has the following steps:

Step 1. Determine the best rating f_i^+ and the worst rating f_i^- for all the attributes. For example, it the attribute i represents a benefit, then

$$f_i^+ = \min_j f_{ij},\ f_i^- = \min_j f_{ij} \tag{16}$$

Naturally, a candidate having scores $(f_1^+, f_2^+, \cdots, f_m^+)$ would be positive ideal whereas a candidate having scores $(f_1^-, f_2^-, \cdots, f_m^-)$ would be a negative ideal candidate. It is assumed that such a positive ideal candidate does not exist; otherwise, the decision would be trivial.

Step 2. Compute the values and S_j and $R_j (j = 1, \cdots, n)$ which represent the average and the worst group scores of the alternatives O_j, respectively, with the relations

$$S_j = \sum_{i=1}^{n} w_i \frac{(f_i^+ - f_{ij})}{(f_i^+ - f_i^-)}, S_j \in [0,1] \tag{17}$$

$$R_j = \max_i \left[w_i \frac{(f_i^+ - f_{ij})}{(f_i^+ - f_i^-)} \right], R_j = [0,1] \tag{18}$$

Here, w_j ($\sum_{i=1}^{m} w_i = 1, w_i \in [0,1], i = 1, 2, \cdots, m$) is the relative importance weights of the criteria set by the decision maker. The smaller values of S_j and R_j correspond to the better, average and worse group scores of alternatives O_j, respectively.

Step 3. Compute the Q_j values for $j = 1, 2, \cdots, m$ with the relation

$$Q_j = \frac{\alpha(S_j - S^+)}{(S^- - S^+)} + \frac{(1-\alpha)(R_j - R^+)}{(R^- - R^+)}, \tag{19}$$

where

$$S^+ = \min_j S_j, S^- = \max_j S_j \tag{20}$$

$$R^+ = \min_j R_j, R^- = \max_j R_j \tag{21}$$

and α is the weight of decision making strategy "the majority of attribute" (or "the maximum group utility"). The compromise can be selected with "voting by majority" ($\alpha > 0.5$), with "consensus" ($\alpha = 0.5$), with "veto" ($\alpha < 0.5$).

Step 4. Rank the alternatives by sorting each S, R and Q values in a decreasing order. The result is a set of three ranking lists denoted as $S_{[.]}$, $R_{[.]}$ and $Q_{[.]}$.

Step 5. Propose the alternative O_{j1} corresponding to $O_{[1]}$ (the smallest among Q_j values) as compromise solution if

C1. The alternative O_{j1} has an acceptable advantage; in other words, $Q_{[2]} - Q_{[1]} \geq DQ$ where $DQ = \frac{1}{(m-1)}$, and m is the number of alternatives.

C2. The alternative O_{j1} is stable within the decision making process; in other words, it is also the best ranked in $S_{[.]}$ or $R_{[.]}$.

If one of the above conditions is not satisfied, then a set of compromise solutions is proposed, which consists of:

- Alternatives O_{j1} and O_{j2} where $Q_{j2} = Q_{[2]}$ if only the condition is not satisfied, or
- Alternatives $O_{j1}, O_{j2}, \cdots, O_{jk}$ if the condition C_1 is not satisfied; and O_{jk} is determined by the relation $Q_k - Q_{[1]} < DQ$ for the maximum k where $Q_{jk} = Q_{[k]}$ (the positions of these alternatives are in closeness).

3. VIKOR Model for MCGDM Problems with TFNNs

Let $\{\varphi_1, \varphi_2, \ldots \varphi_m\}$ be a group of alternatives, $\{d_1, d_2, \ldots d_t\}$ be a list of experts with weighting vector being $\{v_1, v_2, \ldots v_t\}$, and $\{c_1, c_2, \ldots c_n\}$ be a list of criteria with weighting vector being $\{\omega_1, \omega_2, \ldots \omega_n\}$, which thereby satisfies $\omega_i \in [0,1], v_\lambda \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1, \sum_{\lambda=1}^{t} v_\lambda = 1$. Construct the evaluation matrixes $\eta^\lambda = \left[\eta_{ij}^\lambda\right]_{m \times n}$ where $\eta_{ij}^\lambda = \left\{ \left(\left(\phi_{ij}^L\right)^\lambda, \left(\phi_{ij}^M\right)^\lambda, \left(\phi_{ij}^U\right)^\lambda \right), \left(\left(\varphi_{ij}^L\right)^\lambda, \left(\varphi_{ij}^M\right)^\lambda, \left(\varphi_{ij}^U\right)^\lambda \right), \left(\left(\gamma_{ij}^L\right)^\lambda, \left(\gamma_{ij}^M\right)^\lambda, \left(\gamma_{ij}^U\right)^\lambda \right) \right\}$ means the estimate results of the alternative $\varphi_i (i = 1, 2, \ldots, m)$ based on the criterion $c_j (j = 1, 2, \ldots, n)$ by expert d_λ. Let $\left(\left(\phi_{ij}^L\right)^\lambda, \left(\phi_{ij}^M\right)^\lambda, \left(\phi_{ij}^U\right)^\lambda \right) \in [0,1]$ denote the degree of truth-membership (TMD),

$\left(\left(\varphi_{ij}^{L}\right)^{\lambda},\left(\varphi_{ij}^{M}\right)^{\lambda},\left(\varphi_{ij}^{U}\right)^{\lambda}\right) \in [0,1]$ denote the degree of indeterminacy-membership (IMD) and $\left(\left(\gamma_{ij}^{L}\right)^{\lambda},\left(\gamma_{ij}^{M}\right)^{\lambda},\left(\gamma_{ij}^{U}\right)^{\lambda}\right) \in [0,1]$ denote the degree of falsity-membership (FMD) $0 \leq \left(\phi_{ij}^{U}\right)^{\lambda} + \left(\varphi_{ij}^{U}\right)^{\lambda} + \left(\gamma_{ij}^{U}\right)^{\lambda} \leq 3$ $i = 1,2,\ldots,m, j = 1,2,\ldots,n, \lambda = 1,2,\cdots,t$.

Considering both the TFNNs theories and the traditional VIKOR model, we try to propose a TFNNs VIKOR model to study MCGDM problems effectively. The model can be depicted as follows:

Step 1. Construct the decision matrixes $\eta^{\lambda} = \left[\eta_{ij}^{\lambda}\right]_{m\times n}$, and utilize overall values of $\eta^{\lambda} = \left[\eta_{ij}^{\lambda}\right]_{m\times n}$ to $\eta = \left[\eta_{ij}\right]_{m\times n}$ by using equal (14) or (15);

Step 2. Compute the positive ideal solution (PIS) φ^{+} and the negative ideal solution (NIS) φ^{-};

$$\varphi^{+} = \left\{\left(\left(\phi_j^L\right)^+,\left(\phi_j^M\right)^+,\left(\phi_j^U\right)^+\right),\left(\left(\varphi_j^L\right)^+,\left(\varphi_j^M\right)^+,\left(\varphi_j^U\right)^+\right),\left(\left(\gamma_j^L\right)^+,\left(\gamma_j^M\right)^+,\left(\gamma_j^U\right)^+\right)\right\} \quad (22)$$

$$\varphi^{-} = \left\{\left(\left(\phi_j^L\right)^-,\left(\phi_j^M\right)^-,\left(\phi_j^U\right)^-\right),\left(\left(\varphi_j^L\right)^-,\left(\varphi_j^M\right)^-,\left(\varphi_j^U\right)^-\right),\left(\left(\gamma_j^L\right)^-,\left(\gamma_j^M\right)^-,\left(\gamma_j^U\right)^-\right)\right\} \quad (23)$$

For benefit attribute

$$\left\{\begin{array}{c}\left(\left(\phi_j^L\right)^+,\left(\phi_j^M\right)^+,\left(\phi_j^U\right)^+\right),\\ \left(\left(\varphi_j^L\right)^+,\left(\varphi_j^M\right)^+,\left(\varphi_j^U\right)^+\right),\\ \left(\left(\gamma_j^L\right)^+,\left(\gamma_j^M\right)^+,\left(\gamma_j^U\right)^+\right)\end{array}\right\} = \left\{\begin{array}{c}\left(\max\limits_i\left(\phi_{ij}^L\right),\max\limits_i\left(\phi_{ij}^M\right),\max\limits_i\left(\phi_{ij}^U\right)\right),\\ \left(\min\limits_i\left(\varphi_{ij}^L\right),\min\limits_i\left(\varphi_{ij}^M\right),\min\limits_i\left(\varphi_{ij}^U\right)\right),\\ \left(\min\limits_i\left(\gamma_{ij}^L\right),\min\limits_i\left(\gamma_{ij}^M\right),\min\limits_i\left(\gamma_{ij}^U\right)\right)\end{array}\right\} \quad (24)$$

$$\left\{\begin{array}{c}\left(\left(\phi_j^L\right)^-,\left(\phi_j^M\right)^-,\left(\phi_j^U\right)^-\right),\\ \left(\left(\varphi_j^L\right)^-,\left(\varphi_j^M\right)^-,\left(\varphi_j^U\right)^-\right),\\ \left(\left(\gamma_j^L\right)^-,\left(\gamma_j^M\right)^-,\left(\gamma_j^U\right)^-\right)\end{array}\right\} = \left\{\begin{array}{c}\left(\min\limits_i\left(\phi_{ij}^L\right),\min\limits_i\left(\phi_{ij}^M\right),\min\limits_i\left(\phi_{ij}^U\right)\right),\\ \left(\max\limits_i\left(\varphi_{ij}^L\right),\max\limits_i\left(\varphi_{ij}^M\right),\max\limits_i\left(\varphi_{ij}^U\right)\right),\\ \left(\max\limits_i\left(\gamma_{ij}^L\right),\max\limits_i\left(\gamma_{ij}^M\right),\max\limits_i\left(\gamma_{ij}^U\right)\right)\end{array}\right\} \quad (25)$$

For cost attribute

$$\left\{\begin{array}{c}\left(\left(\phi_j^L\right)^+,\left(\phi_j^M\right)^+,\left(\phi_j^U\right)^+\right),\\ \left(\left(\varphi_j^L\right)^+,\left(\varphi_j^M\right)^+,\left(\varphi_j^U\right)^+\right),\\ \left(\left(\gamma_j^L\right)^+,\left(\gamma_j^M\right)^+,\left(\gamma_j^U\right)^+\right)\end{array}\right\} = \left\{\begin{array}{c}\left(\min\limits_i\left(\phi_{ij}^L\right),\min\limits_i\left(\phi_{ij}^M\right),\min\limits_i\left(\phi_{ij}^U\right)\right),\\ \left(\max\limits_i\left(\varphi_{ij}^L\right),\max\limits_i\left(\varphi_{ij}^M\right),\max\limits_i\left(\varphi_{ij}^U\right)\right),\\ \left(\max\limits_i\left(\gamma_{ij}^L\right),\max\limits_i\left(\gamma_{ij}^M\right),\max\limits_i\left(\gamma_{ij}^U\right)\right)\end{array}\right\} \quad (26)$$

$$\left\{\begin{array}{c}\left(\left(\phi_j^L\right)^-,\left(\phi_j^M\right)^-,\left(\phi_j^U\right)^-\right),\\ \left(\left(\varphi_j^L\right)^-,\left(\varphi_j^M\right)^-,\left(\varphi_j^U\right)^-\right),\\ \left(\left(\gamma_j^L\right)^-,\left(\gamma_j^M\right)^-,\left(\gamma_j^U\right)^-\right)\end{array}\right\} = \left\{\begin{array}{c}\left(\max\limits_i\left(\phi_{ij}^L\right),\max\limits_i\left(\phi_{ij}^M\right),\max\limits_i\left(\phi_{ij}^U\right)\right),\\ \left(\min\limits_i\left(\varphi_{ij}^L\right),\min\limits_i\left(\varphi_{ij}^M\right),\min\limits_i\left(\varphi_{ij}^U\right)\right),\\ \left(\min\limits_i\left(\gamma_{ij}^L\right),\min\limits_i\left(\gamma_{ij}^M\right),\min\limits_i\left(\gamma_{ij}^U\right)\right)\end{array}\right\} \quad (27)$$

Step 3. Based on Equation (11) and the attribute weighting vector ω_j, we can calculate the values of χ_i and ψ_i which express the average and the worst group scores of φ_i.

$$\chi_i = \sum_{j=1}^{n} \omega_j \frac{d\left(\left\{\begin{array}{l}\left(\left(\phi_j^L\right)^+, \left(\phi_j^M\right)^+, \left(\phi_j^U\right)^+\right),\\ \left(\left(\varphi_j^L\right)^+, \left(\varphi_j^M\right)^+, \left(\varphi_j^U\right)^+\right),\\ \left(\left(\gamma_j^L\right)^+, \left(\gamma_j^M\right)^+, \left(\gamma_j^U\right)^+\right)\end{array}\right\}, \left\{\begin{array}{l}\left(\phi_{ij}^L, \phi_{ij}^M, \phi_{ij}^U\right),\\ \left(\varphi_{ij}^L, \varphi_{ij}^M, \varphi_{ij}^U\right),\\ \left(\gamma_{ij}^L, \gamma_{ij}^M, \gamma_{ij}^U\right)\end{array}\right\}\right)}{d\left(\left\{\begin{array}{l}\left(\left(\phi_j^L\right)^+, \left(\phi_j^M\right)^+, \left(\phi_j^U\right)^+\right),\\ \left(\left(\varphi_j^L\right)^+, \left(\varphi_j^M\right)^+, \left(\varphi_j^U\right)^+\right),\\ \left(\left(\gamma_j^L\right)^+, \left(\gamma_j^M\right)^+, \left(\gamma_j^U\right)^+\right)\end{array}\right\}, \left\{\begin{array}{l}\left(\left(\phi_j^L\right)^-, \left(\phi_j^M\right)^-, \left(\phi_j^U\right)^-\right),\\ \left(\left(\varphi_j^L\right)^-, \left(\varphi_j^M\right)^-, \left(\varphi_j^U\right)^-\right),\\ \left(\left(\gamma_j^L\right)^-, \left(\gamma_j^M\right)^-, \left(\gamma_j^U\right)^-\right)\end{array}\right\}\right)} \tag{28}$$

$$\psi_i = \max_j \left\{ \omega_j \frac{d\left(\left\{\begin{array}{l}\left(\left(\phi_j^L\right)^+, \left(\phi_j^M\right)^+, \left(\phi_j^U\right)^+\right),\\ \left(\left(\varphi_j^L\right)^+, \left(\varphi_j^M\right)^+, \left(\varphi_j^U\right)^+\right),\\ \left(\left(\gamma_j^L\right)^+, \left(\gamma_j^M\right)^+, \left(\gamma_j^U\right)^+\right)\end{array}\right\}, \left\{\begin{array}{l}\left(\phi_{ij}^L, \phi_{ij}^M, \phi_{ij}^U\right),\\ \left(\varphi_{ij}^L, \varphi_{ij}^M, \varphi_{ij}^U\right),\\ \left(\gamma_{ij}^L, \gamma_{ij}^M, \gamma_{ij}^U\right)\end{array}\right\}\right)}{d\left(\left\{\begin{array}{l}\left(\left(\phi_j^L\right)^+, \left(\phi_j^M\right)^+, \left(\phi_j^U\right)^+\right),\\ \left(\left(\varphi_j^L\right)^+, \left(\varphi_j^M\right)^+, \left(\varphi_j^U\right)^+\right),\\ \left(\left(\gamma_j^L\right)^+, \left(\gamma_j^M\right)^+, \left(\gamma_j^U\right)^+\right)\end{array}\right\}, \left\{\begin{array}{l}\left(\left(\phi_j^L\right)^-, \left(\phi_j^M\right)^-, \left(\phi_j^U\right)^-\right),\\ \left(\left(\varphi_j^L\right)^-, \left(\varphi_j^M\right)^-, \left(\varphi_j^U\right)^-\right),\\ \left(\left(\gamma_j^L\right)^-, \left(\gamma_j^M\right)^-, \left(\gamma_j^U\right)^-\right)\end{array}\right\}\right)} \right\} \tag{29}$$

where d is the normalized Hamming distance and $0 \leq \omega_j \leq 1$ means the weight of attributes which satisfies $\sum_{i=1}^{n} \omega_i = 1$.

Step 4. Compute the values of Ω_i based on the results of χ_i and ψ_i, the calculating formula is characterized as follows:

$$\Omega_i = \alpha \frac{(\chi_i - \chi^+)}{(\chi^- - \chi^+)} + (1 - \alpha) \frac{(\psi_i - \psi^+)}{(\psi^- - \psi^+)} \tag{30}$$

where

$$\chi^+ = \min_i \chi_i, \chi^- = \max_i \chi_i \tag{31}$$

$$\psi^+ = \min_i \psi_i, \psi^- = \max_i \psi_i \tag{32}$$

where α means the coefficient of decision making strategic. $\alpha > 0.5$ depicts "the maximum group utility", $\alpha = 0.5$ depicts equality and $\alpha < 0.5$ depicts the minimum regret.

Step 5. To choose the best alternative in accordance with the values of Ω_i, the alternative with minimum value is the best choice.

4. Numerical Example

4.1. Calculating Steps Based on MCGDM Problems

In this section we present a numerical example to show potential evaluation of emerging technology commercialization with TFNNs in order to illustrate the method proposed in this paper. There is a panel with five possible emerging technology enterprises $\varphi_i (i = 1, 2, 3, 4, 5)$ to select from. The experts select four criteria to evaluate the five possible emerging technology enterprises: ① c_1 stands for the technical advancement; ② c_2 stands for the potential market and market risk; ③ c_3 stands for the industrialization infrastructure, human resources and financial conditions; ④ c_4 stands for the employment creation and the development of science and technology. The five possible emerging technology enterprises $\varphi_i(i = 1, 2, 3, 4, 5)$ are to be evaluated using the TFNNs with the four criteria by three experts $d_\lambda(\lambda = 1, 2, 3)$ (criteria weight $\omega = (0.42, 0.13, 0.25, 0.30)$, experts weight $v = (0.35, 0.45, 0.20)$.), which are given in Tables 1–3.

Table 1. Triangular fuzzy neutrosophic numbers (TFNNs) evaluation matrix by d_1.

	c_1	c_2	c_3	c_4
φ_1	$\{(0.6,0.8,0.9),(0.2,0.3,0.5),(0.1,0.3,0.4)\}$	$\{(0.3,0.5,0.7),(0.4,0.5,0.6),(0.2,0.4,0.5)\}$	$\{(0.5,0.6,0.7),(0.1,0.2,0.3),(0.2,0.3,0.4)\}$	$\{(0.4,0.7,0.9),(0.5,0.6,0.8),(0.2,0.4,0.6)\}$
φ_2	$\{(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)\}$	$\{(0.2,0.4,0.6),(0.3,0.5,0.7),(0.2,0.3,0.5)\}$	$\{(0.5,0.6,0.9),(0.6,0.7,0.8),(0.5,0.6,0.7)\}$	$\{(0.4,0.5,0.7),(0.3,0.4,0.6),(0.5,0.6,0.8)\}$
φ_3	$\{(0.3,0.5,0.6),(0.2,0.4,0.5),(0.5,0.6,0.8)\}$	$\{(0.2,0.3,0.4),(0.4,0.5,0.6),(0.6,0.7,0.8)\}$	$\{(0.7,0.8,0.9),(0.3,0.5,0.6),(0.2,0.4,0.5)\}$	$\{(0.4,0.5,0.8),(0.5,0.7,0.9),(0.2,0.3,0.4)\}$
φ_4	$\{(0.2,0.5,0.7),(0.3,0.6,0.8),(0.1,0.2,0.3)\}$	$\{(0.5,0.7,0.8),(0.3,0.4,0.5),(0.2,0.5,0.7)\}$	$\{(0.4,0.6,0.8),(0.3,0.4,0.5),(0.2,0.3,0.6)\}$	$\{(0.5,0.7,0.9),(0.4,0.6,0.8),(0.2,0.3,0.5)\}$
φ_5	$\{(0.7,0.8,0.9),(0.2,0.3,0.4),(0.2,0.3,0.4)\}$	$\{(0.5,0.7,0.8),(0.4,0.5,0.8),(0.2,0.4,0.5)\}$	$\{(0.5,0.6,0.7),(0.2,0.4,0.5),(0.1,0.4,0.6)\}$	$\{(0.3,0.4,0.9),(0.2,0.4,0.5),(0.1,0.5,0.6)\}$

Table 2. TFNNs evaluation matrix by d_2.

	c_1	c_2	c_3	c_4
φ_1	$\{(0.5,0.7,0.8),(0.1,0.2,0.4),(0.1,0.2,0.3)\}$	$\{(0.2,0.4,0.6),(0.3,0.4,0.5),(0.1,0.3,0.4)\}$	$\{(0.4,0.5,0.6),(0.2,0.4,0.5),(0.3,0.5,0.7)\}$	$\{(0.3,0.5,0.8),(0.4,0.5,0.7),(0.3,0.4,0.5)\}$
φ_2	$\{(0.4,0.5,0.7),(0.3,0.6,0.8),(0.4,0.6,0.7)\}$	$\{(0.3,0.4,0.5),(0.2,0.4,0.6),(0.3,0.4,0.5)\}$	$\{(0.4,0.5,0.7),(0.5,0.6,0.9),(0.3,0.4,0.6)\}$	$\{(0.2,0.4,0.6),(0.2,0.3,0.4),(0.1,0.3,0.5)\}$
φ_3	$\{(0.4,0.7,0.9),(0.3,0.5,0.8),(0.6,0.8,0.9)\}$	$\{(0.1,0.3,0.5),(0.2,0.4,0.7),(0.5,0.8,0.9)\}$	$\{(0.2,0.4,0.5),(0.3,0.5,0.7),(0.2,0.4,0.6)\}$	$\{(0.4,0.5,0.7),(0.5,0.8,0.9),(0.2,0.3,0.6)\}$
φ_4	$\{(0.3,0.4,0.7),(0.3,0.7,0.9),(0.2,0.4,0.5)\}$	$\{(0.2,0.8,0.9),(0.4,0.5,0.6),(0.4,0.5,0.7)\}$	$\{(0.5,0.7,0.9),(0.3,0.4,0.5),(0.2,0.3,0.4)\}$	$\{(0.4,0.5,0.6),(0.2,0.3,0.4),(0.1,0.2,0.3)\}$
φ_5	$\{(0.5,0.6,0.7),(0.1,0.4,0.5),(0.2,0.3,0.6)\}$	$\{(0.6,0.7,0.9),(0.4,0.5,0.7),(0.3,0.4,0.5)\}$	$\{(0.5,0.6,0.8),(0.3,0.4,0.5),(0.4,0.5,0.6)\}$	$\{(0.3,0.4,0.7),(0.1,0.4,0.5),(0.1,0.3,0.6)\}$

Table 3. TFNNs evaluation matrix by d_3.

	c_1	c_2	c_3	c_4
φ_1	$\{(0.5,0.6,0.8),(0.1,0.2,0.6),(0.1,0.2,0.4)\}$	$\{(0.3,0.4,0.6),(0.4,0.5,0.7),(0.2,0.3,0.4)\}$	$\{(0.4,0.5,0.8),(0.3,0.4,0.5),(0.3,0.6,0.8)\}$	$\{(0.4,0.6,0.8),(0.4,0.6,0.7),(0.3,0.4,0.5)\}$
φ_2	$\{(0.4,0.5,0.6),(0.5,0.6,0.7),(0.5,0.6,0.7)\}$	$\{(0.3,0.4,0.6),(0.2,0.4,0.5),(0.3,0.4,0.5)\}$	$\{(0.3,0.5,0.7),(0.2,0.6,0.9),(0.3,0.4,0.7)\}$	$\{(0.3,0.4,0.6),(0.2,0.3,0.5),(0.2,0.3,0.5)\}$

Table 3. *Cont.*

	c_1	c_2	c_3	c_4
φ_3	$\{(0.5,0.7,0.8),(0.4,0.5,0.7),(0.7,0.8,0.9)\}$	$\{(0.2,0.3,0.5),(0.2,0.4,0.5),(0.5,0.7,0.9)\}$	$\{(0.3,0.4,0.5),(0.3,0.4,0.6),(0.2,0.4,0.5)\}$	$\{(0.3,0.5,0.7),(0.5,0.7,0.9),(0.2,0.3,0.4)\}$
φ_4	$\{(0.3,0.4,0.5),(0.3,0.8,0.9),(0.1,0.4,0.5)\}$	$\{(0.2,0.5,0.8),(0.4,0.5,0.9),(0.4,0.6,0.7)\}$	$\{(0.5,0.6,0.9),(0.3,0.4,0.6),(0.2,0.3,0.5)\}$	$\{(0.3,0.5,0.7),(0.2,0.3,0.5),(0.1,0.2,0.4)\}$
φ_5	$\{(0.5,0.6,0.8),(0.1,0.4,0.6),(0.2,0.3,0.5)\}$	$\{(0.6,0.7,0.8),(0.4,0.5,0.6),(0.3,0.4,0.5)\}$	$\{(0.5,0.6,0.7),(0.3,0.4,0.6),(0.4,0.5,0.7)\}$	$\{(0.3,0.4,0.5),(0.2,0.4,0.5),(0.1,0.3,0.4)\}$

Step 1. Utilize overall values of $\eta^{\lambda} = \left[\eta_{ij}^{\lambda}\right]_{m\times n}$ to $\eta = [\eta_{ij}]_{m\times n}$ using the TFNNWA operator; the aggregation results are listed in Table 4 as follows.

Table 4. The aggregation values by TFNNWA operator.

	c_1	c_2
φ_1	$\{(0.5376,0.7243,0.8431),(0.1275,0.2305,0.4690),(0.1000,0.2305,0.3514)\}$	$\{(0.2566,0.4371,0.6383),(0.3514,0.4522,0.5700),(0.1464,0.3318,0.4325)\}$
φ_2	$\{(0.4371,0.5376,0.6822),(0.3675,0.5629,0.7043),(0.3782,0.5206,0.6222)\}$	$\{(0.2665,0.4000,0.5577),(0.2305,0.4325,0.6106),(0.2603,0.3617,0.5000)\}$
φ_3	$\{(0.3894,0.6413,0.8134),(0.2757,0.4624,0.6608),(0.5805,0.7234,0.8637)\}$	$\{(0.1565,0.3000,0.4671),(0.2549,0.4325,0.6201),(0.5329,0.7434,0.8637)\}$
φ_4	$\{(0.2665,0.4371,0.6677),(0.3000,0.6812,0.8637),(0.1366,0.3138,0.4181)\}$	$\{(0.3213,0.7231,0.8536),(0.3617,0.4624,0.6105),(0.3138,0.5186,0.7000)\}$
φ_5	$\{(0.5819,0.6862,0.8117),(0.1275,0.3617,0.4796),(0.2000,0.3000,0.5020)\}$	$\{(0.5675,0.7000,0.8536),(0.4000,0.5000,0.7112),(0.2603,0.4000,0.5000)\}$
	c_3	c_4
φ_1	$\{(0.4371,0.5376,0.6851),(0.1702,0.3138,0.4181),(0.2603,0.4337,0.5911)\}$	$\{(0.3569,0.6001,0.8431),(0.4325,0.5527,0.7335),(0.2603,0.4000,0.5329)\}$
φ_2	$\{(0.4195,0.5376,0.7958),(0.4437,0.6333,0.8637),(0.3587,0.4610,0.6531)\}$	$\{(0.2957,0.4371,0.6383),(0.2305,0.3318,0.4820),(0.2018,0.3824,0.5894)\}$
φ_3	$\{(0.4474,0.5915,0.7153),(0.3000,0.4782,0.6431),(0.2000,0.4000,0.5428)\}$	$\{(0.3812,0.5000,0.7397),(0.5000,0.7434,0.9000),(0.2000,0.3000,0.4801)\}$
φ_4	$\{(0.4671,0.6486,0.8725),(0.3000,0.4000,0.5186),(0.2000,0.3000,0.4820)\}$	$\{(0.4195,0.5819,0.7675),(0.2549,0.3824,0.5331),(0.1275,0.2305,0.3800)\}$
φ_5	$\{(0.5000,0.6000,0.7500),(0.2603,0.4000,0.5186),(0.2462,0.4624,0.6188)\}$	$\{(0.3000,0.4000,0.7738),(0.1464,0.4000,0.5000),(0.1000,0.3587,0.5533)\}$

Step 2. Compute the values of φ^{+} (PIS) and φ^{-} (NIS), for all benefit attributes and based on the Formulas (24) and (25), we can obtain the (PIS) φ^{+} and (NIS) φ^{-} as follows.

$$\varphi^{+}=\left\{\begin{array}{c}\{(0.5819,0.7243,0.8431),(0.1275,0.2305,0.4690),(0.1000,0.2305,0.3514)\},\\ \{(0.5675,0.7231,0.8536),(0.2305,0.4325,0.5700),(0.1464,0.3318,0.4325)\},\\ \{(0.5000,0.6486,0.8725),(0.1702,0.3138,0.4181),(0.2000,0.3000,0.4820)\},\\ \{(0.4195,0.6001,0.8431),(0.1464,0.3318,0.4820),(0.1000,0.2305,0.3800)\}\end{array}\right\}$$

$$\varphi^{-}=\left\{\begin{array}{c}\{(0.2665,0.4371,0.6677),(0.3675,0.6812,0.8637),(0.5805,0.7234,0.8637)\},\\ \{(0.1565,0.3000,0.4671),(0.4000,0.5000,0.7112),(0.5329,0.7434,0.8637)\},\\ \{(0.4195,0.5376,0.6851),(0.4437,0.6333,0.8637),(0.3587,0.4624,0.6531)\},\\ \{(0.2957,0.4000,0.6383),(0.5000,0.7434,0.9000),(0.2603,0.4000,0.5894)\}\end{array}\right\}$$

Step 3. Based on Equation (11) and the attribute weighting vector ω_j, calculate the values of χ_i and ψ_i.

$$\chi_1 = 0.3101, \chi_2 = 0.6959, \chi_3 = 0.7621, \chi_4 = 0.3877, \chi_5 = 0.3039,$$

$$\psi_1 = 0.1738, \psi_2 = 0.2683, \psi_3 = 0.2963, \psi_4 = 0.2486, \psi_5 = 0.1038.$$

Step 4. Compute the values of Ω_i based on the results of χ_i and ψ_i; the calculating values are listed as follows. (Let $\alpha = 0.6$)

$$\Omega_1 = 0.1534, \Omega_2 = 0.8550, \Omega_3 = 1.0000, \Omega_4 = 0.4106, \Omega_5 = 0.0000.$$

Step 5. To choose the best alternative by rank the values of Ω_i, the ranking of Ω_i is $\Omega_5 > \Omega_1 > \Omega_4 > \Omega_2 > \Omega_3$, and the best alternative is φ_5.

4.2. Comparative Analyses

In this section, we compare our proposed extended TFNNs VIKOR model with the TFNNWA and TFNNWG operators defined by Biswas [21].

Based on the values of Table 4 and attributes weighting vector $\omega = (0.42, 0.13, 0.25, 0.30)^T$, we can utilize overall η_{ij} to η_i by TFNNWA and TFNNWG operators.

Calculate results η_i by TFNNWA operator:

$$\begin{array}{l}\eta_1 = \{(0.4720,0.6616,0.8270),(0.1835,0.3051,0.4956),(0.1413,0.2884,0.4196)\}\\ \eta_2 = \{(0.4071,0.5302,0.7247),(0.2852,0.4513,0.6269),(0.2672,0.4113,0.5744)\}\\ \eta_3 = \{(0.4064,0.5970,0.7779),(0.2930,0.4935,0.6851),(0.3026,0.4654,0.6354)\}\\ \eta_4 = \{(0.3941,0.6060,0.8109),(0.2595,0.4589,0.6196),(0.1344,0.2689,0.4126)\}\\ \eta_5 = \{(0.5302,0.6414,0.8251),(0.1500,0.3602,0.4843),(0.1508,0.3246,0.5081)\}\end{array}$$

Calculate results η_i by TFNNWG operator:

$$\begin{array}{l}\eta_1 = \{(0.3854,0.5761,0.7590),(0.2812,0.4078,0.5965),(0.2060,0.3674,0.5071)\}\\ \eta_2 = \{(0.3321,0.4569,0.6516),(0.3634,0.5475,0.7355),(0.2672,0.4113,0.5744)\}\\ \eta_3 = \{(0.3238,0.5054,0.6977),(0.3755,0.5954,0.7831),(0.4438,0.6137,0.7741)\}\\ \eta_4 = \{(0.3154,0.5166,0.7381),(0.3200,0.5653,0.7461),(0.1873,0.3436,0.4993)\}\\ \eta_5 = \{(0.4336,0.5449,0.7733),(0.2185,0.4286,0.5624),(0.2096,0.3963,0.5793)\}\end{array}$$

Calculating the alternative scores $s(\eta_i)$ by score functions of TFNNs as listed in Table 5.

Table 5. Alternative scores $s(\eta_i)$ by TFNNWA and TFNNWG operators.

TFNNWA Operator	TFNNWG Operator
$s(\eta_1) = 0.6277, s(\eta_2) = 0.5431,$ $s(\eta_3) = 0.5299, s(\eta_4) = 0.5961,$ $s(\eta_5) = 0.6078.$	$s(\eta_1) = 0.5692, s(\eta_2) = 0.4928,$ $s(\eta_3) = 0.4507, s(\eta_4) = 0.5444,$ $s(\eta_5) = 0.5546.$

The ranking of alternatives by TFNNWA and TFNNWG operators are listed in Table 6.

Table 6. Rank of alternatives by TFNNWA and TFNNWG operators.

	Order
TFNNWA	$\varphi_1 > \varphi_5 > \varphi_4 > \varphi_2 > \varphi_3$
TFNNWG	$\varphi_1 > \varphi_5 > \varphi_4 > \varphi_2 > \varphi_3$
TFNNs VIKOR	$\varphi_5 > \varphi_1 > \varphi_4 > \varphi_2 > \varphi_3$

Comparing the values of our proposed TFNNs VIKOR method with those of TFNNWA and TFNNWG operators, the results are slightly different in their ranking of the alternatives and the best alternatives are not same. The TFNNs VIKOR method can consider the conflicting attributes and can be more reasonable and scientific in the application of MCGDM problems.

5. Conclusions

In our article, we proposed the TFNNs VIKOR method based on the fundamental theories of TFNNs and the original VIKOR model. Firstly, we introduced the concepts, operation formulas and the distance calculating method of TFNNs. Then we reviewed some aggregation operators of TFNNs. Thereafter, the calculating steps of the VIKOR model for TFNNs MCGDM problems were simply presented using our proposed method, which is more scientific and reasonable for considering the conflicting attributes. Furthermore, a numerical example for potential evaluation of emerging technology commercialization has been proposed to illustrate the new method and some comparisons were also conducted to further illustrate the advantages of the new method.

In the future, our proposed TFNN VIKOR model can be applied to risk analysis, MCGDM problems [46–57] and many other uncertain and fuzzy environments [58–74].

Author Contributions: J.W., G.W. and M.L. conceived and worked together to achieve this work, J.W. compiled the computing program by Matlab and analyzed the data, J.W. and G.W. wrote the paper. Finally, all the authors have read and approved the final manuscript.

Funding: The work was supported by the National Natural Science Foundation of China under Grant No. 71571128 and the Humanities and Social Sciences Foundation of Ministry of Education of the People's Republic of China (16XJA630005) and the Construction Plan of Scientific Research Innovation Team for Colleges and Universities in Sichuan Province (15TD0004).

Conflicts of Interest: The authors declare no conflict of interest.

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Vector Similarity Measures of Q-Linguistic Neutrosophic Variable Sets and Their Multi-Attribute Decision Making Method

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Received: 30 September 2018; Accepted: 16 October 2018; Published: 22 October 2018

Abstract: Since language is used for thinking and expressing habits of humans in real life, the linguistic evaluation for an objective thing is expressed easily in linguistic terms/values. However, existing linguistic concepts cannot describe linguistic arguments regarding an evaluated object in two-dimensional universal sets (TDUSs). To describe linguistic neutrosophic arguments in decision making problems regarding TDUSs, this study proposes a Q-linguistic neutrosophic variable set (Q-LNVS) for the first time, which depicts its truth, indeterminacy, and falsity linguistic values independently corresponding to TDUSs, and vector similarity measures of Q-LNVSs. Thereafter, a linguistic neutrosophic multi-attribute decision-making (MADM) approach by using the presented similarity measures, including the cosine, Dice, and Jaccard measures, is developed under Q-linguistic neutrosophic setting. Lastly, the applicability and effectiveness of the presented MADM approach is presented by an illustrative example under Q-linguistic neutrosophic setting.

Keywords: Q-linguistic neutrosophic variable set; vector similarity measure; cosine measure; Dice measure; Jaccard measure; decision making

1. Introduction

Since language is used for thinking and expressing habits of humans in real life, the linguistic evaluation for an objective thing is expressed easily in linguistic terms/values [1]. Hence, they were applied to linguistic fuzzy reason [1] and linguistic decision-making (DM) problems [2–9]. Because of linguistic uncertainty and hesitancy in the linguistic evaluation for an objective thing, there exist the representations of interval/uncertain linguistic numbers or hesitant linguistic numbers. Hence, on the one hand, interval/uncertain linguistic numbers were proposed and applied to (group) DM problems in uncertain linguistic setting [10–14]. On the other hand, hesitant linguistic variables (LVs) and hesitant uncertain LVs were presented and applied in (group) DM problems in hesitant (uncertain) linguistic setting [15–19]. In addition, a linguistic cubic variable was put forward based on combining an interval LV with a unique LV and used for DM problems in linguistic cubic setting [20,21]. Further, a linguistic cubic hesitant fuzzy number/variable was presented to depict the hybrid information of its uncertain linguistic argument and its hesitant linguistic argument and utilized for DM problems in linguistic cubic hesitant fuzzy setting [22]. By combining a neutrosophic number with language, a neutrosophic linguistic number and some weighted aggregation operators of neutrosophic linguistic numbers [23] were introduced for neutrosophic linguistic number DM problems, and then the similarity measure and expected value of hesitant neutrosophic linguistic numbers [24] were further presented for DM problems with hesitant neutrosophic linguistic numbers.

In real life environments, the truth, indeterminacy, and falsity linguistic arguments regarding an objective thing are presented in a human's thinking and expressing process and linguistic neutrosophic variables/numbers (LNVs) were presented to depict truth, falsity, and indeterminacy linguistic degrees independently [25]. Then, some aggregation operators of LNVs [25,26], cosine measures of LNVs [27], and correlation coefficients of LNVs [28] were proposed, respectively, for DM problems in LNV setting. Regarding the combination of a neutrosophic linguistic number and an LNV, linguistic neutrosophic uncertain numbers and their weighted aggregation operators were presented for DM in uncertain linguistic setting [29]. By the hybrid form of an interval LV (an uncertain linguistic argument) and a single-valued LNV (an argument of confident degree), single-valued linguistic neutrosophic interval LVs, and their weighted aggregation operators were proposed for DM along with uncertain/interval linguistic arguments and their linguistic neutrosophic confident degrees [30]. Regarding hesitant LNV environment, similarity measures between hesitant LNVs were presented by the least common multiple cardinality and applied to hesitant linguistic neutrosophic DM problems [31]. By the hybrid form of LNV [25] and linguistic cubic numbers [20], linguistic neutrosophic cubic numbers and their aggregation operators were introduced for linguistic neutrosophic cubic DM problems [32,33].

However, the various linguistic concepts are all described in a unique universal set, and then in some decision situations there exist the assessment problems of alternatives over two-dimensional universal sets (TDUSs). For example, suppose a person would like to purchase a house in a group of four houses (a set of four alternatives $H = \{H_1, H_2, H_3, H_4\}$). In his/her attractive evaluation of houses, the price (x_1), environment (x_2), and traffic (x_3) of the four houses are considered as a universal set $X = \{x_1, x_2, x_3\}$, and selecting two cities c_1 and c_2 are considered as another universal set $C = \{c_1, c_2\}$. Obviously, the above various linguistic arguments cannot represent such an assessment problem for each alternative H_j (j = 1, 2, 3, 4) over the TDUSs $X = \{x_1, x_2, x_3\}$ and $C = \{c_1, c_2\}$ in linguistic DM setting. Then, a Q-neutrosophic set and a Q-neutrosophic soft set were put forward regarding TDUSs and applied to Q-neutrosophic soft DM problems [34]. Although they can express and handle the assessment problems with TDUSs in neutrosophic DM environments, they cannot carry out linguistic neutrosophic DM problems over TDUSs. To solve this problem, this study presents a Q-linguistic neutrosophic variable set (Q-LNVS) for the first time to express the linguistic evaluation problems of the truth, falsity, and indeterminacy over TDUSs from the predefined linguistic term set (LTS). It then puts forward the vector similarity measures of Q-LNVSs, including the cosine, Dice, and Jaccard measures of Q-LNVSs, and then establishes a multi-attribute DM approach of Q-LNVSs by the vector similarity measures of Q-LNVs to solve linguistic neutrosophic DM problems along with TDUSs. It is obvious that the proposed DM approach shows the advantage of carrying out the linguistic neutrosophic DM problems regarding TDUSs, which existing linguistic neutrosophic DM approaches [25–28] and Q-linguistic neutrosophic soft DM approaches [34] cannot solve.

The framework of this study is organized below. The second section proposes Q-LNVSs and vector similarity measures between Q-LNVSs, including the cosine, Dice, and Jaccard measures of Q-LNVSs. The third section develops a multi-attribute DM approach of Q-LNVSs by using the vector similarity measures in Q-linguistic neutrosophic setting. An illustrative example and its sensitivity analysis to weights are presented in the fourth section. The last section contains conclusions and future study.

2. Vector Similarity Measures of Q-LNVSs

First, we present the concept of Q-LNVS to depict a linguistic neutrosophic evaluation problem by the truth, falsity, and indeterminacy linguistic arguments over TDUSs in linguistic setting.

Definition 1. *Set* $X = \{x_1, x_2, \ldots, x_n\}$ *and* $Q = \{q_1, q_2, \ldots, q_m\}$ *as TDUSs and let LTS be* $S = \{s_l \mid l \in [0, g]\}$*, where* $g + 1$ *is an odd cardinality. Then a Q-LNVS L in X and Q is defined by the following form:*

$$L = \left\{ \begin{array}{l} \langle (x_i, q_j), s_t(x_i, q_j), s_u(x_i, q_j), s_v(x_i, q_j) \rangle \big| x_i \in X, q_j \in Q, \\ s_t(x_i, q_j), s_i(x_i, q_j), s_f(x_i, q_j) \in S, j = 1, 2, \ldots, m; i = 1, 2, \ldots, n \end{array} \right\}$$

where $s_t(x_i, q_j)$, $s_u(x_i, q_j)$, $s_v(x_i, q_j)$ *are the truth, indeterminacy, and falsity LVs, respectively, in TDUSs for* t, u, $v \in [0, g]$.

Then, the basic linguistic element $\langle (x_i, q_j), s_t(x_i, q_j), s_u(x_i, q_j), s_v(x_i, q_j) \rangle$ in L is simply denoted by $l_{ij} = \left\langle (x_i, q_j), s_{t_{ij}}, s_{u_{ij}}, s_{v_{ij}} \right\rangle$, which is called a Q-linguistic neutrosophic element (Q-LNE).

Example 1. *Suppose a person would like to buy a house from a city. There is a set of two potential houses* $H = \{L_1, L_2\}$ *in two cities. Then, set their price* (x_1)*, environment* (x_2)*, and traffic* (x_3) *as a universal set* $X = \{x_1, x_2, x_3\}$ *and set the two cities as another universal set* $Q = \{q_1, q_2\}$*. Based on the predefined LTS* $S = \{s_0$ *= extremely low,* s_1 *= very low,* s_2 *= low,* s_3 *= slightly low,* s_4 *= medium,* s_5 *= slightly high,* s_6 *= high,* s_7 *= very high,* s_8 *= extremely high\}, the two Q-LNE sets obtained from S are given as follows:*

$$L_1 = \{<(x_1, q_1), s_6, s_1, s_2>, <(x_1, q_2), s_5, s_2, s_3>, <(x_2, q_1), s_4, s_3, s_2>, <(x_2, q_2), s_7, s_1, s_3>, \\ <(x_3, q_1), s_6, s_2, s_1>, <(x_3, q_2), s_6, s_1, s_1>\},$$

$$L_2 = \{<(x_1, q_1), s_3, s_4, s_5>, <(x_1, q_2), s_4, s_2, s_1>, <(x_2, q_1), s_4, s_2, s_2>, <(x_2, q_2), s_5, s_1, s_2>, \\ <(x_3, q_1), s_5, s_2, s_1>, <(x_3, q_2), s_6, s_3, s_2>\}.$$

In the following, we give the vector similarity measures between Q-LNVSs, including the cosine, Dice, and Jaccard measures of Q-LNVSs.

Definition 2. *Let TDUSs be* $X = \{x_1, x_2, \ldots, x_n\}$ *and* $Q = \{q_1, q_2, \ldots, q_p\}$ *and let* $l_{ij}^1 = \left\langle (x_i, q_j), s_{t_{ij}^1}, s_{u_{ij}^1}, s_{v_{ij}^1} \right\rangle$ *and* $l_{ij}^2 = \left\langle (x_i, q_j), s_{t_{ij}^2}, s_{u_{ij}^2}, s_{v_{ij}^2} \right\rangle$ ($j = 1, 2, \ldots, p$; $i = 1, 2, \ldots, n$) *be two groups of Q-LNEs in two Q-LNVSs* L_1 *and* L_2 *regarding the LTS* $S = \{s_l \mid l \in [0, g]\}$*. Then, the cosine, Dice, and Jaccard measures of the Q-LNVSs* L_1 *and* L_2 *are defined, respectively, as follows:*

$$C(L_1, L_2) = s_{\frac{g\sum_{j=1}^{p}\sum_{i=1}^{n}(t_{ij}^1 t_{ij}^2 + u_{ij}^1 v_{ij}^2 + v_{ij}^1 v_{ij}^2)}{\sqrt{\sum_{j=1}^{p}\sum_{i=1}^{n}[(t_{ij}^1)^2 + (u_{ij}^1)^2 + (v_{ij}^1)^2]} \times \sqrt{\sum_{j=1}^{p}\sum_{i=1}^{n}[(t_{ij}^2)^2 + (u_{ij}^2)^2 + (v_{ij}^2)^2]}}}, \tag{1}$$

$$D(L_1, L_2) = s_{\frac{2g\sum_{j=1}^{p}\sum_{i=1}^{n}(t_{ij}^1 t_{ij}^2 + u_{ij}^1 v_{ij}^2 + v_{ij}^1 v_{ij}^2)}{\sum_{j=1}^{p}\sum_{i=1}^{n}[(t_{ij}^1)^2 + (u_{ij}^1)^2 + (v_{ij}^1)^2] + \sum_{j=1}^{p}\sum_{i=1}^{n}[(t_{ij}^2)^2 + (u_{ij}^2)^2 + (v_{ij}^2)^2]}}, \tag{2}$$

$$J(L_1, L_2) = s_{\frac{g\sum_{j=1}^{p}\sum_{i=1}^{n}(t_{ij}^1 t_{ij}^2 + u_{ij}^1 v_{ij}^2 + v_{ij}^1 v_{ij}^2)}{\sum_{j=1}^{p}\sum_{i=1}^{n}[(t_{ij}^1)^2 + (u_{ij}^1)^2 + (v_{ij}^1)^2] + \sum_{j=1}^{p}\sum_{i=1}^{n}[(t_{ij}^2)^2 + (u_{ij}^2)^2 + (v_{ij}^2)^2] - \sum_{j=1}^{p}\sum_{i=1}^{n}(t_{ij}^1 t_{ij}^2 + u_{ij}^1 v_{ij}^2 + v_{ij}^1 v_{ij}^2)}}. \tag{3}$$

Obviously, the above cosine, Dice, and Jaccard measures satisfy the following properties:

(1) $s_0 \le C(L_1, L_2), D(L_1, L_2), J(L_1, L_2) \le s_g$;
(2) $C(L_1, L_2) = C(L_2, L_1)$, $D(L_1, L_2) = D(L_2, L_1)$, $J(L_1, L_2) = J(L_2, L_1)$;
(3) $C(L_1, L_2) = D(L_1, L_2) = J(L_1, L_2) = s_g$ if and only if $L_1 = L_2$.

When the importance of elements x_i ($i = 1, 2, \ldots, n$) and q_j ($j = 1, 2, \ldots, p$) is taken into account, the weight vectors corresponding to $X = \{x_1, x_2, \ldots, x_n\}$ and $Q = \{q_1, q_2, \ldots, q_p\}$ are given as $w = \{w_1, w_2, \ldots, w_n\}$ and $w = \{w_1, w_2, \ldots, w_p\}$, respectively. Thus, the weighted cosine, Dice, and Jaccard measures of L_1 and L_2 can be presented, respectively, as follows:

$$C_w(L_1, L_2) = s_{\frac{g\sum_{j=1}^{p}\omega_j\sum_{i=1}^{n} w_i(t_{ij}^1 t_{ij}^2 + u_{ij}^1 v_{ij}^2 + v_{ij}^1 v_{ij}^2)}{\sqrt{\sum_{j=1}^{p}\omega_j\sum_{i=1}^{n} w_i[(t_{ij}^1)^2+(u_{ij}^1)^2+(v_{ij}^1)^2]}\times\sqrt{\sum_{j=1}^{p}\omega_j\sum_{i=1}^{n} w_i[(t_{ij}^2)^2+(u_{ij}^2)^2+(v_{ij}^2)^2]}}}, \tag{4}$$

$$D_w(L_1, L_2) = s_{\frac{2g\sum_{j=1}^{p}\omega_j\sum_{i=1}^{n} w_i(t_{ij}^1 t_{ij}^2 + u_{ij}^1 v_{ij}^2 + v_{ij}^1 v_{ij}^2)}{\sum_{j=1}^{p}\omega_j\sum_{i=1}^{n} w_i[(t_{ij}^1)^2+(u_{ij}^1)^2+(v_{ij}^1)^2]+\sum_{j=1}^{p}\omega_j\sum_{i=1}^{n} w_i[(t_{ij}^2)^2+(u_{ij}^2)^2+(v_{ij}^2)^2]}}, \tag{5}$$

$$J_w(L_1, L_2) = s_{\frac{g\sum_{j=1}^{p}\omega_j\sum_{i=1}^{n} w_i(t_{ij}^1 t_{ij}^2 + u_{ij}^1 v_{ij}^2 + v_{ij}^1 v_{ij}^2)}{\sum_{j=1}^{p}\omega_j\sum_{i=1}^{n} w_i[(t_{ij}^1)^2+(u_{ij}^1)^2+(v_{ij}^1)^2]+\sum_{j=1}^{p}\omega_j\sum_{i=1}^{n} w_i[(t_{ij}^2)^2+(u_{ij}^2)^2+(v_{ij}^2)^2]-\sum_{j=1}^{p}\omega_j\sum_{i=1}^{n} w_i(t_{ij}^1 t_{ij}^2 + u_{ij}^1 v_{ij}^2 + v_{ij}^1 v_{ij}^2)}}. \tag{6}$$

Example 2. *Let us consider two Q-LNE sets L_1 = {<(x_1, q_1), s_6, s_1, s_2>, <(x_1, q_2), s_5, s_2, s_3>, <(x_2, q_1), s_4, s_3, s_2>, <(x_2, q_2), s_7, s_1, s_3>, <(x_3, q_1), s_6, s_2, s_1>, <(x_3, q_2), s_6, s_1, s_1>} and L_2 = {<(x_1, q_1), s_3, s_4, s_5>, <(x_1, q_2), s_4, s_2, s_1>, <(x_2, q_1), s_4, s_2, s_2>, <(x_2, q_2), s_5, s_1, s_2>, <(x_3, q_1), s_5, s_2, s_1>, <(x_3, q_2), s_6, s_3, s_2>} in the LTS $S = \{s_0, s_1, s_2, \ldots, s_8\}$ with $g = 8$ and the TDUSs $X = \{x_1, x_2, x_3\}$ and $Q = \{q_1, q_2\}$. Suppose the weight vectors for $X = \{x_1, x_2, x_3\}$ and $Q = \{q_1, q_2\}$ are given as $w = (0.4, 0.25, 0.35)$ and $\omega = (0.4, 0.6)$, respectively. Then, we compute the measure values of $C_w(L_1, L_2)$, $D_w(L_1, L_2)$, $J_w(L_1, L_2)$.*

By using Equations (4)–(6), their calculational processes are shown as follows:

$$\begin{aligned} C_w(L_1, L_2) &= s_{\frac{g\sum_{j=1}^{2}\omega_j\sum_{i=1}^{3} w_i(t_{ij}^1 t_{ij}^2 + u_{ij}^1 v_{ij}^2 + v_{ij}^1 v_{ij}^2)}{\sqrt{\sum_{j=1}^{2}\omega_j\sum_{i=1}^{3} w_i[(t_{ij}^1)^2+(u_{ij}^1)^2+(v_{ij}^1)^2]}\times\sqrt{\sum_{j=1}^{2}\omega_j\sum_{i=1}^{3} w_i[(t_{ij}^2)^2+(u_{ij}^2)^2+(v_{ij}^2)^2]}}} \\ &= s_{\frac{8\left\{\begin{array}{l} 0.4[0.4(6\times 3+1\times 4+2\times 5)+0.25(4\times 4+3\times 2+2\times 2)+0.35(6\times 5+2\times 2+1\times 1)]+ \\ 0.6[0.4(5\times 4+2\times 2+3\times 1)+0.25(7\times 5+1\times 1+3\times 2)+0.35(6\times 6+1\times 3+1\times 2)] \end{array}\right\}}{\sqrt{\left\{\begin{array}{l} 0.4[0.4(6^2+1^2+2^2)+0.25(4^2+3^2+2^2)+0.35(6^2+2^2+1^2)]+ \\ 0.6[0.4(5^2+2^2+3^2)+0.25(7^2+1^2+3^2)+0.35(6^2+1^2+1^2)] \end{array}\right\}}\times\sqrt{\left\{\begin{array}{l} 0.4[0.4^2(3^2+4^2+5^2)+0.25(4^2+2^2+2^2)+0.35(5^2+2^2+1^2)]+ \\ 0.6[0.4^2(4^2+2^2+1^2)+0.25(5^2+1^2+2^2)+0.35(6^2+3^2+2^2)] \end{array}\right\}}}} \\ &= s_{7.0913}, \end{aligned}$$

$$\begin{aligned} D_w(L_1, L_2) &= s_{\frac{2g\sum_{j=1}^{2}\omega_j\sum_{i=1}^{3} w_i(t_{ij}^1 t_{ij}^2 + u_{ij}^1 v_{ij}^2 + v_{ij}^1 v_{ij}^2)}{\sum_{j=1}^{2}\omega_j\sum_{i=1}^{3} w_i[(t_{ij}^1)^2+(u_{ij}^1)^2+(v_{ij}^1)^2]+\sum_{j=1}^{2}\omega_j\sum_{i=1}^{3} w_i[(t_{ij}^2)^2+(u_{ij}^2)^2+(v_{ij}^2)^2]}} \\ &= s_{\frac{2\times 8\left\{\begin{array}{l} 0.4[0.4(6\times 3+1\times 4+2\times 5)+0.25(4\times 4+3\times 2+2\times 2)+0.35(6\times 5+2\times 2+1\times 1)]+ \\ 0.6[0.4(5\times 4+2\times 2+3\times 1)+0.25(7\times 5+1\times 1+3\times 2)+0.35(6\times 6+1\times 3+1\times 2)] \end{array}\right\}}{\left\{\begin{array}{l} 0.4[0.4(6^2+1^2+2^2)+0.25(4^2+3^2+2^2)+0.35(6^2+2^2+1^2)]+ \\ 0.6[0.4(5^2+2^2+3^2)+0.25(7^2+1^2+3^2)+0.35(6^2+1^2+1^2)] \end{array}\right\}+\left\{\begin{array}{l} 0.4[0.4^2(3^2+4^2+5^2)+0.25(4^2+2^2+2^2)+0.35(5^2+2^2+1^2)]+ \\ 0.6[0.4^2(4^2+2^2+1^2)+0.25(5^2+1^2+2^2)+0.35(6^2+3^2+2^2)] \end{array}\right\}}} \\ &= s_{7.0777}, \end{aligned}$$

$$\begin{aligned} J_w(L_1, L_2) &= s_{\frac{g\sum_{j=1}^{2}\omega_j\sum_{i=1}^{3} w_i(t_{ij}^1 t_{ij}^2 + u_{ij}^1 v_{ij}^2 + v_{ij}^1 v_{ij}^2)}{\sum_{j=1}^{2}\omega_j\sum_{i=1}^{3} w_i[(t_{ij}^1)^2+(u_{ij}^1)^2+(v_{ij}^1)^2]+\sum_{j=1}^{2}\omega_j\sum_{i=1}^{3} w_i[(t_{ij}^2)^2+(u_{ij}^2)^2+(v_{ij}^2)^2]-\sum_{j=1}^{2}\omega_j\sum_{i=1}^{3} w_i(t_{ij}^1 t_{ij}^2 + u_{ij}^1 v_{ij}^2 + v_{ij}^1 v_{ij}^2)}} \\ &= s_{\frac{8\left\{\begin{array}{l} 0.4[0.4(6\times 3+1\times 4+2\times 5)+0.25(4\times 4+3\times 2+2\times 2)+0.35(6\times 5+2\times 2+1\times 1)]+ \\ 0.6[0.4(5\times 4+2\times 2+3\times 1)+0.25(7\times 5+1\times 1+3\times 2)+0.35(6\times 6+1\times 3+1\times 2)] \end{array}\right\}}{\left\{\begin{array}{l} \left\{\begin{array}{l} 0.4[0.4(6^2+1^2+2^2)+0.25(4^2+3^2+2^2)+0.35(6^2+2^2+1^2)]+ \\ 0.6[0.4(5^2+2^2+3^2)+0.25(7^2+1^2+3^2)+0.35(6^2+1^2+1^2)] \end{array}\right\} \\ \left\{\begin{array}{l} 0.4[0.4^2(3^2+4^2+5^2)+0.25(4^2+2^2+2^2)+0.35(5^2+2^2+1^2)]+ \\ 0.6[0.4^2(4^2+2^2+1^2)+0.25(5^2+1^2+2^2)+0.35(6^2+3^2+2^2)] \end{array}\right\} \\ -\left\{\begin{array}{l} 0.4[0.4(6\times 3+1\times 4+2\times 5)+0.25(4\times 4+3\times 2+2\times 2)+0.35(6\times 5+2\times 2+1\times 1)]+ \\ 0.6[0.4(5\times 4+2\times 2+3\times 1)+0.25(7\times 5+1\times 1+3\times 2)+0.35(6\times 6+1\times 3+1\times 2)] \end{array}\right\} \end{array}\right\}}} \\ &= s_{6.3460}, \end{aligned}$$

Obviously, the above vector measure values still belong to the LTS S.

3. DM Approach Based on the Vector Similarity Measures

This section proposes a Q-linguistic neutrosophic multi-attribute DM approach based on the Dice, cosine, and Jaccard measures (the three vector measures) of Q-LNVSs in Q-LNVS setting.

Suppose there is a multi-attribute DM problem, in which $L = \{L_1, L_2, \ldots, L_m\}$ is denoted by a set of m alternatives. Then, TDUSs (two kinds of attribute sets) are specified as $X = \{x_1, x_2, \ldots, x_n\}$ and $Q = \{q_1, q_2, \ldots, q_p\}$, respectively, and then their corresponding weigh vectors are given as $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ and $\boldsymbol{w} = (w_1, w_2, \ldots, w_p)$. Whereas, a decision maker is required to assess the alternatives L_k ($k = 1, 2, \ldots, m$) on the attributes x_i ($i = 1, 2, \ldots, n$) and q_j ($j = 1, 2, \ldots, p$) by Q-LNEs regarding the given LTS $S = \{s_l \mid l \in [0, g]\}$ with the odd cardinality $g + 1$. In the assessment process, the decision maker gives the truth, falsity, and indeterminacy linguistic values for x_i and q_j on an alternative L_k by corresponding linguistic terms in S, which are constructed as a Q-LNE $l_{ij}^k = \left\langle (x_i, q_j), s_{t_{ij}^k}, s_{u_{ij}^k}, s_{v_{ij}^k} \right\rangle$ ($j = 1, 2, \ldots, p$; $i = 1,2, \ldots, n$; $k = 1, 2, \ldots, m$). Hence, all the Q-LNEs provided by the decision maker can be composed of a decision matrix of Q-LNEs $L = \left(l_{ij}^k\right)_{m \times nq}$.

Thus, the proposed DM method using the vector similarity measures of Q-LNVSs is applied to the multi-attribute DM problem with Q-LNVS information. Whereas, the DM steps are depicted in detail below:

Step 1: Since $l_{ij}^* = \left\langle (x_i, q_j), s_{t_{ij}^*}, s_{u_{ij}^*}, s_{v_{ij}^*} \right\rangle = \left\langle \left\langle (x_i, q_j), \max_k \left(s_{t_{ij}^k}\right), \min_k \left(s_{u_{ij}^k}\right), \min_k \left(s_{v_{ij}^k}\right) \right\rangle \right\rangle$ is an ideal Q-LNE as the best Q-LNE, we can establish the following ideal Q-LNVS:

$$L^* = \left\{ \left\langle (x_i, q_j), s_{t_{ij}^*}, s_{u_{ij}^*}, s_{v_{ij}^*} \right\rangle \middle| x_i \in X, q_j \in Q, k = 1, 2, \ldots, m, j = 1, 2, \ldots, p, i = 1, 2, \ldots, n \right\}. \quad (7)$$

Step 2: By applying Equations (4)–(6), the cosine/Dice/Jaccard measure between L_k ($k = 1, 2, \ldots, m$) and L^* is given by using the following formula:

$$C_w(L_k, L^*) = s_{\frac{g\sum_{j=1}^{p} \omega_j \sum_{i=1}^{n} w_i (t_{ij}^k t_{ij}^* + u_{ij}^k v_{ij}^* + v_{ij}^k v_{ij}^*)}{\sqrt{\sum_{j=1}^{p} \omega_j \sum_{i=1}^{n} w_i [(t_{ij}^k)^2 + (u_{ij}^k)^2 + (v_{ij}^k)^2]} \times \sqrt{\sum_{j=1}^{p} \omega_j \sum_{i=1}^{n} w_i [(t_{ij}^*)^2 + (u_{ij}^*)^2 + (v_{ij}^*)^2]}}}, \quad (8)$$

or

$$D_w(L_k, L^*) = s_{\frac{2 \times g\sum_{j=1}^{p} \omega_j^2 \sum_{i=1}^{n} w_i^2 (t_{ij}^k t_{ij}^* + u_{ij}^k v_{ij}^* + v_{ij}^k v_{ij}^*)}{\sum_{j=1}^{p} \omega_j^2 \sum_{i=1}^{n} w_i^2 [(t_{ij}^k)^2 + (u_{ij}^k)^2 + (v_{ij}^k)^2] + \sum_{j=1}^{p} \omega_j^2 \sum_{i=1}^{n} w_i^2 [(t_{ij}^*)^2 + (u_{ij}^*)^2 + (v_{ij}^*)^2]}}, \quad (9)$$

or

$$J_w(L_k, L^*) = s_{\frac{g\sum_{j=1}^{p} \omega_j \sum_{i=1}^{n} w_i (t_{ij}^k t_{ij}^* + u_{ij}^k v_{ij}^* + v_{ij}^k v_{ij}^*)}{\sum_{j=1}^{p} \omega_j \sum_{i=1}^{n} w_i [(t_{ij}^k)^2 + (u_{ij}^k)^2 + (v_{ij}^k)^2] + \sum_{j=1}^{p} \omega_j \sum_{i=1}^{n} w_i [(t_{ij}^*)^2 + (u_{ij}^*)^2 + (v_{ij}^*)^2] - \sum_{j=1}^{p} \omega_j \sum_{i=1}^{n} w_i (t_{ij}^k t_{ij}^* + u_{ij}^k v_{ij}^* + v_{ij}^k v_{ij}^*)}} \quad (10)$$

Step 3: According to the linguistic values of the vector similarity measures, the alternatives are ranked and the best alternative L_{k^*} is chosen regarding the biggest linguistic value for X and Q.

Step 4: Based on $x_j \in X$ (j =1, 2, ... , p) or $q_i \in Q$ (i =1, 2, ... , n), we need to calculate the measure values between $L_{k^*}(x_i, q_j)$ and $L^*(x_i, q_j)$:

$$C(L_{k^*}(x_i, q_j), L^*(x_i, q_j)) = s_{\frac{g\sum_{i=1}^{n} (t_{ij}^k t_{ij}^* + u_{ij}^k v_{ij}^* + v_{ij}^k v_{ij}^*)}{\sqrt{\sum_{i=1}^{n} [(t_{ij}^k)^2 + (u_{ij}^k)^2 + (v_{ij}^k)^2]} \times \sqrt{\sum_{i=1}^{n} [(t_{ij}^*)^2 + (u_{ij}^*)^2 + (v_{ij}^*)^2]}}} \quad \text{for } j = 1, 2, \ldots, p, \quad (11)$$

or

$$C(L_{k^*}(x_i, q_j), L^*(x_i, q_j)) = s_{\frac{g\sum_{j=1}^{p} (t_{ij}^k t_{ij}^* + u_{ij}^k v_{ij}^* + v_{ij}^k v_{ij}^*)}{\sqrt{\sum_{j=1}^{p} [(t_{ij}^k)^2 + (u_{ij}^k)^2 + (v_{ij}^k)^2]} \times \sqrt{\sum_{j=1}^{p} [(t_{ij}^*)^2 + (u_{ij}^*)^2 + (v_{ij}^*)^2]}}} \quad \text{for } i = 1, 2, \ldots, n; \quad (12)$$

$$D(L_{k^*}(x_i, q_j), L^*(x_i, q_j)) = s_{\frac{2g\sum_{i=1}^{n} (t_{ij}^k t_{ij}^* + u_{ij}^k v_{ij}^* + v_{ij}^k v_{ij}^*)}{\sum_{i=1}^{n} [(t_{ij}^k)^2 + (u_{ij}^k)^2 + (v_{ij}^k)^2] + \sum_{i=1}^{n} [(t_{ij}^*)^2 + (u_{ij}^*)^2 + (v_{ij}^*)^2]}} \quad \text{for } j = 1, 2, \ldots, p, \quad (13)$$

or

$$D\big(L_{k^*}(x_i,q_j),L^*(x_i,q_j)\big) = s_{\frac{2g\sum_{j=1}^{p}(t_{ij}^k t_{ij}^* + u_{ij}^k v_{ij}^* + v_{ij}^k v_{ij}^*)}{\sum_{j=1}^{p}[(t_{ij}^k)^2+(u_{ij}^k)^2+(v_{ij}^k)^2]+\sum_{j=1}^{p}[(t_{ij}^*)^2+(u_{ij}^*)^2+(v_{ij}^*)^2]}} \quad \text{for } i = 1,2,\ldots,n; \tag{14}$$

$$J\big(L_{k^*}(x_i,q_j),L^*(x_i,q_j)\big) = s_{\frac{g\sum_{i=1}^{n}(t_{ij}^k t_{ij}^* + u_{ij}^k v_{ij}^* + v_{ij}^k v_{ij}^*)}{\sum_{i=1}^{n}[(t_{ij}^k)^2+(u_{ij}^k)^2+(v_{ij}^k)^2]+\sum_{i=1}^{n}[(t_{ij}^*)^2+(u_{ij}^*)^2+(v_{ij}^*)^2]-\sum_{i=1}^{n}(t_{ij}^k t_{ij}^* + u_{ij}^k v_{ij}^* + v_{ij}^k v_{ij}^*)}} \quad \text{for } j = 1,2,\ldots,p, \tag{15}$$

or

$$J\big(L_{k^*}(x_i,q_j),L^*(x_i,q_j)\big) = s_{\frac{g\sum_{j=1}^{p}(t_{ij}^k t_{ij}^* + u_{ij}^k v_{ij}^* + v_{ij}^k v_{ij}^*)}{\sum_{j=1}^{p}[(t_{ij}^k)^2+(u_{ij}^k)^2+(v_{ij}^k)^2]+\sum_{j=1}^{p}[(t_{ij}^*)^2+(u_{ij}^*)^2+(v_{ij}^*)^2]-\sum_{j=1}^{p}(t_{ij}^k t_{ij}^* + u_{ij}^k v_{ij}^* + v_{ij}^k v_{ij}^*)}} \quad \text{for } i = 1,2,\ldots,n. \tag{16}$$

Step 5: According to the linguistic values of $C\big(L_{k^*}(x_i,q_j),L^*(x_i,q_j)\big)$ or $D\big(L_{k^*}(x_i,q_j),L^*(x_i,q_j)\big)$ or $J\big(L_{k^*}(x_i,q_j),L^*(x_i,q_j)\big)$ for X or Q (depending on some actual situation), we can determine the best one x_{i^*} or q_{j^*} corresponding to the biggest linguistic value.

Step 6: End.

4. Illustrative Example and Sensitivity Analysis to Weights

4.1. Illustrative Example

Suppose a person would like to buy a house in one of two cities. There are four potential houses (alternatives) of L_k (k = 1, 2, 3, 4) in two cities. Then, set their price (x_1), environment (x_2), and traffic (x_3) as a universal set $X = \{x_1, x_2, x_3\}$ and set the two cities as another universal set $Q = \{q_1, q_2\}$. Thus, the Q-LNEs can indicate the influence of both the three attributes of houses and the two cities on his/her buying attractive degree of a house. Herewith, the two weigh vectors of X and Q are given as w = (0.4, 0.25, 0.35) and w = (0.4, 0.6), respectively. Whereas, the alternative L_k (k = 1, 2, 3, 4) are assessed over the TDUSs $X = \{x_1, x_2, x_3\}$ and $Q = \{q_1, q_2\}$ from the given LTS $S = \{s_0$ = extremely low, s_1 = very low, s_2 = low, s_3 = slightly low, s_4 = medium, s_5 = slightly high, s_6 = high, s_7 = very high, s_8 = extremely high$\}$ with g = 8. In the assessment process, the decision maker/buyer can give the truth, indeterminacy, and falsity values for x_i and q_j on an alternative L_k by corresponding linguistic terms in S, and then establish Q-LNEs $l_{ij}^k = \left\langle (x_i,q_j), s_{t_{ij}^k}, s_{u_{ij}^k}, s_{v_{ij}^k} \right\rangle$ (j =1, 2; i = 1,2, 3; k = 1, 2, 3, 4), which are constructed as the DM matrix of Q-LNEs:

$$L = \begin{matrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{matrix} \begin{bmatrix} \langle (x_1,q_1),s_6,s_2,s_1\rangle & \langle (x_1,q_2),s_7,s_2,s_3\rangle & \langle (x_2,q_1),s_5,s_2,s_1\rangle & \langle (x_2,q_2),s_4,s_1,s_1\rangle & \langle (x_3,q_1),s_5,s_2,s_2\rangle & \langle (x_3,q_2),s_7,s_2,s_2\rangle \\ \langle (x_1,q_1),s_6,s_1,s_2\rangle & \langle (x_1,q_2),s_7,s_1,s_1\rangle & \langle (x_2,q_1),s_6,s_1,s_1\rangle & \langle (x_2,q_2),s_5,s_1,s_2\rangle & \langle (x_3,q_1),s_6,s_1,s_3\rangle & \langle (x_3,q_2),s_7,s_2,s_1\rangle \\ \langle (x_1,q_1),s_5,s_2,s_3\rangle & \langle (x_1,q_2),s_6,s_2,s_4\rangle & \langle (x_2,q_1),s_4,s_1,s_1\rangle & \langle (x_2,q_2),s_5,s_2,s_2\rangle & \langle (x_3,q_1),s_4,s_2,s_2\rangle & \langle (x_3,q_2),s_5,s_2,s_3\rangle \\ \langle (x_1,q_1),s_5,s_1,s_1\rangle & \langle (x_1,q_2),s_6,s_3,s_5\rangle & \langle (x_2,q_1),s_5,s_3,s_3\rangle & \langle (x_2,q_2),s_6,s_2,s_4\rangle & \langle (x_3,q_1),s_5,s_1,s_1\rangle & \langle (x_3,q_2),s_6,s_2,s_3\rangle \end{bmatrix}.$$

Thus, the proposed multi-attribute DM approach can be used for this Q-linguistic neutrosophic DM problem. The DM steps are depicted below:

Firstly, we establish the ideal alternative from the DM matrix L by the ideal Q-LNE set:

$$L^* = \{\langle (x_1,q_1),s_6,s_1,s_1\rangle, \langle (x_1,q_2),s_7,s_1,s_1\rangle, \langle (x_2,q_1),s_6,s_1,s_1\rangle, \langle (x_2,q_2),s_6,s_1,s_1\rangle, \langle (x_3,q_1),s_6,s_1,s_1\rangle, \langle (x_3,q_2),s_7,s_2,s_1\rangle\}.$$

Then, by Equations (8)–(10), the measure results and ranking of the four alternatives are given in Table 1.

Table 1. Measure results and ranking of the four alternatives.

Measure Method	Measure Value between L_k (k = 1, 2, 3, 4) and L^*	Ranking
$C_w(L_k, L^*)$	$s_{7.7472}, s_{7.9088}, s_{7.3465}, s_{7.2437}$	$L_2 > L_1 > L_3 > L_4$
$D_w(L_k, L^*)$	$s_{7.7470}, s_{7.9087}, s_{7.3207}, s_{7.2387}$	$L_2 > L_1 > L_3 > L_4$
$J_w(L_k, L^*)$	$s_{7.5095}, s_{7.8194}, s_{6.7478}, s_{6.6097}$	$L_2 > L_1 > L_3 > L_4$

Based on Table 1, all the ranking orders are identical regarding the cosine, Dice, and Jaccard measures. Then, the best alternative is L_2.

Next, the measure values of Equations (11), (13), and (15) regarding Q, and the best city regarding L_2 are given in Table 2.

Table 2. Measure results regarding Q and the best city.

Measure Method	Measure Result	The Best City
$C(L_2(x_i, q_1), L^*(x_i, q_1)), C(L_2(x_i, q_2), L^*(x_i, q_2))$	$s_{7.8409}, s_{7.9457}$	q_2
$D(L_2(x_i, q_1), L^*(x_i, q_1)), D(L_2(x_i, q_2), L^*(x_i, q_2))$	$s_{7.8326}, s_{7.9424}$	q_2
$J(L_2(x_i, q_1), L^*(x_i, q_1)), J(L_2(x_i, q_2), L^*(x_i, q_2))$	$s_{7.6721}, s_{7.8857}$	q_2

Lastly, the results based on Table 2 indicate that the buyer should buy the house L_2 in the best city q_2.

4.2. Sensitivity Analysis to Weights

To indicate the influence of the weights on ranking orders in the illustrative example, we consider that the two weigh vectors of X and Q are given as w = (1/3, 1/3, 1/3) and w = (1/2, 1/2), respectively, to analyze the sensitivity of the weights with respect to the ranking orders of the four alternatives. In this case, by Equations (8)–(10) the measure results and ranking of the four alternatives are indicated in Table 3.

Table 3. Measure results and ranking of the four alternatives with w = (1/3, 1/3, 1/3) and w = (1/2, 1/2).

Measure Method	Measure Value between L_k (k = 1, 2, 3, 4) and L^*	Ranking
$C_w(L_k, L^*)$	$s_{7.7470}, s_{7.8918}, s_{7.3878}, s_{7.2922}$	$L_2 > L_1 > L_3 > L_4$
$D_w(L_k, L^*)$	$s_{7.7430}, s_{7.8917}, s_{7.3448}, s_{7.2892}$	$L_2 > L_1 > L_3 > L_4$
$J_w(L_k, L^*)$	$s_{7.5019}, s_{7.7863}, s_{6.7888}, s_{6.6944}$	$L_2 > L_1 > L_3 > L_4$

In this case, there exists the same ranking order in Tables 1 and 3 regarding the cosine, Dice, and Jaccard measures. Then, the best alternative is sill L_2, which means the buyer should also buy house L_2 in the best city q_2 based on Table 2. It is obvious that all the ranking orders imply the decision robustness based on the cosine, Dice, and Jaccard measures regarding the change of weights in this illustrative example, which also show no sensitivity of all the ranking orders with respect to the change of the weights. In the actual DM applications, however, one of three vector measures can be selected by decision makers' preference or actual requirements.

However, existing various linguistic neutrosophic DM approaches [25–28] cannot handle the DM problems in Q-LNVS setting; while our proposed DM method can carry out both the existing DM problems with LNV information [25–28] and the DM problems with Q-LNVS information, which shows its advantage in Q-LNVS setting because the LNV set is a special case of Q-LNVS under a universal set. Furthermore, the existing Q-neutrosophic soft DM approach [34] cannot deal with the DM problems with Q-LNVS information because the Q-neutrosophic soft set [34] cannot express Q-linguistic neutrosophic information. Hence, our proposed Q-linguistic neutrosophic DM method provides a new way for linguistic neutrosophic DM with TDUSs.

5. Conclusions

This study presented the concept of Q-LNVS for the first time to describe the truth, falsity, and indeterminacy linguistic arguments in TDUSs, and then the cosine, Dice, and Jaccard measures of Q-LNVSs in vector space. Next, a Q-linguistic neutrosophic multi-attribute DM approach in Q-LNVS setting was established by using the cosine, Dice, and Jaccard measures of Q-LNVSs to solve linguistic neutrosophic DM problems regarding TDUSs. Lastly, the application of the developed DM approach was given by an illustrative example in Q-LNVS setting. The decision results show that the established multi-attribute DM approach of Q-LNVSs can solve linguistic neutrosophic DM problems regarding TDUSs (two-dimensional attribute sets) in Q-LNVS setting, which indicates its main advantage and contribution. Based on the first study, the three vector measures of Q-LNVSs will be further used for medical diagnosis, data mining, and clustering analysis for future research in Q-LNVS setting.

Author Contributions: J.Y. proposed the Q-LNVS concept and the vector similarity measures; Z.F. and W.C. gave the DM approach and the calculation and analysis; all the authors wrote the manuscript and revised the final version.

Funding: This paper was supported by the National Natural Science Foundation of China (Nos. 61703280, 51872186).

Conflicts of Interest: The authors declare no conflicts of interest.

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New Multigranulation Neutrosophic Rough Set with Applications

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Received: 3 September 2018; Accepted: 6 October 2018; Published: 2 November 2018

Abstract: After the neutrosophic set (NS) was proposed, NS was used in many uncertainty problems. The single-valued neutrosophic set (SVNS) is a special case of NS that can be used to solve real-word problems. This paper mainly studies multigranulation neutrosophic rough sets (MNRSs) and their applications in multi-attribute group decision-making. Firstly, the existing definition of neutrosophic rough set (we call it type-I neutrosophic rough set (NRS_I) in this paper) is analyzed, and then the definition of type-II neutrosophic rough set (NRS_{II}), which is similar to NRS_I, is given and its properties are studied. Secondly, a type-III neutrosophic rough set (NRS_{III}) is proposed and its differences from NRS_I and NRS_{II} are provided. Thirdly, single granulation NRSs are extended to multigranulation NRSs, and the type-I multigranulation neutrosophic rough set ($MNRS_I$) is studied. The type-II multigranulation neutrosophic rough set ($MNRS_{II}$) and type-III multigranulation neutrosophic rough set ($MNRS_{III}$) are proposed and their different properties are outlined. We found that the three kinds of MNRSs generate tcorresponding NRSs when all the NRs are the same. Finally, $MNRS_{III}$ in two universes is proposed and an algorithm for decision-making based on $MNRS_{III}$ is provided. A car ranking example is studied to explain the application of the proposed model.

Keywords: inclusion relation; neutrosophic rough set; multi-attribute group decision-making (MAGDM); multigranulation neutrosophic rough set (MNRS); two universes

1. Introduction

Many theories have been applied to solve problems with imprecision and uncertainty. Fuzzy set (FS) theories [1–3] use the degree of membership to solve the fuzziness. Rough set (RS) theories [4–7] deal with uncertainty by lower and upper approximation (LUA). Soft set theories [8–10] deal with uncertainty by using a parametrized set. However, all these theories have their own restrictions. Smarandache proposed the concept of the neutrosophic set (NS) [11], which was a generalization of the intuitionistic fuzzy set (IFS). To address real-world uncertainty problems, Wang et al. proposed the single-valued neutrosophic set (SVNS) [12]. Many theories about neutrosophic sets were studied and extended single-valued neutrosophic set [13–15]. Zhang et al. [16] analyzed two kinds of inclusion relations of the NS and then proposed the type-3 inclusion relation of NS. The combinations of the FS and RS are popular and produce many interesting results [17]. Broumi and Smarandache [18] combined the RS and NS, then produced a rough NS and studied its qualities. Yang et al. [19] combined the SVNS and RS, then produced the SVNRS (single-valued neutrosophic rough set) and studied its qualities.

From the view point of granular computing, the RS uses upper and lower approximations to solve uncertainty problems, shown by single granularity. However, with the complexity of real-word problems, we often encounter multiple relationship concepts. Qian and Liang [20] proposed a multigranularity rough set (MGRS). Many scholars have generalized MGRS and acquired some interesting consequences [21–26]. Zhang et al. [27] proposed non-dual MGRSs and investigated their qualities.

Few articles have been published about the combination of NSs and multigranulation rough sets. In this paper, we study three kinds of neutrosophic rough sets (NRSs) and multigranulation neutrosophic rough sets (MNRSs) that are based on three kinds of inclusion relationships of NS and corresponding union and intersection relationships [11,12,16]. Their different properties are discussed. We found that MNRSs degenerate to corresponding NRSs when the NRs are the same. Yang et al. [19] defined the NRS_I and considered its properties. Bo et al. [28] proposed $MNRS_I$ and discussed its properties. In this paper, we study NRS_{II} and $MNRS_{II}$. We also study NRS_{III} and $MNRS_{III}$, which are based on a type-3 inclusion relationship and corresponding union and intersection relationships. Finally, we use $MNRS_{III}$ on two universes to solve multi-attribute group decision-making (MAGDM) problems.

The structure of this article is as follows: In Section 2, some basic notions and operations of NRS_I and NRS_{II} are introduced. In Section 3, the definition of NRS_{III} is proposed and its qualities are investigated, and the differences between NRS_I, NRS_{II}, and NRS_{III} are illustrated using an example. In Section 4, $MNRS_I$ and $MNRS_{II}$ are discussed. In Section 5, $MNRS_{III}$ is proposed and its differences from $MNRS_I$ and $MNRS_{II}$ are studied. In Section 6, $MNRS_{III}$ on two universes is proposed and an application to solve the MAGDM problem is outlined. Finally, Section 7 provides our conclusions and outlook.

2. Preliminary

In this chapter, we look back at several basic concepts of type-I NRS, then propose the definition and properties of type-II NRS.

Definition 1. *[12] A single valued neutrosophic set A in X is denoted by:*

$$A = \{(x, T_A(x), I_A(x), F_A(x))|x \in X\}, \tag{1}$$

where TA(x), IA(x), FA(x) ∈ [0, 1] for each point x in X and satisfies the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

*For convenience, "*SVNS*" is abbreviated to "*NS*" later. Here, NS(X) denotes the set of all* SVNS *in X.*

Definition 2. *[29] A neutrosophic relation* (NR) *is a neutrosophic fuzzy subset of* $X \times Y$*, that is,* $\forall x \in X$, $y \in Y$,

$$R(x, y) = (T_R, I_R, F_R), \tag{2}$$

where T_R: $X \times Y \to [0, 1]$, I_R: $X \times Y \to [0, 1]$, *and* F_R: $X \times Y \to [0, 1]$ *and satisfies* $0 \leq T_R + I_R + F_R \leq 3$. $NR(X \times Y)$ *denotes all the* NRs *in* $X \times Y$.

Definition 3. *[19] Suppose (U, R) is a neutrosophic approximation space* (NAS)*.* $\forall A \in NS(U)$*, the* LUA *of A, denoted by* $\underline{R}(A)$ *and* $\overline{R}(A)$*, is defined as:* $\forall x \in U$,

$$\underline{R}(A) = \bigcap_{y \in U} (R^c(x, y) \cup A(y)), \ \overline{R}(A) = \bigcup_{y \in U} (R(x, y) \cap A(y)). \tag{3}$$

The pair $(\underline{R}(A), \overline{R}(A))$ *is called the* SVNRS *of A. In this paper, we called it type-I neutrosophic rough set* (NRS_I)*. Because the definition of* NRS_I *is based on the type-1 operator of* NS*, the definition can be written as:*

$$\underline{NRS_I}(A) = \bigcap_{1\, y \in U} (R^c(x, y) \cup_1 A(y)), \ \overline{NRS_I}(A) = \bigcup_{1\, y \in U} (R(x, y) \cap_1 A(y)). \tag{4}$$

Proposition 1. *[19] Suppose (U, R) is an* NAS*.* $\forall A, B \in NS(U)$*, we have:*

(1) If $A \subseteq_1 B$, then $\underline{NRS_I}(A) \subseteq_1 \underline{NRS_I}(B)$ and $\overline{NRS_I}(A) \subseteq_1 \overline{NRS_I}(B)$.
(2) $\underline{NRS_I}(A \cap_1 B) = \underline{NRS_I}(A) \cap_1 \underline{NRS_I}(B)$, $\overline{NRS_I}(A \cup_1 B) = \overline{NRS_I}(A) \cup_1 \overline{NRS_I}(B)$.
(3) $\underline{NRS_I}(A) \cup_1 \underline{NRS_I}(B) \subseteq_1 \underline{NRS_I}(A \cup_1 B)$, $\overline{NRS_I}(A \cap_1 B) \subseteq_1 \overline{NRS_I}(A) \cap_1 \overline{NRS_I}(B)$.

According to the $\mathrm{NRS_I}$, we can get the definition and properties of $\mathrm{NRS_{II}}$, which is based on the type-2 operator of NS.

Definition 4. *Suppose (U, R) is an* NAS. *∀A ∈ NS(U), the type-II* LUA *of A, is defined as:*

$$\underline{NRS_{II}}(A) = \underset{y \in U}{\cap_2}(R^c(x,y) \cup_2 A(y)),\ \overline{NRS_{II}}(A) = \underset{y \in U}{\cup_2}(R(x,y) \cap_2 A(y)) \tag{5}$$

The pair $\left(\underline{NRS_{II}}(A), \overline{NRS_{II}}(A)\right)$ *is called* $\mathrm{NRS_{II}}$ *of A.*

Proposition 2. *Suppose (U, R) is an* NAS. *∀A, B ∈ NS(U), we have:*

(1) If $A \subseteq_2 B$, then $\underline{NRS_{II}}(A) \subseteq_2 \underline{NRS_{II}}(B)$, $\overline{NRS_{II}}(A) \subseteq_2 \overline{NRS_{II}}(B)$.
(2) $\underline{NRS_{II}}(A \cap_2 B) = \underline{NRS_{II}}(A) \cap_2 \underline{NRS_{II}}(B)$, $\overline{NRS_{II}}(A \cup_2 B) = \overline{NRS_{II}}(A) \cup_2 \overline{NRS_{II}}(B)$.
(3) $\underline{NRS_{II}}(A) \cup_2 \underline{NRS_{II}}(B) \subseteq_2 \underline{NRS_{II}}(A \cup_2 B)$, $\overline{NRS_{II}}(A \cap_2 B) \subseteq_2 \overline{NRS_{II}}(A) \cap_2 \overline{NRS_{II}}(B)$.

Definition 5. [22] *Suppose A, B are two* NSs, *then the Hamming distance between A and B is defined as:*

$$d_N(A,B) = \sum_{i=1}^{n}\{|T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| + |F_A(x_i) - F_B(x_i)|\}. \tag{6}$$

3. Type-III NRS

In this chapter, we introduce a new NRS, type-III NRS ($\mathrm{NRS_{III}}$). We provide the differences between the three kinds of NRSs. The properties of $\mathrm{NRS_{III}}$ are also given.

Definition 6. *Suppose (U, R) is an* NAS. *∀A ∈ NS(U), the type-III* LUA *of A, is defined as:*

$$\underline{NRS_{III}}(A) = \underset{y \in U}{\cap_3}(R^c(x,y) \cup_3 A(y)),\ \overline{NRS_{III}}(A) = \underset{y \in U}{\cup_3}(R(x,y) \cap_3 A(y)).$$

The pair $\left(\underline{NRS_{III}}(A), \overline{NRS_{III}}(A)\right)$ *is called* $\mathrm{NRS_{III}}$ *of A.*

Proposition 3. *Suppose (U, R) is an* NAS. *∀A, B ∈ NS(U), we have:*

(1) If $A \subseteq_3 B$, then $\underline{NRS_{III}}(A) \subseteq_3 \underline{NRS_{III}}(B)$, $\overline{NRS_{III}}(A) \subseteq_3 \overline{NRS_{III}}(B)$.
(2) $\underline{NRS_{III}}(A \cap_3 B) \subseteq_3 \underline{NRS_{III}}(A) \cap_3 \underline{NRS_{III}}(B)$, $\underline{NRS_{III}}(A) \cup_3 \underline{NRS_{III}}(B) \subseteq_3 \underline{NRS_{III}}(A \cup_3 B)$.
(3) $\overline{NRS_{III}}(A \cap_3 B) \subseteq_3 \overline{NRS_{III}}(A) \cap_3 \overline{NRS_{III}}(B)$, $\overline{NRS_{III}}(A) \cup_3 \overline{NRS_{III}}(B) \subseteq_3 \overline{NRS_{III}}(A \cup_3 B)$.

Proof. (1) Assume $A \subseteq_3 B$,

Case 1: If $T_A(x) < T_B(x)$, $F_A(x) \geq F_B(x)$, then:

$$T_{\underline{NRS_{III}}(A)}(x) = \underset{y \in U}{\wedge}[F_R(x,y) \vee T_A(y)] \leq \underset{y \in U}{\wedge}[F_R(x,y) \vee T_B(y)] = T_{\underline{NRS_{III}}(B)}(x)$$

$$F_{\underline{NRS_{III}}(A)}(x) = \underset{y \in U}{\vee}[T_R(x,y) \wedge F_A(y)] \geq \underset{y \in U}{\vee}[T_R(x,y) \wedge F_B(y)] = F_{\underline{NRS_{III}}(B)}(x).$$

Hence,

$$\underline{NRS_{III}}(A) \subseteq_3 \underline{NRS_{III}}(B).$$

Case 2: If $T_A(x) = T_B(x)$, $F_A(x) > F_B(x)$, then:

$$T_{\underline{NRS_{III}}(A)}(x) = \underset{y\in U}{\wedge}[F_R(x,y) \vee T_A(y)] = \underset{y\in U}{\wedge}[F_R(x,y) \vee T_B(y)] = T_{\underline{NRS_{III}}(B)}(x)$$

$$F_{\underline{NRS_{III}}(A)}(x) = \underset{y\in U}{\vee}[T_R(x,y) \wedge F_A(y)] \geq \underset{y\in U}{\vee}[T_R(x,y) \wedge F_B(y)] = F_{\underline{NRS_{III}}(B)}(x).$$

Hence,

$$\underline{NRS_{III}}(A) \subseteq_3 \underline{NRS_{III}}(B).$$

Case 3: suppose $T_A(x) = T_B(x)$, $F_A(x) = F_B(x)$ and $I_A(x) \leq I_B(x)$, then:

$$T_{\underline{NRS_{III}}(A)}(x) = \underset{y\in U}{\wedge}[F_R(x,y) \vee T_A(y)] = \underset{y\in U}{\wedge}[F_R(x,y) \vee T_B(y)] = T_{\underline{NRS_{III}}(B)}(x)$$

$$F_{\underline{NRS_{III}}(A)}(x) = \underset{y\in U}{\vee}[T_R(x,y) \wedge F_A(y)] = \underset{y\in U}{\vee}[T_R(x,y) \wedge F_B(y)] = F_{\underline{NRS_{III}}(B)}(x)$$

$$I_{\underline{NRS_{III}}(A)}(x) = \begin{cases} I_A(y_j), & R^c(x,y_j) \subseteq_3 A(y_j) \subseteq_3 A(y_k), y_k, y_j \in U \\ I_{R^c}(x,y_j), & A(y_j) \subseteq_3 R^c(x,y_j) \\ 1, & else \end{cases}$$

$$I_{\underline{MNRS_{III}}^o(B)}(x) = \begin{cases} I_B(y_j), & R_i{}^c(x,y_j) \subseteq_3 B(y_j) \subseteq_3 B(y_k), y_k, y_j \in U \\ I_{R_i{}^c}(x,y_j), & B(y_j) \subseteq_3 R_i{}^c(x,y_j) \\ 1, & else \end{cases}.$$

Hence, $I_{\underline{NRS_{III}}(A)}(x) \leq I_{\underline{NRS_{III}}(B)}(x)$. So $\underline{NRS_{III}}(A) \subseteq_3 \underline{NRS_{III}}(B)$.
Summing up the above, if $A \subseteq_3 B$, then $\underline{NRS_{III}}(A) \subseteq_3 \underline{NRS_{III}}(B)$.
Similarly, we can get $\overline{NRS_{III}}(A) \subseteq_3 \overline{NRS_{III}}(B)$.
(2) According the Definition 6, we have:

$$\begin{aligned} \underline{NRS_{III}}(A \cap_3 B) &= \underset{y\in U}{\cap_3}[R^c(x,y) \cup_3 (A \cap_3 B)(y)] \\ &\subseteq_3 \left[\underset{y\in U}{\cap_3}(R^c(x,y) \cup_3 A(y))\right] \cap_3 \left[\underset{y\in U}{\cap_3}(R^c(x,y) \cup_3 B(y))\right] \\ &= \underline{NRS_{III}}(A) \cap_3 \underline{NRS_{III}}(B). \end{aligned}$$

Similarly,

$$\begin{aligned} \underline{NRS_{III}}(A) \cup_3 \underline{NRS_{III}}(B) &= \left[\underset{y\in U}{\cap_3}(R^c(x,y) \cup_3 A(y))\right] \cup_3 \left[\underset{y\in U}{\cap_3}(R^c(x,y) \cup_3 B(y))\right] \\ &\subseteq_3 \underset{y\in U}{\cap_3}[R^c(x,y) \cup_3 (A \cup_3 B)(y)] \\ &= \underline{NRS_{III}}(A \cup_3 B). \end{aligned}$$

(3) The proof is similar to that of Case 2. □

Example 1. *Define* NAS *(U, R), where U = {x_1, x_2} and R is given in Table 1.*

Table 1. A neutrosophic relation R.

R	x_1	x_2
x_1	(0.4, 0.6, 0.7)	(0.2, 0.2, 0.9)
x_2	(0.7, 0.1, 0.4)	(0.8, 0.8, 0.6)

Suppose A is an NS and A = {(x_1, 0.8, 0.2, 0.1), (x_2, 0.4, 0.9, 0.5). Then, by Definitions 3, 4 and 6, we can get:

$$\begin{array}{rl}\underline{NRS_I}(A)(x_1) = (0.8, 0.8, 0.2), & \underline{NRS_I}(A)(x_2) = (0.6, 0.2, 0.5),\\ \overline{NRS_I}(A)(x_1) = (0.4, 0.6, 0.7), & \overline{NRS_I}(A)(x_2) = (0.7, 0.2, 0.4),\\ \underline{NRS_{II}}(A)(x_1) = (0.8, 0.4, 0.2), & \underline{NRS_{II}}(A)(x_2) = (0.6, 0.9, 0.5),\\ \overline{NRS_{II}}(A)(x_1) = (0.4, 0.2, 0.7), & \overline{NRS_{II}}(A)(x_2) = (0.7, 0.8, 0.4),\\ \underline{NRS_{III}}(A)(x_1) = (0.8, 1, 0.2), & \underline{NRS_{III}}(A)(x_2) = (0.6, 0, 0.5),\\ \overline{NRS_{III}}(A)(x_1) = (0.4, 0.6, 0.7), & \overline{NRS_{III}}(A)(x_2) = (0.7, 0.1, 0.4).\end{array}$$

4. Type-I and Type-II MNRS

We have proposed a kind of multigranulation neutrosophic rough set [30] (we called it type-I multigranulation neutrosophic rough set in this paper). $MNRS_I$ is based on a type-1 operator of NRs. In this chapter, we define the type-II multigranulation neutrosophic rough set ($MNRS_{II}$), which is based on a type-2 operator of NRs.

Definition 7. *[28] Suppose U is a non-empty finite universe, and R_i ($1 \le i \le m$) is a binary* NR *on U. We call the tuple ordered set (U, R_i) the multigranulation neutrosophic approximation space* (MNAS).

Definition 8. *[28] Suppose (U, R_i) is an* MNAS. *$\forall A \in NS(U)$, the type-I optimistic* LUA *of A, represented by $\underline{MNRS_I}^o(A)$ and $\overline{MNRS_I}^o(A)$, is defined as:*

$$\underline{MNRS_I}^o(A)(x) = \overset{m}{\underset{i=1}{\cup_1}} \left(\underset{y \in U}{\cap_1} (R_i{}^c(x, y) \cup_1 A(y)) \right)$$

$$\overline{MNRS_I}^o(A)(x) = \overset{m}{\underset{i=1}{\cap_1}} \left(\underset{y \in U}{\cup_1} (R_i(x, y) \cap_1 A(y)) \right).$$

Then, A is named a definable NS *when $\underline{MNRS_I}^o(A) = \overline{MNRS_I}^o(A)$. Alternatively, we name the pair $\left(\underline{MNRS_I}^o(A), \overline{MNRS_I}^o(A)\right)$ an optimistic* $MNRS_I$.

Definition 9. *[30] Suppose (U, R_i) is an* MNAS. *$\forall A \in NS(U)$, the type-I pessimistic* LUA *of A, represented by $\underline{MNRS_I}^p(A)$ and $\overline{MNRS_I}^p(A)$, is defined as:*

$$\underline{MNRS_I}^p(A)(x) = \overset{m}{\underset{i=1}{\cap_1}} \left(\underset{y \in U}{\cap_1} (R_i{}^c(x, y) \cup_1 A(y)) \right)$$

$$\overline{MNRS_I}^p(A)(x) = \overset{m}{\underset{i=1}{\cup_1}} \left(\underset{y \in U}{\cup_1} (R_i(x, y) \cap_1 A(y)) \right).$$

Similarly, A is named a definable NS *when $\underline{MNRS_I}^p(A) = \overline{MNRS_I}^p(A)$. Alternatively, we name the pair $\left(\underline{MNRS_I}^p(A), \overline{MNRS_I}^p(A)\right)$ a pessimistic* $MNRS_I$.

Definition 10. *Suppose (U, R_i) is an* MNAS. *$\forall A \in NS(U)$, the type-II optimistic* LUA *of A, represented by $\underline{MNRS_{II}}^o(A)$ and $\overline{MNRS_{II}}^o(A)$, is defined as:*

$$\underline{MNRS_{II}}^o(A)(x) = \overset{m}{\underset{i=1}{\cup_2}} \left(\underset{y \in U}{\cap_2} (R_i{}^c(x, y) \cup_2 A(y)) \right)$$

$$\overline{MNRS_{II}}^o(A)(x) = \overset{m}{\underset{i=1}{\cap_2}} \left(\underset{y \in U}{\cup_2} (R_i(x, y) \cap_2 A(y)) \right).$$

Then, A is named a definable NS *when* $\underline{MNRS_{II}}^{o}(A) = \overline{MNRS_{II}}^{o}(A)$. *Alternatively, we name the pair* $\left(\underline{MNRS_{II}}^{o}(A), \overline{MNRS_{II}}^{o}(A)\right)$ *an optimistic* MNRS_{II}.

Definition 11. *Suppose (U, R_i) is an* MNAS. *∀A ∈ NS(U), the type-II pessimistic* LUA *of A, represented by* $\underline{MNRS_{II}}^{p}(A)$ *and* $\overline{MNRS_{II}}^{p}(A)$, *is defined as:*

$$\underline{MNRS_{II}}^{p}(A)(x) = \overset{m}{\underset{i=1}{\cap_2}}\left(\underset{y\in U}{\cap_2}(R_i{}^{c}(x,y) \cup_2 A(y))\right)$$

$$\overline{MNRS_{II}}^{p}(A)(x) = \overset{m}{\underset{i=1}{\cup_2}}\left(\underset{y\in U}{\cup_2}(R_i(x,y) \cap_2 A(y))\right).$$

Similarly, A is named a definable NS *when* $\underline{MNRS_{II}}^{p}(A) = \overline{MNRS_{II}}^{p}(A)$. *Alternatively, we name the pair* $\left(\underline{MNRS_{II}}^{p}(A), \overline{MNRS_{II}}^{p}(A)\right)$ *a pessimistic* MNRS_{II}.

Proposition 4. *Suppose (U, R_i) is an* MNAS. *∀A, B ∈ NS(U), then:*

(1) $\underline{MNRS_{II}}^{o}(A) = \sim \overline{MNRS_{II}}^{o}(\sim A)$, $\underline{MNRS_{II}}^{p}(A) = \sim \overline{MNRS_{II}}^{p}(\sim A)$.
(2) $\overline{MNRS_{II}}^{o}(A) = \sim \underline{MNRS_{II}}^{o}(\sim A)$, $\overline{MNRS_{II}}^{p}(A) = \sim \underline{MNRS_{II}}^{p}(\sim A)$.
(3) $\underline{MNRS_{II}}^{o}(A \cap_2 B) = \underline{MNRS_{II}}^{o}(A) \cap_2 \underline{MNRS_{II}}^{o}(B)$, $\underline{MNRS_{II}}^{p}(A \cap_2 B) = \underline{MNRS_{II}}^{p}(A) \cap_2 \underline{MNRS_{II}}^{p}(B)$.
(4) $\overline{MNRS_{II}}^{o}(A \cup_2 B) = \overline{MNRS_{II}}^{o}(A) \cup_2 \overline{MNRS_{II}}^{o}(B)$, $\overline{MNRS_{II}}^{p}(A \cup_2 B) = \overline{MNRS_{II}}^{p}(A) \cup_2 \overline{MNRS_{II}}^{p}(B)$.
(5) $A \subseteq_2 B \Rightarrow \underline{MNRS_{II}}^{o}(A) \subseteq_2 \underline{MNRS_{II}}^{o}(B), \underline{MNRS_{II}}^{p}(A) \subseteq_2 \underline{MNRS_{II}}^{p}(B)$.
(6) $A \subseteq_2 B \Rightarrow \overline{MNRS_{II}}^{o}(A) \subseteq_2 \overline{MNRS_{II}}^{o}(B), \overline{MNRS_{II}}^{p}(A) \subseteq_2 \overline{MNRS_{II}}^{p}(B)$.
(7) $\underline{MNRS_{II}}^{o}(A) \cup_2 \underline{MNRS_{II}}^{o}(B) \subseteq_2 \underline{MNRS_{II}}^{o}(A \cup_2 B)$, $\underline{MNRS_{II}}^{p}(A) \cup_2 \underline{MNRS_{II}}^{p}(B) \subseteq_2 \underline{MNRS_{II}}^{p}(A \cup_2 B)$.
(8) $\overline{MNRS_{II}}^{o}(A \cap_2 B) \subseteq_2 \overline{MNRS_{II}}^{o}(A) \cap_2 \overline{MNRS_{II}}^{o}(B)$, $\overline{MNRS_{II}}^{p}(A \cap_2 B) \subseteq_2 \overline{MNRS_{II}}^{p}(A) \cap_2 \overline{MNRS_{II}}^{p}(B)$.

Proof. Equations (1), (2), (5), and (6) are obviously according to Definitions 10 and 11. Next, we will prove Equations (3), (4), (7), and (8).

(3) By Definition 10,

$$\begin{aligned}
\underline{MNRS_{II}}^{o}(A \cap_2 B)(x) &= \overset{m}{\underset{i=1}{\cup_2}}\left(\underset{y\in U}{\cap_2}(R_i{}^{c}(x,y) \cup_2 (A \cap_2 B)(y))\right) \\
&= \overset{n}{\underset{i=1}{\cup_2}}\left(\underset{y\in U}{\cap_2}((R_i{}^{c}(x,y) \cup_2 A(y)) \cap (R_i{}^{c}(x,y) \cup_2 B(y)))\right) \\
&= \left(\overset{n}{\underset{i=1}{\cup_2}}\left(\underset{y\in U}{\cap_2}(R_i{}^{c}(x,y) \cup_2 A(y))\right)\right) \cap_2 \left(\overset{n}{\underset{i=1}{\cup_2}}\left(\underset{y\in U}{\cap_2}(R_i{}^{c}(x,y) \cup_2 B(y))\right)\right) \\
&= \underline{MNRS_{II}}^{o}A(x) \cap_2 \underline{MNRS_{II}}^{o}B(y).
\end{aligned}$$

Similarly, from Definition 11, we can get the following:

$$\underline{MNRS_{II}}^{p}(A \cap_2 B) = \underline{MNRS_{II}}^{p}(A) \cap_2 \underline{MNRS_{II}}^{p}(B).$$

(4) The proof is similar to that of Equation (3).

(7) By Definition 10, we can get:

$$\begin{aligned}
T_{\underline{MNRS_{II}}^o(A\cup_2 B)}(x) &= \max_{i=1}^{m}\min_{y\in U}\{\max[F_{R_i}(x,y),(\max(T_A(y),T_B(y)))]\} \\
&= \max_{i=1}^{m}\min_{y\in U}\{\max[(\max(F_{R_i}(x,y),T_A(y))),(\max(F_{R_i}(x,y),T_B(y)))]\} \\
&\geq \max\left\{\left[\max_{i=1}^{m}\min_{y\in U}(\max(F_{R_i}(x,y),T_A(y)))\right],\left[\max_{i=1}^{m}\min_{y\in U}(\max(F_{R_i}(x,y),T_B(y)))\right]\right\} \\
&= \max\left(T_{\underline{MNRS_{II}}^o(A)}(x),T_{\underline{MNRS_{II}}^o(B)}(x)\right).
\end{aligned}$$

$$\begin{aligned}
I_{\underline{MNRS_{II}}^o(A\cup_2 B)}(x) &= \max_{i=1}^{m}\min_{y\in U}\{\max[(1-I_{R_i}(x,y)),(\max(I_A(y),I_B(y)))]\} \\
&= \max_{i=1}^{m}\min_{y\in U}\{\max[(\max((1-I_{R_i}(x,y)),I_A(y))),(\max((1-I_{R_i}(x,y)),I_B(y)))]\} \\
&\geq \max\left\{\left[\max_{i=1}^{m}\min_{y\in U}(\max((1-I_{R_i}(x,y)),I_A(y)))\right],\left[\max_{i=1}^{m}\min_{y\in U}(\max((1-I_{R_i}(x,y)),I_B(y)))\right]\right\} \\
&= \max\left(I_{\underline{MNRS_{II}}^o(A)}(x),I_{\underline{MNRS_{II}}^o(B)}(x)\right).
\end{aligned}$$

$$\begin{aligned}
F_{\underline{MNRS_{II}}^o(A\cup_2 B)}(x) &= \min_{i=1}^{m}\max_{y\in U}\{\min[T_{R_i}(x,y),(\min(F_A(y),F_B(y)))]\} \\
&= \min_{i=1}^{m}\max_{y\in U}\{\min[\min(T_{R_i}(x,y),F_A(y))],[\min(T_{R_i}(x,y),F_B(y))]\} \\
&\leq \min\left\{\left[\min_{i=1}^{m}\max_{y\in U}(\min(T_{R_i}(x,y),F_A(y)))\right],\left[\min_{i=1}^{m}\max_{y\in U}(\min(T_{R_i}(x,y),F_B(y)))\right]\right\} \\
&= \min\left(F_{\underline{MNRS_{II}}^o(A)}(x),F_{\underline{MNRS_{II}}^o(B)}(x)\right).
\end{aligned}$$

Hence, $\underline{MNRS_{II}}^o(A)\cup_2\underline{MNRS_{II}}^o(B)\subseteq_2\underline{MNRS_{II}}^o(A\cup_2 B)$.

Additionally, according to Definition 11, we can get $\underline{MNRS_{II}}^p(A)\cup_2\underline{MNRS_{II}}^p(B)\subseteq_2\underline{MNRS_{II}}^p(A\cup_2 B)$.

(8) The proof is similar to that of Equation (7). □

Remark 1. *Note that if the* NRs *are the same one, then the optimistic (pessimistic)* MNRS_{II} *degenerates into* NRS_{II} *in Section 2.*

5. Type-III MNRS

In this chapter, MNRS_{III}, which is based on a type-3 inclusion relation and corresponding union and intersection relations, is proposed and their characterizations are provided.

Definition 12. *Suppose (U, R_i) is an* MNAS. *∀A ∈ NS(U), the type-III optimistic* LUA *of A, represented by* $\underline{MNRS_{III}}^o(A)$ *and* $\overline{MNRS_{III}}^o(A)$*, is defined as:*

$$\underline{MNRS_{III}}^o(A)(x)=\bigcup_{i=1}^{m}{}_3\left(\bigcap_{y\in U}{}_3\,(R_i{}^c(x,y)\cup_3 A(y))\right)$$

$$\overline{MNRS_{III}}^o(A)(x)=\bigcap_{i=1}^{m}{}_3\left(\bigcup_{y\in U}{}_3\,(R_i(x,y)\cap_3 A(y))\right).$$

Then, A is named a definable NS *when* $\underline{MNRS_{III}}^o(A)=\overline{MNRS_{III}}^o(A)$. *Alternatively, we name the pair* $\left(\underline{MNRS_{III}}^o(A),\overline{MNRS_{III}}^o(A)\right)$ *an optimistic* MNRS_{III}.

Definition 13. *Suppose (U, R_i) is an* MNAS. *∀A ∈ NS(U), the type-III pessimistic* LUA *of A, represented by* $\underline{MNRS_{III}}^p(A)$ *and* $\overline{MNRS_{III}}^p(A)$*, is defined as:*

$$\underline{MNRS_{III}}^p(A)(x)=\bigcap_{i=1}^{m}{}_3\left(\bigcap_{y\in U}{}_3\,(R_i{}^c(x,y)\cup_3 A(y))\right)$$

$$\overline{MNRS_{III}}^{p}(A)(x) = \overset{m}{\underset{i=1}{\cup_3}} \left(\underset{y \in U}{\cup_3} (R_i(x,y) \cap_3 A(y)) \right).$$

Similarly, A is named a definable NS *when* $\underline{MNRS_{III}}^{p}(A) = \overline{MNRS_{III}}^{p}(A)$. *Alternatively, we name the pair* $\left(\underline{MNRS_{III}}^{p}(A), \overline{MNRS_{III}}^{p}(A)\right)$ *a pessimistic* MNRS_{III}.

Proposition 5. *Suppose (U, R_i) is an* MNAS. *$\forall A, B \in NS(U)$, then:*

(1) $\underline{MNRS_{III}}^{o}(A) = \sim \overline{MNRS_{III}}^{o}(\sim A)$, $\underline{MNRS_{III}}^{p}(A) = \sim \overline{MNRS_{III}}^{p}(\sim A)$.

(2) $\overline{MNRS_{III}}^{o}(A) = \sim \underline{MNRS_{III}}^{o}(\sim A)$, $\overline{MNRS_{III}}^{p}(A) = \sim \underline{MNRS_{III}}^{p}(\sim A)$.

(3) $A \subseteq_3 B \Rightarrow \underline{MNRS_{III}}^{o}(A) \subseteq_3 \underline{MNRS_{III}}^{o}(B), \underline{MNRS_{III}}^{p}(A) \subseteq_3 \underline{MNRS_{III}}^{p}(B)$.

(4) $A \subseteq_3 B \Rightarrow \overline{MNRS_{III}}^{o}(A) \subseteq_3 \overline{MNRS_{III}}^{o}(B), \overline{MNRS_{III}}^{p}(A) \subseteq_3 \overline{MNRS_{III}}^{p}(B)$.

(5) $\underline{MNRS_{III}}^{o}(A \cap_3 B) \subseteq_3 \underline{MNRS_{III}}^{o}(A) \cap_3 \underline{MNRS_{III}}^{o}(B)$, $\underline{MNRS_{III}}^{p}(A \cap_3 B) \subseteq_3 \underline{MNRS_{III}}^{p}(A) \cap_3 \underline{MNRS_{III}}^{p}(B)$.

(6) $\overline{MNRS_{III}}^{o}(A) \cup_3 \overline{MNRS_{III}}^{o}(B) \subseteq_3 \overline{MNRS_{III}}^{o}(A \cup_3 B)$, $\overline{MNRS_{III}}^{p}(A) \cup_3 \overline{MNRS_{III}}^{p}(B) \subseteq_3 \overline{MNRS_{III}}^{p}(A \cup_3 B)$.

(7) $\underline{MNRS_{III}}^{o}(A) \cup_3 \underline{MNRS_{III}}^{o}(B) \subseteq_3 \underline{MNRS_{III}}^{o}(A \cup_3 B)$, $\underline{MNRS_{III}}^{p}(A) \cup_3 \underline{MNRS_{III}}^{p}(B) \subseteq_3 \underline{MNRS_{III}}^{p}(A \cup_3 B)$.

(8) $\overline{MNRS_{III}}^{o}(A \cap_3 B) \subseteq_3 \overline{MNRS_{III}}^{o}(A) \cap_3 \overline{MNRS_{III}}^{o}(B)$, $\overline{MNRS_{III}}^{p}(A \cap_3 B) \subseteq_3 \overline{MNRS_{III}}^{p}(A) \cap_3 \overline{MNRS_{III}}^{p}(B)$.

Proof. Equations (1) and (2) can be directly derived from Definitions 12 and 13. We only provide the proof of Equations (3)–(8).

(3) Suppose $A \subseteq_3 B$, then:

Case 1: If $T_A(x) < T_B(x)$, $F_A(x) \geq F_B(x)$, then:

$$T_{\underline{MNRS_{III}}^{o}(A)}(x) = \overset{m}{\underset{i=1}{\vee}} \underset{y \in U}{\wedge} \left[F_{R_i}(x,y) \vee T_A(y)\right] \leq \overset{m}{\underset{i=1}{\vee}} \underset{y \in U}{\wedge} \left[F_{R_i}(x,y) \vee T_B(y)\right] = T_{\underline{MNRS_{III}}^{o}(B)}(x)$$

$$F_{\underline{MNRS_{III}}^{o}(A)}(x) = \overset{m}{\underset{i=1}{\wedge}} \underset{y \in U}{\vee} \left[T_{R_i}(x,y) \wedge F_A(y)\right] \geq \overset{m}{\underset{i=1}{\wedge}} \underset{y \in U}{\vee} \left[T_{R_i}(x,y) \wedge F_B(y)\right] = F_{\underline{MNRS_{III}}^{o}(B)}(x).$$

Hence, $\underline{MNRS_{III}}^{o}(A) \subseteq_3 \underline{MNRS_{III}}^{o}(B)$.

Case 2: If $T_A(x) = T_B(x)$, $F_A(x) > F_B(x)$, then:

$$T_{\underline{MNRS_{III}}^{o}(A)}(x) = \overset{m}{\underset{i=1}{\vee}} \underset{y \in U}{\wedge} \left[F_{R_i}(x,y) \vee T_A(y)\right] = \overset{m}{\underset{i=1}{\vee}} \underset{y \in U}{\wedge} \left[F_{R_i}(x,y) \vee T_B(y)\right] = T_{\underline{MNRS_{III}}^{o}(B)}(x)$$

$$F_{\underline{MNRS_{III}}^{o}(A)}(x) = \overset{m}{\underset{i=1}{\wedge}} \underset{y \in U}{\vee} \left[T_{R_i}(x,y) \wedge F_A(y)\right] \geq \overset{m}{\underset{i=1}{\wedge}} \underset{y \in U}{\vee} \left[T_{R_i}(x,y) \wedge F_B(y)\right] = F_{\underline{MNRS_{III}}^{o}(B)}(x).$$

Hence, $\underline{MNRS_{III}}^{o}(A) \subseteq_3 \underline{MNRS_{III}}^{o}(B)$.

Case 3: suppose $T_A(x) = T_B(x)$, $F_A(x) = F_B(x)$ and $I_A(x) \leq I_B(x)$, then:

$$T_{\underline{MNRS_{III}}^{o}(A)}(x) = \overset{m}{\underset{i=1}{\vee}} \underset{y \in U}{\wedge} \left[F_{R_i}(x,y) \vee T_A(y)\right] = \overset{m}{\underset{i=1}{\vee}} \underset{y \in U}{\wedge} \left[F_{R_i}(x,y) \vee T_B(y)\right] = T_{\underline{MNRS_{III}}^{o}(B)}(x)$$

$$F_{\underline{MNRS_{III}}^{o}(A)}(x) = \overset{m}{\underset{i=1}{\wedge}} \underset{y \in U}{\vee} \left[T_{R_i}(x,y) \wedge F_A(y)\right] \geq \overset{m}{\underset{i=1}{\wedge}} \underset{y \in U}{\vee} \left[T_{R_i}(x,y) \wedge F_B(y)\right] = F_{\underline{MNRS_{III}}^{o}(B)}(x)$$

$$I_{\underline{MNRS_{III}}^{o}(A)}(x) = \begin{cases} I_A(y_j), R_i{}^c(x,y_j) \subseteq_3 A(y_j) \subseteq_3 A(y_k), y_k, y_j \in U \\ I_{R_i{}^c}(x,y_j), A(y_j) \subseteq_3 R_i{}^c(x,y_j) \\ 0, else \end{cases}$$

$$I_{\underline{MNRS_{III}}^{o}(B)}(x) = \begin{cases} I_B(y_j), R_i{}^c(x, y_j) \subseteq_3 B(y_j) \subseteq_3 B(y_k), y_k, y_j \in U \\ I_{R_i{}^c}(x, y_j), B(y_j) \subseteq_3 R_i{}^c(x, y_j) \\ 0, else \end{cases}.$$

Hence, $I_{\underline{MNRS_{III}}^{o}(A)}(x) \leq I_{\underline{MNRS_{III}}^{o}(B)}(x)$. So, $\underline{MNRS_{III}}^{o}(A) \subseteq_3 \underline{MNRS_{III}}^{o}(B)$.
Summing up the above, if $A \subseteq_3 B$, then $\underline{MNRS_{III}}^{o}(A) \subseteq_3 \underline{MNRS_{III}}^{o}(B)$.
Similarly, we can get $\underline{MNRS_{III}}^{p}(A) \subseteq_3 \underline{MNRS_{III}}^{p}(B)$.
(4) The proof is similar to that of Equation (3).
(5) From Definition 12, we have:

$$\begin{aligned} \underline{MNRS_{III}}^{o}(A \cap_3 B) &= \overset{m}{\underset{i=1}{\cup_3}} \left(\underset{y \in U}{\cap_3} (R_i{}^c(x, y) \cup_3 (A(y) \cap_3 B(y))) \right) \\ &\subseteq_3 \overset{m}{\underset{i=1}{\cup_3}} \left(\underset{y \in U}{\cap_3} ((R_i{}^c(x, y) \cup_3 A(y)) \cap_3 (R_i{}^c(x, y) \cup_3 B(y))) \right) \\ &\subseteq_3 \left(\overset{m}{\underset{i=1}{\cup_3}} \left(\underset{y \in U}{\cap_3} (R_i{}^c(x, y) \cup_3 A(y)) \right) \right) \cap_3 \left(\overset{m}{\underset{i=1}{\cup_3}} \left(\underset{y \in U}{\cap_3} (R_i{}^c(x, y) \cup_3 B(y)) \right) \right) \\ &= \underline{MNRS_{III}}^{o}(A) \cap_3 \underline{MNRS_{III}}^{o}(B). \end{aligned}$$

Similarly, from Definition 13, we can get $\underline{MNRS_{III}}^{p}(A \cap_3 B) \subseteq_3 \underline{MNRS_{III}}^{p}(A) \cap_3 \underline{MNRS_{III}}^{p}(B)$.
(6) From Definition 12, we have:

$$\begin{aligned} \overline{MNRS_{III}}^{o}(A) \cup_3 \overline{MNRS_{III}}^{o}(B) &= \left(\overset{m}{\underset{i=1}{\cap_3}} \left(\underset{y \in U}{\cup_3} (R_i(x, y) \cap_3 A(y)) \right) \right) \cup_3 \left(\overset{m}{\underset{i=1}{\cap_3}} \left(\underset{y \in U}{\cup_3} (R_i(x, y) \cap_3 B(y)) \right) \right) \\ &\subseteq_3 \overset{m}{\underset{i=1}{\cap_3}} \left(\underset{y \in U}{\cup_3} ((R_i(x, y) \cap_3 A(y)) \cup_3 (R_i(x, y) \cap_3 B(y))) \right) \\ &\subseteq_3 \overset{m}{\underset{i=1}{\cap_3}} \left(\underset{y \in U}{\cup_3} (R_i(x, y) \cap_3 (A(y) \cup_3 B(y))) \right) \\ &= \overline{MNRS_{III}}^{o}(A \cup_3 B). \end{aligned}$$

Similarly, from Definition 13, we can get $\overline{MNRS_{III}}^{p}(A \cup_3 B) = \overline{MNRS_{III}}^{p}(A) \cup_3 \overline{MNRS_{III}}^{p}(B)$.
(7) From Definition 12, we have:

$$\begin{aligned} \underline{MNRS_{III}}^{o}(A \cup_3 B) &= \overset{m}{\underset{i=1}{\cup_3}} \left(\underset{y \in U}{\cap_3} (R_i{}^c(x, y) \cup_3 (A \cup_3 B)(y)) \right) \\ &= \overset{m}{\underset{i=1}{\cup_3}} \left(\underset{y \in U}{\cap_3} (R_i{}^c(x, y) \cup_3 (A(y) \cup_3 B(y))) \right) \\ &\supseteq_3 \overset{m}{\underset{i=1}{\cup_3}} \left(\left(\left[\underset{y \in U}{\cap_3} (R_i{}^c(x, y) \cup_3 A(y)) \right] \cup_3 \left[\underset{y \in U}{\cap_3} (R_i{}^c(x, y) \cup_3 B(y)) \right] \right) \right) \\ &= \left(\overset{m}{\underset{i=1}{\cup_3}} \left[\underset{y \in U}{\cap_3} (R_i{}^c(x, y) \cup_3 A(y)) \right] \right) \cup_3 \left(\overset{m}{\underset{i=1}{\cup_3}} \left[\underset{y \in U}{\cap_3} (R_i{}^c(x, y) \cup_3 B(y)) \right] \right) \\ &= \underline{MNRS_{III}}^{o}(A) \cup_3 \underline{MNRS_{III}}^{o}(B). \end{aligned}$$

Hence, $\underline{MNRS_{III}}^{o}(A) \cup_3 \underline{MNRS_{III}}^{o}(B) \subseteq_3 \underline{MNRS_{III}}^{o}(A \cup_3 B)$.

Additionally, from Definition 13, we can get $\underline{MNRS_{III}}^{p}(A) \cup_3 \underline{MNRS_{III}}^{p}(B) \subseteq_3 \underline{MNRS_{III}}^{p}(A \cup_3 B)$.

(8) From Definition 12, we have:

$$\begin{aligned}\overline{MNRS_{III}}^{o}(A \cap_3 B) &= \overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in U}{\cup_3}(R_i(x,y) \cap_3 (A \cap_3 B)(y))\right)\\ &= \overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in U}{\cup_3}(R_i(x,y) \cap_3 (A(y) \cap_3 B(y)))\right)\\ &\subseteq_3 \overset{m}{\underset{i=1}{\cap_3}}\left(\left[\underset{y\in U}{\cup_2}(R_i(x,y) \cap_3 A(y))\right] \cap_3 \left[\underset{y\in U}{\cup_3}(R_i(x,y) \cap_3 B(y))\right]\right)\\ &= \left(\overset{m}{\underset{i=1}{\cap_3}}\left[\underset{y\in U}{\cup_3}(R_i(x,y) \cap_3 A(y))\right]\right) \cap_3 \left(\overset{m}{\underset{i=1}{\cap_3}}\left[\underset{y\in U}{\cup_3}(R_i(x,y) \cap_3 B(y))\right]\right)\\ &= \overline{MNRS_{III}}^{o}(A) \cap_3 \overline{MNRS_{III}}^{o}(B).\end{aligned}$$

Hence, $\overline{MNRS_{III}}^{o}(A \cap_3 B) \subseteq_3 \overline{MNRS_{III}}^{o}(A) \cap_3 \overline{MNRS_{III}}^{o}(B)$.

Similarly, from Definition 13, we can get $\overline{MNRS_{III}}^{p}(A \cap_3 B) \subseteq_3 \overline{MNRS_{III}}^{p}(A) \cap_3 \overline{MNRS_{III}}^{p}(B)$.

□

Remark 2. *Note that if the* NRs *are the same one, then the optimistic (pessimistic)* MNRS$_{\text{III}}$ *degenerates into* NRS$_{\text{III}}$ *in Section 3.*

6. Type-III MNRS in Two Universes with Its Applications

In this chapter, we propose the concept of MNRS$_{\text{III}}$ in two universes and use it to deal with the MAGDM problem.

Definition 14. *[28] Suppose U, V are two non-empty finite universes, and $R_i \in NS(U \times V)$ ($1 \leq i \leq m$) is a binary* NR. *We call (U, V, R_i) the* MNAS *in two universes.*

Definition 15. *Suppose (U, V, R_i) is an* MNAS *in two universes. $\forall A \in NS(V)$ and $x \in U$, the type-III optimistic* LUA *of A in (U, V, R_i), represented by $\underline{MNRS_{III}}^{o}(A)$ and $\overline{MNRS_{III}}^{o}(A)$, is defined as:*

$$\underline{MNRS_{III}}^{o}(A)(x) = \overset{m}{\underset{i=1}{\cup_3}}\left(\underset{y\in V}{\cap_3}(R_i^{\,c}(x,y) \cup_3 A(y))\right)$$

$$\overline{MNRS_{III}}^{o}(A)(x) = \overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in V}{\cup_3}(R_i(x,y) \cap_3 A(y))\right).$$

Then, A is named a definable NS *in two universes when $\underline{MNRS_{III}}^{o}(A) = \overline{MNRS_{III}}^{o}(A)$. Alternatively, we name the pair $\left(\underline{MNRS_{III}}^{o}(A), \overline{MNRS_{III}}^{o}(A)\right)$ an optimistic* MNRS$_{\text{III}}$ *in two universes.*

Definition 16. *Suppose (U, V, R_i) is an* MNAS *in two universes. $\forall A \in NS(V)$ and $x \in U$, the type-III pessimistic* LUA *of A in (U, V, R_i), denoted by $\underline{MNRS_{III}}^{p}(A)$ and $\overline{MNRS_{III}}^{p}(A)$, is defined as follows:*

$$\underline{MNRS_{III}}^{p}(A)(x) = \overset{m}{\underset{i=1}{\cap_3}}\left(\underset{y\in V}{\cap_3}(R_i^{\,c}(x,y) \cup_3 A(y))\right)$$

$$\overline{MNRS_{III}}^{p}(A)(x) = \overset{m}{\underset{i=1}{\cup_3}}\left(\underset{y\in V}{\cup_3}(R_i(x,y) \cap_3 A(y))\right).$$

Similarly, A is named a definable NS *when $\underline{MNRS_{III}}^{p}(A) = \overline{MNRS_{III}}^{p}(A)$. Alternatively, we name the pair $\left(\underline{MNRS_{III}}^{p}(A), \overline{MNRS_{III}}^{p}(A)\right)$ a pessimistic* MNRS$_{\text{III}}$ *in two universes.*

Remark 3. *Note that if the two domains are the same, then the optimistic (pessimistic)* $\mathrm{MNRS_{III}}$ *in two universes degenerates into the optimistic (pessimistic)* $\mathrm{MNRS_{III}}$ *in a single universe in Section 5.*

The MAGDM problem is becoming more and more generally present in our daily life. MAGDM means to select or rank all the feasible alternatives in various criterions. There are many ways to solve the MAGDM problem, but we use MNRS to solve it in this paper. Next, we give the basic description of the considered MAGDM problem.

For the car-ranking question, suppose $U = \{x_1, x_2, \ldots, x_n\}$ is the decision set and $V = \{y_1, y_2, \ldots, y_m\}$ is the criteria set in which x_1 represents "very popular", x_2 represents "popular", x_3 represents "less popular", ..., x_n represents "not popular", y_1 represents the vehicle type", y_2 represents the size of the space, y_3 represents the ride height, y_4 represents quality, and ..., y_m represents length of durability. Then, l selection experts make evaluations about the criteria sets according to their own experiences. Here, the evaluations were shown by NRs. Next, we calculate the degree of popularity for a given car. Therefore, we need to use MGNRS to solve the above problem. For the MAGDM problem under a multigranulation neutrosophic environment, the optimistic lower approximation can be regarded as an optimistic risk decision, and the optimistic upper approximation can be regarded as an optimistic conservative decision. Additionally, the pessimistic lower approximation can be regarded as a pessimistic risk decision and the pessimistic upper approximation can be regarded as a pessimistic conservative decision. According to the distance of neutrosophic sets, we define the difference function $d_N(A, B)(x_i) = (1/3)(|T_A(x_i) - T_B(x_i)| + |I_A(x_i) - I_B(x_i)| + |F_A(x_i) - F_B(x_i)|)$. We used the difference function to represent the distance of optimistic (pessimistic) upper and lower approximation. The smaller the value of the distance is, the better the alternative x_i is, because the risk decision and the conservative decision are close. By comparing the distance value, all alternatives can be ranked and we can choose the optimal alternative. In this paper, we only used three kinds of optimistic upper and lower approximation to decision-making.

Next, we show the process of the above car-ranking question based on MGNRSs over two universes. Let $R_l \in NR(U \times V)$ be NRs from U to V, where $\forall(x_i, y_j) \in U \times V$, $R_l(x_i, y_j)$ denotes the degree of popularity for criteria set y_j ($y_j \in V$). R_l can be obtained according to experts' experience. Given a car A, according to the unconventional questionnaire (suppose there are three options—"like", "not like", and "neutral" to choose for each of the criteria sets, and everyone can choose one or more options), then we can get the popularity of every criterion as described by an NS A in the universe V according to the questionnaire. By use of the following Algorithm 1, we can determine the degree of popularity of the given car A.

Algorithm 1 Decision algorithm

Input Multigranulation neutrosophic decision information systems (U, V, $\mathbf{R}$).

Output The degree of popularity of the given car.

Step 1 Computing three kinds of optimistic multigranulation LUA $\underline{MNRS_I}^o(A)$, $\overline{MNRS_I}^o(A)$, $\underline{MNRS_{II}}^o(A)$, $\overline{MNRS_{II}}^o(A)$, $\underline{MNRS_{III}}^o(A)$, $\overline{MNRS_{III}}^o(A)$.

Step 2 Calculate $d(\underline{MNRS_I}^o(x_i), \overline{MNRS_I}^o(x_i))$, $d(\underline{MNRS_{II}}^o(x_i), \overline{MNRS_{II}}^o(x_i))$ and $d(\underline{MNRS_{III}}^o(x_i), \overline{MNRS_{III}}^o(x_i))$.

Step 3 The best choice is to select x_h (which means that the most welcome degree is x_h) if $d(\underline{MNRS}^o(x_h), \overline{MNRS}^o(x_h)) = \min_{i\in\{1,2,\cdots,n\}} d(\underline{MNRS}^o(x_i), \overline{MNRS}^o(x_i))$.

Step 4 If h has two or more values, then each x_h will be the best choice. In this case, the car may have two or more popularities and each x_k will be regarded as the most possible popularity; otherwise, we use other methods to make a decision.

Next, we use an example to explain the algorithm.

Let $U = \{x_1, x_2, x_3, x_4\}$ be the decision set, in which x_1 denotes "very popular", x_2 denotes "popular", x_3 denotes "less popular", and x_4 denotes "not popular". Let $V = \{y_1, y_2, y_3, y_4, y_5\}$ be criteria sets, in which y_1 denotes the vehicle type, y_2 denotes the size of the space, y_3 denotes the ride height, y_4 denotes quality, and y_5 denotes length of durability.

Suppose that R_1, R_2, and R_3 are given by three invited experts. They provide their evaluations for all criteria y_j with respect to decision set elements x_i. The evaluation R_1, R_2, and R_3 are NRs between attribute set V and decision evaluation set U., that is., there are $R_1, R_2, R_3 \in NR(U \times V)$.

Suppose three experts present their judgment (the neutrosophic relation R_1, R_2, and R_3) for the attribute and decision sets in Tables 2–4:

Table 2. Neutrosophic relation R_1.

R_1	y_1	y_2	y_3	y_4	y_5
x_1	(0.8, 0.6, 0.5)	(0.2, 0.3, 0.9)	(0, 0, 1)	(0.7, 0.5, 0.6)	(0, 0, 1)
x_2	(0.6, 0.4, 0.6)	(0.9, 0.3, 0.4)	(1, 0, 0)	(0, 0, 1)	(0.3, 0.6, 0.7)
x_3	(0.2, 0.5, 0.9)	(0.6, 0.7, 0.5)	(0.8, 0.7, 0.8)	(0, 0, 1)	(1, 0, 0)
x_4	(0.6, 0.4, 0.7)	(0, 0, 1)	(0, 0, 1)	(0.9, 0.8, 0.1)	(0, 0, 1)

Table 3. Neutrosophic relation R_2.

R_2	y_1	y_2	y_3	y_4	y_5
x_1	(0.9, 0.3, 0.6)	(0, 0, 1)	(0, 0, 1)	(0.5, 0.6, 0.5)	(0.2, 0.3, 0.9)
x_2	(0.3, 0.7, 0.8)	(0.7, 0.5, 0.6)	(0.9, 0.1, 0.1)	(0, 0, 1)	(0.4, 0.5, 0.8)
x_3	(0.1, 0.6, 0.8)	(0.3, 0.6, 0.5)	(0.7, 0.3, 0.6)	(0, 0, 1)	(1, 0, 0)
x_4	(0.7, 0.5, 0.6)	(0, 0, 1)	(0, 0, 1)	(1, 0, 0)	(0, 0, 1)

Table 4. Neutrosophic relation R_3.

R_3	y_1	y_2	y_3	y_4	y_5
x_1	(0.6, 0.9, 0.4)	(0.1, 0.1, 0.8)	(0.1, 0, 0.9)	(0.8, 0.4, 0.8)	(0, 0, 1)
x_2	(0.5, 0.6, 0.6)	(0.6, 0.2, 0.7)	(1, 0, 0)	(0, 0, 1)	(0, 0, 1)
x_3	(0.1, 0.4, 0.7)	(0.2, 0.2, 0.7)	(0.5, 0.7, 0.6)	(0, 0, 1)	(0.9, 0.1, 0.2)
x_4	(0.6, 0.3, 0.4)	(0, 0, 1)	(0, 0, 1)	(0.7, 0.5, 0.4)	(0, 0, 1)

Suppose A is a car and each criterion in V is as follows:

$$A = \{(y_1, 0.9, 0.2, 0.2), (y_2, 0.2, 0.7, 0.8), (y_3, 0, 1, 0.3), (y_4, 0.7, 0.6, 0.3), (y_5, 0.1, 0.8, 0.9)\}.$$

Then, we can calculate the three kinds of optimistic LUAs of A as follow:

$$\begin{gathered}
\underline{MNRS_I}^o(A)(x_1) = (0.8, 1, 0.3),\ \underline{MNRS_I}^o(A)(x_2) = (0.1, 0.9, 0.6),\\
\underline{MNRS_I}^o(A)(x_3) = (0.2, 0.8, 0.9),\ \underline{MNRS_I}^o(A)(x_4) = (0.7, 1, 0.3),\\
\overline{MNRS_I}^o(A)(x_1) = (0.7, 0.6, 0.5),\ \overline{MNRS_I}^o(A)(x_2) = (0.3, 0.6, 0.3),\\
\overline{MNRS_I}^o(A)(x_3) = (0.2, 0.6, 0.8),\ \overline{MNRS_I}^o(A)(x_4) = (0.7, 0.5, 0.4),\\
\underline{MNRS_{II}}^o(A)(x_1) = (0.8, 0.6, 0.3),\ \underline{MNRS_{II}}^o(A)(x_2) = (0.1, 0.6, 0.6),\\
\underline{MNRS_{II}}^o(A)(x_3) = (0.2, 0.6, 0.9),\ \underline{MNRS_{II}}^o(A)(x_4) = (0.7, 0.6, 0.3),\\
\overline{MNRS_{II}}^o(A)(x_1) = (0.7, 0.4, 0.5),\ \overline{MNRS_{II}}^o(A)(x_2) = (0.3, 0.2, 0.3),\\
\overline{MNRS_{II}}^o(A)(x_3) = (0.2, 0.6, 0.8),\ \overline{MNRS_{II}}^o(A)(x_4) = (0.7, 0.2, 0.4),\\
\underline{MNRS_{III}}^o(A)(x_1) = (0.8, 0, 0.3),\ \underline{MNRS_{III}}^o(A)(x_2) = (0.1, 0, 0.6),\\
\underline{MNRS_{III}}^o(A)(x_3) = (0.2, 0.9, 0.9),\ \underline{MNRS_{III}}^o(A)(x_4) = (0.7, 0.6, 0.3),\\
\overline{MNRS_{III}}^o(A)(x_1) = (0.7, 1, 0.5),\ \overline{MNRS_{III}}^o(A)(x_2) = (0.3, 0, 0.3),\\
\overline{MNRS_{III}}^o(A)(x_3) = (0.2, 0.7, 0.8)\ \overline{MNRS_{III}}^o(A)(x_4) = (0.7, 0.5, 0.4).
\end{gathered}$$

Therefore, we can get:

$d(\underline{MNRS_I}^o(x_1), \overline{MNRS_I}^o(x_1)) = 0.7/3, d(\underline{MNRS_I}^o(x_2), \overline{MNRS_I}^o(x_2)) = 0.8/3,$
$d(\underline{MNRS_I}^o(x_3), \overline{MNRS_I}^o(x_3)) = 0.1, d(\underline{MNRS_I}^o(x_4), \overline{MNRS_I}^o(x_4)) = 0.2,$
$d(\underline{MNRS_{II}}^o(x_1), \overline{MNRS_{II}}^o(x_1)) = 0.5/3, d(\underline{MNRS_{II}}^o(x_2), \overline{MNRS_{II}}^o(x_2)) = 0.3,$
$d(\underline{MNRS_{II}}^o(x_3), \overline{MNRS_{II}}^o(x_3)) = 0.1/3, d(\underline{MNRS_{II}}^o(x_4), \overline{MNRS_{II}}^o(x_4)) = 0.5/3,$
$d(\underline{MNRS_{III}}^o(x_1), \overline{MNRS_{III}}^o(x_1)) = 1.3/3, d(\underline{MNRS_{III}}^o(x_2), \overline{MNRS_{III}}^o(x_2)) = 0.5/3,$
$d(\underline{MNRS_{III}}^o(x_3), \overline{MNRS_{III}}^o(x_3)) = 0.1, d(\underline{MNRS_{III}}^o(x_4), \overline{MNRS_{III}}^o(x_4)) = 0.2/3.$

Thus, for the type-I and type-II MNRS, the optimistic best choice is to select x_3, that is, this car is less popular; for the type-III MNRS, the optimistic best choice is to select x_4, that is, this car is not popular.

7. Conclusions

NRS and MNRS are extensions of the Pawlak rough set theory. In this paper, we analysed the NRS_I and NRS_{II}, we proposed model NRS_{III}, and used an example to outline the differences between the three kinds of NRS. We gave the definition of $MNRS_{III}$, which is based on the type-3 operator relation of NS, and considered their properties. Furthermore, we proposed $MNRS_{III}$ in two universes and we presented an algorithm of the MAGDM problem based on it.

In the future, we will be researching other types of fusions of MGRSs and NSs. We will also study the applications of concepts in this paper to some algebraic systems (for example, pseudo-BCI algebras, neutrosophic triplet groups, see [30,31]).

Author Contributions: X.Z. and C.B. initiated the research and wrote the paper, S.S. participated in some of the research work, and F.S. supervised the research work and provided helpful suggestions.

Funding: This work was supported by the National Natural Science Foundation of China (Grant No. 61573240) and the Graduate Student Innovation Project of Shanghai Maritime University 2017ycx082.

Conflicts of Interest: The authors declare no conflict of interest.

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A Multi-Criteria Group Decision-Making Method with Possibility Degree and Power Aggregation Operators of Single Trapezoidal Neutrosophic Numbers

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Received: 4 October 2018; Accepted: 27 October 2018; Published: 2 November 2018

Abstract: Single valued trapezoidal neutrosophic numbers (SVTNNs) are very useful tools for describing complex information, because of their advantage in describing the information completely, accurately and comprehensively for decision-making problems. In the paper, a method based on SVTNNs is proposed for dealing with multi-criteria group decision-making (MCGDM) problems. Firstly, the new operations SVTNNs are developed for avoiding evaluation information aggregation loss and distortion. Then the possibility degrees and comparison of SVTNNs are proposed from the probability viewpoint for ranking and comparing the single valued trapezoidal neutrosophic information reasonably and accurately. Based on the new operations and possibility degrees of SVTNNs, the single valued trapezoidal neutrosophic power average (SVTNPA) and single valued trapezoidal neutrosophic power geometric (SVTNPG) operators are proposed to aggregate the single valued trapezoidal neutrosophic information. Furthermore, based on the developed aggregation operators, a single valued trapezoidal neutrosophic MCGDM method is developed. Finally, the proposed method is applied to solve the practical problem of the most appropriate green supplier selection and the rank results compared with the previous approach demonstrate the proposed method's effectiveness.

Keywords: single valued trapezoidal neutrosophic number; multi-criteria group decision making; possibility degree; power aggregation operators

1. Introduction

Multi-criteria decision-making (MCDM) problems are important issues in practice and many MCDM methods have been proposed to deal with such issues. Due to the vagueness of human being thinking and the increased complexity of the objects, there are always much uncertainty, incomplete, indeterminate and inconsistent information in evaluating objects. Traditionally, vagueness information is always described by fuzzy sets (FSs) [1] using the membership function, intuitionistic fuzzy sets (IFSs) [2] using membership and non-membership functions and hesitant fuzzy sets (HFSs) [3] using one/several possible membership degrees. Many fuzzy methods are proposed, for example, Medina [4] extends the fuzzy soft set by Multi-adjoint concept lattices, Pozna & Precup [5] proposed the operator and application to a fuzzy model, Jane et al. [6] proposed fuzzy S-tree for medical image retrieval and Kumar & Jarial [7] proposed a hybrid clustering method based on an improved artificial bee colony and fuzzy c-means algorithm. However, fuzzy sets cannot deal with the indeterminate information and inconsistent information which exists commonly in complex MCDM problems. As a generalization of the IFSs [2], neutrosophic sets (NSs) [8–10] are proposed to deal with the uncertainty, incomplete, indeterminate and inconsistent information by using the truth-membership, indeterminacy-membership and falsity-membership functions.

Due to the advantages of handling uncertainty, imprecise, incomplete, indeterminate and inconsistent information existing in real world, NSs have attracted many researchers' attentions However NSs are proposed from the philosophical point of view, it is difficult to be directly applied in real scientific and engineering areas without specific descriptions. Therefore, in accordance with the real demand difference, three main subsets of NSs were proposed, namely single valued neutrosophic sets (SVNSs) [11], interval neutrosophic sets (INSs) [12] and multi-valued neutrosophic set (MVNSs) [13]. Based on the aforementioned sets by specifying the NSs, many MCDM methods were developed, which can be classified as three main aspects: aggregation operators, measures and the extension of classic decision-making methods. These methods have been successfully applied in many areas, such as medical diagnosis [14,15], medical treatment [16], neural networks [17], supplier selection [18,19] and green product development [20].

With regard to aggregation operators of SVNSs, Liu and Wang [21] proposed a single-valued neutrosophic normalized weighted Bonferroni mean operator, Liu et al. [22] proposed the generalized neutrosophic operators, Sahin [23] developed the neutrosophic weighted operators. Considering real situations, INSs is more suitable and flexible for describing incomplete information than SVNs. Sun et al. [24] introduced the interval neutrosophic number Choquet integral operator, Ye [25] proposed the interval neutrosophic number ordered weighted operators, Zhang et al. [26] proposed the interval neutrosophic number weighted operators. All of these methods demonstrate the effectiveness.

In respect of measures, Sahin and Kucuk [27] proposed the subset-hood measure for SVNSs, Ye [28–30] and Wu et al. [31] developed some measures of SVNSs including the weighted correlation coefficient [28], cross-entropy [29,31], similarity measure [30]. Broumi and Smarandache proposed the correlation coefficient [32] and cosine similarity measure [33] distance [34] of INSs, Ye [35] proposed the similarity measures between INSs, Sahin and Karabacak [36] developed the inclusion measure for INSs. All of these measures are verified by real cases and demonstrate the effectiveness as well.

In respect of the extension of classic decision-making methods, Zhang and Wu [19] developed an extended TOPSIS method for the MCDM with incomplete weight information under a single valued neutrosophic environment; Biswas et al. [37] developed the entropy based grey relational analysis method to deal with MCDM problems in which all the criteria weight information described by SVNSs is unknown; Peng et al. [38] developed the outranking approach for MCDM problems based on ELECTRE method; and Sahin and Yigider [39] developed a MCGDM method based on the TOPSIS method for dealing with supplier selection problems. Chi and Liu [40] developed the extended TOPSIS method for deal MCDM problems based on INSs.

Peng et al. [13] firstly defined MVN and developed the approach for solving MCGDM problems based on the multi-valued neutrosophic power weighted operators. Wang and Li [41] proposed the Hamming distance between multi-valued neutrosophic numbers (MVNN) and the extended TODIM method for dealing with MCDM problems. Wu et al. [42] proposed the novel MCDM methods based on several cross-entropy measures of MVNSs.

However, these subsets of NSs cannot describe the assessment information with different dimensions. For overcoming the shortcomings and improving the flexibility and practicality of these sets, by extending the concept of trapezoidal intuitionistic fuzzy numbers (TrIFNs) [43], single valued trapezoidal neutrosophic numbers (SVTNNs) [44] are proposed for improving the ability to describe complex indeterminate and inconsistent information. Then, SVTNNs attract the attention of some researchers on them as very useful tools on describing evaluation information. Based on SVTNNs, Ye [44] developed the MCDM method on the basis of trapezoidal neutrosophic weighted arithmetic averaging (TNWAA) operator or trapezoidal neutrosophic weighted geometric averaging (TNWGA) operator. However, the correlation of trapezoidal numbers and three membership degrees has been ignored and the indeterminate-membership degree is regarded to be equal to falsity-membership degree in these operators, which will lead to information distortion and loss. Meanwhile, it does not take into account the information about the relationships among the assessment information being aggregated, which always exists in the process of solving MCDM problems. To overcome this shortcoming, motivated by the ideal of power aggregation operators [45,46], considering the relationship among the information being aggregated and the possibility degree widely used as a very useful tool to aggregate and rank uncertain data from the probability viewpoint, in this paper we propose the possibility degrees of SVTNNs, single trapezoidal neutrosophic power average (SVTNPA) and single valued trapezoidal neutrosophic power geometric (SVTNPG) operators to deal with MCGDM problems. The prominent characteristics of these proposed operators are taking into account relationship among the aggregation information and overcome the drawbacks of the existing operator of SVTNNs. Then, we utilize these operators and possibility degrees to develop a novel single valued trapezoidal neutrosophic MCGDM method.

The motivation and main attribution of the paper are presented as below:

(1) The novel operation laws of SVTNNs are conducted to overcome the lack of operation laws of SVTNNs appeared in previous paper.
(2) Based on the novel operations of SVTNNs, the SVTNPA and SVTNPG operators are developed.
(3) Based on the concept of the possibility degree, the possibility degree of SVTNNs is defined and presented.
(4) Based on possibility degree of SVTNNs, SVTNPA and SVTNPG operators, a novel method for solving MCGDM problems under single trapezoidal neutrosophic environment is developed.

The rest of the paper is organized as follows. In Section 2, we introduce some basic concepts and operators related to subsets of NS. In Section 3, we propose new operations, possibility degrees and comparison of SVTNNs. SVTNPA and SVTNPG operators are developed in Section 4. The method for solving MCGDM problems under single trapezoidal neutrosophic environment is developed in Section 5. An illustrative example for selecting the most appropriate green supplier for Shanghai General Motors Company is provided in Section 6. Meanwhile a comparison with other method is presented to show the effectiveness of the proposed approach. Finally, conclusions are drawn in Section 7.

2. Preliminaries

In this section, some basic concepts, definitions of SVTNNs and two aggregation operators are introduced, which are laying groundwork of latter analysis.

2.1. NS and SVNS

Definition 1 ([14]). *Let X be a space of points (objects), with a generic element in X denoted by x. A NS A in X is characterized by three membership functions, namely truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$ and falsity-membership function $F_A(x)$, where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]^-0, 1^+[$, i.e., $T_A(x) : X \to]^-0, 1^+[$, $I_A(x) : X \to]^-0, 1^+[$ and $F_A(x) : X \to]^-0, 1^+[$. Therefore, it is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ and $^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.*

The neutrosophic set needs to be specified from a technical point of view, otherwise it is difficult to apply in the real scientific and engineering areas. Therefore, Wang et al. [13] proposed the concept SVNS as an instance of neutrosophic set for easily operating and conveniently applying in practical issues.

Definition 2 ([13]). *Let X be a space of points (objects). A SVNS A in X can be expressed as follows:*

$$A = \{x, \langle T_A(x), I_A(x), F_A(x) \rangle | x \in X\},$$

where $T_A(x) \in [0,1]$, $I_A(x) \in [0,1]$ *and* $F_A(x) \in [0,1]$.

Obviously, the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ satisfies the condition $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

2.2. The Trapezoidal Fuzzy Number and SVTNNs

Definition 3 ([43,47]). *Let* $\tilde{a}$ *be a trapezoidal fuzzy number* $\tilde{a} = (a_1, a_2, a_3, a_4)$ *and* $a_1 \le a_2 \le a_3 \le a_4$. *Then its membership function* $\mu_{\tilde{a}}(x) : R \to [0,1]$ *can be defined as follows:*

$$\mu_{\tilde{a}}(x) = \begin{cases} (x - a_1)\mu_{\tilde{a}}/(a_2 - a_1), & a_1 \le x < a_2; \\ \mu_{\tilde{a}}, & a_2 \le x \le a_3; \\ (a_4 - x)\mu_{\tilde{a}}/(a_4 - a_3), & a_3 < x \le a_4; \\ 0, & otherwise. \end{cases}$$

Because of the great validity and feasibility of trapezoidal fuzzy numbers and SVNSs in decision-making problems, Ye [44] developed the SVTNNs by combining the two concepts.

Definition 4 ([44]). *Let U be a space of points (objects). Then a SVTNN* α *can be represented as*

$$\alpha = \langle [a_1, a_2, a_3, a_4], (T(\alpha), I(\alpha), F(\alpha)) \rangle$$

whose truth-membership $T(\alpha)$, *indeterminacy-membership* $I(\alpha)$ *and falsity-membership* $F(\alpha)$ *can be described as follows:*

$$T(\alpha) = \begin{cases} (x - a_1)T(\alpha)/(a_2 - a_1), & a_1 \le x < a_2; \\ T(\alpha), & a_2 \le x \le a_3; \\ (a_4 - x)T(\alpha)/(a_4 - a_3), & a_3 < x \le a_4; \\ 0, & otherwise. \end{cases}$$

$$I(\alpha) = \begin{cases} (x - a_1)I(\alpha)/(a_2 - a_1), & a_1 \le x < a_2; \\ I(\alpha), & a_2 \le x \le a_3; \\ (a_4 - x)I(\alpha)/(a_4 - a_3), & a_3 < x \le a_4; \\ 0, & otherwise. \end{cases}$$

$$F(\alpha) = \begin{cases} (x - a_1)F(\alpha)/(a_2 - a_1), & a_1 \le x < a_2; \\ F(\alpha), & a_2 \le x \le a_3; \\ (a_4 - x)F(\alpha)/(a_4 - a_3), & a_3 < x \le a_4; \\ 0, & otherwise. \end{cases}$$

Especially, if $a_1 \ge 0$ and $a_4 > 0$, then $\alpha = \langle [a_1, a_2, a_3, a_4], (T(\alpha), I(\alpha), F(\alpha)) \rangle$ becomes a positive SVTNN. If $I(\alpha) = 1 - T(\alpha) - F(\alpha)$, then the SVTNN is a TrIFN. And if $I(\alpha) = 0$, $F(\alpha) = 0$, then the SVTNN becomes a trapezoidal fuzzy number, that is $\alpha = \langle [a_1, a_2, a_3, a_4], T(\alpha) \rangle$.

Example 1. *Let* $\alpha_1 = \langle [0.3, 0.4, 0.7, 0.8], (0.8, 0.2, 0.4) \rangle$ *be a SVTNN. Then its truth-membership* $T(\alpha_1)$, *indeterminacy-membership* $I(\alpha_1)$ *and falsity-membership* $F(\alpha_1)$ *can be obtained, respectively, as follows:*

$$T(\alpha_1) = \begin{cases} 8(x - 0.3), & 0.3 \le x < 0.4; \\ 0.8, & 0.4 \le x \le 0.7; \\ 8(0.8 - x), & 0.7 < x \le 0.8; \\ 0, & otherwise. \end{cases}$$

$$I(\alpha_1) = \begin{cases} 2(x-0.3), & 0.3 \le x < 0.4; \\ 0.2, & 0.4 \le x \le 0.7; \\ 2(0.8-x), & 0.7 < x \le 0.8; \\ 0, & otherwise. \end{cases}$$

$$F(\alpha_1) = \begin{cases} 4(x-0.3), & 0.3 \le x < 0.4; \\ 0.4, & 0.4 \le x \le 0.7; \\ 4(0.8-x), & 0.7 < x \le 0.8; \\ 0, & otherwise. \end{cases}$$

2.3. PA and PG Operators

The power average (PA) operator was firstly proposed by Yager [45]; then, based on PA operator, Xu and Yager [46] developed the power geometric (PG) operator.

Definition 5 ([45,46]). *Let $\widetilde{h} = \{h_1, h_2, \cdots, h_n\}$ a collection of positive real numbers, then PA operator and PG operator can be defined, respectively, as follows:*

$$PA(h_1, h_2, \cdots, h_n) = \sum_{i=1}^{n} \frac{(1+G(h_i))h_i}{\sum_{i=1}^{n}(1+G(h_i))}$$

$$PG(h_1, h_2, \cdots, h_n) = \prod_{i=1}^{n} \left(h_i^{((1+G(h_i))/\sum_{i=1}^{n}(1+G(h_i)))} \right)$$

where $G(h_i) = \sum_{j=1, j \ne i}^{n} Sup(h_i, h_j)$, $i = 1, 2, \cdots, n$. $Sup(h_i, h_j)$ is the support for h_i from h_j, satisfying the following properties:

(1) $Sup(h_i, h_j) \in [0, 1]$.
(2) $Sup(h_i, h_j) = Sup(h_j, h_i)$.
(3) *If* $|h_i - h_j| \le |a - b|$, *then* $Sup(h_i, h_j) \ge Sup(a, b)$, *where a and b are two positive real numbers.*

3. New Operations and Comparison of SVTNNs

In this section, new operations and comparison method of SVTNNs are proposed for overcoming the limitations in Reference [44] which can avoid information loss and distortion effectively.

3.1. The New Operations of SVTNNs

In order to aggregate different SVTNNs in decision-making process, Ye [44] defined the operations of SVTNNs.

Definition 6 ([44]). *Let* $\alpha = \langle [a_1, a_2, a_3, a_4], (T(\alpha), I(\alpha), F(\alpha)) \rangle$ *and* $\beta = \langle [b_1, b_2, b_3, b_4], (T(\beta), I(\beta), F(\beta)) \rangle$ *be two positive SVTNNs,* $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$, $0 \le b_1 \le b_2 \le b_3 \le b_4 \le 1$, $\zeta \ge 0$. *Then the operations of SVTNNs can be defined as follows:*

(1) $\alpha + \beta = \langle [a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4], (T(\alpha) + T(\beta) - T(\alpha)T(\beta), I(\alpha)I(\beta), F(\alpha)F(\beta)) \rangle$;
(2) $\alpha\beta = \langle [a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4], (T(\alpha)T(\beta), I(\alpha) + I(\beta) - I(\alpha)I(\beta), F(\alpha) + F(\beta) - F(\alpha)F(\beta)) \rangle$;
(3) $\zeta\alpha = \left\langle [\zeta a_1, \zeta a_2, \zeta a_3, \zeta a_4], \left(1 - (1 - T(\alpha))^{\zeta}, (I(\alpha))^{\zeta}, (F(\alpha))^{\zeta}\right) \right\rangle$;
(4) $\alpha^{\zeta} = \left\langle [a_1{}^{\zeta}, a_2{}^{\zeta}, a_3{}^{\zeta}, a_4{}^{\zeta}], \left((T(\alpha))^{\zeta}, 1 - (1 - I(\alpha))^{\zeta}, 1 - (1 - F(\alpha))^{\zeta}\right) \right\rangle$;

However, there are some shortcomings in Definition 7.

(1) The trapezoidal fuzzy numbers and three membership degrees of SVTNNs are considered as two separate parts and operated individually in the operation $\alpha + \beta$, which ignore the correlation among them and cannot reflect the actual results.

Example 2. *Let* $\alpha_1 = \langle [0.5, 0.6, 0.7, 0.8], (0, 0, 1) \rangle$ *and* $\alpha_2 = \langle [0.2, 0.3, 0.4, 0.5], (1, 0, 0) \rangle$ *be two SVTNNs.*

$$\alpha_1 + \alpha_2 = \langle [0.5, 0.6, 0.7, 0.8], (0, 0, 1) \rangle + \langle [0.2, 0.3, 0.4, 0.5], (1, 0, 0) \rangle = \langle [0.7, 0.9, 1.1, 1.3], (1, 0, 0) \rangle;$$

This result is inaccurate since the falsity-membership of α_1, the correlations among trapezoidal fuzzy numbers and the membership degrees of α_1 and α_2 are not considered. Thus, the operations would be unreasonable.

(2) The three membership degrees of SVTNNs are also operated as the trapezoidal fuzzy numbers in the operation $\zeta\alpha$, which can produce the repeat operation and make the result bias.

Example 3. *Let* $\alpha_1 = \langle [0.03, 0.05, 0.07, 0.09], (0.3, 0.5, 0.5) \rangle$ *be a SVTNN,* $\zeta = 10$*. Then the result* $\zeta\alpha_1$ *can be obtained by using Definition 6.*

$$10\alpha_1 = \langle [0.3, 0.5, 0.7, 0.9], (0.9718, 0.001, 0.001) \rangle$$

The three membership degrees of these SVTNNs are operated repeatedly which make the result distort significantly and conflict with common sense.

For overcoming the limitations existing in the operations proposed by Ye [44], motivated by the operations on triangular intuitionistic fuzzy numbers proposed by Wang et al. [48], new operations of SVTNNs are defined as below.

Definition 7. *Let* $\alpha = \langle [a_1, a_2, a_3, a_4], (T(\alpha), I(\alpha), F(\alpha)) \rangle$ *and* $\beta = \langle [b_1, b_2, b_3, b_4], (T(\beta), I(\beta), F(\beta)) \rangle$ *be two positive SVTNNs,* $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$, $0 \le b_1 \le b_2 \le b_3 \le b_4 \le 1$, $\zeta \ge 0$*. Then the new operations of SVTNNs can be defined as follows:*

(1) $neg(\alpha) = \langle [1 - a_4, 1 - a_3, 1 - a_2, 1 - a_1], (T(\alpha), I(\alpha), F(\alpha)) \rangle$;

(2) $\alpha \oplus \beta = \left\langle [a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4], \left(\frac{\varphi(\alpha)T(\alpha) + \varphi(\beta)T(\beta)}{\varphi(\alpha) + \varphi(\beta)}, \frac{\varphi(\alpha)I(\alpha) + \varphi(\beta)I(\beta)}{\varphi(\alpha) + \varphi(\beta)}, \frac{\varphi(\alpha)F(\alpha) + \varphi(\beta)F(\beta)}{\varphi(\alpha) + \varphi(\beta)} \right) \right\rangle$, *where* $\varphi(\alpha) = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6}$, $\varphi(\beta) = \frac{b_1 + 2b_2 + 2b_3 + b_4}{6}$;

(3) $\alpha \otimes \beta = \langle [a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4], (T(\alpha)T(\beta), I(\alpha) + I(\beta) - I(\alpha)I(\beta), F(\alpha) + F(\beta) - F(\alpha)F(\beta)) \rangle$;

(4) $\zeta\alpha = \langle [\zeta a_1, \zeta a_2, \zeta a_3, \zeta a_4], (T(\alpha), I(\alpha), F(\alpha)) \rangle$;

(5) $\alpha^{\zeta} = \left\langle [a_1{}^{\zeta}, a_2{}^{\zeta}, a_3{}^{\zeta}, a_4{}^{\zeta}], \left((T(\alpha))^{\zeta}, 1 - (1 - I(\alpha))^{\zeta}, 1 - (1 - F(\alpha))^{\zeta} \right) \right\rangle$;

$\zeta\alpha$, $\alpha \oplus \beta$, $\alpha \otimes \beta$ and α^{ζ} do not appear alone in application due to the meaninglessness of their results. Only in the aggregation process do $\alpha \oplus \beta$ and/or $\alpha \otimes \beta$ being combined with $\zeta\alpha$ and/or α^{ζ} make sense.

Example 4. *Let* $\alpha_1 = \langle [0.5, 0.6, 0.7, 0.8], (0, 0, 1) \rangle$ *and* $\alpha_2 = \langle [0.2, 0.3, 0.4, 0.5], (1, 0, 0) \rangle$ *be two SVTNNs,* $\zeta = 2$*, the following results can be obtained based on Definition 7.*

(1) $neg(\alpha_1) = \langle [0.5, 0.6, 0.8, 0.9], (0.4, 0.1, 0.5) \rangle$;

(2) $\alpha_1 \oplus \alpha_2 = \langle [0.3, 0.5, 1.0, 1.2], (0.64, 0.22, 0.26) \rangle$;

(3) $\alpha_1 \otimes \alpha_2 = \langle [0.02, 0.06, 0.24, 0.35], (0.32, 0.37, 0.55) \rangle$;

(4) $2\alpha_1 = \langle [0.2, 0.4, 0.8, 1.0], (0.4, 0.1, 0.5) \rangle$;

(5) $\alpha_1{}^2 = \langle[0.04, 0.09, 0.25, 0.36], (0.16, 0.19, 0.75)\rangle$.

Compared with the operations proposed by Ye [44], the new operations of SVTNNs have some excellent advantages on reflecting the effect of all truth, indeterminacy and falsity membership degrees of SVTNNs on aggregation results and taking into account the correlation of the trapezoidal fuzzy numbers and three membership degrees of SVTNNs, which can avoid information loss and distortion effectively.

In terms of the corresponding operations of SVTNNs, the following theorem can be easily proved.

Theorem 1. *Let α_1, α_2, α_3 be three SVTNNs and $\zeta \geq 0$. Then the following equations must be true and easy to proof.*

(1) $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$;
(2) $(\alpha_1 \oplus \alpha_2) \oplus \alpha_3 = \alpha_1 \oplus (\alpha_2 \oplus \alpha_3)$;
(3) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$;
(4) $(\alpha_1 \otimes \alpha_2) \otimes \alpha_3 = \alpha_1 \otimes (\alpha_2 \otimes \alpha_3)$;
(5) $\zeta\alpha_1 \oplus \zeta\alpha_2 = \zeta(\alpha_2 \oplus \alpha_1)$;
(6) $(\alpha_2 \otimes \alpha_1)^{\tau} = \alpha_1{}^{\tau} \otimes \alpha_2{}^{\tau}$.

3.2. The Possibility Degree

The possibility degree, which is proposed from the probability viewpoint, is a very useful tool to rank uncertain data reasonably and accurately.

Definition 8 ([49,50]). *Let $y = [y_1, y_2] \subseteq [0,1]$ and $z = [z_1, z_2] \subseteq [0,1]$ be two real number intervals with uniform probability distribution, the probability $y \geq z$ can be represented as $p(y \geq z)$, which exists the following properties:*

(1) $0 \leq p(y \geq z) \leq 1$.
(2) $p(y \geq z) + p(z \geq y) = 1$.
(3) *If $y = z$, then $p(y \geq z) = p(z \geq y) = 0.5$.*
(4) *If ξ is an arbitrary interval or number, $p(y \geq z) \geq 0.5$, $p(z \geq \xi) \geq 0.5$, then $p(y \geq \xi) \geq 0.5$.*
(5) *If $\min(y) > \max(z)$, then $p(y \geq z) = 1$.*

Based on the concept of the possibility degree, the possibility degree of two arbitrary positive SVTNNs is presented.

Definition 9. *Let $\alpha = \langle[a_1, a_2, a_3, a_4], (T(\alpha), I(\alpha), F(\alpha))\rangle$ and $\beta = \langle[b_1, b_2, b_3, b_4], (T(\beta), I(\beta), F(\beta))\rangle$ be two positive SVTNNs, $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1$, $0 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq 1$. Then the possibility degree of $\alpha \succ \beta$ $p(\alpha \succ \beta)$ can be defined as follows:*

$$\begin{aligned} p(\alpha \succ \beta) = \; & \tfrac{1}{2+\gamma}\Bigg(\max\left\{1 - \max\left[\frac{\sum_{i=1}^{4}\max(b_i - a_i, 0) + (b_4 - a_1) + 2\max(T(\beta) - T(\alpha), 0)}{\sum_{i=1}^{4}|b_i - a_i| + (b_4 - b_1) + (a_4 - a_1) + 2|T(\beta) - T(\alpha)|}, 0\right], 0\right\} \\ & + \gamma\max\left\{1 - \max\left[\frac{\sum_{i=1}^{4}\max(b_i - a_i, 0) + (b_4 - a_1) + 2\max(I(\beta) - I(\alpha), 0)}{\sum_{i=1}^{4}|b_i - a_i| + (b_4 - b_1) + (a_4 - a_1) + 2|I(\beta) - I(\alpha)|}, 0\right], 0\right\} \\ & + 1 - \max\left\{1 - \max\left[\frac{\sum_{i=1}^{4}\max(b_i - a_i, 0) + (b_4 - a_1) + 2\max(F(\beta) - F(\alpha), 0)}{\sum_{i=1}^{4}|b_i - a_i| + (b_4 - b_1) + (a_4 - a_1) + 2|F(\beta) - F(\alpha)|}, 0\right], 0\right\}\Bigg), \end{aligned}$$

where the value of $\gamma \in [0,1]$ is the coefficient that can reflect the attitudes of decision-makers. $\gamma > 0.5$, $\gamma = 0.5$ and $\gamma < 0.5$ denotes, respectively, the decision-makers' attitude of optimism, compromise and pessimism.

Example 5. *Let* $\alpha_1 = \langle[0.3, 0.4, 0.7, 0.8], (0.8, 0.2, 0.4)\rangle$ *and* $\alpha_2 = \langle[0.2, 0.5, 0.6, 0.7], (0.6, 0.1, 0.3)\rangle$ *be two SVTNNs,* $\gamma = 0.5$*. The result of* $p(\alpha_1 \succ \alpha_2)$ *can be obtained as follows.*

Because

$$\begin{aligned}&\frac{\sum_{i=1}^{4}\max(b_i-a_i,0)+(b_4-a_1)+2\max(T(\alpha_2)-T(\alpha_1),0)}{\sum_{i=1}^{4}|b_i-a_i|+(b_4-b_1)+(a_4-a_1)+2|T(\alpha_2)-T(\alpha_1)|}\\ &=\frac{\max(0.2-0.3,0)+\max(0.5-0.4,0)+\max(0.6-0.7,0)+\max(0.7-0.8,0)+(0.7-0.4)+2\max(0.6-0.8,0)}{|0.2-0.3|+|0.5-0.4|+|0.6-0.7|+|0.7-0.8|+(0.7-0.2)+(0.8-0.3)+2|0.6-0.8|}\\ &=\frac{0.4}{1.8}=0.222;\end{aligned}$$

$$\frac{\sum_{i=1}^{4}\max(b_i-a_i,0)+(b_4-a_1)+2\max(I(\alpha_2)-I(\alpha_1),0)}{\sum_{i=1}^{4}|b_i-a_i|+(b_4-b_1)+(a_4-a_1)+2|I(\alpha_2)-I(\alpha_1)|}=\frac{0.4}{1.6}=0.25;$$

$$\frac{\sum_{i=1}^{4}\max(b_i-a_i,0)+(b_4-a_1)+2\max(F(\alpha_2)-F(\alpha_1),0)}{\sum_{i=1}^{4}|b_i-a_i|+(b_4-b_1)+(a_4-a_1)+2|F(\alpha_2)-F(\alpha_1)|}=\frac{0.4}{1.6}=0.25$$

Therefore, we can obtain

$$\begin{aligned}p(\alpha_1\succ\alpha_2)&=\tfrac{1}{2.5}(\max\{1-\max[0.222,0],0\}+0.5\times\max\{1-\max[0.25,0],0\}+1-\max\{1-\max[0.25,0],0\})\\&=\tfrac{1}{2.5}(0.778+0.5\times0.75+0.25)\\&=0.561.\end{aligned}$$

Theorem 2. *Let* $\alpha = \langle[a_1, a_2, a_3, a_4], (T(\alpha), I(\alpha), F(\alpha))\rangle$ *and* $\beta = \langle[b_1, b_2, b_3, b_4], (T(\beta), I(\beta), F(\beta))\rangle$ *be two positive SVTNNs,* $0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$, $0 \le b_1 \le b_2 \le b_3 \le b_4 \le 1$*. Then the following properties must be true.*

(1) $0 \le p(\alpha \succ \beta) \le 1$.
(2) $p(\alpha \succ \beta) + p(\beta \succ \alpha) = 1$.
(3) *If* $a_i = b_i$, $i = 1,2,3,4$, $T(\alpha) = T(\beta)$, $I(\alpha) = I(\beta)$ *and* $F(\alpha) = F(\beta)$*, then* $p(\alpha \succ \beta) = p(\beta \succ \alpha) = 0.5$.
(4) *If* ξ *is an arbitrary positive SVTNN,* $p(\alpha \succ \beta) \ge 0.5$, $p(\beta \succ \xi) \ge 0.5$*, then* $p(\alpha \ge \xi) \ge 0.5$.
(5) *If* $a_1 \ge b_4$, $T(\alpha) \ge T(\beta)$, $I(\alpha) \ge I(\beta)$ *and* $F(\alpha) \le F(\beta)$*, then* $p(\alpha \succ \beta) = 1$.

Now we prove the property (2), the proofs of other properties are similar to the proof the property (2), thus, they are omitted.

Proof. Let $x(\alpha,\beta) = \frac{\sum_{i=1}^{4}\max(b_i-a_i,0)+(b_4-a_1)+2\max(T(\beta)-T(\alpha),0)}{\sum_{i=1}^{4}|b_i-a_i|+(b_4-b_1)+(a_4-a_1)+2|T(\beta)-T(\alpha)|}$, $y(\alpha,\beta) = \frac{\sum_{i=1}^{4}\max(b_i-a_i,0)+(b_4-a_1)+2\max(I(\beta)-I(\alpha),0)}{\sum_{i=1}^{4}|b_i-a_i|+(b_4-b_1)+(a_4-a_1)+2|I(\beta)-I(\alpha)|}$, $z(\alpha,\beta) = \frac{\sum_{i=1}^{4}\max(b_i-a_i,0)+(b_4-a_1)+2\max(F(\beta)-F(\alpha),0)}{\sum_{i=1}^{4}|b_i-a_i|+(b_4-b_1)+(a_4-a_1)+2|F(\beta)-F(\alpha)|}$.

Then

$$p(\alpha\succ\beta)=\tfrac{1}{3}(\max\{1-\max[x(\alpha,\beta),0],0\}+\gamma\max\{1-\max[y(\alpha,\beta),0],0\}+1-\max\{1-\max[z(\alpha,\beta),0],0\}).$$

Because

$$\begin{aligned}&x(\alpha,\beta)+x(\beta,\alpha)\\&=\frac{\sum_{i=1}^{4}\max(b_i-a_i,0)+(b_4-a_1)+2\max(T(\beta)-T(\alpha),0)}{\sum_{i=1}^{4}|b_i-a_i|+(b_4-b_1)+(a_4-a_1)+2|T(\beta)-T(\alpha)|}+\frac{\sum_{i=1}^{4}\max(a_i-b_i,0)+(a_4-b_1)+2\max(T(\alpha)-T(\beta),0)}{\sum_{i=1}^{4}|a_i-b_i|+(a_4-a_1)+(b_4-b_1)+2|T(\alpha)-T(\beta)|}\\&=\frac{\sum_{i=1}^{4}|a_i-b_i|+(a_4-b_1)+(b_4-a_1)+2|T(\alpha)-T(\beta)|}{\sum_{i=1}^{4}|a_i-b_i|+(a_4-a_1)+(b_4-b_1)+2|T(\alpha)-T(\beta)|}=1;\end{aligned}$$

$$y(\alpha,\beta)+y(\beta,\alpha)=1;\ z(\alpha,\beta)+z(\beta,\alpha)=1.$$

We can obtain $\max\{1-\max[x(\alpha,\beta),0],0\} + \max\{1-\max[x(\beta,\alpha),0],0\} = 1$; $\max\{1-\max[y(\alpha,\beta),0],0\}+\max\{1-\max[y(\beta,\alpha),0],0\} = 1$; $1-\max\{1-\max[z(\alpha,\beta),0],0\} + 1-\max\{1-\max[z(\beta,\alpha),0],0\} = 1$.

Therefore,

$$
\begin{aligned}
& p(\alpha \succ \beta) + P(\beta \succ \alpha) \\
= & \tfrac{1}{2+\lambda}(\max\{1-\max[x(\alpha,\beta),0],0\} + \lambda\max\{1-\max[y(\alpha,\beta),0],0\} + 1 - \max\{1-\max[z(\alpha,\beta),0],0\}) \\
& + \tfrac{1}{2+\lambda}(\max\{1-\max[x(\beta,\alpha),0],0\} + \lambda\max\{1-\max[y(\beta,\alpha),0],0\} + 1 - \max\{1-\max[z(\beta,\alpha),0],0\}) \\
= & \tfrac{1}{2+\lambda}(1+\lambda+1) = 1.
\end{aligned}
$$

□

The proof of the property (2) is completed now.

3.3. The Comparison Method of SVTNNs

In this subsection, based on the concept of the possibility degree of two arbitrary positive SVTNNs defined in Definition 9, the new comparison method for two SVTNNs is presented.

For comparing different SVTNNs in decision-making process, Ye [44] defined the score function and comparison of SVTNNs.

Definition 10 [44]. *Let* $\alpha = \langle [a_1, a_2, a_3, a_4], (T(\alpha), I(\alpha), F(\alpha)) \rangle$ *and* $\beta = \langle [b_1, b_2, b_3, b_4], (T(\beta), I(\beta), F(\beta)) \rangle$ *be two SVTNNs. Then the score degree of* α $S(\alpha)$ *can be defined as follows:*

$$S(\alpha) = \frac{1}{12}(a_1 + a_2 + a_3 + a_4) \times (2 + T(\alpha) - I(\alpha) - F(\alpha)).$$

If $S(\alpha) > S(\beta)$, then $\alpha \succ \beta$; if $S(\alpha) < S(\beta)$, then $\alpha \prec \beta$; if $S(\alpha) = S(\beta)$, then $\alpha \sim \beta$.

However, the score function is operated by assuming that the parameters of trapezoidal fuzzy numbers own same weight, which cannot reflect the different importance for the four parameters of a trapezoidal fuzzy number and make aggregating result bias.

Example 6. *Let* $\alpha_1 = \langle [0.1, 0.3, 0.5, 0.6], (0.6, 0, 0.4) \rangle$ *and* $\alpha_2 = \langle [0, 0.4, 0.5, 0.6], (0.6, 0, 0.4) \rangle$ *be two SVTNNs.*

$$S(\alpha_1) = \frac{1}{12}(0.1 + 0.3 + 0.5 + 0.6) \times (2 + 0.6 - 0 - 0.4) = 0.275; S(\alpha_2) = 0.275.$$

We cannot compare these two SVTNNs using the above function but it is easy to know that α_1 is superior to α_2.

Meanwhile, the function operates the indeterminacy-membership degree as like the false-membership degree, which does not take the preference of decision-makers into consideration.

Example 7. *Let* $\alpha_1 = \langle [0.2, 0.3, 0.4, 0.5], (0.6, 0, 0.4) \rangle$ *and* $\alpha_2 = \langle [0.2, 0.3, 0.4, 0.5], (0.6, 0.4, 0) \rangle$ *be two SVTNNs.*

$$S(\alpha_1) = \frac{1}{12}(0.2 + 0.3 + 0.4 + 0.5) \times (2 + 0.6 - 0 - 0.4) = 0.257; S(\alpha_2) = 0.257.$$

$S(\alpha_1) = S(\alpha_2)$ indicates that α_1 is equal to α_2. However, it is obvious that α_2 is superior to α_1.

These shortcomings existing in the score function given in Definition 10 may make the comparison results of SVTNNs unacceptable. For overcoming the limitations of Definition 10, based on the concept of the possibility degree of two arbitrary positive SVTNNs defined in Definition 9, we propose a new comparison method.

Definition 11. *Let* α *and* β *be two positive SVTNNs,* γ *be an arbitrary positive SVTNN and then the comparison method can be defined as follows.*

(1) *If* $p(\alpha \succ \gamma) > p(\beta \succ \gamma)$, *then* $\alpha \succ \beta$, *i.e.,* α *is superior to* β.
(2) *If* $p(\alpha \succ \gamma) = p(\beta \succ \gamma)$, *then* $\alpha \sim \beta$, *i.e.,* α *is equal to* β.
(3) *If* $p(\alpha \succ \gamma) < p(\beta \succ \gamma)$, *then* $\alpha \prec \beta$, *i.e.,* β *is superior to* α.

Example 8. *Let* $\lambda = 0.5$. *When using the data of Example 4 and the following can be obtained.*

$$p(\alpha_1 \succ \alpha_2) = 0.508;\ p(\alpha_2 \succ \alpha_1) = 0.492, \text{so } \alpha_1 \succ \alpha_2.$$

When using the data of Example 5 and the following can be obtained.

$$p(\alpha_1 \succ \alpha_2) = 0.329; p(\alpha_2 \succ \alpha_1) = 0.671, so\ \alpha_2 \succ \alpha_1.$$

Thus, the results of the above two examples are consistent with our common sense. Because the score function can overcome the shortcoming existing in Reference [44] by calculating the indeterminacy-membership degree by taking into account the preference of decision-makers, the results are more grounded in reality than the results obtained by using the score degree proposed by Ye [44].

4. Single Valued Trapezoidal Neutrosophic Power Aggregation Operators

In this section, the SVTNPA and SVTNPG operators based on the new operations of SVTNNs are developed.

Definition 12. *Let* $\alpha_i = \langle [a_{i1}, a_{i2}, a_{i3}, a_{i4}], (T(\alpha_i), I(\alpha_i), F(\alpha_i)) \rangle$ *be a collection of positive SVTNNs. Then the single valued trapezoidal neutrosophic power average (SVTNPA) operator can be defined as follows:*

$$\begin{aligned} SVTNPA(\alpha_1, \alpha_2, \cdots, \alpha_n) &= \frac{1+G(\alpha_1)}{\sum_{i=1}^n (1+G(\alpha_i))}\alpha_1 \oplus \frac{1+G(\alpha_2)}{\sum_{i=1}^n (1+G(\alpha_i))}\alpha_2 \oplus \cdots \oplus \frac{1+G(\alpha_n)}{\sum_{i=1}^n (1+G(\alpha_i))}\alpha_n \\ &= \oplus_{i=1}^n \left(\frac{1+G(\alpha_i)}{\sum_{i=1}^n (1+G(\alpha_i))}\alpha_i \right), \end{aligned}$$

where $G(\alpha_i) = \sum_{j=1, j \neq i}^n Sup(\alpha_i, \alpha_j)$, $Sup(\alpha_i, \alpha_j)$ *is the support for* α_i *from* α_j, satisfying the following properties.

(1) $Sup(\alpha_i, \alpha_j) \in [0, 1]$.
(2) $Sup(\alpha_i, \alpha_j) = Sup(\alpha_j, \alpha_i)$.
(3) *If* $|p(\alpha_i \succ \alpha_j) - p(\alpha_j \succ \alpha_i)| < |p(\pi \succ \nu) - p(\nu \succ \pi)|$, *then* $Sup(\alpha_i, \alpha_j) > Sup(\pi, \nu)$, *where* π *and* ν *are two positive SVTNNs,* $p(\alpha_i \succ \alpha_j)$, $p(\alpha_j \succ \alpha_i)$, $p(\pi \succ \nu)$ *and* $p(\nu \succ \pi)$ *are the possibility degree of* $\alpha_i \succ \alpha_j$, $\alpha_j \succ \alpha_i$, $\pi \succ \nu$ *and* $\nu \succ \pi$.

The support for α_i from α_j can be obtained using the function $Sup(\alpha_i, \alpha_j) = 1 - |p(\alpha_i \succ \alpha_j) - p(\alpha_j \succ \alpha_i)|$. Obviously, the closer the values of the score of α_i and α_j, the more they support each other.

Theorem 3. *Let* $\alpha_i = \langle [a_{i1}, a_{i2}, a_{i3}, a_{i4}], (T(\alpha_i), I(\alpha_i), F(\alpha_i)) \rangle (i = 1, 2, \cdots, n)$ *be a collection of positive SVTNNs. The aggregated result, obtained by using the SVTNPA operator, is also a positive SVTNN, and*

$$\begin{aligned} SVTNPA(\alpha_1, \alpha_2, \cdots, \alpha_n) &= \oplus_{i=1}^n \left(\frac{1+G(\alpha_i)}{\sum_{i=1}^n (1+G(\alpha_i))}\alpha_i \right) \\ &= \langle ([\textstyle\sum_{i=1}^n (\tau(\alpha_i) a_{i1}), \sum_{i=1}^n (\tau(\alpha_i) a_{i2}), \sum_{i=1}^n (\tau(\alpha_i) a_{i3}), \sum_{i=1}^n (\tau(\alpha_i) a_{i4})], \\ &\quad \left(\frac{\sum_{i=1}^n (\varphi(\alpha_i) T(\alpha_i))}{\sum_{i=1}^n \varphi(\alpha_i)}, \frac{\sum_{i=1}^n (\varphi(\alpha_i) I(\alpha_i))}{\sum_{i=1}^n \varphi(\alpha_i)}, \frac{\sum_{i=1}^n (\varphi(\alpha_i) F(\alpha_i))}{\sum_{i=1}^n \varphi(\alpha_i)} \right) \rangle, \end{aligned}$$

where $G(\alpha_i) = \sum_{j=1, j\neq i}^{n} Sup(\alpha_i, \alpha_j)$, $Sup(\alpha_i, \alpha_j) = 1 - |p(\alpha_i \succ \alpha_j) - p(\alpha_j \succ \alpha_i)|$ *is the support for* α_i *from* α_j, $\tau(\alpha_i) = \frac{1+G(\alpha_i)}{\sum_{i=1}^{n}(1+G(\alpha_i))}$, $\varphi(\alpha_i) = \frac{1}{6}(\tau(\alpha_i)a_{i1} + 2\tau(\alpha_i)a_{i2} + 2\tau(\alpha_i)a_{i3} + \tau(\alpha_i)a_{i4})$, $p(\alpha_i \succ \alpha_j)$ *and* $p(\alpha_j \succ \alpha_i)$ *are the score functions of* $\alpha_i \succ \alpha_j$, $\alpha_j \succ \alpha_i$.

Proof. According to Definition 8, the aggregated result is also a positive SVTNN. Therefore, Theorem 3 can be easily proven by using a mathematical induction on n.

(1) For $n = 2$, since

$$\frac{1+G(\alpha_1)_1}{\sum_{i=1}^{2}(1+G(\alpha_i))}\alpha_1 = \langle [\tau(\alpha_1)a_{11}, \tau(\alpha_1)a_{12}, \tau(\alpha_1)a_{13}, \tau(\alpha_1)a_{14}], (T(\alpha_1), I(\alpha_1), F(\alpha_1)) \rangle;$$

$$\frac{1+G(\alpha_2)}{\sum_{i=1}^{2}(1+G(\alpha_i))}\alpha_2 = \langle [\tau(\alpha_2)a_{21}, \tau(\alpha_2)a_{22}, \tau(\alpha_2)a_{23}, \tau(\alpha_2)a_{24}], (T(\alpha_2), I(\alpha_2), F(\alpha_2)) \rangle.$$

Then

$$\begin{aligned} & SVTNPA(\alpha_1, \alpha_2) = \tfrac{1+G(\alpha_1)}{\sum_{i=1}^{2}(1+G(\alpha_i))}\alpha_1 \oplus \tfrac{1+G(\alpha_2)}{\sum_{i=1}^{2}(1+G(\alpha_i))}\alpha_2 \\ & = \langle ([\tau(\alpha_1)a_{11} + \tau(\alpha_2)a_{21}, \tau(\alpha_1)a_{12} + \tau(\alpha_2)a_{22}, \tau(\alpha_1)a_{13} + \tau(\alpha_2)a_{23}, \tau(\alpha_1)a_{14} + \tau(\alpha_2)a_{24}], \\ & \left(\tfrac{\varphi(\alpha_1)T(\alpha_1)+\varphi(\alpha_2)T(\alpha_2)}{\varphi(\alpha_1)+\varphi(\alpha_2)}, \tfrac{\varphi(\alpha_1)I(\alpha_1)+\varphi(\alpha_2)I(\alpha_2)}{\varphi(\alpha_1)+\varphi(\alpha_2)}, \tfrac{\varphi(\alpha_1)I(\alpha_1)+\varphi(\alpha_2)I(\alpha_2)}{\varphi(\alpha_1)+\varphi(\alpha_2)} \right) \rangle. \end{aligned}$$

□

(2) If we hold $n = k$, then

$$\begin{aligned} SVTNPA(\alpha_1, \alpha_2, \cdots, \alpha_n) &= \oplus_{i=1}^{k}\left(\tfrac{1+G(\alpha_i)}{\sum_{i=1}^{k}(1+G(\alpha_i))}\alpha_i \right) \\ &= \Big\langle \Big(\Big[\textstyle\sum_{i=1}^{k}(\tau(\alpha_i)a_{i1}), \sum_{i=1}^{k}(\tau(\alpha_i)a_{i2}), \sum_{i=1}^{k}(\tau(\alpha_i)a_{i3}), \sum_{i=1}^{k}(\tau(\alpha_i)a_{i4}) \Big], \\ & \left(\tfrac{\sum_{i=1}^{k}(\varphi(\alpha_i)T(\alpha_i))}{\sum_{i=1}^{k}\varphi(\alpha_i)}, \tfrac{\sum_{i=1}^{k}(\varphi(\alpha_i)I(\alpha_i))}{\sum_{i=1}^{k}\varphi(\alpha_i)}, \tfrac{\sum_{i=1}^{k}(\varphi(\alpha_i)F(\alpha_i))}{\sum_{i=1}^{k}\varphi(\alpha_i)} \right) \Big\rangle. \end{aligned}$$

When $n = k+1$, by the operations described in Definition 10, we have

$$\begin{aligned} SVTNPA(\alpha_1, \alpha_2, \cdots, \alpha_n) &= \oplus_{i=1}^{k}\left(\tfrac{1+G(\alpha_i)}{\sum_{i=1}^{n}(1+G(\alpha_i))}\alpha_i \right) \oplus \tfrac{1+G(\alpha_{k+1})}{\sum_{i=1}^{n}(1+G(\alpha_i))}\alpha_{k+1} \\ &= \Big\langle \Big(\Big[\textstyle\sum_{i=1}^{k+1}(\tau(\alpha_i)a_{i1}), \sum_{i=1}^{k+1}(\tau(\alpha_i)a_{i2}), \sum_{i=1}^{k+1}(\tau(\alpha_i)a_{i3}), \sum_{i=1}^{k+1}(\tau(\alpha_i)a_{i4}) \Big], \\ & \left(\tfrac{\sum_{i=1}^{k+1}(\varphi(\alpha_i)T(\alpha_i))}{\sum_{i=1}^{k+1}\varphi(\alpha_i)}, \tfrac{\sum_{i=1}^{k+1}(\varphi(\alpha_i)I(\alpha_i))}{\sum_{i=1}^{k+1}\varphi(\alpha_i)}, \tfrac{\sum_{i=1}^{k+1}(\varphi(\alpha_i)F(\alpha_i))}{\sum_{i=1}^{k+1}\varphi(\alpha_i)} \right) \Big\rangle. \end{aligned}$$

□

So, $n = k+1$, Theorem 2 is also right.

According to (1) and (2), we can get Theorem 3 hold for any n.

Example 9. *Let* $\alpha_1 = \langle [0.3, 0.4, 0.7, 0.8], (0.8, 0.2, 0.4) \rangle$, $\alpha_2 = \langle [0.2, 0.5, 0.6, 0.7], (0.6, 0.1, 0.3) \rangle$, $\alpha_3 = \langle [0.3, 0.4, 0.5, 0.6], (0.7, 0.3, 0.3) \rangle$ *and* $\alpha_4 = \langle [0.3, 0.5, 0.5, 0.7], (0.6, 0.2, 0.3) \rangle$ *be four positive SVTNNs,* $\lambda = 0.8$. *Then* $SVTNPA(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ *can be calculated as follows.*

Because $p(\alpha_1 \succ \alpha_2) = 0.566$, $p(\alpha_1 \succ \alpha_3) = 0.541$, $p(\alpha_1 \succ \alpha_4) = 0.547$, $p(\alpha_2 \succ \alpha_3) = 0.452$, $p(\alpha_2 \succ \alpha_4) = 0.467$, $p(\alpha_3 \succ \alpha_4) = 0.530$, we can obtain the following results.

$$\begin{aligned} G(\alpha_1) &= \textstyle\sum_{j=1, j\neq i}^{3} Sup(\alpha_1, \alpha_j) \\ &= \textstyle\sum_{j=2}^{4} \left(1 - \left|p(\alpha_1 \succ \alpha_j) - p(\alpha_j \succ \alpha_1)\right|\right) \\ &= (1 - |0.566 - 0.434|) + (1 - |0.541 - 0.459|) + (1 - |0.547 - 0.453|) \\ &= 2.692, \end{aligned}$$

$$G(\alpha_2) = 2.707,\ G(\alpha_3) = 2.762,\ G(\alpha_4) = 2.779.$$

$$\tau(\alpha_1) = \frac{1+G(\alpha_1)}{\sum_{i=1}^{4}(1+G(\alpha_i))} = \frac{1+2.692}{(1+2.692)+(1+2.707)+(1+2.762)+(1+2.779)} = 0.247,$$
$$\tau(\alpha_2) = 0.248,\ \tau(\alpha_3) = 0.252,\ \tau(\alpha_4) = 0.253.$$

$$\varphi(\alpha_1) = \tfrac{1}{6}(\tau(\alpha_1)a_{11} + 2\tau(\alpha_1)a_{12} + 2\tau(\alpha_1)a_{13} + \tau(\alpha_1)a_{14}) = \tfrac{1}{6} \times 0.247 \times (0.3 + 2 \times 0.4 + 2 \times 0.7 + 0.8) = 0.136,$$
$$\varphi(\alpha_2) = 0.128,\ \varphi(\alpha_3) = 0.113,\ \varphi(\alpha_4) = 0.127.$$

Therefore, $SVTNPA(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \langle [0.275, 0.450, 0.574, 0.699], (0.676,\ 0.197, 0.327) \rangle$.

Theorem 4. *Let* $\alpha_i = \langle [a_{i1}, a_{i2}, a_{i3}, a_{i4}], (T(\alpha_i), I(\alpha_i), F(\alpha_i)) \rangle (i = 1, 2, \cdots, n)$ *be a collection of positive SVTNNs. If* $Sup(\alpha_i, \alpha_j) = c$ $(c \in [0,1], i \neq j, j = 1, 2, \cdots, n)$, *then SVTNPA operator reduces to single valued trapezoidal neutrosophic average (SVTNA) operator as follows:*

$$SVTNPA(\alpha_1, \alpha_2, \cdots, \alpha_n) = SVTNA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \oplus_{i=1}^{n} \left(\frac{1}{n} \alpha_i \right)$$

Proof. Because $Sup(\alpha_i, \alpha_j) = c$ $(c \in [0,1], i \neq j, j = 1, 2, \cdots, n)$, we have $G(\alpha_i) = \sum_{j=1, j\neq i}^{n} Sup(\alpha_i, \alpha_j) = (n-1)c$.

Therefore,

$$SVTNPA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \oplus_{i=1}^{n} \left(\frac{1+G(\alpha_i)_i}{\sum_{i=1}^{n}(1+G(\alpha_i))} \alpha_i \right) = \oplus_{i=1}^{n} \left(\frac{1+(n-1)c}{\sum_{i=1}^{n}(1+(n-1)c)} \alpha_i \right) = \oplus_{i=1}^{n} \left(\frac{1}{n} \alpha_i \right).$$

□

Finally, we can get $SVTNPA(\alpha_1, \alpha_2, \cdots, \alpha_n) = SVTNA(\alpha_1, \alpha_2, \cdots, \alpha_n) = \oplus_{i=1}^{n} (\alpha_i)$ and the proof of Theorem 4 is completed now.

Definition 13. *Let* $\alpha_i = \langle [a_{i1}, a_{i2}, a_{i3}, a_{i4}], (T(\alpha_i), I(\alpha_i), F(\alpha_i)) \rangle (i = 1, 2, \cdots, n)$ *be a collection of positive SVTNNs. Then the single valued trapezoidal neutrosophic power geometric (SVTNPG) operator can be defined as follows:*

$$SVTNPG(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha_1^{\frac{1+G(\alpha_1)}{\sum_{i=1}^{n}(1+G(\alpha_i))}} \otimes \alpha_2^{\frac{1+G(\alpha_2)}{\sum_{i=1}^{n}(1+G(\alpha_i))}} \otimes \cdots \otimes \alpha_n^{\frac{1+G(\alpha_n)}{\sum_{i=1}^{n}(1+G(\alpha_i))}} = \otimes_{i=1}^{n} \left(\alpha_i^{\frac{1+G(\alpha_i)}{\sum_{i=1}^{n}(1+G(\alpha_i))}} \right),$$

where $G(\alpha_i) = \sum_{j=1, j\neq i}^{n} Sup(\alpha_i, \alpha_j)$, $Sup(\alpha_i, \alpha_j)$ *is the support for* α_i *from* α_j.

Theorem 5. *Let* $\alpha_i = \langle [a_{i1}, a_{i2}, a_{i3}, a_{i4}], (T(\alpha_i), I(\alpha_i), F(\alpha_i)) \rangle (i = 1, 2, \cdots, n)$ *be a collection of positive SVTNNs. The aggregated result, obtained by using the SVTNPG operator, is also a positive SVTNN, and*

$$\begin{aligned} &SVTNPG(\alpha_1, \alpha_2, \cdots, \alpha_n) = \otimes_{i=1}^{n} \left(\alpha_i^{\tau(\alpha_i)} \right) \\ &= \left\langle \left[\prod_{i=1}^{n} a_{i1}^{\tau(\alpha_i)}, \prod_{i=1}^{n} a_{i2}^{\tau(\alpha_i)}, \prod_{i=1}^{n} a_{i3}^{\tau(\alpha_i)}, \prod_{i=1}^{n} a_{i4}^{\tau(\alpha_i)} \right], \left(\prod_{i=1}^{n} (T(\alpha_i))^{\tau(\alpha_i)}, 1 - \prod_{i=1}^{n} \left((1 - I(\alpha_i))^{\tau(\alpha_i)} \right), \right.\right. \\ &\left.\left. 1 - \prod_{i=1}^{n} \left((1 - F(\alpha_i))^{\tau(\alpha_i)} \right) \right) \right\rangle, \end{aligned}$$

where $\tau(\alpha_i) = \frac{1+G(\alpha_i)}{\sum_{i=1}^{n}(1+G(\alpha_i))}$, $G(\alpha_i) = \sum_{j=1, j\neq i}^{n} Sup(\alpha_i, \alpha_j)$, $Sup(\alpha_i, \alpha_j) = 1 - \left|p(\alpha_i \succ \alpha_j) - p(\alpha_j \succ \alpha_i)\right|$ *is the support for* α_i *from* α_j, $p(\alpha_i \succ \alpha_j)$ *and* $p(\alpha_j \succ \alpha_i)$ *are the score functions of* $\alpha_i \succ \alpha_j$, $\alpha_j \succ \alpha_i$.

The proof of Theorem 5 can refer to Theorem 3.

Example 10. *Use the data of Example 9. Then* $SVTNPG(\alpha_1, \alpha_2, \alpha_3)$ *can be calculated as follows.*

According to Example 9, we can get $\tau(\alpha_1) = 0.247$, $\tau(\alpha_2) = 0.248$, $\tau(\alpha_3) = 0.252$, $\tau(\alpha_4) = 0.253$;

$$\text{So } SVTNPG(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \langle[0.271, 0.447, 0.569, 0.696], (0.669,\ 0.204, 0.326)\rangle.$$

Theorem 6. *Let* $\alpha_i = \langle[a_{i1}, a_{i2}, a_{i3}, a_{i4}], (T(\alpha_i), I(\alpha_i), F(\alpha_i))\rangle (i = 1, 2, \cdots, n)$ *be a collection of positive SVTNNs. If* $Sup(\alpha_i, \alpha_j) = c\ (c \in [0, 1], i \neq j, j = 1, 2, \cdots, n)$ *, then SVTNPG operator reduces to single valued trapezoidal neutrosophic geometric (SVTNG) operator as follows:*

$$SVTNPG(\alpha_1, \alpha_2, \cdots, \alpha_n) = SVTNG(\alpha_1, \alpha_2, \cdots, \alpha_n) = \oplus_{i=1}^{n}\left({\alpha_i}^{1/n}\right).$$

The proof of Theorem 6 can refer to Theorem 4.

5. A MCGDM Method Based on Possibility Degree and Power Aggregation Operators under Single Valued Trapezoidal Neutrosophic Environment

In this section, the possibility degrees of SVTNNs, single trapezoidal neutrosophic power weighted aggregation operators are applied to MCGDM problems single valued trapezoidal neutrosophic information.

For a MCGDM problems with single valued trapezoidal neutrosophic information, assume that the set of alternatives is $B = \{B_1, B_2, \cdots, B_m\}$, $D = \{D_1, D_2, \cdots, D_t\}$ is the set of decision-makers who evaluate the alternatives according to the criteria $C = \{C_1, C_2, \cdots, C_n\}$. The evaluation information $\alpha_{ij}^{y}(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n; y = 1, 2, \cdots, t)$ which is described by positive SVTNNs, can be given by decision-makers $D_y(y = 1, 2, \cdots, t)$ when they assess the alternatives $B_i(i = 1, 2, \cdots, m)$ with respect to the criteria $C_j(j = 1, 2, \cdots, n)$ and then the decision matrices $R_y = (\alpha_{ij}^{y})_{m\times n}$ are obtained. A method of determining the ranking of the alternatives is introduced here and the decision-making procedures are shown as follows.

Step 1. Normalize the decision matrices.

Normalize the decision-making information α_{ij}^{y} in the matrices $R_y = (\alpha_{ij}^{y})_{m\times n}$. The criteria can be classified into the benefit type and the cost type. For the benefit-type criterion, the form of the evaluation information needs no change; but for the cost-type criterion, the negation operator is used.

The normalization of the decision matrices can be represented as follows:

$$\begin{cases} \widetilde{\alpha}_{ij}^{y} = \alpha_{ij}^{y} & , C_j \in B_T \\ \widetilde{\alpha}_{ij}^{y} = neg(\alpha_{ij}^{y}) & , C_j \in C_T \end{cases},$$

where B_T is the set of benefit-type criteria and C_T is the set of cost-type criteria.

The normalized decision matrices are denoted as $\overline{R}_y = (\widetilde{\alpha}_{ij}^{y})_{m\times n}$.

Step 2. Aggregate the values of alternatives on each criterion to get the collective SVTNNs.

Based on the Definitions 12 or 13, the collective SVTNNs α_{ij} or $\widetilde{\alpha}_{ij}$ can be gotten by SVTNPA or SVTNPG operator, the aggregation values of decision-makers on each alternative are as follows:

$$\alpha_{iy} = SVTNPA(\widetilde{\alpha}_{i1}^{y}, \widetilde{\alpha}_{i2}^{y}, \cdots, \widetilde{\alpha}_{in}^{y}) \text{ or } \widetilde{\alpha}_{iy} = SVTNPG(\widetilde{\alpha}_{i1}^{y}, \widetilde{\alpha}_{i2}^{y}, \cdots, \widetilde{\alpha}_{in}^{y}).$$

Then the collective preference matrix $P = (\alpha_{iy})_{m \times y}$ or $\widetilde{P} = (\widetilde{\alpha}_{iy})_{m \times y}$ can be obtained.

Step 3. Aggregate the values of alternative on each decision-maker to get the overall SVTNNs.

Based on the Definitions 12 or 13, the overall SVTNNs α_{ij} or $\widetilde{\alpha}_{ij}$ can be gotten by SVTNPA or SVTNPG operator, the aggregation values of alternative on each decision-maker are as follows:

$$\beta_i = SVTNPA(\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{it}) \text{ or } \widetilde{\beta}_i = SVTNPA(\widetilde{\alpha}_{i1}, \widetilde{\alpha}_{i2}, \cdots, \widetilde{\alpha}_{it}).$$

Then the coverall preference matrix $K = (\beta_i)$ or $\widetilde{K} = (\widetilde{\beta}_i)$ can be obtained.

Step 4. Calculate the possibility degrees of the assessment values of each alternative superior than other alternatives' values.

Based on Definition 9, the possibility degrees of $\beta_i \succ \beta_{i'}(i \neq i')$ or $\widetilde{\beta}_i \succ \widetilde{\beta}_{i'}(i \neq i')$ can be obtained. The matrix of $p(\beta_i \succ \beta_{i'})$ or $p(\widetilde{\beta}_i \succ \widetilde{\beta}_{i'})$ can be represented as $U = (p(\beta_i \succ \beta_{i'}))_{m \times m}$ or $\widetilde{U} = \left(p(\widetilde{\beta}_i \succ \widetilde{\beta}_{i'})\right)_{m \times m}$.

Step 5. Calculate the collective possibility degree index of each alternative to derive the overall values of the alternatives.

Aggregate U or $\widetilde{U}$ to get the overall possibility degree index $p(B_i)$ of the alternative B_i by using the following functions:

$$p(B_i) = \frac{1}{m-1} \sum\nolimits_{i'=1, i' \neq i}^{m} p(\beta_i \succ \beta_{i'}) \text{ or } \widetilde{p}(B_i) = \frac{1}{m-1} \sum\nolimits_{i'=1, i' \neq i}^{m} p(\widetilde{\beta}_i \succ \widetilde{\beta}_{i'}).$$

Then the overall possibility degree index matrix $Q = (p(B_i))^T$ or $\widetilde{Q} = (\widetilde{p}(B_i))^T$ can be obtained.

Step 6. Rank the alternatives and select the best one.

According to the results obtained in Step 5, rank the alternatives by the overall values in descending order and the first order alternative is the best.

6. Illustrative Example

In this section, a green supplier selection problem is used to illustrate the validity and effectiveness of the developed method.

6.1. Background

The following case background is adapted from [51].

In recent years, more and more people pay attention the serious environmental problems caused badly by the rapid economic development of all over the world. The green supply chain management becomes imperative under this situation because of its advantages on the sustainable development of economics and protection of environment. Meanwhile, it can bring tremendous economic benefit and competitive strengthen for the enterprises.

Motivated by the advantages of green supply chain management, Shanghai General Motors (SGM) Company wants to select the most appropriate green supplier as its cooperative alliance. After pre-evaluation, four suppliers become the final alternatives for further evaluation, including Howden Hua Engineering Company (B_1), Sino Trunk (B_2), Taikai Electric Group Company (B_3) and Shantui construction machinery Company (B_4). SGM employs four experts ($D_y(y = 1,2,3,4)$) coming from the departments of production, purchasing, quality inspection, engineering to form a group of decision-makers for evaluating the four suppliers $B_i(i = 1,2,3,4)$ according the product quality (C_1), technology capability (C_2), pollution control (C_3) and environment management (C_4).

The four experts $D_y (y = 1, 2, 3, 4)$ give their assessment information about the four green suppliers $B_i (i = 1, 2, 3, 4)$ according to the four criteria ($C_j (j = 1, 2, 3, 4)$). Assume that the four experts' attitudes on evaluating the four green suppliers are neutral, that is $\lambda = 0.5$. The assessment information $\alpha_{ij}^y (i = 1, 2, 3, 4; j = 1, 2, 3, 4; y = 1, 2, 3, 4)$ is described by SVTNNs and the decision matrices are shown in R_1, R_2, R_3 and R_4.

$$R_1 = \begin{array}{c} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \\ \langle[0.6,0.7,0.8,0.9],(0.6,0.3,0.2)\rangle & \langle[0.3,0.4,0.5,0.6],(0.6,0.2,0.4)\rangle & \langle[0.5,0.6,0.7,0.9],(0.3,0.3,0.4)\rangle & \langle[0.6,0.7,0.8,0.9],(0.5,0.3,0.3)\rangle \\ \langle[0.3,0.4,0.5,0.6],(0.7,0.2,0.3)\rangle & \langle[0.4,0.5,0.6,0.7],(0.5,0.2,0.3)\rangle & \langle[0.4,0.5,0.6,0.8],(0.7,0.2,0.3)\rangle & \langle[0.5,0.7,0.8,0.9],(0.4,0.1,0.6)\rangle \\ \langle[0.3,0.4,0.5,0.6],(0.4,0.3,0.2)\rangle & \langle[0.3,0.4,0.5,0.7],(0.6,0.1,0.3)\rangle & \langle[0.2,0.4,0.5,0.6],(0.5,0.3,0.3)\rangle & \langle[0.6,0.7,0.8,0.9],(0.5,0.4,0.2)\rangle \\ \langle[0.1,0.2,0.4,0.5],(0.8,0.2,0.1)\rangle & \langle[0.3,0.4,0.6,0.7],(0.2,0.5,0.4)\rangle & \langle[0.5,0.6,0.7,0.8],(0.6,0.3,0.1)\rangle & \langle[0.4,0.5,0.6,0.7],(0.7,0.1,0.2)\rangle \end{array}$$

$$R_2 = \begin{array}{c} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \\ \langle[0.1,0.3,0.4,0.5],(0.5,0.2,0.4)\rangle & \langle[0.2,0.4,0.6,0.8],(0.8,0.1,0.2)\rangle & \langle[0.2,0.5,0.6,0.8],(0.7,0.3,0.1)\rangle & \langle[0.1,0.4,0.5,0.6],(0.6,0.2,0.3)\rangle \\ \langle[0.2,0.4,0.6,0.8],(0.6,0.1,0.3)\rangle & \langle[0.4,0.6,0.8,1.0],(0.7,0.2,0.2)\rangle & \langle[0.4,0.6,0.8,1.0],(0.5,0.2,0.3)\rangle & \langle[0.3,0.5,0.6,0.7],(0.8,0.1,0.1)\rangle \\ \langle[0.2,0.4,0.6,1.0],(0.6,0.3,0.2)\rangle & \langle[0.2,0.4,0.6,0.8],(0.8,0.1,0.2)\rangle & \langle[0.1,0.2,0.6,0.8],(0.6,0.2,0.2)\rangle & \langle[0.1,0.2,0.3,0.5],(0.6,0.2,0.4)\rangle \\ \langle[0.2,0.3,0.4,0.7],(0.5,0.2,0.3)\rangle & \langle[0.1,0.2,0.3,0.5],(0.6,0.4,0.2)\rangle & \langle[0.1,0.3,0.5,0.7],(0.7,0.2,0.2)\rangle & \langle[0.1,0.2,0.4,0.5],(0.5,0.1,0.3)\rangle \end{array}$$

$$R_3 = \begin{array}{c} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \\ \langle[0.5,0.7,0.8,0.9],(0.6,0.1,0.3)\rangle & \langle[0.4,0.5,0.6,0.8],(0.7,0.2,0.3)\rangle & \langle[0.4,0.6,0.7,0.8],(0.3,0.7,0.1)\rangle & \langle[0.3,0.5,0.6,0.8],(0.5,0.3,0.3)\rangle \\ \langle[0.6,0.7,0.8,0.9],(0.7,0.2,0.2)\rangle & \langle[0.1,0.3,0.5,0.6],(0.4,0.5,0.2)\rangle & \langle[0.3,0.5,0.6,0.7],(0.4,0.3,0.3)\rangle & \langle[0.1,0.2,0.4,0.5],(0.7,0.2,0.1)\rangle \\ \langle[0.7,0.8,0.9,1.0],(0.6,0.2,0.2)\rangle & \langle[0.3,0.4,0.6,0.7],(0.5,0.4,0.2)\rangle & \langle[0.1,0.2,0.6,0.8],(0.5,0.2,0.3)\rangle & \langle[0.1,0.2,0.4,0.5],(0.6,0.2,0.3)\rangle \\ \langle[0.4,0.5,0.7,0.9],(0.5,0.2,0.3)\rangle & \langle[0.1,0.2,0.3,0.4],(0.4,0.5,0.1)\rangle & \langle[0.1,0.3,0.5,0.6],(0.6,0.2,0.2)\rangle & \langle[0.1,0.2,0.3,0.5],(0.5,0.4,0.2)\rangle \end{array}$$

$$R_4 = \begin{array}{c} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \\ \langle[0.4,0.5,0.7,0.8],(0.4,0.2,0.5)\rangle & \langle[0.4,0.5,0.6,0.7],(0.6,0.1,0.4)\rangle & \langle[0.5,0.6,0.7,0.9],(0.3,0.4,0.4)\rangle & \langle[0.4,0.7,0.8,1.0],(0.3,0.1,0.6)\rangle \\ \langle[0.5,0.6,0.7,0.9],(0.3,0.3,0.5)\rangle & \langle[0.5,0.6,0.7,0.8],(0.4,0.3,0.3)\rangle & \langle[0.4,0.6,0.7,0.8],(0.7,0.1,0.3)\rangle & \langle[0.5,0.6,0.8,0.9],(0.5,0.3,0.4)\rangle \\ \langle[0.3,0.5,0.6,0.8],(0.4,0.2,0.2)\rangle & \langle[0.2,0.4,0.5,0.8],(0.6,0.3,0.2)\rangle & \langle[0.2,0.4,0.5,0.6],(0.5,0.2,0.3)\rangle & \langle[0.3,0.5,0.6,0.8],(0.4,0.3,0.2)\rangle \\ \langle[0.1,0.2,0.4,0.6],(0.6,0.2,0.3)\rangle & \langle[0.3,0.5,0.6,0.7],(0.5,0.5,0.1)\rangle & \langle[0.5,0.6,0.7,0.8],(0.4,0.2,0.3)\rangle & \langle[0.2,0.4,0.6,0.7],(0.5,0.4,0.1)\rangle \end{array}$$

6.2. The Procedures of Single Valued Trapezoidal Neutrosophic MCGDM Method

The proposed MCGDM method is used for determining the ranking of the green suppliers.

Step 1. Normalize the decision matrices.

The four criteria $C_j (j = 1, 2, 3, 4)$ are regarded as the benefit-type criterion, so the decision matrices change nothing.

Step 2. Aggregate the values of the four alternatives on each criterion to get the collective SVTNNs.

Use the SVTNPA or SVTNPG operator to aggregate the values of four alternatives on each criterion, the collective SVTNNs are obtained shown in P and $\widetilde{P}$.

$$P = \begin{array}{c} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \\ \langle[0.50,0.60,0.70,0.82],(0.49,0.28,0.32)\rangle & \langle[0.15,0.40,0.53,0.68],(0.67,0.20,0.23)\rangle & \langle[0.40,0.58,0.68,0.83],(0.53,0.32,0.25)\rangle & \langle[0.42,0.58,0.70,0.85],(0.39,0.20,0.48)\rangle \\ \langle[0.40,0.53,0.63,0.75],(0.55,0.17,0.40)\rangle & \langle[0.33,0.53,0.70,0.88],(0.64,0.16,0.23)\rangle & \langle[0.27,0.42,0.57,0.67],(0.56,0.29,0.21)\rangle & \langle[0.47,0.60,0.73,0.85],(0.47,0.25,0.38)\rangle \\ \langle[0.34,0.47,0.57,0.70],(0.50,0.29,0.24)\rangle & \langle[0.15,0.30,0.53,0.78],(0.66,0.20,0.23)\rangle & \langle[0.30,0.40,0.62,0.75],(0.55,0.25,0.24)\rangle & \langle[0.25,0.45,0.55,0.75],(0.47,0.25,0.22)\rangle \\ \langle[0.32,0.42,0.57,0.67],(0.57,0.27,0.20)\rangle & \langle[0.13,0.25,0.40,0.60],(0.58,0.22,0.25)\rangle & \langle[0.17,0.30,0.45,0.60],(0.51,0.29,0.22)\rangle & \langle[0.28,0.43,0.58,0.70],(0.48,0.33,0.20)\rangle \end{array}$$

$$\widetilde{P} = \begin{array}{c} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \\ \langle[0.48,0.58,0.69,0.81],(0.48,0.28,0.33)\rangle & \langle[0.14,0.39,0.52,0.67],(0.64,0.20,0.26)\rangle & \langle[0.39,0.57,0.67,0.82],(0.50,0.38,0.25)\rangle & \langle[0.42,0.57,0.70,0.84],(0.38,0.21,0.48)\rangle \\ \langle[0.39,0.51,0.62,0.74],(0.56,0.18,0.39)\rangle & \langle[0.31,0.52,0.69,0.87],(0.64,0.15,0.23)\rangle & \langle[0.20,0.38,0.55,0.66],(0.53,0.31,0.20)\rangle & \langle[0.47,0.60,0.72,0.85],(0.45,0.25,0.38)\rangle \\ \langle[0.32,0.46,0.56,0.69],(0.49,0.28,0.25)\rangle & \langle[0.14,0.29,0.51,0.76],(0.65,0.20,0.25)\rangle & \langle[0.21,0.33,0.60,0.72],(0.55,0.26,0.25)\rangle & \langle[0.25,0.45,0.55,0.75],(0.47,0.25,0.23)\rangle \\ \langle[0.27,0.39,0.56,0.66],(0.52,0.29,0.21)\rangle & \langle[0.12,0.25,0.39,0.59],(0.57,0.24,0.25)\rangle & \langle[0.14,0.28,0.42,0.57],(0.49,0.34,0.20)\rangle & \langle[0.24,0.40,0.57,0.70],(0.49,0.34,0.21)\rangle \end{array}$$

Step 3. Aggregate the values of the four alternatives on each green supplier to get the overall SVTNNs by using the SVTNPA or SVTNPG operator.

The coverall preference matrix shown in K or $\widetilde{K}$.

$$K = \begin{array}{c} B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{pmatrix} [0.36,0.54,0.65,0.79],(0.51,0.25,0.32) \\ [0.37,0.52,0.66,0.79],(0.55,0.22,0.31) \\ [0.26,0.40,0.57,0.74],(0.54,0.25,0.23) \\ [0.22,0.35,0.50,0.64],(0.53,0.28,0.21) \end{pmatrix}$$

$$\widetilde{K} = \begin{matrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{matrix} \begin{pmatrix} [0.32, 0.52, 0.64, 0.78], (0.50, 0.27, 0.34) \\ [0.33, 0.50, 0.64, 0.77], (0.54, 0.23, 0.31) \\ [0.22, 0.37, 0.55, 0.73], (0.54, 0.25, 0.25) \\ [0.18, 0.32, 0.48, 0.63], (0.52, 0.30, 0.22) \end{pmatrix}$$

Step 4. Calculate the possibility degrees of the assessment values of each alternative superior than other alternatives' values to get the possibility degrees matrix U or $\widetilde{U}$.

$$U = \begin{matrix} & B_1 & B_2 & B_3 & B_4 \\ B_1 & - & 0.48 & 0.51 & 0.54 \\ B_2 & 0.52 & - & 0.52 & 0.54 \\ B_3 & 0.48 & 0.48 & - & 0.53 \\ B_4 & 0.47 & 0.46 & 0.47 & - \end{matrix}$$

$$\widetilde{U} = \begin{matrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{matrix} \begin{pmatrix} - & 0.48 & 0.51 & 0.53 \\ 0.52 & - & 0.53 & 0.54 \\ 0.49 & 0.47 & - & 0.52 \\ 0.47 & 0.46 & 0.48 & - \end{pmatrix}$$

Step 5. Calculate the collective possibility degree index of each alternative to derive the overall values of the alternatives.

Aggregate U or $\widetilde{U}$ to get the overall possibility degree index and the overall possibility degree index matrix Q or $\widetilde{Q}$.

$$Q = \begin{matrix} B_1 & B_2 & B_3 & B_4 \\ (0.512 & 0.526 & 0.497 & 0.465) \end{matrix} \qquad Q = \begin{matrix} B_1 & B_2 & B_3 & B_4 \\ (0.510 & 0.528 & 0.494 & 0.468) \end{matrix}$$

Step 6. Rank the green suppliers and select the best one.

The ranking of the four green suppliers is $B_2 \succ B_1 \succ B_3 \succ B_4$. Therefore, SGM Company will choose Sino Trunk as its cooperative alliance.

The rankings of green suppliers using the SVTNPA operators for different values of λ are shown in Figure 1. In general, larger values of λ are associated with relatively pessimistic decision-makers; thus, the alternatives were associated with relatively overall possibility degree index. In contrast, lower values of λ are associated with relatively optimistic decision-makers. When the decision-makers do not indicate any preferences, the most commonly-used value (λ = 0.5) is used.

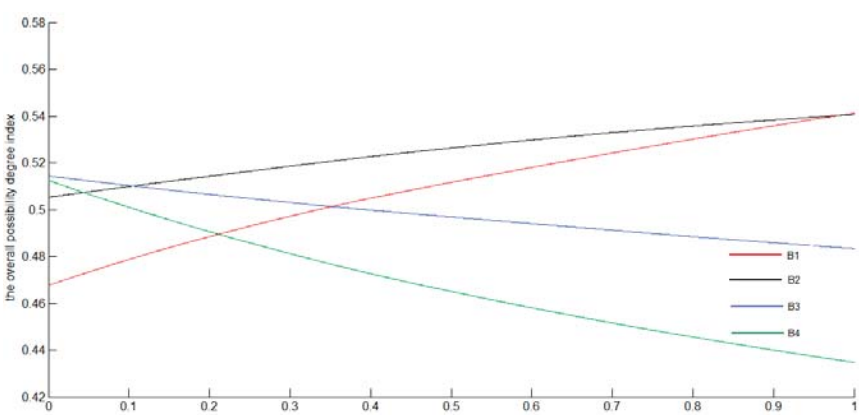

Figure 1. Rankings of various green suppliers for different values of λ.

6.3. Comparison Analysis and Discussion

In order to validate the accuracy of the proposed single valued trapezoidal neutrosophic MCGDM method, a comparative study is conducted based on the illustrative example in this paper and the method used for comparison was proposed by Ye [44].

When resolving the above example using the approach described in Reference [44], which involves the use of t trapezoidal neutrosophic weighted arithmetic averaging (TNWAA) operator or trapezoidal neutrosophic weighted geometric averaging (TNWGA) operator with known weights to comprehensively analyze green suppliers, the weights of the decision-makers and criteria can be generated using the PA operator ($w_{ij} = \frac{1+G(\alpha_{ij})}{\sum_{j=1}^{n}(1+G(\alpha_{ij}))}$, $G(\alpha_{ij}) = \sum_{j=1, j\neq j'}^{n}\left(1 - \left|S(\alpha_{ij}) - S(\alpha_{ij'})\right|\right)$) and $S(\alpha_{ij})$ [44] is the score function value of the SVTNN a_{ij}. The overall values of four alternatives on each criterion obtained by using TNWAA operator are shown as the matrix M, the matrix $\widetilde{M}$ got by using TNWGA operator.

$$M = \begin{array}{c} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \\ \left(\begin{array}{c} \langle[0.52,0.62,0.72,0.85],(0.51,0.27,0.32)\rangle \\ \langle[0.40,0.53,0.63,0.75],(0.54,0.17,0.36)\rangle \\ \langle[0.34,0.46,0.57,0.69],(0.51,0.24,0.25)\rangle \\ \langle[0.32,0.42,0.57,0.67],(0.63,0.24,0.17)\rangle \end{array}\right. & \begin{array}{c} \langle[0.15,0.40,0.53,0.68],(0.68,0.18,0.22)\rangle \\ \langle[0.32,0.52,0.70,0.87],(0.67,0.14,0.21)\rangle \\ \langle[0.15,0.30,0.53,0.78],(0.66,0.19,0.24)\rangle \\ \langle[0.13,0.25,0.40,0.60],(0.58,0.20,0.24)\rangle \end{array} & \begin{array}{c} \langle[0.40,0.57,0.67,0.82],(0.50,0.26,0.22)\rangle \\ \langle[0.27,0.42,0.57,0.67],(0.57,0.28,0.19)\rangle \\ \langle[0.28,0.38,0.62,0.74],(0.55,0.24,0.25)\rangle \\ \langle[0.17,0.30,0.45,0.60],(0.50,0.30,0.18)\rangle \end{array} & \left.\begin{array}{c} \langle[0.43,0.58,0.70,0.85],(0.42,0.17,0.47)\rangle \\ \langle[0.47,0.60,0.73,0.85],(0.50,0.23,0.37)\rangle \\ \langle[0.25,0.45,0.55,0.75],(0.48,0.24,0.22)\rangle \\ \langle[0.27,0.42,0.57,0.70],(0.51,0.30,0.17)\rangle \end{array}\right) \end{array}$$

$$\widetilde{M} = \begin{array}{c} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \\ \left(\begin{array}{c} \langle[0.51,0.62,0.72,0.84],(0.48,0.28,0.33)\rangle \\ \langle[0.39,0.52,0.62,0.74],(0.47,0.18,0.39)\rangle \\ \langle[0.31,0.45,0.56,0.69],(0.50,0.28,0.26)\rangle \\ \langle[0.27,0.39,0.56,0.66],(0.50,0.30,0.21)\rangle \end{array}\right. & \begin{array}{c} \langle[0.14,0.40,0.52,0.67],(0.65,0.20,0.25)\rangle \\ \langle[0.31,0.52,0.69,0.86],(0.64,0.15,0.23)\rangle \\ \langle[0.14,0.28,0.51,0.76],(0.65,0.20,0.25)\rangle \\ \langle[0.12,0.25,0.39,0.59],(0.57,0.23,0.25)\rangle \end{array} & \begin{array}{c} \langle[0.39,0.57,0.67,0.82],(0.50,0.38,0.25)\rangle \\ \langle[0.21,0.38,0.56,0.66],(0.53,0.31,0.20)\rangle \\ \langle[0.20,0.32,0.59,0.72],(0.54,0.26,0.25)\rangle \\ \langle[0.14,0.28,0.42,0.57],(0.49,0.34,0.20)\rangle \end{array} & \left.\begin{array}{c} \langle[0.42,0.57,0.70,0.84],(0.38,0.21,0.48)\rangle \\ \langle[0.47,0.60,0.72,0.85],(0.45,0.25,0.38)\rangle \\ \langle[0.24,0.45,0.55,0.74],(0.47,0.25,0.23)\rangle \\ \langle[0.23,0.39,0.56,0.70],(0.50,0.34,0.21)\rangle \end{array}\right) \end{array}$$

The collective values of the four green suppliers can also be obtained by using the TNWAA operator as the matrix U or the matrix $\widetilde{U}$ by using the TNWGA operator.

$$U = \begin{array}{c} B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \left(\begin{array}{c} [0.37,0.52,0.66,0.78],(0.57,0.20,0.27) \\ \langle[0.38,0.54,0.66,0.80],(0.55,0.22,0.29)\rangle \\ \langle[0.26,0.40,0.57,0.74],(0.56,0.23,0.24)\rangle \\ \langle[0.23,0.35,0.50,0.64],(0.56,0.26,0.19)\rangle \end{array}\right)$$

$$\widetilde{U} = \begin{array}{c} B_1 \\ B_2 \\ B_3 \\ B_4 \end{array} \left(\begin{array}{c} \langle[0.33,0.50,0.64,0.77],(0.51,0.23,0.31)\rangle \\ \langle[0.33,0.53,0.65,0.79],(0.49,0.27,0.34)\rangle \\ \langle[0.22,0.37,0.55,0.73],(0.54,0.25,0.25)\rangle \\ \langle[0.18,0.32,0.48,0.63],(0.51,0.30,0.22)\rangle \end{array}\right)$$

Finally, the score values $s_i(i = 1,2,3,4)$ of each green supplier can be obtained by using the score degree function show in the matrix H or $\widetilde{H}$.

$$H = \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \left(\begin{array}{cccc} 0.410 & 0.404 & 0.342 & 0.301 \end{array}\right) \end{array} \qquad \widetilde{H} = \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \left(\begin{array}{cccc} 0.371 & 0.362 & 0.317 & 0.267 \end{array}\right) \end{array}$$

So, the ranking is $B_1 \succ B_2 \succ B_3 \succ B_4$ and the best green supplier obtained by using the approach in Reference [44] is B_1. The ranking results of different methods can be shown in Table 1.

Table 1. The ranking results of different methods.

Methods	Operators	Ranking of Alternatives
The method in Reference [44]	NNTWA operator NNTWG operator	$B_1 \succ B_2 \succ B_4 \succ B_3$ $B_4 \succ B_2 \succ B_1 \succ B_3$
The proposed method	SVTNPA operator and the possibility degrees SVTNNs SVTNPG operator and the possibility degrees SVTNNs	$B_2 \succ B_1 \succ B_3 \succ B_4$ $B_2 \succ B_1 \succ B_4 \succ B_3$

From Table 1, it can be seen results of the ranking on the four green suppliers obtained by the proposed single trapezoidal neutrosophic MCGDM method in this paper is quite different from that the ranking obtained by the method introduced in Reference [44]. The main reasons are summarized as follows.

(a) The new operations of SVTNNs defined in this paper, which take the conservative and reliable principle, can take account of the correlation between trapezoidal fuzzy numbers and three membership degrees of SVTNNs. However, the operations in Reference [44] divide the trapezoidal fuzzy numbers and three membership degrees of SVTNNs into two parts and calculate them separately, which make aggregating results deviate from the reality.
(b) The new comparison of SVTNNs proposed in this paper has some crucial advantages over comparison of SVTNNs based on the score degree function in Reference [44], which can take the preference of decision-makers into consideration.
(c) The relationship among the aggregation information, which exists in the aggregation process of in practical MCDM problems, is ignored [44]. Whereas, the SVTNPA and SVTNPG operators, which can effectively take the relationship among the assessment information being aggregated into consideration and in this paper, the advantages of the possibility degree of SVTNNs are combined to rank the uncertain information reasonably and accurately from the probability viewpoint. Hence, the ranking result of this paper is more objective and reasonable than that obtained by using the operators in Reference [44].

7. Conclusions

In order to improve the reasonability and effectiveness of the methods on dealing with single valued trapezoidal neutrosophic MCGDM problems, also overcome the limitations of the existing approaches. In this paper, a single valued trapezoidal neutrosophic MCGDM method is proposed form the possibility degree of SVTNNs and the single valued trapezoidal neutrosophic power aggregation operators. Firstly, the new operations of SVTNNs are proposed for avoiding information loss and distortion, the possibility degrees of SVTNNs are proposed from the probability viewpoint. Based on the proposed operations and possibility degrees, SVTNPA and SVTNPG operators are proposed. Furthermore, a single valued trapezoidal neutrosophic MCGDM method based on SVTNPA, SVTNPG operator and the possibility degrees of SVTNNs is developed. The prominent advantages of the proposed method are not only its ability to effectively deal with the preference information expressed by SVTNNs but also the consideration of the relationship among the information being aggregated in the process on dealing with the practical MCGDM problems and the advantage of the possibility degrees of SVTNNs, which can avoid information loss and distortion, is combined. Thus, the final results are more scientific and reasonable. Finally, the method is applied to a practical problem on selecting the most appropriate green supplier for SGM Company, meanwhile, the comparison with other method is carried on and demonstrates its feasibility and effectiveness in dealing with MCGDM problems.

In future research, the developed method will be extended to other domains, such as personnel selection and medical diagnosis.

Author Contributions: X.W. proposed the revised operation rules, J.Q. gave the MDGDM approaches and SVTNPA, SVTNPG operator, all the author wrote the manuscript and revised the final version.

Funding: This work is supported by the National Natural Science Foundation of China (Grant No. 7170011164, 71571193), the Natural Science Foundation of Hubei Province under Grant No. (2016CFB519) and the Project of Youth and Middle-aged Innovation Team of Science and Technology of Hubei Province Grant No. (T201711).

Conflicts of Interest: The authors declare that there is no conflict of interests regarding the publication of this paper.

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Neutrosophic Triplet Non-Associative Semihypergroups with Application

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Received: 8 October 2018; Accepted: 27 October 2018; Published: 8 November 2018

Abstract: In this paper, we extended the idea of a neutrosophic triplet set to non-associative semihypergroups and define neutrosophic triplet $\mathcal{LA}$-semihypergroup. We discuss some basic results and properties. At the end, we provide an application of the proposed structure in Football.

Keywords: LA-semihypergroups; neutrosophic triplet set; neutro-homomorphism

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1. Introduction

The study of origin and features of neutralities lies in the scope of a new branch of philosophy known as Neutrosophy. In 1995, Smarandache (for the first time) used the idea of Neutrosophy and developed neutrosophic logic which is a more practical and realistic approach, to handle imprecise and vague information. He introduced the concept of (T-truth, I-indeterminacy, F-falsity) memberships. According to Smarandache Neutrosophic, logics generalizes the all previous logics such as fuzzy logic [1], intuitionistic fuzzy logic [2] and interval valued fuzzy logic [3]. Kandasamy and Smarandache [4] developed many neutrosophic algebraic structures, neutrosophic bigroup, neutrosophic vector space, neutrosophic groups and so on, based on neutrosophic logic. For practical applications, we refer the readers to [5–9]. For neutrosophic triplet sets, we refer the readers [10–13]. In 2016, Smarandache and Ali [14] gave the concept of Neutrosophic triplet groups which is a very useful addition in the theory of groups.

Hyperstucture theory was brought-out by Marty [15] in 1934, when he defined hypergroup, set about analyzing their properties and exerted them to a group. Several papers and books have been compiled in this direction, see references [16–18]. In 1990, in Greece, a congress was organized by Thomas Vougiouklis on hyperstructure, which was first named algebraic hyper structures and its applications algebraic hyper structures(AHA); however actually was the fourth, because there had been three more congresses in Italy by Corsini, on the same topic but random names. During this congress, Vougiouklis [19] presented the concept of weak structure, presently known as Hv-structure. A number of writers have gone through various aspects of Hv-structure. For instance, references [20–27]. Another book by Davvaz and Fotea in 2007 has been devoted especially to the study of hyperring theory [28].

Kazim and Naseeruddin [29] in 1970, presented the concept of left almost semigroups (LA-semigroups) and shifted the discussion toward non-associative structures. According to them, a groupoid S is called $\mathcal{LA}$-semigroups, if it is satisfies the left invertive law: $(w_1w_2)w_3 = (w_3w_2)w_1$ for all $w_1, w_2, w_3 \in S$. After that, researchers started working in this direction such as, references [30–32] and Yusuf gave the idea of left almost rings [33]. Hila and Dine [34] in 2011, shifted the non-associative structures to non-associative hyperstructures and furnished the idea of $\mathcal{LA}$-semihypergroup, which is generalization of semigroup, semihypergroup, $\mathcal{LA}$-semigroup by using left invertive law with the help of Marty's hyperoperation.Yaqoob et al. [35] expanded the work of Hila and Dine. Yousafzai et al. in [36] and Amjad et al. [37] tried to generalize different aspects of left almost semihypergroups. The concept of Hv-$\mathcal{LA}$-semigroup was laid by Gulistan et al. [38] in 2015. The idea of partially ordered left almost semihypergroups was developed by Naveed et al. [39] in 2015. Rehman et al. [40], initiated the study of $\mathcal{LA}$-hyperrings and discussed its hyperideals and hypersystems in 2017. Nawaz et al. introduced the concept of left almost semihyperrings [41]. Yaqoob et al. [42] gave the idea of left almost polygroups in 2018.

In this paper, we extended the idea of neutrosophic triplet set to non-associative semihypergroups. We define neutrosophic triplet $\mathcal{LA}$-semihypergroup. In neutrosophic triplet $\mathcal{LA}$-semihypergroup every element "w" has left neut(w) and left anti(w). In neutrosophic triplet $\mathcal{LA}$-semihypergroup left neut(w) of an element "w" may or may not be equal to left identity. We also defined the neutro-homomorphism on $\mathcal{LA}$-semihypergroups. At the end, we present an application of the proposed structure in football.

2. Preliminaries

This section of paper consists of some basic definitions, which are directly used in our work.

Definition 1 ([34])**.** *Let $\mathcal{H}$ be a non void set and $\circ : \mathcal{H} * \mathcal{H} \longrightarrow P^{\bullet}(\mathcal{H})$ be a hyperoperation, where $P^{\bullet}(\mathcal{H})$ is the family non-void subset of $\mathcal{H}$. The pair $(\mathcal{H}, *)$ is called hypergroupoid.*

For any two non-void subsets W_1 and W_2 of $\mathcal{H}$, then

$$W_1 * W_2 = \bigcup_{w_1 \in W_1, w_2 \in W_2} w_1 * w_2.$$

Definition 2 ([34])**.** *An $\mathcal{LA}$-semihypergroup is the hypergroupoid $(\mathcal{H}, *)$ with*

$$(w_1 * w_2) * w_3 = (w_3 * w_2) * w_1 \tag{1}$$

for all, $w_1, w_2, w_3 \in \mathcal{H}$. The equation (1) is called left invertive law.

Definition 3 ([35])**.** *An element e of an LA-semihypergroup $\mathcal{H}$ is called left identity (resp., pure left identity) if for all $w_1 \in \mathcal{H}$, $w_1 \in e * w_1$ (resp., $w_1 = e * w_1$). An element e of an LA-semihypergroup $\mathcal{H}$ is called right identity (resp., pure right identity) if for all $w_1 \in \mathcal{H}$, $w_1 \in w_1 * e$ (resp., $w_1 = e * w_1$). An element e of an LA-semihypergroup $\mathcal{H}$ is called identity (resp., pure right identity) if for all $w_1 \in \mathcal{H}$, $w_1 \in w_1 * e \cap e * w_1$ (resp., $w_1 = w_1 * e \cap e * w_1$).*

Definition 4 ([35])**.** *An $\mathcal{LA}$-smihypergroup with pure left identity satisfies the following property*

$$w_1 * (w_2 * w_3) = w_2 * (w_1 * w_3).$$

Definition 5 ([14])**.** *Let N be a non-void set with a binary operation $*$ and $w_1 \in N$. Then w_1 is said to be neutrosophic triplet if there exist an element $neut(w_1) \in N$ such that*

$$w_1 * neut(w_1) = neut(w_1) * w_1 = w_1,$$

where $neut(w_1)$ is different from unity element. Also there exist $anti(w_1) \in N$ such that

$$w_1 * anti(w_1) = anti(w_1) * w_1 = neut(w_1).$$

If there are more $anti(w_1)'s$ for a given w_1, one takes that $anti(w_1) = w_2$ that $anti(w_1)$ in its turn forms a neutrosophic triplet, i.e., there exists $neut(w_2)$ and $anti(w_2)$. We denote the neutrosophic triplet w_1 by $(w_1, neut(w_1), anti(w_1))$. By $neut(w_1)$, we means neutral of w_1.

Example 1 ([14]). *Consider Z_6 under multiplication modulo 6. Then 2 is a neutrosophic triplet, because $neut\,(2) = 4$, as $2 \times 4 = 8$. Similarly $anti\,(2) = 2$ because $2 \times 2 = 4$. Thus 2 is a neutrosophic triplet, which is denoted by $(2,4,2)$. Similarly 4 is a neutrosophic triplet because $neut\,(a) = anti\,(4) = 4$. So 4 is represented by as $(4,4,4)$. 3 is not a neutrosophic triplet as $neut(3) = 5$ but $anti(3)$ does not exist in Z_6 and 0 is a trivial neutrosophic triplet as $neut\,(0) = anti\,(0) = 0$. This is denoted by $(0,0,0)$.*

3. Neutrosophic Triplet $\mathcal{LA}$-Semihypergroups

In this section, we defined the neutrosophic triplet $\mathcal{LA}$-semihypergroup and some results on neutrosophic triplet $\mathcal{LA}$-semihypergroup are provided.

Definition 6. *Let $\mathcal{H}$ be a non void set with a binary hyperoperation $*$ and $w_1 \in \mathcal{H}$. Then $\mathcal{H}$ is called*

1. left neutrosophic triplet set if for every $w_1 \in \mathcal{H}$, there exist $neut\,(w_1)$ and $anti(w_1)$ such that

$$w_1 \in neut\,(w_1) * w_1,$$
$$neut(w_1) \in anti\,(w_1) * w_1.$$

2. right neutrosophic triplet set if for every $w_1 \in \mathcal{H}$, there exist $neut\,(w_1)$ and $anti(w_1)$ such that

$$w_1 \in w_1 * neut\,(w_1)\,,$$
$$neut(w_1) \in w_1 * anti\,(w_1)\,.$$

3. neutrosophic triplet set if for every $w_1 \in \mathcal{H}$, there exist $neut\,(w_1)$ and $anti(w_1)$ such that

$$w_1 \in (neut\,(w_1) * w_1) \cap (w_1 * neut\,(w_1)),$$
$$neut(w_1) \in (anti\,(w_1) * w_1) \cap (w_1 * anti\,(w_1)).$$

Definition 7. *Let $\mathcal{H}$ be a set with a binary hyperoperation $*$ and $w_1 \in \mathcal{H}$. Then $\mathcal{H}$ is called*

1. pure left neutrosophic triplet set if for every $w_1 \in \mathcal{H}$, there exist $neut\,(w_1)$ and $anti\,(w_1)$ such that

$$w_1 = neut\,(w_1) * w_1,$$
$$neut(w_1) = anti\,(w_1) * w_1.$$

2. pure right neutrosophic triplet set if for every $w_1 \in \mathcal{H}$, there exist $neut\,(w_1)$ and $anti(w_1)$ such that

$$w_1 = w_1 * neut\,(w_1)\,,$$
$$neut(w_1) = w_1 * anti\,(w_1)\,.$$

3. pure neutrosophic triplet set if for every $w_1 \in \mathcal{H}$, there exist $neut\,(w_1)$ and $anti(w_1)$ such that

$$w_1 = (neut\,(w_1) * w_1) \cap (w_1 * neut\,(w_1)),$$
$$neut\,(w_1) = (anti\,(w_1) * w_1) \cap (w_1 * anti\,(w_1))\,.$$

Example 2. *Let* $\mathcal{H} = \{w_1, w_2, w_3\}$ *be a se with hyperoperation defined as follows:*

$*$	w_1	w_2	w_3
w_1	w_3	$\{w_1, w_2\}$	w_1
w_2	$\{w_1, w_2\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$
w_3	w_2	$\{w_1, w_3\}$	w_2

A Cayley table 1

Here (w_1, w_2, w_2) , (w_2, w_2, w_2) *and* (w_3, w_2, w_3) *are neutrosophic triplets.*

Definition 8. *Let* $(\mathcal{H}, *)$ *be a left (resp., right, left pure, right pure) neutrosophic triplet set. Then* $\mathcal{H}$ *is called left (resp., right, left pure, right pure) neutrosophic triplet* $\mathcal{LA}$*-semihypergroup, if the following conditions are satisfied:*

1. $(\mathcal{H}, *)$ is well defined.
2. $(\mathcal{H}, *)$ satisfies the left invertive law.

Example 3. *Let* $\mathcal{H} = \{w_1, w_2, w_3, w_4, w_5\}$ *be a set with the hyperoperation defined as follows:*

$*$	w_1	w_2	w_3	w_4	w_5
w_1	w_1	w_1	w_1	w_1	w_1
w_2	w_1	$\{w_3, w_5\}$	w_3	$\{w_1, w_4\}$	$\{w_3, w_5\}$
w_3	w_1	w_3	w_3	$\{w_1, w_4\}$	w_3
w_4	w_1	$\{w_1, w_4\}$	$\{w_1, w_4\}$	w_4	$\{w_1, w_4\}$
w_5	w_1	$\{w_2, w_5\}$	w_3	$\{w_1, w_4\}$	$\{w_2, w_5\}$

A Cayley table 2

Here $(\mathcal{H}, *)$ *is an* $\mathcal{LA}$*-semihypergroup, as the element of* $\mathcal{H}$ *satisfies the left invertive law. Here*(w_1, w_1, w_1) , (w_2, w_5, w_5) , (w_3, w_3, w_3) , (w_4, w_4, w_4) *and* (w_5, w_2, w_5) *are left neutrosophic triplets. Hence* $(\mathcal{H}, *)$ *is a left neutrosophic triplet* $\mathcal{LA}$*-semihypergroup.*

Definition 9. *Let* $(\mathcal{H}, *)$ *be neutrosophic (resp., pure neutrosophic) triplet set. Then* $\mathcal{H}$ *is said to be neutrosophic (resp., pure neutrosophic) triplet* $\mathcal{LA}$*-semihypergroup, if the following condition are satisfied:*

1. $(\mathcal{H}, *)$ is a well defined.
2. $(\mathcal{H}, *)$ satisfies the left invertive law.

Example 4. *Let* $\mathcal{H} = \{w_1, w_2, w_3\}$ *and the hyperoperation defined in the table as follows:*

$*$	w_1	w_2	w_3
w_1	$\{w_1, w_2\}$	$\{w_1, w_2\}$	w_3
w_2	$\mathcal{H}$	$\mathcal{H}$	w_3
w_3	w_3	w_3	w_3

A Cayley table 3

Here $(\mathcal{H}, *)$ is an $\mathcal{LA}$-semihypergroup, as the element of $\mathcal{H}$ satisfies the left invertive law. Here (w_1, w_2, w_1), (w_2, w_1, w_2) and (w_3, w_3, w_3) are neutrosophic triplets. Hence $(\mathcal{H}, *)$ is a neutrosophic triplet $\mathcal{LA}$-semihypergroup.

Remark 1. *Neut(w_2) of an element "w_2" is not unique under the hyperoperation $*$ in $\mathcal{H}$ and depend on elements and hyperoperation. By the Example 4 $neut(w_2) = w_1, w_2$. Similarly $anti(w_2) = w_1, w_2$ of an element "w_2" is not unique and depends on the element and the hyperoperation $*$.*

Remark 2. *Left neut of an element is could be different from left identity.*

Definition 10. *Let $(\mathcal{H}, *)$ be a neutrosophic $\mathcal{LA}$-semihypergroup. An element $w_1 \in \mathcal{H}$, then there exist pure left $neut(w_1)$ such that $w_1 = neut\,(w_1) * w_1$ and pure left $anti(w_1)$ such that $neut(w_1) = anti\,(w_1) * w_1$.*

Proposition 1. *Let $(\mathcal{H}, *)$ be a pure left neutrosophic triplet $\mathcal{LA}$-semihypergroup with pure left identity. Then $w_2 * w_1 = w_3 * w_1$ if and only if*

$$neut\,(w_1) * w_2 = neut\,(w_1) * w_3.$$

Proof. Suppose that $w_2 * w_3 = w_3 * w_1$ for $w_1, w_2, w_3 \in \mathcal{H}$. Since $(\mathcal{H}, *)$ is a pure left neutrosophic $\mathcal{LA}$ semihypergroup, so $anti\,(w_1) \in \mathcal{H}$. Multiply $anti\,(w_1)$ to the right side of $w_2 * w_1 = w_3 * w_1$

$$\begin{aligned}(w_2 * w_1) * anti\,(w_1) &= (w_3 * w_1) * anti\,(w_1)\\ (anti\,(a) * w_1) * w_2 &= (anti\,(w_1) * w_1) * w_3\\ neut\,(w_1) * w_2 &= neut\,(w_1) * w_3.\end{aligned}$$

Conversely, let $neut\,(w_1) * w_2 = neut\,(w_1) * w_3$. Multiply to both right sides by w_1

$$\begin{aligned}neut\,(w_1) * (w_2 * w_1) &= neut\,(w_1) * (w_3 * w_1)\\ w_2 * (neut\,(w_1) * w_1) &= w_3 * (neut\,(w_1) * w_1)\\ w_2 * w_1 &= w_3 * w_1.\end{aligned}$$

This completes the proof. □

Proposition 2. *Let $(\mathcal{H}, *)$ be a pure right neutrosophic triplet $\mathcal{LA}$-semihypergroup with pure left identity. Then $w_2 * neut\,(w_1) = w_3 * neut\,(w_1)$ if $w_2 * anti\,(w_1) = w_3 * anti(w_1)$ for all $w_1, w_2, w_3 \in \mathcal{H}$.*

Proof. Suppose $(\mathcal{H}, *)$ is a pure right neutrosophic triplet $\mathcal{LA}$-semihypergroup with pure left identity and $w_2 * anti\,(w_1) = w_3 * anti(w_1)$ for $w_1, w_2, w_3 \in \mathcal{H}$. Multiply w_1 to the left side of $w_2 * anti\,(w_1) = w_3 * anti(w_1)$,

$$\begin{aligned}w_1 * (w_2 * anti\,(w_1)) &= w_1 * (w_3 * anti(w_1))\\ w_2 * (w_1 * anti\,(w_1)) &= w_3 * (w_1 * anti(w_1))\\ w_2 * neut(w_1) &= w_3 * neut\,(w_1) \quad \text{(because } neut(w_1) = w_1 * anti\,(w_1)\text{)}.\end{aligned}$$

Therefore,

$$w_2 * neut\,(w_1) = w_3 * neut(w_1).$$

□

Proposition 3. *Let $(\mathcal{H}, *)$ be a pure right neutrosophic triplet $\mathcal{LA}$-semihypergroup. Then $neut(w_1) * w_2 = neut\,(w_1) * w_3$ if $w_2 * anti\,(w_1) = w_3 * anti(w_1)$ for all $w_1, w_2, w_3 \in \mathcal{H}$.*

Proof. Suppose $(\mathcal{H},*)$ is a pure right neutrosophic triplet $\mathcal{LA}$-semihypergroup and $w_2 * anti\,(w_1) = w_3 * anti(w_1)$ for $w_1, w_2, w_3 \in \mathcal{H}$. Multiply w_1 to the right side of $w_2 * anti\,(w_1) = w_3 * anti(w_1)$,

$$\begin{aligned}(w_2 * anti\,(w_1)) * w_1 &= (w_3 * anti(w_1)) * w_1\\ (w_1 * anti\,(w_1)) * w_2 &= (w_1 * anti(w_1)) * w_3\\ neut(w_1) * w_2 &= neut\,(w_1) * w_3 \text{ (because } neut(w_1) = w_1 * anti\,(w_1)\text{)}.\end{aligned}$$

Therefore,

$$neut(w_1) * w_2 = neut\,(w_1) * w_3.$$

□

Theorem 1. *Let $(\mathcal{H},*)$ be a pure right neutrosophic triplet idempotent $\mathcal{LA}$-semihypergroup. Then $neut(w_1) * neut\,(w_1) = neut\,(w_1)$.*

Proof. Consider $neut(w_1) * neut\,(w_1) = neut\,(w_1)$. Multiply first with w_1 to the right and then again multiply with w_1 to the right, i.e.,

$$\begin{aligned}((neut(w_1) * neut\,(w_1)) * w_1) * w_1 &= (neut\,(w_1) * w_1) * w_1\\ ((w_1 * neut\,(w_1)) * neut(w_1)) * w_1 &= (w_1 * w_1) * neut\,(w_1)\\ (w_1 * neut(w_1)) * (w_1 * neut\,(w_1)) &= w_1 * neut\,(w_1)\\ w_1 * w_1 &= w_1\\ w_1 &= w_1.\end{aligned}$$

This shows that

$$neut(w_1) * neut\,(w_1) = neut\,(w_1).$$

□

Theorem 2. *Let $(\mathcal{H},*)$ be a pure right neutrosophic triplet idempotent $\mathcal{LA}$-semihypergroup with pure left identity. Then $neut(w_1) * anti\,(w_1) = anti\,(w_1)$.*

Proof. Let $(\mathcal{H},*)$ be a pure right neutrosophic triplet $\mathcal{LA}$-semihypergroup with pure left identity. Multiply w_1 to the left of both side $neut(w_1) * anti\,(w_1) = anti\,(w_1)$, i.e.,

$$\begin{aligned}w_1 * ((neut(w_1) * anti\,(w_1)) &= w_1 * anti\,(w_1)\\ neut(w_1) * (w_1 * anti\,(w_1)) &= neut(w_1)\\ neut(w_1) * neut(w_1) &= neut\,(w_1)\\ neut\,(w_1) &= neut\,(w_1) \text{ (By Theorem 1)}\end{aligned}$$

This shows that

$$neut(w_1) * anti\,(w_1) = anti\,(w_1).$$

□

Theorem 3. *Let $(\mathcal{H},*)$ be a pure right neutrosophic triplet $\mathcal{LA}$-semihypergroup. Then*

1. $neut(w_1) * neut\,(w_2) = neut(w_1 * w_2)$ for all $w_1, w_2 \in \mathcal{H}$.
2. $ant\,(w_1) * anti\,(w_2) = anti\,(w_1 * w_2)$ for all $w_1, w_2 \in \mathcal{H}$.

Proof. 1. Consider the left hand side $neut(w_1) * neut\,(w_2)$. Multiply first with w_2 to the right and then again multiply with w_1 to the right, i.e.,

$$\begin{aligned}((neut(w_1) * neut\,(w_2)) * w_2) * w_1 &= ((w_2 * neut\,(w_2)) * neut(w_1)) * w_1 \\ &= (w_2 * neut(w_1)) * w_1 \\ &= (w_1 * neut(w_1)) * w_2 \\ &= w_1 * w_2.\end{aligned}$$

So

$$((neut(w_1) * neut\,(w_2)) * w_2) * w_1 = w_1 * w_2 \tag{2}$$

Now consider the right side $neut(w_1 * w_2)$. Multiply first with w_2 to the right and then again multiply with w_1 to the right, i.e.,

$$\begin{aligned}(neut(w_1 * w_2) * w_2) * w_1 &= (w_1 * w_2) * neut(w_1 * w_2) \\ &= w_1 * w_2.\end{aligned}$$

So

$$(neut(w_1 * w_2) * w_2) * w_1 = w_1 * w_2 \tag{3}$$

From the Equations (2) and (3) it is clear that $neut(w_1) * neut\,(w_2) = neut(w_1 * w_2)$.

2. Consider the left hand side $anti(w_1) * anti\,(w_2)$. Multiply first with w_2 to the right and then again multiply with w_1 to the right, i.e.,

$$\begin{aligned}((anti(w_1) * anti\,(w_2)) * w_2) * w_1 &= ((w_2 * anti\,(w_2)) * anti(w_1)) * w_1 \\ &= (neut(w_2) * anti(w_1)) * w_1 \\ &= (w_1 * anti(w_1)) * neut(w_2) \\ &= neut(w_1) * neut(w_2) \\ &= neut(w_1 * w_2).\end{aligned}$$

So

$$((anti(w_1) * anti\,(w_2)) * w_2) * w_1 = neut(w_1 * w_2) \tag{4}$$

Now consider the right side $anti(w_1 * w_2)$. Multiply first with w_2 to the right and then again multiply with w_1 to the right, i.e.,

$$\begin{aligned}(anti(w_1 * w_2) * w_2) * w_1 &= (w_1 * w_2) * anti(w_1 * w_2) \\ &= neut(w_1 * w_2)\end{aligned}$$

So

$$(anti(w_1 * w_2) * w_2) * w_1 = neut(w_1 * w_2) \tag{5}$$

From the Equations (4) and (5) it is clear that

$$anti(w_1) * anti\,(w_2) = anti(w_1 * w_2).$$

□

In the following example, we show that in a left neutrosophic triplet $\mathcal{LA}$-semihypergroup

$$neut(w_1) * neut\,(w_2) \neq neut(w_1 * w_2) \tag{6}$$

$$\text{and}\quad anti(w_1) * anti\,(w_2) \neq anti(w_1 * w_2). \tag{7}$$

Example 5. *Let $\mathcal{H} = \{w_1, w_2, w_3, w_4\}$ be a set with the hyperoperation defined as follow*

$*$	w_1	w_2	w_3	w_4
w_1	$\{w_1, w_3, w_4\}$	$\{w_2, w_4\}$	$\mathcal{H}$	$\mathcal{H}$
w_2	$\{w_1, w_2, w_4\}$	$\{w_1, w_2\}$	$\{w_2, w_4\}$	$\{w_2, w_3, w_4\}$
w_3	$\{w_1, w_2, w_3\}$	$\{w_1, w_3, w_4\}$	$\{w_2, w_4\}$	$\{w_2, w_3, w_4\}$
w_4	$\mathcal{H}$	$\{w_2, w_3, w_4\}$	$\mathcal{H}$	$\{w_1, w_3\}$

A Cayley table 4

All the elements of $\mathcal{H}$ satisfies the left invertive law. Here (w_1, w_3, w_4), (w_2, w_4, w_1), (w_3, w_1, w_4) and (w_4, w_2, w_3) are left neutrosophic triplets. Hence $(\mathcal{H}, *)$ is a left neutrosophic triplet $\mathcal{LA}$-semihypergroup. Now

$$\begin{aligned} neut(w_1) * neut\,(w_2) &\neq neut(w_1 * w_2) \\ w_3 * w_4 &\neq neut(\{w_2, w_4\}) \\ \{w_2, w_3, w_4\} &\neq neut(w_2) \cup neut(w_4) \\ \{w_2, w_3, w_4\} &\neq \{w_2, w_4\}. \end{aligned}$$

Also

$$\begin{aligned} anti(w_1) * anti\,(w_2) &\neq anti(w_1 * w_2) \\ w_4 * w_1 &\neq anti(\{w_2, w_4\}) \\ \mathcal{H} &\neq anti(w_2) \cup anti(w_4) \\ \mathcal{H} &\neq \{w_1, w_3\}. \end{aligned}$$

Hence this shows that $neut(w_1) * neut\,(w_2) \neq neut(w_1 * w_2)$ and $anti(w_1) * anti\,(w_2) \neq anti(w_1 * w_2)$.

Theorem 4. *Let $(\mathcal{H}, *)$ be a pure left neutrosophic $\mathcal{LA}$-semihypergroup. Then $neut\,(anti\,(w_1)) = neut\,(w_1)$.*

Proof. Let $neut\,(anti\,(w_1)) = neut\,(w_1)$. If we put $anti\,(w_1) = w_2$, then

$$\begin{aligned} neut\,(w_2) &= neut\,(w_1)\,.\text{ Post multiply by } w_2 \\ neut\,(w_2) * w_2 &= neut\,(w_1) * w_2 \\ w_2 &= neut\,(w_1) * w_2 \\ anti(w_1) &= neut\,(w_1) * anti(w_1), \text{ as } w_2 = anti\,(w_1) \\ anti\,(w_1) &= anti\,(w_1)\,.\text{ By Theorem 1 } neut\,(w_1) * anti(w_1) = anti(w_1) \end{aligned}$$

Hence $neut\,(anti\,(w_1)) = neut\,(w_1)$. □

Theorem 5. *Let $(\mathcal{H}, *)$ be a pure left neutrosophic $\mathcal{LA}$-semihypergroup. Then $anti\,(anti\,(w_1)) = w_1$.*

Proof. Consider $anti\,(anti\,(w_1)) = w_1$. Post multiplying both sides $anti\,(w_1)$

$$\begin{aligned} anti\,(anti\,(w_1)) * anti\,(w_1) &= w_1 * anti\,(w_1) \\ neut\,(anti\,(w_1)) &= (neut\,(w_1) * w_1) * anti\,(w_1) \\ neut\,(anti\,(w_1)) &= (anti\,(w_1) * w_1) * neut\,(w_1) \text{ by left invertive law} \\ neut\,(anti\,(w_1)) &= neut\,(w_1) * neut\,(w_1) \\ neut\,(anti\,(w_1)) &= neut\,(w_1) \text{ by Theorem 1 } neut\,(w_1) * neut\,(w_1) = neut\,(w_1) \\ neut\,(w_1) &= neut\,(w_1) \text{ by Theorem 4} \end{aligned}$$

Hence $anti\,(anti\,(w_1)) = w_1$. □

Definition 11. *Let $(\mathcal{H}, *)$ be a neutrosophic $\mathcal{LA}$-semihypergroup and let K be a subset of $\mathcal{H}$. Then, K is called neutrosophic triplet $\mathcal{LA}$-subsemihypergroup, if K itself is a neutrosophic triplet $\mathcal{LA}$-semihypergroup.*

Example 6. *Let $\mathcal{H} = \{w_1, w_2, w_3, w_4, w_5\}$ be a set with the hyperoperation defined in the table as follow*

$*$	w_1	w_2	w_3	w_4	w_5
w_1	$\{w_1, w_3\}$	w_2	$\{w_2, w_3\}$	$\{w_4, w_5\}$	w_5
w_2	$\{w_2, w_3\}$	$\{w_2, w_3\}$	$\{w_2, w_3\}$	$\{w_4, w_5\}$	w_5
w_3	$\{w_2, w_3\}$	$\{w_2, w_3\}$	$\{w_2, w_3\}$	$\{w_4, w_5\}$	w_5
w_4	$\{w_4, w_5\}$	$\{w_4, w_5\}$	$\{w_4, w_5\}$	w_4	w_5
w_5	w_5	w_5	w_5	w_5	w_5

A Cayley table 5

Here $(\mathcal{H}, *)$ is an $\mathcal{LA}$-semihypergroup, because the element of $\mathcal{H}$ satisfies the left invertive law. Here (w_1, w_1, w_1), (w_2, w_3, w_3), (w_3, w_2, w_3), (w_4, w_4, w_4) and (w_5, w_4, w_4) are neutrosophic triplet. Hence $(\mathcal{H}, *)$ is a neutrosophic triplet $\mathcal{LA}$-semihypergroup. Let $K = \{w_1, w_2, w_3\}$ be subset of $\mathcal{H}$. As K is a neutrosophic $\mathcal{LA}$-semihypergroup under the $*$. Then K is called neutrosophic triplet $\mathcal{LA}$-subsemihypergroup of $\mathcal{H}$.

Lemma 1. *Let K be a non-empty subset of a neutrosophic triplet $\mathcal{LA}$-semihypergroup $\mathcal{H}$. The following are equivalent.*

1. K is a neutrosophic triplet $\mathcal{LA}$-semihypergroup.
2. For all $w_1, w_2 \in K$, $w_1 * w_2 \in K$.

Proof. The proof is straightforward. □

Definition 12. *Let $(\mathcal{H}_1, *_1)$ and $(\mathcal{H}_2, *_2)$ are two neutrosophic triplet $\mathcal{LA}$-semihypergroups. Let $f : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a mapping. Then f is called neutro-homomorphism if for all $w_1, w_2 \in \mathcal{H}_1$, we have*

1. $f(w_1 *_1 w_2) = f\,(w_2) *_2 f\,(w_2)$,
2. $f\,(net\,(w_1)) = neut\,(f\,(w_1))$,
3. $f\,(anti\,(w_1)) = anti\,(f\,(w_1))$.

Theorem 6. *Let $f : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a neutro-homomorphism. Where $\mathcal{H}_1$and $\mathcal{H}_2$ are two neutrosophic triplet $\mathcal{LA}$-semihypergroup. Let*

1. The image of f is a neutrosophic triplet $\mathcal{LA}$-subsemihypergroup of $\mathcal{H}_2$.

2. The inverse image of f is a neutrosophic $\mathcal{LA}$-subsemihypergroup of $\mathcal{H}_1$.

Proof. The proof is straightforward. □

Remark 3. *We have the following key points;*

1. Every neutrosophic triplet $\mathcal{LA}$-semihypergroup is an $\mathcal{LA}$-semihypergroup, but the reverse may or may not true.
2. In neutrosophic triplet $\mathcal{LA}$-semihypergroup, every element must have a left $neut\,(.)$, but in an $\mathcal{LA}$-semihypergroup the left $neut\,(.)$ of an element may or may not exist.
3. In neutrosophic $\mathcal{LA}$-semihypergroup, every element must have left $anti\,(.)$, but in an $\mathcal{LA}$-semihypergroup the element may or may not have semihypergroup.
4. In neutrosophic $\mathcal{LA}$-semihypergroup pure left $neut\,(.)$ is not equal to pure left Identity.

4. Application

Neutrosophic triplet $\mathcal{LA}$-semihypergroups has many applications in different areas. Here, we present an application of neutrosophic triplet $\mathcal{LA}$-semihypergroup in football. We can use different versions of neut and anti elements like left, right, pure left and pure right that we may see in different situations. The interesting prospect of this newly defined structure is that it is not comutative, so any change from the left and same types of change from the rigth of a certain element may affect the final results with respect to neut and anti.

Consider a Football team; the centre midfield player "C_m" having a degree of performance d_1. The players "C_{ml2}" and "C_{mr1}" are the midfield player having degree of performance d_1. Thus using Definition 6, the $neut\,(C_m) \in \{C_{ml2}, C_{mr1}\}$. The players "$C_{ml1}$" and "$C_{mr2}$" are having better degree of performance d_2, thus using Definition 6, the $neut\,(C_m) \in anti\,(C_m) * Cm \cap C_m * anti\,(C_m) = C_{ml1} * C_m \cap C_m * C_{mr2} = \{d_1, d_2\}$ Neutrosophic triplet $\mathcal{LA}$-semihypergroup can help the coach to select the players for filling the position in the playground, when a player gets injured. The major advantage of neutrosophic triplet $\mathcal{LA}$-semihypergroup is that if we have a centre mid player and this player has the other players having the same performance on the right side as neut of it and it has one player on the left having better performance than it as shown in the following Figure 1.

If the performance of a player playing on the left side and right side of a centre mid player is equal to performance of a centre mid player then the structure reduces to a duplet structure. Similarly, we can find many applications in different directions.

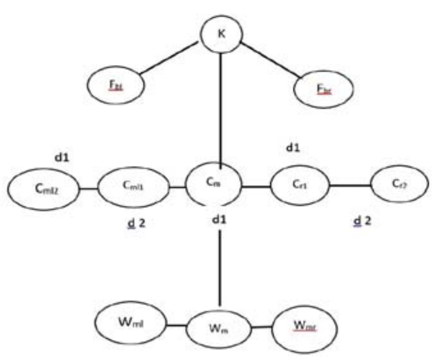

Figure 1. A view of football match.

5. Conclusions

In this paper, we apply the idea of neutrosophic triplet sets at the very useful non-associative hyperstructures, namely $\mathcal{LA}$-semihypergroups. We define neutrosophic triplet set (left, right, pure left, pure right). We discuss some basic results and an application of the proposed structure at the end. In future, we are aiming to extend this idea and give more interesting results.

Author Contributions: All authors contributed equally.

Funding: Universiti Kebangsaan Malaysia- Grant No. GUP-2017-105.

Acknowledgments: We are indebted to Universiti Kebangsaan Malaysia for providing financial support under the grant GUP-2017-105, and authors are highly grateful to referees for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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Generalized Q-Neutrosophic Soft Expert Set for Decision under Uncertainty

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Received: 22 October 2018; Accepted: 7 November 2018; Published: 9 November 2018

Abstract: Neutrosophic triplet structure yields a symmetric property of truth membership on the left, indeterminacy membership in the centre and false membership on the right, as do points of object, centre and image of reflection. As an extension of a neutrosophic set, the Q-neutrosophic set was introduced to handle two-dimensional uncertain and inconsistent situations. We extend the soft expert set to generalized Q-neutrosophic soft expert set by incorporating the idea of soft expert set to the concept of Q-neutrosophic set and attaching the parameter of fuzzy set while defining a Q-neutrosophic soft expert set. This pattern carries the benefits of Q-neutrosophic sets and soft sets, enabling decision makers to recognize the views of specialists with no requirement for extra lumbering tasks, thus making it exceedingly reasonable for use in decision-making issues that include imprecise, indeterminate and inconsistent two-dimensional data. Some essential operations namely subset, equal, complement, union, intersection, AND and OR operations and additionally several properties relating to the notion of generalized Q-neutrosophic soft expert set are characterized. Finally, an algorithm on generalized Q-neutrosophic soft expert set is proposed and applied to a real-life example to show the efficiency of this notion in handling such problems.

Keywords: algorithm; decision making; expert set; generalized neutrosophic set; neutrosophic sets; Q-neutrosophic; soft sets.

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1. Introduction

Zadeh established the concept of fuzzy set [1] as a way to handle uncertain information, by assigning a number to each element that shows the degree of membership of the element. Intuitionistic fuzzy set [2] is another way to handle uncertainty that assigns two numbers to each element. These numbers show the the degree of membership and the degree of nonmembership of the element. However, these theories fail to handle an indeterminate environment, hence Smarandache established the idea of neutrosophy [3] as an extension of fuzzy set and intuitionistic fuzzy set to mitigate such situations. Neutrosophic set (NS) [4] is recognized via three independent membership functions that depict the degrees of truth (T), indeterminacy (I), and falsity (F). Soft set [5] is another commonly used method in handling uncertainties. It has been extended extensively to fuzzy soft set [6], vague soft set [7–9] and neutrosophic soft set [10]. Although these concepts are widely applicable to different life branches, they lack the ability to handle two-dimensional problems. This motivates the definition of Q-fuzzy soft set [11,12] that served the uncertainty and two-dimensionality simultaneously. Recently, this was extended to the theory of Q-neutrosophic soft set (Q-NSS) [13] by extending the theory of Q-fuzzy soft set to a neutrosophic set. Q-NSS is a tri-component two-dimensional set, enabling it to address inconsistent, indeterminate and imprecise data in which the indeterminacy is measured unequivocally and truth, indeterminacy and falsity memberships are independent. Relations between Q-NSSs were studied in [13] while their measures of distance, similarity and entropy were discussed in [14].

Decision-making is an exploration point of research especially in uncertain environments. Recently, many researchers applied neutrosophic sets to real-life decision making problems containing uncertain, indeterminate and incompatible information [15–25].

Recently, the need for models incorporating opinions of experts and validating information supplied by observers has been recognized. In such cases, it is pertinent to have the conclusions of a specialist to approve the information obtained from observers before these data can be utilized to make a decision. The lack of this feature was one of the major problems that was inherent in fuzzy and soft sets, and their hybrid models, including Q-NSSs. The concept of soft expert sets (SESs) [26] is the first model in literature to deal with this issue, by presenting the opinions of the experts, with no extra task. Although SES was considered a novel idea at the time of its initiation, it does not have the capacity to represent the uncertainty that appears in most real issues. Many generalizations of the SES model were introduced to overcome this issue such as fuzzy soft expert sets [27], neutrosophic soft expert sets [28], neutrosophic vague soft expert set [17], complex neutrosophic soft expert set [18], vague soft expert sets [29–31], generalized neutrosophic soft expert set [32] and Q-neutrosophic soft expert set (Q-NSES) [33]. Q-NSES has the capacity to handle indeterminacy and two-dimensionality simultaneously, since it incorporates the elements of both soft expert set and Q-neutrosophic set. The structure of this concept enables it to provide the opinions of experts to activate the data obtained from individuals and able to present the ideas within a two-dimensional indeterminate environment which makes it suitable to describe many real problems.

In this study, we redefine the operations of Q-NSES [33] and introduce the conception of generalized Q-neutrosophic soft expert set (GQ-NSES) as an extension of Q-NSES by attaching the parameterization of fuzzy sets while defining a Q-NSES. The proposed concept is more practical as it includes uncertainty in the selection of a fuzzy set corresponding to each value of the parameter. We will introduce some concepts related to this model along with basic operations relevant to GQ-NSESs, namely the union, intersection and complement operations. The commutative and associative laws of these operations will be proposed and an application of GQ-NSES in decision-making will be illustrated.

2. Preliminaries

We review some basic ideas of soft set, neutrosophic set and Q-neutrosophic soft expert set that are related to the study in this work.

2.1. Neutrosophic Set

In the following, we recall the notion of neutrosophic set [4] with the operations of subset, complement, intersection and union [3].

Definition 1 (see [4])**.** *A neutrosophic set Γ on the universe X is defined as*

$$\Gamma = \{\langle x, (T_\Gamma(x), I_\Gamma(x), F_\Gamma(x))\rangle : x \in X\}, \textit{where } T, I, F : X \to]^-0, 1^+[$$

and

$$^-0 \le T_\Gamma(x) + I_\Gamma(x) + F_\Gamma(x) \le 3^+.$$

Definition 2 (see [3])**.** *Let Γ and Ψ be two neutrosophic sets. Then, we say that Γ is a subset of Ψ denoted by $\Gamma \subseteq \Psi$ if and only if $T_\Gamma(x) \le T_\Psi(x)$, $I_\Gamma(x) \ge I_\Psi(x)$ and $F_\Gamma(x) \ge F_\Psi(x)$ for all $x \in X$.*

Definition 3 (see [3])**.** *The union of two neutrosophic sets Γ and Ψ in the universe X is denoted by $\Gamma \cup \Psi = \Lambda$, where*

$$\Lambda = \{\langle x, (\max\{T_\Gamma(x), T_\Psi(x)\}, \min\{I_\Gamma(x), I_\Psi(x)\}, \min\{F_\Gamma(x), F_\Psi(x)\})\rangle : x \in X\}.$$

Definition 4 (see [3]). *The intersection of two neutrosophic sets* Γ *and* Ψ *in the universe* X *is denoted by* $\Gamma \cap \Psi = \Lambda$, *where*

$$\Lambda = \{\langle x, (\min\{T_\Gamma(x), T_\Psi(x)\}, \max\{I_\Gamma(x), I_\Psi(x)\}, \max\{F_\Gamma(x), F_\Psi(x)\})\rangle : x \in X\}.$$

Definition 5 (see [3]). *The complement of a neutrosophic set* Γ *in the universe* X *is denoted by* Γ^c, *where*

$$\Gamma^c = \{\langle x, (1 - T_\Gamma(x), 1 - I_\Gamma(x), 1 - F_\Gamma(x))\rangle : x \in X\}.$$

The neutrosophic empty set Γ_0 *in the universe* X *is* $\Gamma_0 = \{\langle x, (0, 1, 1)\rangle : x \in X\}$.

2.2. Q-Neutrosophic Soft Expert Set

Abu Qamar and Hassan [13] proposed Q-neutrosophic set (Q-NS) for dealing with two-dimensional inconsistent, indeterminate and uncertain information.

Definition 6 (see [13]). *Let* X *be a universal set and* Q *be a nonempty set. A Q-neutrosophic set* Γ_Q *in* X *and* Q *is an object of the form*

$$\Gamma_Q = \{\Big\langle (x,q), T_{\Gamma_Q}(x,q), I_{\Gamma_Q}(x,q), F_{\Gamma_Q}(x,q) \Big\rangle : x \in X, q \in Q\},$$

where $T_{\Gamma_Q}, I_{\Gamma_Q}, F_{\Gamma_Q} : X \times Q \to]^-0, 1^+[$ *are the true membership function, indeterminacy membership function and false membership function, respectively, with* $^-0 \leq T_{\Gamma_Q} + I_{\Gamma_Q} + F_{\Gamma_Q} \leq 3^+$.

Hassan et al. raised the notion of Q-neutrosophic soft expert [33].

Let X be a universe, Q be a nonempty set, E a set of parameters, U a set of experts (agents), and $O = \{1 = agree, 0 = disagree\}$ a set of opinions. Let $Z = E \times U \times O$ and $A \subseteq Z$.

Definition 7 (see [33]). *A pair* $(\widehat{\Gamma}_Q, A)$ *is called a Q-NSES over* X, *where* $\widehat{\Gamma}_Q$ *is a mapping given by* $\widehat{\Gamma}_Q : A \to QNSES$ *such that QNSES is the set of all QNSES over* U.

In the following, we refined the basic operations of Q-NSES introduced in [33].

Definition 8. *Let* $(\widehat{\Gamma}_Q, A)$ *and* $(\widehat{\Psi}_Q, B)$ *be two Q-NSESs over* X. *Then,* $(\widehat{\Gamma}_Q, A)$ *is said to be Q-NSE subset of* $(\widehat{\Psi}_Q, B)$, *denoted by* $(\widehat{\Gamma}_Q, A)\widehat{\subseteq}(\widehat{\Psi}_Q, B)$ *if* $A \subseteq B$ *and* $T_{\widehat{\Gamma}_{Q(a)}}(x,q) \leq T_{\widehat{\Psi}_{Q(a)}}(x,q), I_{\widehat{\Gamma}_{Q(a)}}(x,q) \geq I_{\widehat{\Psi}_{Q(a)}}(x,q), F_{\widehat{\Gamma}_{Q(a)}}(x,q) \geq F_{\widehat{\Psi}_{Q(a)}}(x,q)$ $\forall a \in A, (x,q) \in X \times Q$.

Definition 9. *The complement of* $(\widehat{\Gamma}_Q, A)$ *is defined as* $(\widehat{\Gamma}_Q, A)^c = (\widehat{\Gamma}^c_Q, A)$, *where* $\widehat{\Gamma}^c_Q = A \to P(X \times Q)$ *and*

$$\widehat{\Gamma}^c_Q = \Big\{\Big\langle a, T_{\widehat{\Gamma}^c_Q(a)}(x,q), I_{\widehat{\Gamma}^c_Q(a)}(x,q), F_{\widehat{\Gamma}^c_Q(a)}(x,q) \Big\rangle : a \in A, (x,q) \in X \times Q\Big\},$$

such that $\forall a \in A, (x,q) \in X \times Q$

$$T_{\widehat{\Gamma}^c_Q(a)}(x,q) = 1 - T_{\widehat{\Gamma}_Q}(x,q),$$
$$I_{\widehat{\Gamma}^c_Q(a)}(x,q) = 1 - I_{\widehat{\Gamma}_Q}(x,q),$$
$$F_{\widehat{\Gamma}^c_Q(a)}(x,q) = 1 - F_{\widehat{\Gamma}_Q}(x,q).$$

Now, we propose the union and intersection of two Q-NSESs.

Definition 10. *The union of* $(\widehat{\Gamma}_Q, A)$ *and* $(\widehat{\Psi}_Q, B)$ *is a Q-NSES* $(\widehat{\Upsilon}_Q, C)$*, defined as* $(\widehat{\Gamma}_Q, A)\widehat{\cup}\,(\widehat{\Psi}_Q, B) = (\widehat{\Upsilon}_Q, C)$*, where* $C = A \cup B$ *and for all* $c \in C$ *and* $(x, q) \in X \times Q$ *the truth, indeterminacy and falsity memberships of* $(\widehat{\Upsilon}_Q, C)$ *are as follows:*

$$T_{\widehat{\Upsilon}_Q(c)}(x,q) = \begin{cases} T_{\widehat{\Gamma}_Q(c)}(x,q) & \text{if } c \in A - B, \\ T_{\widehat{\Psi}_Q(c)}(x,q) & \text{if } c \in B - A, \\ \max\{T_{\widehat{\Gamma}_Q(c)}(x,q), T_{\widehat{\Psi}_Q(c)}(x,q)\} & \text{if } c \in A \cap B, \end{cases}$$

$$I_{\widehat{\Upsilon}_Q(c)}(x,q) = \begin{cases} I_{\widehat{\Gamma}_Q(c)}(x,q) & \text{if } c \in A - B, \\ I_{\widehat{\Psi}_Q(c)}(x,q) & \text{if } c \in B - A, \\ \min\{I_{\widehat{\Gamma}_Q(c)}(x,q), I_{\widehat{\Psi}_Q(c)}(x,q)\} & \text{if } c \in A \cap B, \end{cases}$$

and

$$F_{\widehat{\Upsilon}_Q(c)}(x,q) = \begin{cases} F_{\widehat{\Gamma}_Q(c)}(x,q) & \text{if } c \in A - B, \\ F_{\widehat{\Psi}_Q(c)}(x,q) & \text{if } c \in B - A, \\ \min\{F_{\widehat{\Gamma}_Q(c)}(x,q), F_{\widehat{\Psi}_Q(c)}(x,q)\} & \text{if } c \in A \cap B. \end{cases}$$

Definition 11. *The intersection of* $(\widehat{\Gamma}_Q, A)$ *and* $(\widehat{\Psi}_Q, B)$ *is a Q-NSES* $(\widehat{\Upsilon}_Q, C)$*, defined as* $(\widehat{\Gamma}_Q, A)\widehat{\cap}(\widehat{\Psi}_Q, B) = (\widehat{\Upsilon}_Q, C)$*, where* $C = A \cap B$ *and for all* $c \in C$ *and* $(x, q) \in X \times Q$ *the truth, indeterminacy and falsity memberships of* $(\widehat{\Upsilon}_Q, C)$ *are as follows:*

$$\begin{aligned} T_{\widehat{\Upsilon}_Q(c)}(x,q) &= \min\{T_{\widehat{\Gamma}_Q(c)}(x,q), T_{\widehat{\Psi}_Q(c)}(x,q)\}, \\ I_{\widehat{\Upsilon}_Q(c)}(x,q) &= \max\{I_{\widehat{\Gamma}_Q(c)}(x,q), I_{\widehat{\Psi}_Q(c)}(x,q)\}, \\ F_{\widehat{\Upsilon}_Q(c)}(x,q) &= \max\{F_{\widehat{\Gamma}_Q(c)}(x,q), F_{\widehat{\Psi}_Q(c)}(x,q)\}. \end{aligned}$$

Definition 12. *If* $(\widehat{\Gamma}_Q, A)$ *and* $(\widehat{\Psi}_Q, B)$ *are two Q-neutrosophic soft expert sets on X, then* $(\widehat{\Gamma}_Q, A)$ *AND* $(\widehat{\Psi}_Q, B)$ *is the Q-neutrosophic soft expert set denoted by* $(\widehat{\Gamma}_Q, A)\widehat{\wedge}(\widehat{\Psi}_Q, B)$ *and defined by* $(\widehat{\Gamma}_Q, A)\widehat{\wedge}(\widehat{\Psi}_Q, B) = (\widehat{\Lambda}_Q, A \times B)$*, where* $\widehat{\Lambda}_Q(a, b) = \widehat{\Gamma}_Q(a)\widehat{\cap}\widehat{\Psi}_Q(b)$ *for all* $(a, b) \in A \times B$ *is the operation of intersection of two Q-neutrosophic sets on X.*

Definition 13. *If* $(\widehat{\Gamma}_Q, A)$ *and* $(\widehat{\Psi}_Q, B)$ *are two Q-neutrosophic soft expert sets on X, then* $(\widehat{\Gamma}_Q, A)$ *OR* $(\widehat{\Psi}_Q, B)$ *is the Q-neutrosophic soft expert set denoted by* $(\widehat{\Gamma}_Q, A)\widehat{\vee}(\widehat{\Psi}_Q, B)$ *and defined by* $(\widehat{\Gamma}_Q, A)\widehat{\vee}\,(\widehat{\Psi}_Q, B) = (\widehat{\Lambda}_Q, A \times B)$*, where* $\widehat{\Lambda}_Q(a, b) = \widehat{\Gamma}_Q(a)\widehat{\cup}\widehat{\Psi}_Q(b)$ *for all* $(a, b) \in A \times B$ *is the operation of union of two Q-neutrosophic sets on X.*

3. Generalized Q-Neutrosophic Soft Expert Set

In this section, we propose the generalized Q-neutrosophic soft expert set (GQ-NSES) and proceed to introduce several concepts related to this model. We will put forward the operations of union, intersection and complement of GQ-NSESs, and proceed with the properties of the commutative and associative laws of these operations.

We begin by proposing the definition of GQ-NSES, followed by an illustrative example.

Let X be a universe, Q be a nonempty set, E a set of parameters, U a set of experts (agents), and $O = \{1 = agree, 0 = disagree\}$ a set of opinions. Let $Z = E \times U \times O$, $A \subseteq Z$ and f be a fuzzy set; that is, $f : A \to [0, 1]$.

Definition 14. *A pair* $(\widehat{\Gamma}^f_Q, A)$ *is called a GQ-NSES over X, where* $\widehat{\Gamma}^f_Q$ *is a mapping given by*

$$\widehat{\Gamma}^f_Q : A \to P(X \times Q) \times I,$$

where $P(X \times Q)$ *denotes the power Q-neutrosophic soft expert set. For all* $a \in A$, $\widehat{\Gamma}_Q$ *is referred to as the Q-neutrosophic expert value set of the parameter a, i.e,*

$$\widehat{\Gamma}_Q(a) = \left\{ \left\langle (x,q), T_{\widehat{\Gamma}_Q}(x,q), I_{\widehat{\Gamma}_Q}(x,q), F_{\widehat{\Gamma}_Q}(x,q) \right\rangle \right\}$$

presents the degree of belongingness, indeterminacy belongingness and non-belongingness of elements of X in $\widehat{\Gamma}_Q$, *where* $\forall (x,q) \in X \times Q, \forall a \in A$, $T_{\widehat{\Gamma}_Q}, I_{\widehat{\Gamma}_Q}, F_{\widehat{\Gamma}_Q}$ *representing the membership functions of truth, indeterminacy and falsity, respectively. The values* $T_{\widehat{\Gamma}_Q}, I_{\widehat{\Gamma}_Q}, F_{\widehat{\Gamma}_Q} \in [0,1]$ *and*

$$0 \leq T_{\widehat{\Gamma}_Q}(x,q) + I_{\widehat{\Gamma}_Q}(x,q) + F_{\widehat{\Gamma}_Q}(x,q) \leq 3.$$

The GQ-NSES $(\widehat{\Gamma}^f_Q, A)$ *is a parametrized family of Q-neutrosophic soft expert sets on X, which has the degree of preference of the approximate value set which represented by* $f(a)$ *for each paremeter a. The GQ-NSES can be written as:*

$$(\widehat{\Gamma}^f_Q, A) = \left\{ \left\langle a, (\widehat{\Gamma}_Q(a), f(a)) \right\rangle : a \in A, \widehat{\Gamma}_Q(a) \in P(X \times Q), f(a) \in [0,1] \right\}.$$

In short, for each parameter a, $\widehat{\Gamma}^f_Q(a)$ *gives not only the extent to which each element in X belongs, indeterminacy belong or not belong to* $\widehat{\Gamma}_Q$ *but also indicates how much such belonging is preferred.*

Example 1. *Suppose a company wants to fill a position to be chosen by an expert committee. There are three candidates* $X = \{x_1, x_2, x_3\}$ *with two types of qualifications* $Q = \{q_1 = master, q_2 = doctorate\}$ *and the hiring committee takes into consideration a set of parameters* $E = \{e_1 = computer\ knowledge, e_2 = experience\}$. *Let* $U = \{u_1, u_2\}$ *be the set of two committee members. Then, we can view the GQ-NSES* $(\widehat{\Gamma}^f_Q, A)$ *as consisting of the following collection of approximation:*

$$\begin{aligned}
(\widehat{\Gamma}^f_Q, A) = \Big\{ & \Big\langle (e_1,u_1,1), \Big([(x_1,q_1),0.1,0.2,0.6], [(x_1,q_2),0.4,0.3,0.7], [(x_2,q_1),0.5,0.2,0.1], \\
& [(x_2,q_2),0.7,0.2,0.3], [(x_3,q_1),0.8,0.3,0.1], [(x_3,q_2),0.2,0.3,0.6], 0.7 \Big) \Big\rangle, \\
& \Big\langle (e_1,u_2,1), \Big([(x_1,q_1),0.6,0.4,0.2], [(x_1,q_2),0.5,0.3,0.2], [(x_2,q_1),0.5,0.4,0.3], \\
& [(x_2,q_2),0.9,0.4,0.2], [(x_3,q_1),0.6,0.8,0.4], [(x_3,q_2),0.4,0.1,0.5], 0.5 \Big) \Big\rangle, \\
& \Big\langle (e_2,u_1,1), \Big([(x_1,q_1),0.7,0.2,0.3], [(x_1,q_2),0.8,0.4,0.6], [(x_2,q_1),0.3,0.6,0.9], \\
& [(x_2,q_2),0.1,0.3,0.3], [(x_3,q_1),0.4,0.7,0.5], [(x_3,q_2),0.8,0.7,0.4], 0.4 \Big) \Big\rangle \\
& \Big\langle (e_2,u_2,1), \Big([(x_1,q_1),0.7,0.6,0.5], [(x_1,q_2),0.7,0.4,0.1], [(x_2,q_1),0.8,0.6,0.3], \\
& [(x_2,q_2),0.7,0.6,0.2], [(x_3,q_1),0.3,0.4,0.2], [(x_3,q_2),0.5,0.3,0.7], 0.6 \Big) \Big\rangle, \\
& \Big\langle (e_1,u_1,0), \Big([(x_1,q_1),0.5,0.5,0.3], [(x_1,q_2),0.8,0.8,0.4], [(x_2,q_1),0.7,0.1,0.3], \\
& [(x_2,q_2),0.9,0.7,0.5], [(x_3,q_1),0.9,0.6,0.5], [(x_3,q_2),0.6,0.3,0.3], 0.8 \Big) \Big\rangle, \\
& \Big\langle (e_1,u_2,0), \Big([(x_1,q_1),0.2,0.2,0.5], [(x_1,q_2),0.7,0.4,0.9], [(x_2,q_1),0.8,0.7,0.5], \\
& [(x_2,q_2),0.8,0.3,0.3], [(x_3,q_1),0.3,0.2,0.8], [(x_3,q_2),0.5,0.5,0.5], 0.2 \Big) \Big\rangle,
\end{aligned}$$

$$\Big\langle (e_2,u_1,0),\Big([(x_1,q_1),0.4,0.8,0.8],[(x_1,q_2),0.9,0.6,0.6],[(x_2,q_1),0.9,0.7,0.5],$$
$$[(x_2,q_2),0.2,0.1,0.6],[(x_3,q_1),0.4,0.7,0.9],[(x_3,q_2),0.8,0.2,0.1],0.3\Big)\Big\rangle,$$
$$\Big\langle (e_2,u_2,0),\Big([(x_1,q_1),0.9,0.4,0.5],[(x_1,q_2),0.7,0.1,0.6],[(x_2,q_1),0.1,0.8,0.4],$$
$$[(x_2,q_2),0.5,0.7,0.8],[(x_3,q_1),0.4,0.4,0.4],[(x_3,q_2),0,6,0.8,0.2],0.3\Big)\Big\rangle\Big\}.$$

Each element of the GQ-NSES implies the opinion of each expert based on each parameter about the candidates with their qualifications and the degree of preference of the approximate value set. For example, $[(x_1,q_1),0.1,0.2,0.6]$ under the parameter $(e_1,u_1,1)$ shows the degree to which expert u_1 agree that the candidate x_1 with a master qualification q_1 has a computer knowledge, whereas $[(x_1,q_1),0.5,0.5,0.3]$ under the parameter $(e_1,u_1,0)$ shows the degree to which expert u_1 disagree that the candidate x_1 with a master qualification q_1 has a computer knowledge.

Now, we present the ideas of the subset of two GQ-NSESs and the equality of two GQ-NSESs.

Definition 15. *Let $(\widehat{\Gamma}^f_Q, A)$ and $(\widehat{\Psi}^g_Q, B)$ be two GQ-NSESs over X. Then, $(\widehat{\Gamma}^f_Q, A)$ is said to be GQ-NSE subset of $(\widehat{\Psi}^g_Q, B)$, denoted by $(\widehat{\Gamma}^f_Q, A) \sqsubseteq (\widehat{\Psi}^g_Q, B)$ if $A \subseteq B$ and for $a \in A$, the following conditions are satisfied:*

1. *$f(a)$ is a fuzzy subset of $g(a)$, that is $f(a) \leq g(a)$,*
2. *$\widehat{\Gamma}_Q(a)$ is a Q-neutrosophic soft expert subset of $\widehat{\Psi}_Q(a)$, that is $T_{\widehat{\Gamma}_Q(a)}(x,q) \leq T_{\widehat{\Psi}_Q(a)}(x,q)$, $I_{\widehat{\Gamma}_Q(a)}(x,q) \geq I_{\widehat{\Psi}_Q(a)}(x,q)$, $F_{\widehat{\Gamma}_Q(a)}(x,q) \geq F_{\widehat{\Psi}_Q(a)}(x,q) \forall (x,q) \in X \times Q$.*

Definition 16. *Let $(\widehat{\Gamma}^f_Q, A)$ and $(\widehat{\Psi}^g_Q, B)$ be two GQ-NSESs over X. Then, $(\widehat{\Gamma}^f_Q, A)$ is said to be equal to $(\widehat{\Psi}^g_Q, B)$, denoted by $(\widehat{\Gamma}^f_Q, A) \widehat{=} (\widehat{\Psi}^g_Q, B)$ if $(\widehat{\Gamma}^f_Q, A)$ is a GQ-NSE subset of $(\widehat{\Psi}^g_Q, B)$ and $(\widehat{\Psi}^g_Q, B)$ is a GQ-NSE subset of $(\widehat{\Gamma}_Q, A)$.*

Next, we give the definitions of an agree- GQ-NSES and a disagree- GQ-NSES.

Definition 17. *An agree- GQ-NSES $(\widehat{\Gamma}^f_Q, A)_1$ over X is a GQ-NSES subset of $(\widehat{\Gamma}_Q, A)$ defined as*

$$(\widehat{\Gamma}^f_Q, A)_1 = \Big\{\widehat{\Gamma}^f_{Q_1}(a) : a \in E \times U \times \{1\}\Big\}.$$

Definition 18. *A disagree- GQ-NSES $(\widehat{\Gamma}^f_Q, A)_0$ over X is a GQ-NSES subset of $(\widehat{\Gamma}^f_Q, A)$ defined as*

$$(\widehat{\Gamma}^f_Q, A)_0 = \Big\{\widehat{\Gamma}^f_{Q_0}(a) : a \in E \times U \times \{1\}\Big\}.$$

In the following, we discuss the operations of complement, union and intersection of GQ-NSESs.

Definition 19. *The complement of $(\widehat{\Gamma}^f_Q, A)$ is defined as*

$$(\widehat{\Gamma}^f_Q, A)^c = ((\widehat{\Gamma}^f_Q)^c, A)$$
$$= \Big\{\Big\langle a, (\widehat{\Gamma}_Q{}^c(a), f^c(a))\Big\rangle : a \in A, \widehat{\Gamma}_Q(a) \in P(X \times Q), f(a) \in [0,1]\Big\},$$

where, $f^c(a) = 1 - f(a)$ and $\widehat{\Gamma}^c_Q = \Big\{\Big\langle a, T_{\widehat{\Gamma}^c_Q(a)}(x,q), I_{\widehat{\Gamma}^c_Q(a)}(x,q), F_{\widehat{\Gamma}^c_Q(a)}(x,q)\Big\rangle : a \in A, (x,q) \in X \times Q\Big\}$, such that $\forall a \in A, (x,q) \in X \times Q$

$$T_{\widehat{\Gamma}^c_Q(a)}(x,q) = 1 - T_{\widehat{\Gamma}_Q}(x,q),$$
$$I_{\widehat{\Gamma}^c_Q(a)}(x,q) = 1 - I_{\widehat{\Gamma}_Q}(x,q),$$
$$F_{\widehat{\Gamma}^c_Q(a)}(x,q) = 1 - F_{\widehat{\Gamma}_Q}(x,q).$$

Example 2. *Consider the approximation given in Example 1, where*

$$\widehat{\Gamma}^f_Q(e_1,u_1,1) = \Big\{\Big([(x_1,p),0.1,0.2,0.6],[(x_1,q),0.4,0.3,0.7],[(x_2,p),0.5,0.2,0.1],$$
$$[(x_2,q),0.7,0.2,0.3],[(x_3,p),0.8,0.3,0.1],[(x_3,q),0.2,0.3,0.6],0.7\Big)\Big\}.$$

By using the GQ-NSES complement, we obtain the complement of the approximation given by

$$(\widehat{\Gamma}^f_Q)^c(e_1,u_1,1) = \Big\{\Big([(x_1,p),0.9,0.8,0.4],[(x_1,q),0.6,0.7,0.3],[(x_2,p),0.5,0.8,0.9],$$
$$[(x_2,q),0.3,0.8,0.7],[(x_3,p),0.2,0.7,0.9],[(x_3,q),0.8,0.7,0.4],0.3\Big)\Big\}.$$

Proposition 1. *If $(\widehat{\Gamma}^f_Q, A)$ is a GQ-NSES over X, then $((\widehat{\Gamma}^f_Q, A)^c)^c = (\widehat{\Gamma}^f_Q, A)$.*

Proof. Suppose that $(\widehat{\Gamma}^f_Q, A)$ is a GQ-NSES over X defined as $(\widehat{\Gamma}^f_Q, A) = \Big\{\Big\langle a, \Big((T_{\widehat{\Gamma}^f_Q(a)}(x,q), I_{\widehat{\Gamma}^f_Q(a)}(x,q), F_{\widehat{\Gamma}^f_Q(a)}(x,q)), f(a)\Big)\Big\rangle : a \in A, (x,q) \in X \times Q\Big\}$. The complement of $(\widehat{\Gamma}^f_Q, A)$ denoted by $(\widehat{\Gamma}^f_Q, A)^c = ((\widehat{\Gamma}^f_Q)^c, A)$ is as defined below:

$$(\widehat{\Gamma}^f_Q)^c = \Big\{\Big\langle a, \Big((T_{(\widehat{\Gamma}^f_Q)^c(a)}(x,q), I_{(\widehat{\Gamma}^f_Q)^c(a)}(x,q), F_{(\widehat{\Gamma}^f_Q)^c(a)}(x,q)), f^c(a)\Big)\Big\rangle : a \in A, (x,q) \in X \times Q\Big\}$$
$$= \Big\{\Big\langle a, \Big((1 - T_{\widehat{\Gamma}^f_Q(a)}(x,q), 1 - I_{\widehat{\Gamma}^f_Q(a)}(x,q), 1 - F_{\widehat{\Gamma}^f_Q(a)}(x,q)), 1 - f(a)\Big)\Big\rangle : a \in A, (x,q) \in X \times Q\Big\}.$$

Thus,

$$((\widehat{\Gamma}^f_Q, A)^c)^c = \Big\{\Big\langle a, \Big(((1 - T_{\widehat{\Gamma}^f_Q(a)}(x,q))^c, (1 - I_{\widehat{\Gamma}^f_Q(a)}(x,q))^c, (1 - F_{\widehat{\Gamma}^f_Q(a)}(x,q))^c), (1 - f(a)))^c\Big)\Big\rangle$$
$$: a \in A, (x,q) \in X \times Q\Big\}$$
$$= \Big\{\Big\langle a, \Big((1 - (1 - T_{\widehat{\Gamma}^f_Q(a)}(x,q)), 1 - (1 - I_{\widehat{\Gamma}^f_Q(a)}(x,q)), 1 - (1 - F_{\widehat{\Gamma}^f_Q(a)}(x,q))), 1 - (1 - f(a))\Big)\Big\rangle$$
$$: a \in A, (x,q) \in X \times Q\Big\}$$
$$= \Big\{\Big\langle a, \Big((T_{\widehat{\Gamma}^f_Q(a)}(x,q), I_{\widehat{\Gamma}^f_Q(a)}(x,q), F_{\widehat{\Gamma}^f_Q(a)}(x,q)), f(a)\Big)\Big\rangle : a \in A, (x,q) \in X \times Q\Big\}$$
$$= (\widehat{\Gamma}^f_Q, A).$$

This completes the proof. □

Now, we define the union and intersection of two GQ-NSESs.

Definition 20. *The union of* $(\widehat{\Gamma}^f_Q, A)$ *and* $(\widehat{\Psi}^g_Q, B)$ *is a GQ-NSES* $(\widehat{\Upsilon}^h_Q, C)$*, defined as* $(\widehat{\Gamma}^f_Q, A)\widehat{\bigcup}\,(\widehat{\Psi}^g_Q, B) = (\widehat{\Upsilon}^h_Q, C)$*, where* $C = A \cup B$ *and for all* $c \in C$ *and* $(x,q) \in X \times Q$ *the truth, indeterminacy and falsity memberships of* $(\widehat{\Upsilon}^h_Q, C)$ *are as follows:*

$$T_{\widehat{\Upsilon}^h_Q(c)}(x,q) = \begin{cases} T_{\widehat{\Gamma}^f_Q(c)}(x,q) & \text{if } c \in A - B, \\ T_{\widehat{\Psi}^g_Q(c)}(x,q) & \text{if } c \in B - A, \\ \max\{T_{\widehat{\Gamma}^f_Q(c)}(x,q), T_{\widehat{\Psi}^g_Q(c)}(x,q)\} & \text{if } c \in A \cap B, \end{cases}$$

$$I_{\widehat{\Upsilon}^h_Q(c)}(x,q) = \begin{cases} I_{\widehat{\Gamma}^f_Q(c)}(x,q) & \text{if } c \in A - B, \\ I_{\widehat{\Psi}^g_Q(c)}(x,q) & \text{if } c \in B - A, \\ \min\{I_{\widehat{\Gamma}^f_Q(c)}(x,q), I_{\widehat{\Psi}^g_Q(c)}(x,q)\} & \text{if } c \in A \cap B, \end{cases}$$

$$F_{\widehat{\Upsilon}^h_Q(c)}(x,q) = \begin{cases} F_{\widehat{\Gamma}^f_Q(c)}(x,q) & \text{if } c \in A - B, \\ F_{\widehat{\Psi}^g_Q(c)}(x,q) & \text{if } c \in B - A, \\ \min\{F_{\widehat{\Gamma}^f_Q(c)}(x,q), F_{\widehat{\Psi}^g_Q(c)}(x,q)\} & \text{if } c \in A \cap B, \end{cases}$$

and $h(c) = \max\{f(c), g(c) : \forall c \in C\}$.

Definition 21. *The intersection of* $(\widehat{\Gamma}^f_Q, A)$ *and* $(\widehat{\Psi}^g_Q, B)$ *is a GQ-NSES* $(\widehat{\Upsilon}^h_Q, C)$*, defined as* $(\widehat{\Gamma}^f_Q, A)\widehat{\bigcap}(\widehat{\Psi}^g_Q, B) = (\widehat{\Upsilon}^h_Q, C)$*, where* $C = A \cap B$ *and for all* $c \in C$ *and* $(x,q) \in X \times Q$ *the truth, indeterminacy and falsity memberships of* $(\widehat{\Upsilon}^h_Q, C)$ *are as follows:*

$$\begin{aligned} T_{\widehat{\Upsilon}^h_Q(c)}(x,q) &= \min\{T_{\widehat{\Gamma}^f_Q(c)}(x,q), T_{\widehat{\Psi}^g_Q(c)}(x,q)\}, \\ I_{\widehat{\Upsilon}^h_Q(c)}(x,q) &= \max\{I_{\widehat{\Gamma}^f_Q(c)}(x,q), I_{\widehat{\Psi}^g_Q(c)}(x,q)\}, \\ F_{\widehat{\Upsilon}^h_Q(c)}(x,q) &= \max\{F_{\widehat{\Gamma}^f_Q(c)}(x,q), F_{\widehat{\Psi}^g_Q(c)}(x,q)\}, \end{aligned}$$

and $h(c) = \min\{f(c), g(c) : \forall c \in C\}$.

Example 3. *Assume that two GQ-NSESs* $(\widehat{\Gamma}^f_Q, A)$ *and* $(\widehat{\Psi}^g_Q, B)$ *are defined as follows:*

$$\begin{aligned} (\widehat{\Gamma}^f_Q, A) = \Big\{ & \Big\langle (e_1,u_1,1), \Big([(x_1,q_1),0.2,0.5,0.4], [(x_1,q_2),0.1,0.3,0.3], [(x_2,q_1),0.8,0.5,0.5], \\ & \qquad [(x_2,q_2),0.8,0.8,0.2], [(x_3,q_1),0.6,0.3,0.5], [(x_3,q_2),0.4,0.3,0.1], 0.4 \Big) \Big\rangle, \\ & \Big\langle (e_1,u_2,1), \Big([(x_1,q_1),0.2,0.4,0.7], [(x_1,q_2),0.5,0.1,0.2], [(x_2,q_1),0.1,0.2,0.6], \\ & \qquad [(x_2,q_2),0.6,0.5,0.2], [(x_3,q_1),0.6,0.4,0.1], [(x_3,q_2),0.7,0.2,0.4], 0.2 \Big) \Big\rangle, \\ & \Big\langle (e_1,u_1,0), \Big([(x_1,q_1),0.3,0.3,0.3], [(x_1,q_2),0.7,0.3,0.5], [(x_2,q_1),0.5,0.6,0.2], \\ & \qquad [(x_2,q_2),0.7,0.3,0.1], [(x_3,q_1),0.3,0.5,0.2], [(x_3,q_2),0.7,0.5,0.6], 0.5 \Big) \Big\rangle, \\ & \Big\langle (e_1,u_2,0), \Big([(x_1,q_1),0.4,0.4,0.3], [(x_1,q_2),0.2,0.3,0.6], [(x_2,q_1),0.1,0.4,0.5], \\ & \qquad [(x_2,q_2),0.6,0.4,0.2], [(x_3,q_1),0.2,0.3,0.6], [(x_3,q_2),0.2,0.1,0.1], 0.4 \Big) \Big\rangle \Big\}, \end{aligned}$$

$$(\widehat{\Psi}^g_Q, A) = \Big\{ \Big\langle (e_1, u_1, 1), \Big([(x_1, q_1), 0.7, 0.3, 0.2], [(x_1, q_2), 0.4, 0.3, 0.1], [(x_2, q_1), 0.4, 0.4, 0.5], [(x_2, q_2), 0.6, 0.5, 0.5], [(x_3, q_1), 0.6, 0.6, 0.2], [(x_3, q_2), 0.3, 0.5, 0.5], 0.4 \Big) \Big\rangle,$$
$$\Big\langle (e_1, u_2, 1), \Big([(x_1, q_1), 0.7, 0.6, 0.3], [(x_1, q_2), 0.6, 0.7, 0.2], [(x_2, q_1), 0.1, 0.5, 0.7], [(x_2, q_2), 0.7, 0.5, 0.1], [(x_3, q_1), 0.3, 0.2, 0.3], [(x_3, q_2), 0.1, 0.1, 0.4], 0.3 \Big) \Big\rangle,$$
$$\Big\langle (e_1, u_1, 0), \Big([(x_1, q_1), 0.5, 0.7, 0.4], [(x_1, q_2), 0.3, 0.4, 0.1], [(x_2, q_1), 0.5, 0.1, 0.6], [(x_2, q_2), 0.8, 0.6, 0.2], [(x_3, q_1), 0.7, 0.8, 0.6], [(x_3, q_2), 0.5, 0.8, 0.6], 0.6 \Big) \Big\rangle,$$
$$\Big\langle (e_1, u_2, 0), \Big([(x_1, q_1), 0.3, 0.2, 0.7], [(x_1, q_2), 0.4, 0.8, 0.1], [(x_2, q_1), 0.1, 0.1, 0.4], [(x_2, q_2), 0.8, 0.3, 0.2], [(x_3, q_1), 0.4, 0.1, 0.2], [(x_3, q_2), 0.2, 0.5, 0.3], 0.6 \Big) \Big\rangle \Big\}.$$

Then,

$$(\widehat{\Gamma}^f_Q, A) \widehat{\bigcup} (\widehat{\Psi}^g_Q, A) = \Big\{ \Big\langle (e_1, u_1, 1), \Big([(x_1, q_1), 0.7, 0.3, 0.2], [(x_1, q_2), 0.4, 0.3, 0.1], [(x_2, q_1), 0.8, 0.4, 0.5], [(x_2, q_2), 0.8, 0.5, 0.2], [(x_3, q_1), 0.6, 0.3, 0.2], [(x_3, q_2), 0.4, 0.3, 0.1], 0.4 \Big) \Big\rangle,$$
$$\Big\langle (e_1, u_2, 1), \Big([(x_1, q_1), 0.7, 0.4, 0.3], [(x_1, q_2), 0.6, 0.1, 0.2], [(x_2, q_1), 0.1, 0.2, 0.6], [(x_2, q_2), 0.7, 0.5, 0.1], [(x_3, q_1), 0.6, 0.2, 0.1], [(x_3, q_2), 0.7, 0.1, 0.4], 0.3 \Big) \Big\rangle,$$
$$\Big\langle (e_1, u_1, 0), \Big([(x_1, q_1), 0.5, 0.3, 0.3], [(x_1, q_2), 0.7, 0.3, 0.1], [(x_2, q_1), 0.5, 0.1, 0.2], [(x_2, q_2), 0.8, 0.3, 0.1], [(x_3, q_1), 0.7, 0.5, 0.2], [(x_3, q_2), 0.7, 0.5, 0.6], 0.6 \Big) \Big\rangle,$$
$$\Big\langle (e_1, u_2, 0), \Big([(x_1, q_1), 0.4, 0.2, 0.3], [(x_1, q_2), 0.4, 0.3, 0.1], [(x_2, q_1), 0.1, 0.1, 0.4], [(x_2, q_2), 0.8, 0.3, 0.2], [(x_3, q_1), 0.4, 0.1, 0.2], [(x_3, q_2), 0.2, 0.1, 0.1], 0.6 \Big) \Big\rangle \Big\}.$$

The standard commutative and associative laws relevant to the operations of union and intersection are satisfied and stated below.

Proposition 2. *Let* $(\widehat{\Gamma}^f_Q, A), (\widehat{\Psi}^g_Q, B)$ *and* $(\widehat{\Upsilon}^h_Q, C)$ *be GQ-NSESs over a universe X. Then, the following properties hold true:*

(1) $(\widehat{\Gamma}^f_Q, A) \widehat{\cup} (\widehat{\Psi}^g_Q, B) = (\widehat{\Psi}^g_Q, B) \widehat{\cup} (\widehat{\Gamma}^f_Q, A)$,
(2) $(\widehat{\Gamma}^f_Q, A) \widehat{\cap} (\widehat{\Psi}^g_Q, B) = (\widehat{\Psi}^g_Q, B) \widehat{\cap} (\widehat{\Gamma}^f_Q, A)$,
(3) $((\widehat{\Gamma}^f_Q, A) \widehat{\cup} (\widehat{\Psi}^g_Q, B)) \widehat{\cup} (\widehat{\Upsilon}^h_Q, C) = (\widehat{\Gamma}^f_Q, A) \widehat{\cup} ((\widehat{\Psi}^g_Q, B) \widehat{\cup} (\widehat{\Upsilon}^h_Q, C))$,
(4) $((\widehat{\Gamma}^f_Q, A) \widehat{\cap} (\widehat{\Psi}^g_Q, B)) \widehat{\cap} (\widehat{\Upsilon}^h_Q, C) = (\widehat{\Gamma}^f_Q, A) \widehat{\cap} ((\widehat{\Psi}^g_Q, B) \widehat{\cap} (\widehat{\Upsilon}^h_Q, C))$.

Proof. (1) We will prove that $(\widehat{\Gamma}^f_Q, A) \widehat{\cup} (\widehat{\Psi}^g_Q, B) = (\widehat{\Psi}^g_Q, B) \widehat{\cup} (\widehat{\Gamma}_Q, A)$ by using Definition 20 and we consider the case when $c \in A \cap B$ as the other cases are trivial:

$$(\widehat{\Gamma}^f_Q, A) \widehat{\bigcup} (\widehat{\Psi}^g_Q, B) = \Big\{ \Big\langle c, \Big(\big(\max\{T_{\widehat{\Gamma}^f_Q(c)}(x,q), T_{\widehat{\Psi}^g_Q(c)}(x,q)\}, \min\{I_{\widehat{\Gamma}^f_Q(c)}(x,q), I_{\widehat{\Psi}^g_Q(c)}(x,q)\}, \min\{F_{\widehat{\Gamma}^f_Q(c)}(x,q), F_{\widehat{\Psi}^g_Q(c)}(x,q)\} \big), \max\{f(c), g(c)\} \Big) \Big\rangle : (x,q) \in X \times Q \Big\}$$
$$= \Big\{ \Big\langle c, \Big(\big(\max\{T_{\widehat{\Psi}^g_Q(c)}(x,q), T_{\widehat{\Gamma}^f_Q(c)}(x,q)\}, \min\{I_{\widehat{\Psi}^g_Q(c)}(x,q), I_{\widehat{\Gamma}^f_Q(c)}(x,q)\}, \min\{F_{\widehat{\Psi}^g_Q(c)}(x,q), F_{\widehat{\Gamma}^f_Q(c)}(x,q)\} \big), \max\{g(c), f(c)\} \Big) \Big\rangle : (x,q) \in X \times Q \Big\}$$
$$= (\widehat{\Psi}^g_Q, B) \widehat{\bigcup} (\widehat{\Gamma}^f_Q, A).$$

(2) The proof is similar to that of part (1).

(3) We want to prove that $((\widehat{\Gamma}^f_Q, A)\widehat{\bigcup}(\widehat{\Psi}^g_Q, B))\widehat{\bigcup}(\widehat{\Upsilon}^h_Q, C) = (\widehat{\Gamma}^f_Q, A)\widehat{\bigcup}((\widehat{\Psi}^g_Q, B)\widehat{\bigcup}(\widehat{\Upsilon}^h_Q, C))$ by using Definition 20 and we consider the case when $c \in A \cap B$ as the other cases are trivial

$$(\widehat{\Gamma}^f_Q, A)\widehat{\bigcup}(\widehat{\Psi}^g_Q, B) = \Big\{\Big\langle c, \Big(\big(\max\{T_{\widehat{\Gamma}^f_Q(c)}(x,q), T_{\widehat{\Psi}^g_Q(c)}(x,q)\}, \min\{I_{\widehat{\Gamma}^f_Q(c)}(x,q), I_{\widehat{\Psi}^g_Q(c)}(x,q)\}, \min\{F_{\widehat{\Gamma}^f_Q(c)}(x,q), F_{\widehat{\Psi}^g_Q(c)}(x,q)\}\big), \max\{f(c), g(c)\}\Big)\Big\rangle : (x,q) \in X \times Q\Big\}.$$

Considering the case when $c \in C$, then we have

$$\begin{aligned}
&((\widehat{\Gamma}^f_Q, A)\widehat{\bigcup}(\widehat{\Psi}^g_Q, B))\widehat{\bigcup}(\widehat{\Upsilon}^h_Q, C)\\
&= \Big\{\Big\langle c, \Big(\big(\max\big\{\max\{T_{\widehat{\Gamma}^f_Q(c)}(x,q), T_{\widehat{\Psi}^g_Q(c)}(x,q)\}, T_{\widehat{\Upsilon}^h_Q(c)}(x,q)\big\},\\
&\quad \min\big\{\min\{I_{\widehat{\Gamma}^f_Q(c)}(x,q), I_{\widehat{\Psi}^g_Q(c)}(x,q)\}, I_{\widehat{\Upsilon}^h_Q(c)}(x,q)\big\},\\
&\quad \min\big\{\min\{F_{\widehat{\Gamma}^f_Q(c)}(x,q), F_{\widehat{\Psi}^g_Q(c)}(x,q)\}, F_{\widehat{\Upsilon}^h_Q(c)}(x,q)\big\}\big), \max\big\{\max\{f(c), g(c)\}, h(c)\big\}\Big)\Big\rangle :\\
&\quad (x,q) \in X \times Q\Big\}\\
&= \Big\{\Big\langle c, \Big(\big(\max\big\{T_{\widehat{\Gamma}^f_Q(c)}(x,q), T_{\widehat{\Psi}^g_Q(c)}(x,q), T_{\widehat{\Upsilon}^h_Q(c)}(x,q)\big\},\\
&\quad \min\big\{I_{\widehat{\Gamma}^f_Q(c)}(x,q), I_{\widehat{\Psi}^g_Q(c)}(x,q), I_{\widehat{\Upsilon}^h_Q(c)}(x,q)\big\},\\
&\quad \min\big\{F_{\widehat{\Gamma}^f_Q(c)}(x,q), F_{\widehat{\Psi}^g_Q(c)}(x,q), F_{\widehat{\Upsilon}^h_Q(c)}(x,q)\big\}\big), \max\big\{f(c), g(c), h(c)\big\}\Big)\Big\rangle :\\
&\quad (x,q) \in X \times Q\Big\}\\
&= \Big\{\Big\langle c, \Big(\big(\max\big\{T_{\widehat{\Gamma}^f_Q(c)}(x,q), \max\{T_{\widehat{\Psi}^g_Q(c)}(x,q), T_{\widehat{\Upsilon}^h_Q(c)}(x,q)\}\big\},\\
&\quad \min\big\{I_{\widehat{\Gamma}^f_Q(c)}(x,q), \min\{I_{\widehat{\Psi}^g_Q(c)}(x,q), I_{\widehat{\Upsilon}^h_Q(c)}(x,q)\}\big\},\\
&\quad \min\big\{F_{\widehat{\Gamma}^f_Q(c)}(x,q), \min\{F_{\widehat{\Psi}^g_Q(c)}(x,q), F_{\widehat{\Upsilon}^h_Q(c)}(x,q)\}\big\}\big), \max\big\{f(c), \max\{g(c), h(c)\}\big\}\Big)\Big\rangle :\\
&\quad (x,q) \in X \times Q\Big\}\\
&= (\widehat{\Gamma}^f_Q, A)\widehat{\bigcup}((\widehat{\Psi}^g_Q, B)\widehat{\bigcup}(\widehat{\Upsilon}^h_Q, C)).
\end{aligned}$$

(4) The proof is similar to that of part (3).

□

Next, we define AND and OR operations of GQ-NSESs.

Definition 22. *If $(\widehat{\Gamma}^f_Q, A)$ and $(\widehat{\Psi}^g_Q, B)$ are two generalized Q-neutrosophic soft expert sets on X, then $(\widehat{\Gamma}^f_Q, A)$ AND $(\widehat{\Psi}^g_Q, B)$ is the generalized Q-neutrosophic soft expert set denoted by $(\widehat{\Gamma}^f_Q, A)\widehat{\bigwedge}(\widehat{\Psi}^g_Q, B)$ and defined by $(\widehat{\Gamma}^f_Q, A)\widehat{\bigwedge}(\widehat{\Psi}^g_Q, B) = (\widehat{\Lambda}^h_Q, A \times B)$, where $\widehat{\Lambda}^h_Q(a,b) = \widehat{\Gamma}^f_Q(a)\widehat{\bigcap}\widehat{\Psi}^f_Q(b)$ and the truth, indeterminacy and falsity memberships of $(\widehat{\Lambda}^h_Q, A \times B)$ are as follows:*

$$\begin{aligned}
T_{\widehat{\Upsilon}^h_Q(a,b)}(x,q) &= \min\{T_{\widehat{\Gamma}^f_Q(a)}(x,q), T_{\widehat{\Psi}^g_Q(b)}(x,q)\},\\
I_{\widehat{\Upsilon}^h_Q(a,b)}(x,q) &= \max\{I_{\widehat{\Gamma}^f_Q(a)}(x,q), I_{\widehat{\Psi}^g_Q(b)}(x,q)\},\\
F_{\widehat{\Upsilon}^h_Q(a,b)}(x,q) &= \max\{F_{\widehat{\Gamma}^f_Q(a)}(x,q), F_{\widehat{\Psi}^g_Q(b)}(x,q)\},
\end{aligned}$$

and $h(a,b) = \min\{f(a), g(b) : \forall a \in A$ and $b \in B\}$.

Definition 23. *If* $(\widehat{\Gamma}^f_Q, A)$ *and* $(\widehat{\Psi}^g_Q, B)$ *are two generalized Q-neutrosophic soft expert sets on X, then* $(\widehat{\Gamma}^f_Q, A)$ *OR* $(\widehat{\Psi}^g_Q, B)$ *is the generalized Q-neutrosophic soft expert set denoted by* $(\widehat{\Gamma}^f_Q, A)\widehat{\vee}(\widehat{\Psi}^g_Q, B)$ *and defined by* $(\widehat{\Gamma}^f_Q, A)\widehat{\vee}\ (\widehat{\Psi}^g_Q, B) = (\widehat{\Lambda}^h_Q, A \times B)$*, where* $\widehat{\Lambda}^h_Q(a,b) = \widehat{\Gamma}^f_Q(a)\widehat{\bigcup}\widehat{\Psi}^g_Q(b)$ *and the truth, indeterminacy and falsity memberships of* $(\widehat{\Lambda}^h_Q, A \times B)$ *are as follows*

$$\begin{aligned}
T_{\widehat{\Upsilon}^h_Q(a,b)}(x,q) &= \max\{T_{\widehat{\Gamma}^f_Q(a)}(x,q), T_{\widehat{\Psi}^g_Q(b)}(x,q)\},\\
I_{\widehat{\Upsilon}^h_Q(a,b)}(x,q) &= \min\{I_{\widehat{\Gamma}^f_Q(a)}(x,q), I_{\widehat{\Psi}^g_Q(b)}(x,q)\},\\
F_{\widehat{\Upsilon}^h_Q(a,b)}(x,q) &= \min\{F_{\widehat{\Gamma}^f_Q(a)}(x,q), F_{\widehat{\Psi}^g_Q(b)}(x,q)\},
\end{aligned}$$

and $h(a,b) = \max\{f(a), g(b) : \forall a \in A \text{ and } b \in B\}$.

Example 4. *Assume that two GQ-NSESs* $(\widehat{\Gamma}^f_Q, A)$ *and* $(\widehat{\Psi}^g_Q, B)$ *are defined as follows:*

$$\begin{aligned}
(\widehat{\Gamma}^f_Q, A) = \Big\{ \Big\langle (e_1,u_1,1), \Big(&[(x_1,q_1),0.2,0.5,0.4], [(x_1,q_2),0.1,0.3,0.3], [(x_2,q_1),0.8,0.5,0.5],\\
&[(x_2,q_2),0.8,0.8,0.2], [(x_3,q_1),0.6,0.3,0.5], [(x_3,q_2),0.4,0.3,0.1], 0.4\Big)\Big\rangle,\\
\Big\langle (e_1,u_1,0), \Big(&[(x_1,q_1),0.3,0.3,0.3], [(x_1,q_2),0.7,0.3,0.5], [(x_2,q_1),0.5,0.6,0.2],\\
&[(x_2,q_2),0.7,0.3,0.1], [(x_3,q_1),0.3,0.5,0.2], [(x_3,q_2),0.7,0.5,0.6], 0.5\Big)\Big\rangle,
\end{aligned}$$

$$\begin{aligned}
(\widehat{\Psi}^g_Q, A) = \Big\{ \Big\langle (e_1,u_1,1), \Big(&[(x_1,q_1),0.7,0.3,0.2], [(x_1,q_2),0.4,0.3,0.1], [(x_2,q_1),0.4,0.4,0.5],\\
&[(x_2,q_2),0.6,0.5,0.5], [(x_3,q_1),0.6,0.6,0.2], [(x_3,q_2),0.3,0.5,0.5], 0.4\Big)\Big\rangle,\\
\Big\langle (e_1,u_1,0), \Big(&[(x_1,q_1),0.5,0.7,0.4], [(x_1,q_2),0.3,0.4,0.1], [(x_2,q_1),0.5,0.1,0.6],\\
&[(x_2,q_2),0.8,0.6,0.2], [(x_3,q_1),0.7,0.8,0.6], [(x_3,q_2),0.5,0.8,0.6], 0.6\Big)\Big\rangle.
\end{aligned}$$

Then,

$$\begin{aligned}
(\widehat{\Gamma}^f_Q, A)\widehat{\bigvee}(\widehat{\Psi}^g_Q, A) = \Big\{ \Big\langle ((e_1,u_1,1),(e_1,u_1,1)), \Big(&[(x_1,q_1),0.7,0.3,0.2], [(x_1,q_2),0.4,0.3,0.1], [(x_2,q_1),0.8,0.4,0.5],\\
&[(x_2,q_2),0.8,0.5,0.2], [(x_3,q_1),0.6,0.3,0.2], [(x_3,q_2),0.4,0.3,0.1], 0.4\Big)\Big\rangle,\\
\Big\langle ((e_1,u_1,1),(e_1,u_1,0)), \Big(&[(x_1,q_1),0.5,0.5,0.4], [(x_1,q_2),0.3,0.3,0.1], [(x_2,q_1),0.8,0.1,0.5],\\
&[(x_2,q_2),0.8,0.6,0.2], [(x_3,q_1),0.7,0.3,0.5], [(x_3,q_2),0.5,0.3,0.1], 0.4\Big)\Big\rangle,\\
\Big\langle ((e_1,u_1,0),(e_1,u_1,1)), \Big(&[(x_1,q_1),0.7,0.3,0.2], [(x_1,q_2),0.7,0.3,0.1], [(x_2,q_1),0.5,0.4,0.2],\\
&[(x_2,q_2),0.7,0.3,0.1], [(x_3,q_1),0.6,0.5,0.2], [(x_3,q_2),0.7,0.5,0.5], 0.5\Big)\Big\rangle,\\
\Big\langle ((e_1,u_1,0),(e_1,u_1,0)), \Big(&[(x_1,q_1),0.5,0.3,0.3], [(x_1,q_2),0.7,0.3,0.1], [(x_2,q_1),0.5,0.1,0.2],\\
&[(x_2,q_2),0.8,0.3,0.1], [(x_3,q_1),0.7,0.5,0.2], [(x_3,q_2),0.7,0.5,0.6], 0.6\Big)\Big\rangle\Big\}.
\end{aligned}$$

Proposition 3. *Let* $(\widehat{\Gamma}^f_Q, A), (\widehat{\Psi}^g_Q, B)$ *and* $(\widehat{\Upsilon}^h_Q, C)$ *be GQ-NSESs over a universe X. Then, the following properties hold true:*

(1) $((\widehat{\Gamma}^f_Q, A)\widehat{\bigwedge}(\widehat{\Psi}^g_Q, B))\widehat{\bigwedge}(\widehat{\Upsilon}^h_Q, C) = (\widehat{\Gamma}^f_Q, A)\widehat{\bigwedge}((\widehat{\Psi}^g_Q, B)\widehat{\bigwedge}(\widehat{\Upsilon}^h_Q, C))$,

(2) $((\widehat{\Gamma}^f_Q, A)\widehat{\bigvee}(\widehat{\Psi}^g_Q, B))\widehat{\bigvee}(\widehat{\Upsilon}^h_Q, C) = (\widehat{\Gamma}^f_Q, A)\widehat{\bigvee}((\widehat{\Psi}^g_Q, B)\widehat{\bigvee}(\widehat{\Upsilon}^h_Q, C))$.

Proof. The proof is similar to the proof of part 3 of Proposition 2. □

4. Application of Generalized Q-Neutrosophic Soft Expert Set

In this section, an application of GQ-NSES in decision making is discussed. The problem we consider is as follows:

An investment company considers several business options to increase its portfolio. The set of possible alternatives is $X = \{x_1 = \text{car company}, x_2 = \text{food company}, x_3 = \text{computer company}\}$, $Q = \{q_1 = local, q_2 = international\}$. The company will choose the best option according to the following three criteria $E = \{e_1 = \text{risk}, e_2 = \text{growth}, e_3 = \text{environmental impact}\}$. Let $U = \{u_1, u_2\}$ be a set of experts. After deliberation, the experts construct the following GQ-NSES:

$$
\begin{aligned}
(\widehat{\Gamma}_Q, A) = \Big\{ & \Big\langle (e_1,u_1,1), \Big([(x_1,q_1),0.4,0.3,0.7], [(x_1,q_2),0.2,0.1,0.4], [(x_2,q_1),0.3,0.3,0.2], \\
& \quad [(x_2,q_2),0.5,0.1,0.5], [(x_3,q_1),0.5,0.6,0.3], [(x_3,q_2),0.8,0.4,0.2], 0.2 \Big) \Big\rangle, \\
& \Big\langle (e_1,u_2,1), \Big([(x_1,q_1),0.4,0.6,0.3], [(x_1,q_2),0.6,0.1,0.1], [(x_2,q_1),0.5,0.6,0.6], \\
& \quad [(x_2,q_2),0.7,0.5,0.6], [(x_3,q_1),0.6,0.7,0.8], [(x_3,q_2),0.2,0.7,0.5], 0.6 \Big) \Big\rangle, \\
& \Big\langle (e_2,u_1,1), \Big([(x_1,q_1),0.3,0.3,0.3], [(x_1,q_2),0.4,0.9,0.2], [(x_2,q_1),0.8,0.2,0.1], \\
& \quad [(x_2,q_2),0.4,0.5,0.5], [(x_3,q_1),0.6,0.7,0.7], [(x_3,q_2),0.5,0.5,0.5], 0.9 \Big) \Big\rangle, \\
& \Big\langle (e_2,u_2,1), \Big([(x_1,q_1),0.9,0.6,0.2], [(x_1,q_2),0.2,0.1,0.8], [(x_2,q_1),0.1,0.8,0.6], \\
& \quad [(x_2,q_2),0.6,0.4,0.8], [(x_3,q_1),0.2,0.2,0.2], [(x_3,q_2),0.9,0.6,0.4], 0.7 \Big) \Big\rangle, \\
& \Big\langle (e_3,u_1,1), \Big([(x_1,q_1),0.5,0.4,0.3], [(x_1,q_2),0.3,0.8,0.9], [(x_2,q_1),0.3,0.9,0.1], \\
& \quad [(x_2,q_2),0.1,0.1,0.4], [(x_3,q_1),0.7,0.1,0.3], [(x_3,q_2),0.4,0.5,0.7], 0.5 \Big) \Big\rangle, \\
& \Big\langle (e_3,u_2,1), \Big([(x_1,q_1),0.6,0.6,0.2], [(x_1,q_2),0.1,0.2,0.2], [(x_2,q_1),0.9,0.4,0.2], \\
& \quad [(x_2,q_2),0.6,0.7,0.4], [(x_3,q_1),0.5,0.5,0.3], [(x_3,q_2),0.4,0.5,0.1], 0,3 \Big) \Big\rangle, \\
& \Big\langle (e_1,u_1,0), \Big([(x_1,q_1),0.5,0.7,0.3], [(x_1,q_2),0.4,0.7,0.9], [(x_2,q_1),0.2,0.8,0.9], \\
& \quad [(x_2,q_2),0.3,0.2,0.8], [(x_3,q_1),0.8,0.1,0.4], [(x_3,q_2),0.7,0.9,0.6], 0.8 \Big) \Big\rangle, \\
& \Big\langle (e_1,u_2,0), \Big([(x_1,q_1),0.7,0.4,0.2], [(x_1,q_2),0.6,0.5,0.8], [(x_2,q_1),0.3,0.4,0.8], \\
& \quad [(x_2,q_2),0.6,0.7,0.9], [(x_3,q_1),0.4,0.3,0.2], [(x_3,q_2),0.6,0.5,0.9], 0.6 \Big) \Big\rangle, \\
& \Big\langle (e_2,u_1,0), \Big([(x_1,q_1),0.2,0.1,0.2], [(x_1,q_2),0.4,0.5,0.8], [(x_2,q_1),0.6,0.6,0.6], \\
& \quad [(x_2,q_2),0.4,0.4,0.4], [(x_3,q_1),0.8,0.2,0.5], [(x_3,q_2),0.3,0.7,0.8], 0.5 \Big) \Big\rangle, \\
& \Big\langle (e_2,u_2,0), \Big([(x_1,q_1),0.1,0.5,0.6], [(x_1,q_2),0.9,0.4,0.4], [(x_2,q_1),0.7,0.6,0.5], \\
& \quad [(x_2,q_2),0.5,0.1,0.2], [(x_3,q_1),0.3,0.2,0.2], [(x_3,q_2),0.7,0.3,0.8], 0.7 \Big) \Big\rangle, \\
& \Big\langle (e_3,u_1,0), \Big([(x_1,q_1),0.1,0.7,0.9], [(x_1,q_2),0.4,0.3,0.8], [(x_2,q_1),0.9,0.9,0.3], \\
& \quad [(x_2,q_2),0.8,0.8,0.1], [(x_3,q_1),0.7,0.4,0.6], [(x_3,q_2),0.4,0.8,0.2], 0.8 \Big) \Big\rangle, \\
& \Big\langle (e_3,u_2,0), \Big([(x_1,q_1),0.7,0.3,0.2], [(x_1,q_2),0.5,0.3,0.3], [(x_2,q_1),0.7,0.4,0.1], \\
& \quad [(x_2,q_2),0.3,0.1,0.9], [(x_3,q_1),0.4,0.6,0.5], [(x_3,q_2),0.6,0.1,0.1], 0.4 \Big) \Big\rangle \Big\}.
\end{aligned}
$$

The experts may use the following algorithm to choose the best option for the investment:

1. Input the GQ-NSES $(\widehat{\Gamma}^f_Q, A)$.
2. Find the values of $|T_{\widehat{\Gamma}^f_Q} + I_{\widehat{\Gamma}^f_Q} - F_{\widehat{\Gamma}^f_Q}|$ for each element $(x, q) \in X \times Q$, where $T_{\widehat{\Gamma}^f_Q}, I_{\widehat{\Gamma}^f_Q}, F_{\widehat{\Gamma}^f_Q}$ representing the truth, indeterminacy and falsity membership functions.
3. Compute the score of each element $(x, q) \in X \times Q$ by taking the sum of the products of the numerical grade of each element with the corresponding values of $f(a)$ (the degree of preference) for the agree-GQ-NSES and disagree-GQ-NSES, denoted by ν_i and η_i, respectively.
4. Find the values of the score $\delta_j = \nu_i - \eta_i$.
5. Find m for which $\delta_m = \max \delta_i$.

Table 1 presents the values of $|T_{\widehat{\Gamma}_Q} + I_{\widehat{\Gamma}_Q} - F_{\widehat{\Gamma}_Q}|$ for each element $(x, q) \in X \times Q$.

Table 1. Values of $|T_{\widehat{\Gamma}^f_Q} + I_{\widehat{\Gamma}^f_Q} - F_{\widehat{\Gamma}^f_Q}|$ for each element $(x, q) \in X \times Q$.

$E \times U \times O$	(x_1, q_1)	(x_1, q_2)	(x_2, q_1)	(x_2, q_2)	(x_3, q_1)	(x_3, q_2)	f(a)
$(e_1, u_1, 1)$	0	−0.1	0.4	0.1	0.8	1	0.2
$(e_1, u_2, 1)$	0.7	0.6	0.5	0.6	0.5	0.4	0.6
$(e_2, u_1, 1)$	0.3	1.1	0.9	0.4	0.6	0.5	0.9
$(e_2, u_2, 1)$	1.3	−0.5	0.3	0.2	0.2	1.1	0.7
$(e_3, u_1, 1)$	0.6	0.2	1.1	−0.2	0.5	0.2	0.5
$(e_3, u_2, 1)$	1	0.1	1.1	0.9	0.7	0.1	0.3
$(e_1, u_1, 0)$	0.9	0.2	0.1	−0.3	0.5	1	0.8
$(e_1, u_2, 0)$	0.9	0.3	−0.1	0.4	0.5	0.2	0.6
$(e_2, u_1, 0)$	0.1	0.1	0.6	0.4	0.5	0.2	0.5
$(e_2, u_2, 0)$	0	0.9	0.8	0.4	0.3	0.2	0.7
$(e_3, u_1, 0)$	−0.1	−0.1	1.5	1.5	0.5	1	0.8
$(e_3, u_2, 0)$	0.8	0.5	1	−0.5	0.5	0.6	0.4

Tables 2 and 3 present the grades for the agree-GQ-NSES and disagree-GQ-NSES, taken from the first six rows and the last six rows of Table 1, respectively.

Table 2. Numerical grades for agree-GQ-NSES.

$E \times U \times \{1\}$	(x_1, q_1)	(x_1, q_2)	(x_2, q_1)	(x_2, q_2)	(x_3, q_1)	(x_3, q_2)	f(a)
$(e_1, u_1, 1)$	0	−0.1	0.4	0.1	0.8	1	0.2
$(e_1, u_2, 1)$	0.7	0.6	0.5	0.6	0.5	0.4	0.6
$(e_2, u_1, 1)$	0.3	1.1	0.9	0.4	0.6	0.5	0.9
$(e_2, u_2, 1)$	1.3	−0.5	0.3	0.2	0.2	1.1	0.7
$(e_3, u_1, 1)$	0.6	0.2	1.1	−0.2	0.5	0.2	0.5
$(e_3, u_2, 1)$	1	0.1	1.1	0.9	0.7	0.1	0.3

Table 3. Numerical grades for disagree-GQ-NSES.

$E \times U \times \{0\}$	(x_1, q_1)	(x_1, q_2)	(x_2, q_1)	(x_2, q_2)	(x_3, q_1)	(x_3, q_2)	f(a)
$(e_1, u_1, 0)$	0.9	0.2	0.1	−0.3	0.5	1	0.8
$(e_1, u_2, 0)$	0.9	0.3	−0.1	0.4	0.5	0.2	0.6
$(e_2, u_1, 0)$	0.1	0.1	0.6	0.4	0.5	0.2	0.5
$(e_2, u_2, 0)$	0	0.9	0.8	0.4	0.3	0.2	0.7
$(e_3, u_1, 0)$	−0.1	−0.1	1.5	1.5	0.5	1	0.8
$(e_3, u_2, 0)$	0.8	0.5	1	−0.5	0.5	0.6	0.4

Let ν_i and η_i represent the score of each numerical grade for the agree-GQ-NSES and disagree-GQ-NSES, respectively. These values are given in Table 4.

Table 4. The score $\delta_j = \nu_i - \eta_i$.

ν_i	η_i	δ_j
$\nu(x_1, q_1) = 2.2$	$\eta(x_1, q_1) = 1.55$	$\delta(x_1, q_1) = 0.65$
$\nu(x_1, q_2) = 2.01$	$\eta(x_1, q_2) = 1.14$	$\delta(x_1, q_2) = 0.87$
$\nu(x_2, q_1) = 2.28$	$\eta(x_2, q_1) = 2.48$	$\delta(x_2, q_1) = -0.2$
$\nu(x_2, q_2) = 1.05$	$\eta(x_2, q_2) = 1.48$	$\delta(x_2, q_2) = -0.43$
$\nu(x_3, q_1) = 1.6$	$\eta(x_3, q_1) = 1.76$	$\delta(x_3, q_1) = -0.16$
$\nu(x_3, q_2) = 1.79$	$\eta(x_3, q_2) = 2.2$	$\delta(x_3, q_2) = -0.41$

As can be seen in Table 4, $\max \delta_i = \delta(x_1, q_2)$. Therefore, the best option is to invest in an international car company.

5. Comparative Analysis

A generalized Q-neutrosophic soft expert model offers better compatibility, accuracy and flexibility than existing models. This can be confirmed by a comparison utilizing generalized Q-neutrosophic soft expert with the strategy utilized in [33] as seen in Table 5. We can note that the proposed method include a degree of preference in the decision process, thus make it highly effective in decision-making. The comparison is similarly conducted as the illustration in Section 4, whereby the ranking order is found to be consistent.

Table 5. Comparison of GQ-NSES to Q-NSES.

Method	GQ-NSES	Q-NSES
True	Yes	Yes
Falsity	Yes	Yes
Indeterminacy	Yes	Yes
Expert	Yes	Yes
Q	Yes	Yes
Degree of preference	Yes	No
Ranking	$(x_1, q_2) > (x_1, q_1) > (x_3, q_1) > (x_2, q_1) > (x_3, q_2) > (x_2, q_2)$	$(x_3, q_1) > (x_1, q_2) > (x_2, q_1) > (x_2, q_2) > (x_3, q_2) > (x_1, q_1)$

From Table 5, it can be seen that the results of the ranking on the six companies obtained by the proposed GQ-NSES method in this paper is different from the ranking obtained by the method introduced in [33]. The main reason is that the proposed method has some crucial advantages over Q-NSES, which can take the preference of decision-makers into consideration.

6. Conclusions

The concept of GQ-NSES was initiated by incorporating the idea of SESs to Q-NSs and attaching a degree of preference corresponding to each parameter. The proposed concept is significantly superior and improved generalization of Q-NSES, which delivers better results, particularly for decision-making problems. The basic operations on GQ-NSES were defined and subsequently the basic properties were proven. An algorithm incorporating GQ-NSES is introduced and applied to a real-life example. The notion of GQ-NSES extended current neutrosophic theories for dealing with indeterminacy and will stimulate further studies on extensions and applied usage. In the future, one could study the measures of distance, similarity and entropy of GQ-NSES by generalizing the results in [14]. Furthermore, the algebraic structures such as group, ring and field of the Q-NSS and Q-NSES and their generalizations may be studied.

Author Contributions: Conceptualization, M.A.Q.; Funding Acquisition, N.H.; Methodology, M.A.Q.; Project Administration, N.H.; Supervision, N.H.; Validation, N.H.; Writing—Original Draft, M.A.Q.; Writing—Review and Editing, N.H.

Funding: This research is funded by Universiti Kebangsaan Malaysia.

Acknowledgments: We thank Universiti Kebangsaan Malaysia for financing this research under GUP-2017-105 grant.

Conflicts of Interest: We declare that there is no conflict of interest.

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Simplified Neutrosophic Sets Based on Interval Dependent Degree for Multi-Criteria Group Decision-Making Problems

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Received: 23 October 2018; Accepted: 12 November 2018; Published: 15 November 2018

Abstract: In this paper, a new approach and framework based on the interval dependent degree for multi-criteria group decision-making (MCGDM) problems with simplified neutrosophic sets (SNSs) is proposed. Firstly, the simplified dependent function and distribution function are defined. Then, they are integrated into the interval dependent function which contains interval computing and distribution information of the intervals. Subsequently, the interval transformation operator is defined to convert simplified neutrosophic numbers (SNNs) into intervals, and then the interval dependent function for SNNs is deduced. Finally, an example is provided to verify the feasibility and effectiveness of the proposed method, together with its comparative analysis. In addition, uncertainty analysis, which can reflect the dynamic change of the final result caused by changes in the decision makers' preferences, is performed in different distribution function situations. That increases the reliability and accuracy of the result.

Keywords: simplified neutrosophic sets (SNSs); interval number; dependent degree; multi-criteria group decision-making (MCGDM)

1. Introduction

In multi-criteria decision making (MCDM) problems, because of the increasing complexity of the socioeconomic situation and the inherent knowledge restrictions of people, the characteristics of many things in the world exhibit fuzzy and uncertain features which are difficult to describe by exact numerical values [1]. Fuzzy sets (FSs) [2–4], which were proposed by Zadeh in 1965, are regarded as an effective way to describe fuzzy information and are widely applied to many decision making problems [5]. However, it is hard to describe the degree of non-membership and this is insufficient in some cases. For this reason, Atanassov introduced intuitionistic fuzzy sets (IFSs) [6–8] which are an extension of FSs. Vague sets are also defined in Reference [9], which are pointed out as being mathematically equivalent to Atanassov's IFSs by Bustince [10]. In recent years, IFSs have been widely used in solving MCDM problems [11–15]. IFSs have also been extended to some other forms and expressions such as interval intuitionistic fuzzy sets [16–19] and interval intuitionistic hesitant fuzzy sets [20,21], etc.

However, for its fixed scope definition, FSs or IFSs theory has been restricted to some uncertainty cases in real problems, especially when the information is incomplete and inconsistent [22]. For example [23], when an expert is asked to evaluate a certain statement, he or she may say that the possibility of true is 0.5, that of false is 0.6, and the degree of not sure is 0.2. The sum of possibility exceeds the scope of FSs and IFSs, and cannot be solved by them.

For this reason, Smarandache initially developed neutrosophic logic and neutrosophic sets (NSs) theory [24–26], which provides a tool to define the possibility and neutrality degree between the affirmative and the negative in most practical situations [27]. An NS is a set in which each element has degrees of truth membership, indeterminacy membership, and falsity membership and it lies in $]0^-, 1^+[$, the nonstandard unit interval [28]. It may be seen as an extension of the standard interval of IFSs and has many practical applications such as medical diagnosis, e-learning, image processing and data mining, etc. [29–35]. For convenience in practical situations, Smarandache [24] and Wang [23] introduced single-valued neutrosophic sets (SVNSs). Then, using similarity and entropy measures, the correlation coefficient of SVNSs were put forward by Majumdar [36] and Ye [37], respectively. Huang [38] developed several new formulas of the distance measures for SVNSs. Thanh [39] built a new recommender system based on a clustering algorithm for SVNSs. Karaaslan [40] defined the correlation coefficient for single valued neutrosophic refined soft sets. Ye [41] found several new similarity measure formulas for SVNSs by building the cotangent function. Recently, the concept of simplified neutrosophic sets (SNSs) and aggregation operators were introduced by Ye [42], which can be characterized by three real numbers in the interval $[0, 1]$. Because its definition is more in line with the needs of many engineering situations, SNSs have quickly been applied to MCDM and MCGDM problems. Ye [43] proposed the similarity measures between SNSs and INSs in MCDM problems. Peng [44] defined the outranking relations with SNSs. Based on the work of Ye [42], Peng [45] redefined some aggregation operators of SNSs by utilizing the t-norm and t-conorm. Ye [46] proposed exponential entropy measures for SNSs and studied their properties.

The methods for solving MCDM or MCGDM problems using SNNs outlined above are proved to be effective and feasible. There are some aspects which need to be promoted or further studied. (1) Most of the operators for SNSs are derived from the operators of fuzzy sets arithmetic such as probability degree, score function, correlation coefficient, and similarity measures, etc. Is there any other method or framework to perform computing on SNSs? (2) In the calculation process, most existing methods need several steps including defining operation laws, choosing aggregation operators, and performing ranking functions. It often becomes complex and difficult to understand. In fact, there is usually no direct correlation between previous steps and latter steps. Therefore, some methods become a combination of various steps and lack algorithm integrality and consistency. (3) Most existing methods directly utilize the three values of an SNN as parameters without adequately considering the implied distribution information. This leads to some operational deficiencies and information loss. (4) Many traditional models are too deterministic and lose their uncertainty information in the calculating process. Hence, stability checking and uncertainty analysis cannot be relied upon for the decision results in the later stages. So we cannot know whether the decision results will change when decision makers' preferences change slightly, and that means we do not know whether the final result is stable and insensitive enough.

To solve these problems, a novel approach and framework for the MCGDM problem with SNNs is proposed. The main advantages and outstanding contributions are shown below. (1) Unlike most of the existing methods, the proposed model represents a novel framework which does not require deriving from fuzzy sets operations. It builds on interval number and interval dependent degree operators. (2) The proposed method does not need those complex definitions and operator steps, and is more concise and intuitive. It has higher computation integration for directly combining two main steps into unified dependent degree formula. (3) The proposed method, which can describe the implied distribution information of an SNN through defining the distribution function in dependent degree formula, shows stronger capabilities of description for SNNs and avoids information loss. (4) As a result of maintaining information flexibility and dynamics by distribution function, the proposed model can analyze the uncertainty and stability of decision results through choosing different distribution function expressions. The method takes into account information integrity, computation simplicity, and dynamic analysis capability.

The rest of the paper is organized as follows. In Section 2, some important concepts including interval number, simplified dependent function, distribution function, and interval dependent function are defined. Subsequently, the specific expressions of the dependent degree function are deduced under the different distribution function. In Section 3, NSs and SNSs are briefly reviewed. Then, an interval transformation operator is defined to transform SNN into interval number. On the basis of the transformation, the interval dependent function of SNNs is deduced. In Section 4, a computing procedure based on the interval dependent degrees of SNNs for MCGDM problems is developed. In Section 5, an illustrative instance and a comparative analysis are adopted to validate the proposed method. Finally, in Section 6, conclusions are given.

2. Interval Number and Interval Dependent Function

In the section, some basic concepts and definitions about simplified dependent function, including interval numbers, interval dependent function, and distribution function are introduced.

2.1. Interval Number

Definition 1. *Interval number. Let* $X = [a, b] = \{x | a \leq x \leq b; a, b \in R\}$*, and then* X *is called an interval number. In particular,* X *will be degenerated into a real number if* $a = b$*. Here,* $X = [0, 1]$ *is called a standard interval.*

Subsequently, the operators of two non-negative interval numbers $X = [a, b]$ and $Y = [c, d]$ are defined as follows:

$$X + Y = [a + c, b + d] \quad (1)$$

$$X - Y = [a - d, b - c] \quad (2)$$

$$\lambda X = [\lambda a, \lambda b](\lambda > 0) \quad (3)$$

$$\frac{1}{X} = \left[\frac{1}{b}, \frac{1}{a}\right] \quad (4)$$

2.2. Simplified Dependent Function and Interval Dependent Function

Definition 2. *Simplified dependent function. Suppose a finite interval* $X = [a, b]$ *and its optimal value is* b*, if* $\forall x \in X$ *and there is a function* $k(x, X)$ *that satisfies the following properties: (1)* $k(x, X)$ *reaches the maximum value* 1 *when* $x = b$*, and reaches the minimum value* 0 *when* $x = a$*. (2) When* $x \in X$ *and* $x \neq a, b$*, then* $0 < k(x, X) < 1$ *holds. (3)* $\forall x_1, x_2 \in X$ *and* $x_1 < x_2$*, then* $k(x_1, X) < k(x_2, X)$ *holds. Then,* $k(x, X)$ *is called the simplified dependent function of* x *on the interval* X*.*
Here, we give some examples of simplified dependent function expressions.

$$k(x, X) = \frac{x - a}{b - a}, (X = [a, b]) \quad (5)$$

$$k(x, X) = \frac{\alpha x}{\alpha - 1 + x}, (X = [0, 1]) \quad (6)$$

$$k(x, X) = \frac{e^{\alpha x} - 1}{e^{\alpha} - 1}, (X = [0, 1]) \quad (7)$$

Figures 1 and 2 show the shapes of Equations (6) and (7), respectively. In Figure 1, the larger the parameter α applied, the steeper the front of the function is and the flatter the back of the function is. That is always used to describe the different psychology status of decision makers to the values near the interval endpoints. Figure 2 shows the contrary situation.

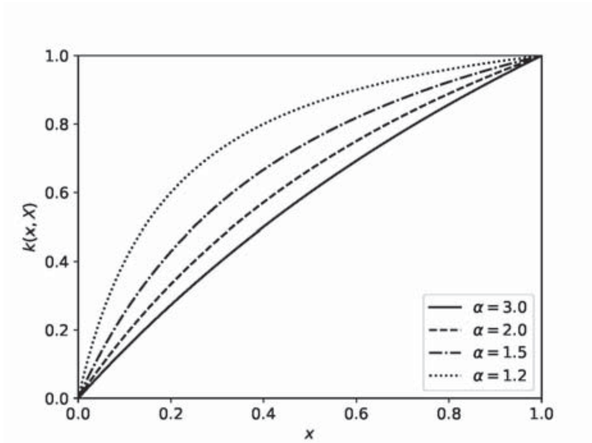


Figure 1. Dependent function of Equation (6).

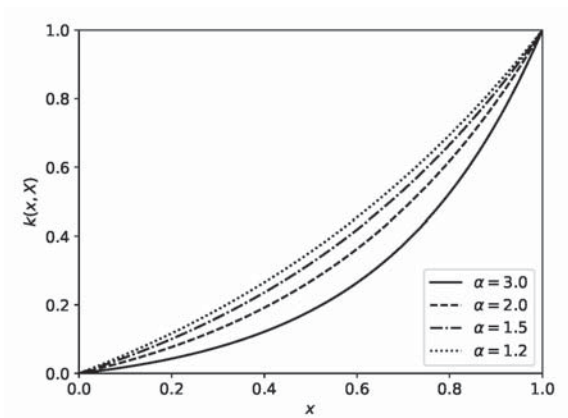


Figure 2. dependent function of Equation (7).

Definition 3. *Interval dependent function. Suppose a finite interval $X = [a, b]$ and its optimal value is b, for a subinterval $X_0 = [a_0, b_0]$ and $X_0 \subseteq X$, then,*

$$k(X_0, X) = \int_{x \in X_0} k(x, X)h(x, X_0)dx, (a_0 \neq b_0) \tag{8}$$

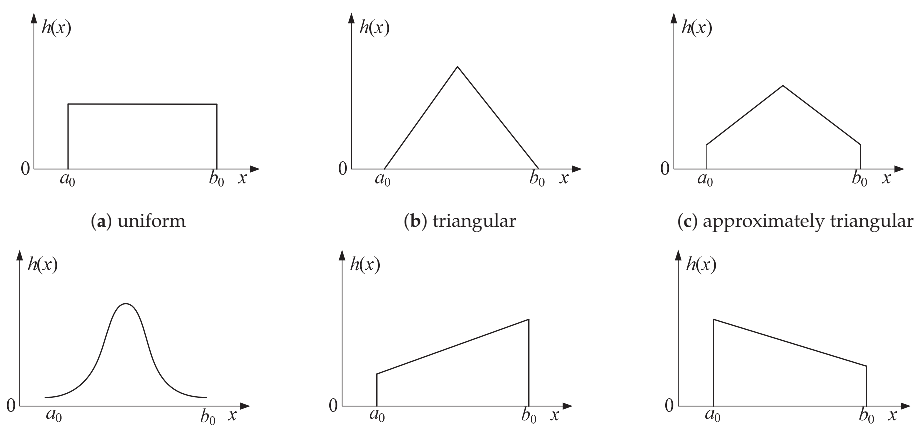
Here, $k(x, X)$ is the simplified dependent function, $h(x, X_0)$ is the probability density function on subinterval X_0. $k(X_0, X)$ is called the interval dependent function of subinterval X_0 on interval X.

$k(X_0, X)$ has the following properties: (1) X_0 will be degenerated into a real number if $a_0 = b_0$, and $k(X_0, X)$ will be degenerated into the simplified dependent function $k(a_0, X)$. Especially, $k(X_0, X)$ reaches the maximum value 1 when $a_0 = b_0 = b$, and reaches the minimum value 0 when $a_0 = b_0 = a$. (2) when $a_0 \neq b_0$, $0 < k(X_0, X) < 1$.

Definition 4. *Distribution function. Distribution function $h(x)$ describes the distribution of x in interval $[a_0, b_0]$. $h(x)$ is in the form of probability density function. Therefore,*

$$\int_{x \in [a_0, b_0]} h(x)dx = 1 \tag{9}$$

Figure 3 gives some commonly used probability density function expressions, such as uniform distribution, triangular distribution, trapezoid distribution, and normal distribution. Among these, Figure 3a indicates that every value in the interval occurred at an equal probability. Figure 3b–d indicate that the probability of occurrence of the middle value in the interval is the maximum. Figure 3e,f, respectively, indicate that the probabilities of all values in the interval are linearly increasing or decreasing. The forms of the distribution functions are various and can be fixed according to different actual application situations. It cannot only be used to describe the distribution of the values in the interval, but also to examine the stability of the interval dependent degree.



(**a**) uniform (**b**) triangular (**c**) approximately triangular

(**d**) normal (**e**) trapezoid1 (**f**) trapezoid2

Figure 3. Distribution function.

Property 1. *When the probability density function $h(x, X_0)$ follows uniform distribution as Figure 3a, then,*

$$\begin{aligned} k(X_0, X) &= \int_{a_0}^{b_0} k(x, X)h(x, X_0)dx \\ &= \int_{a_0}^{b_0} k(x, X)\frac{1}{b_0 - a_0}dx \\ &= \frac{1}{b_0 - a_0}\int_{a_0}^{b_0} k(x, X)dx \end{aligned} \tag{10}$$

Property 2. *When the probability density function $h(x, X_0)$ follows triangular distribution as Figure 3b, then,*

$$\begin{aligned} k(X_0, X) &= \int_{a_0}^{\frac{a_0+b_0}{2}} k(x, X)h(x, X_0)dx + \int_{\frac{a_0+b_0}{2}}^{b_0} k(x, X)h(x, X_0)dx \\ &= \frac{2}{b_0 - a_0}\left[\int_{a_0}^{\frac{a_0+b_0}{2}} k(x, X)\frac{x - a_0}{\frac{a_0+b_0}{2} - a_0}dx + \int_{\frac{a_0+b_0}{2}}^{b_0} k(x, X)\frac{x - b_0}{\frac{a_0+b_0}{2} - b_0}dx\right] \\ &= \frac{4}{(b_0 - a_0)^2}\left[\int_{a_0}^{\frac{a_0+b_0}{2}} k(x, X)(x - a_0)dx + \int_{\frac{a_0+b_0}{2}}^{b_0} k(x, X)(b_0 - x)dx\right] \end{aligned} \tag{11}$$

Property 3. *When the probability density function $h(x, X_0)$ follows approximately triangular distribution as Figure 3c, then,*

$$\begin{aligned} k(X_0, X) &= \int_{a_0}^{\frac{a_0+b_0}{2}} k(x, X)h(x, X_0)dx + \int_{a_0+b_0 2}^{b_0} k(x, X)h(x, X_0)dx \\ &= \int_{a_0}^{\frac{a_0+b_0}{2}} k(x, X)\frac{2b_0 + 4x - 6a_0}{3(b_0 - a_0)^2}dx + \int_{\frac{a_0+b_0}{2}}^{b_0} k(x, X)\frac{6b_0 - 4x - 2a_0}{3(b_0 - a_0)^2}dx \\ &= \frac{2}{3(b_0 - a_0)^2}\left[\int_{a_0}^{\frac{a_0+b_0}{2}} k(x, X)(b_0 + 2x - 3a_0)dx + \int_{\frac{a_0+b_0}{2}}^{b_0} k(x, X)(3b_0 - 2x - a_0)dx\right] \end{aligned} \quad (12)$$

Property 4. *When the probability density function $h(x, X_0)$ follows normal distribution as Figure 3d, then,*

$$\begin{aligned} k(X_0, X) &= \int_{a_0}^{b_0} k(x, X)h(x, X_0)dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{a_0}^{b_0} k(x, X)e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx \quad (\mu = \frac{a_0 + b_0}{2}, \sigma = \frac{|a_0 - b_0|}{6}) \end{aligned} \quad (13)$$

Property 5. *When the probability density function $h(x, X_0)$ follows normal distribution as Figure 3e, then,*

$$\begin{aligned} k(X_0, X) &= \int_{a_0}^{b_0} k(x, X)h(x, X_0)dx \\ &= \int_{a_0}^{b_0} k(x, X)\left(\frac{x - a_0}{b_0 - a_0}\frac{2}{3(b_0 - a_0)} + \frac{2}{3(b_0 - a_0)}\right)dx \\ &= \frac{2}{3(b_0 - a_0)^2}\int_{a_0}^{b_0} k(x, X)(x + b_0 - 2a_0)dx \end{aligned} \quad (14)$$

3. Interval Transformation Operator and Interval Dependent Function of SNS

3.1. NSs and SNSs

Definition 5 ([25])**.** *Let X be a space of points (objects), with a generic element in X, denoted by x. An NS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$, and $F_A(x)$ are standard or non-standard subsets of $]0^-, 1^+[$, that is, $T_A(x) : X \to]0^-, 1^+[$, $I_A(x) : X \to]0^-, 1^+[$ and $F_A(x) : X \to]0^-, 1^+[$. There is no restriction on the sum of $T_A(x), I_A(x)$, and $F_A(x)$, therefore $0^- \le supT_A(x) + supI_A(x) + supF_A(x) \le 3^+$.*

Definition 6 ([25])**.** *An NS A is contained in another NS B, denoted by $A \subseteq B$, if and only if $infT_A(x) \le infT_B(x)$, $supT_A(x) \le supT_B(x)$, $infI_A(x) \ge infI_B(x)$, $supI_A(x) \ge supI_B(x)$, $infF_A(x) \ge infF_B(x)$, and $supF_A(x) \ge supF_B(x)$ for $x \in X$. Since it is difficult to apply NSs to practical problems, Ye (2014a) reduced NSs of non-standard intervals into SNSs of standard intervals that would preserve the operations of NSs.*

Definition 7 ([42])**.** *Let X be a space of points (objects), with a generic element in X, denoted by x. An NS A in X is characterised by $T_A(x)$, $I_A(x)$, and $F_A(x)$, which are subintervals/subsets in the standard interval $[0, 1]$, that is, $T_A(x) : X \to [0, 1]$, $I_A(x) : X \to [0, 1]$ and $F_A(x) : X \to [0, 1]$. Then, a simplification of A is denoted by $A = \{< x, T_A(x), I_A(x), F_A(x) > | x \in X\}$, which is called an SNS. In particular, if X has only one element, $A = < T_A(x), I_A(x), F_A(x) >$ is called an SNN. For convenience, a SNN is denoted by $A = < T_A, I_A, F_A >$. Clearly, SNSs are a subclass of NSs.*

Definition 8 ([42])**.** *An SNS A is contained in another SNS B, denoted by $A \subseteq B$, if and only if $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, and $F_A(x) \ge F_B(x)$ for any $x \in X$.*

Definition 9 ([42])**.** *The complement of an SNS A is denoted by A^C and is defined as*

$$A^C = \{< x, F_A(x), 1 - I_A(x), T_A(x) > | x \in X\} \tag{15}$$

3.2. Interval Transformation Operator of SNNs

Definition 10. *Interval transformation operator Z. For an SNN $A = < T_A, I_A, F_A >$, there is an operator that makes $Z(A) = \{X_0, X\}$, here $X_0 = \left[\frac{T_A}{T_A + I_A + F_A}, \frac{T_A + I_A}{T_A + I_A + F_A}\right]$, $X = [0, 1]$ is the standard interval. Then Z is called interval transformation operator of SNN A.*

The meaning of the operator Z is briefly explained here. From definition 6, there is $T_A, I_A, F_A \in [0, 1]$ holds. So divided by $T_A + I_A + F_A$, SNN A will be mapped into the standard interval $[0, 1]$. X_0 is the range of truth-membership values of A actually. $\frac{T_A}{T_A + I_A + F_A}$ and $\frac{T_A + I_A}{T_A + I_A + F_A}$ are lower bound and upper bound of truth-membership values of A, respectively. $X = [0, 1]$ is the maximum range of truth-membership values and its optimal value is 1 obviously. Although X_0 only represents the range of truth-membership values, it must be decided by T_A, I_A and F_A together.

Example 1. *Assume three SNNs $A = < 0.5, 0.2, 0.2 >$, $B = < 0.5, 0.3, 0.2 >$, and $C = < 0.5, 0.3, 0.3 >$. The following transformation results can be obtained utilizing the operator Z:*

$$Z(A) = \{X_0 = [0.556, 0.778], X = [0, 1]\}$$
$$Z(B) = \{X_0 = [0.5, 0.8], X = [0, 1]\}$$
$$Z(C) = \{X_0 = [0.455, 0.727], X = [0, 1]\}$$

3.3. Interval Dependent Function of SNNs

For a SNN $A = < T_A, I_A, F_A >$, by transformation operator Z, the subinterval $X_0 = \left[\frac{T_A}{T_A + I_A + F_A}, \frac{T_A + I_A}{T_A + I_A + F_A}\right]$ and the standard interval $X = [0, 1]$ are obtained. Then, according to Equation (8), the interval dependent degree of SNN A is:

$$k(A) = k(X_0, X) = \int_{x \in X_0} k(x, X) h(x, X_0) dx, (I_A \neq 0) \tag{16}$$

Here, $k(x, X)$ is the simplified dependent function, $h(x, X_0)$ is the distribution function on subinterval X_0. $k(A)$ is called the interval dependent function of SNN A on the standard interval X.

$k(A)$ has the following properties: (1) X_0 will be degenerated into a real number if $I_A = 0$, and $k(A)$ will be degenerated into the simplified dependent function $k(\frac{T_A}{T_A + I_A + F_A}, X)$. Especially, $k(X_0, X)$ reaches the maximum value 1 when $I_A = F_A = 0$, and reaches the minimum value 0 when $I_A = T_A = 0$. (2) When $I_A \neq 0$, $0 < k(A) < 1$.

The proposed interval dependent function of SNN has the following meanings: Firstly, the proposed function represents a new way of thinking and framework without the need of deriving from fuzzy sets operations. Secondly, the proposed function integrates the distribution function and simplified dependent function into a formula, and so is more concise and intuitive and has higher computation integration. Thirdly, the proposed function can describe the distribution information by defining the inherent distribution function and describe the SNN better. Fourthly, by defining various distribution functions which reflect decision makers' preferences, the proposed model can analyze the uncertainty and sensibility of decision results.

4. The MCGDM Method Based on the Interval Dependent Degrees of SNNs

Suppose that there are m alternatives $A = \{a_1, a_2, ..., a_m\}$ and n criteria $C = \{c_1, c_2, ..., c_n\}$, and the weight vector of criteria is $w = (w_1, w_2, ..., w_n)$, where $w_j \geq 0$ $(j = 1, 2, ..., n)$, $\sum_{j=1}^{n} w_j = 1$. there are l decision-makers $D = \{d_1, d_2, ..., d_l\}$ with its weight vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$. Let $R = (r_{ij}^k)_{m\times n}$ be the decision matrix, where the value of a criterion denoted by SNNs $r_{ij}^k = < T_{r_{ij}^k}, I_{r_{ij}^k}, F_{r_{ij}^k} >$, where $T_{r_{ij}^k}$ indicates the truth-membership function that the alternative a_i satisfies the criterion c_j for the kth decision-maker, $I_{r_{ij}^k}$ represents the determinacy membership function that the alternative a_i satisfies the criterion c_j for the kth decision-maker, and $F_{r_{ij}^k}$ is the falsity-membership function that the alternative a_i satisfies the criterion c_j for the kth decision-maker. The proposed method uses the interval transformation operator and interval dependent degree of SNNs to solve the MCGDM problem mentioned above. A procedure for sorting and choosing the most desirable alternative(s) is provided in the following steps.

Step 1. Normalize the decision matrix. Generally, there are two kinds criteria including maximizing criteria and minimizing criteria in MCDM problems. For the maximizing criteria, it remains unchanged. For the minimizing criteria, it can be transformed into maximizing criteria by taking its complement as $r_{ij}^k = (r_{ij}^k)^C = < F_{r_{ij}^k}, 1 - I_{r_{ij}^k}, T_{r_{ij}^k} >$ in Definition 9.

Step 2. Interval transformation. Performing interval transformation operator $Z(r_{ij}^k)$ to SNN r_{ij}^k according Definition 10. Then, the corresponding subinterval is obtained as

$$X_0^{r_{ij}^k} = \left[\frac{T_{r_{ij}^k}}{T_{r_{ij}^k} + I_{r_{ij}^k} + F_{r_{ij}^k}}, \frac{T_{r_{ij}^k} + I_{r_{ij}^k}}{T_{r_{ij}^k} + I_{r_{ij}^k} + F_{r_{ij}^k}} \right] \tag{17}$$

Step 3. Select the simplified dependent function and distribution function. According to the preference of decision makers and the actual requirements, the forms of dependent function Definition 2 and distribution function as in Definition 4 should be decided.

Step 4. Calculate the interval dependent degree of each SNN of the decision matrix. According to Equation (16), the dependent degree of SNN r_{ij}^k as

$$k(r_{ij}^k) = k(X_0^{r_{ij}^k}, X) = \int_{x \in X_0^{r_{ij}^k}} k(x, X) h(x, X_0^{r_{ij}^k}) dx \tag{18}$$

Here, $X_0^{r_{ij}^k}$ is obtained in step 2, X is the standard interval $[0, 1]$, $k(*)$ function and $h(*)$ function are decided in step 3.

Step 5. Calculate the comprehensive dependent degree of each alternative. The comprehensive dependent degree of each alternative a_i is obtained as

$$K(a_i) = \sum_{k=1}^{l} \left(\lambda_k \sum_{j=1}^{n} \omega_j k(r_{ij}^k) \right) \tag{19}$$

Then, the sorting result is achieved by comparing those comprehensive dependent degrees of all the alternatives.

Step 6. Stability analysis. By selecting different distribution functions in Definition 4, the stability analysis is performed to the decision results. It can be seen whether the sorting result will change under different distributions of truth-membership values.

5. An Illustrative Example

In this section, an example of MCGDM problems is provided to illustrate the feasibility, reliability, and effectiveness of the proposed method.

Consider a MCGDM problem adapted from Reference [47]. There is a company which wants to choose a suitable supplier as its long-term partner. The expert set $D = \{d_1, d_2, d_3\}$ is composed of three experts with their weight vector being $\lambda = (0.4, 0.3, 0.3)$. There are four suppliers comprising set $S = \{s_1, s_2, s_3, s_4\}$. Each supplier has been evaluated on four criteria denoted by supplier set $C = \{c_1, c_2, c_3, c_4\}$ that includes product quality c_1, production capacity c_2, after-sales service c_3, and management ability c_4. The weight vector of the criteria is given as $w = (0.27, 0.27, 0.27, 0.19)$. For each expert, the four possible alternatives are evaluated under all the criteria. The evaluation values are in the form of SNNs $r_{ij}^k (i = 1, 2, 3, 4; j = 1, 2, 3, 4; k = 1, 2, 3)$, as shown in the following Tables 1–3:

Table 1. Evaluation data of expert d_1.

	c_1	c_2	c_3	c_4
s_1	$< 0.65, 0.10, 0.25 >$	$< 0.50, 0.18, 0.32 >$	$< 0.68, 0.12, 0.20 >$	$< 0.50, 0.10, 0.25 >$
s_2	$< 0.83, 0.12, 0.05 >$	$< 0.65, 0.15, 0.20 >$	$< 0.50, 0.10, 0.40 >$	$< 0.67, 0.18, 0.15 >$
s_3	$< 0.67, 0.13, 0.20 >$	$< 0.50, 0.15, 0.35 >$	$< 0.68, 0.12, 0.20 >$	$< 0.50, 0.20, 0.30 >$
s_4	$< 0.66, 0.14, 0.20 >$	$< 0.50, 0.16, 0.34 >$	$< 0.70, 0.10, 0.20 >$	$< 0.50, 0.15, 0.35 >$

Table 2. Evaluation data of expert d_2.

	c_1	c_2	c_3	c_4
s_1	$< 0.90, 0.02, 0.08 >$	$< 0.10, 0.10, 0.80 >$	$< 0.15, 0.15, 0.70 >$	$< 0.10, 0.05, 0.85 >$
s_2	$< 0.75, 0.15, 0.10 >$	$< 0.85, 0.05, 0.10 >$	$< 0.50, 0.10, 0.40 >$	$< 0.68, 0.10, 0.22 >$
s_3	$< 0.50, 0.05, 0.45 >$	$< 0.40, 0.15, 0.45 >$	$< 0.68, 0.12, 0.20 >$	$< 0.15, 0.05, 0.80 >$
s_4	$< 0.50, 0.10, 0.40 >$	$< 0.50, 0.10, 0.40 >$	$< 0.60, 0.10, 0.30 >$	$< 0.50, 0.05, 0.45 >$

Table 3. Evaluation data of expert d_3.

	c_1	c_2	c_3	c_4
s_1	$< 0.65, 0.15, 0.20 >$	$< 0.30, 0.10, 0.60 >$	$< 0.65, 0.20, 0.15 >$	$< 0.50, 0.10, 0.40 >$
s_2	$< 0.85, 0.05, 0.10 >$	$< 0.85, 0.05, 0.10 >$	$< 0.34, 0.16, 0.50 >$	$< 0.60, 0.10, 0.30 >$
s_3	$< 0.61, 0.18, 0.21 >$	$< 0.67, 0.13, 0.20 >$	$< 0.68, 0.22, 0.10 >$	$< 0.30, 0.10, 0.60 >$
s_4	$< 0.62, 0.28, 0.10 >$	$< 0.68, 0.22, 0.10 >$	$< 0.68, 0.12, 0.20 >$	$< 0.50, 0.10, 0.40 >$

5.1. The Decision Making Procedure

In this case, some main parameter values of the proposed method are explained here. For comparison, two simplified dependent functions, which include linear function Equation (5) and nonlinear function Equation (6) with parameter α initialized to 2.0, are adopted. At this point, the function Equation (6) shows a moderate curve. At the beginning, the distribution function uses triangular distribution as in Figure 3b. For the final stability test and uncertainty test, the parameter α and distribution function will change.

Step 1. Normalize the decision matrix. Because all the criteria are maximizing criteria, all the SNNs should remain unchanged.

Step 2. Interval transformation. By interval transformation operator Z, the corresponding subinterval is obtained as $X_0^{r_{ij}^1}$, $X_0^{r_{ij}^2}$, and $X_0^{r_{ij}^3}$ in Tables 4–6.

Table 4. The corresponding subintervals of simplified neutrosophic numbers (SNNs) of d_1.

	c_1	c_2	c_3	c_4
$X_0^{r_{1j}^1}$	$[0.650, 0.750]$	$[0.500, 0.680]$	$[0.680, 0.800]$	$[0.588, 0.706]$
$X_0^{r_{2j}^1}$	$[0.830, 0.950]$	$[0.650, 0.800]$	$[0.500, 0.600]$	$[0.670, 0.850]$
$X_0^{r_{3j}^1}$	$[0.670, 0.800]$	$[0.500, 0.650]$	$[0.680, 0.800]$	$[0.500, 0.700]$
$X_0^{r_{4j}^1}$	$[0.660, 0.800]$	$[0.500, 0.660]$	$[0.700, 0.800]$	$[0.500, 0.650]$

Table 5. The corresponding subintervals of SNNs of d_2.

	c_1	c_2	c_3	c_4
$X_0^{r_{1j}^2}$	$[0.900, 0.920]$	$[0.100, 0.200]$	$[0.150, 0.300]$	$[0.100, 0.150]$
$X_0^{r_{2j}^2}$	$[0.750, 0.900]$	$[0.850, 0.900]$	$[0.500, 0.600]$	$[0.680, 0.780]$
$X_0^{r_{3j}^2}$	$[0.500, 0.550]$	$[0.400, 0.550]$	$[0.680, 0.800]$	$[0.150, 0.200]$
$X_0^{r_{4j}^2}$	$[0.500, 0.600]$	$[0.500, 0.600]$	$[0.600, 0.700]$	$[0.500, 0.550]$

Table 6. The corresponding subintervals of SNNs of d_3.

	c_1	c_2	c_3	c_4
$X_0^{r_{1j}^3}$	$[0.650, 0.800]$	$[0.300, 0.400]$	$[0.650, 0.850]$	$[0.500, 0.600]$
$X_0^{r_{2j}^3}$	$[0.850, 0.900]$	$[0.850, 0.900]$	$[0.340, 0.500]$	$[0.600, 0.700]$
$X_0^{r_{3j}^3}$	$[0.610, 0.790]$	$[0.670, 0.800]$	$[0.680, 0.900]$	$[0.300, 0.400]$
$X_0^{r_{4j}^3}$	$[0.620, 0.900]$	$[0.680, 0.900]$	$[0.680, 0.800]$	$[0.500, 0.600]$

Step 3. Select the simplified dependent function and distribution function. To facilitate a clearer comparison, a linear simplified dependent function as in Equation (5) and a nonlinear simplified dependent function as in Equation (6) $(\alpha = 2.0)$ are selected. In addition, triangular distribution as in Figure 3b is used as the distribution function.

Step 4. Calculate interval dependent degree of each SNN of the decision matrix. According to Equation (18), the dependent degrees $k(r_{ij}^k)$ of SNNs are obtained as in Tables 7 and 8, in which Table 7 shows the dependent degrees with linear simplified dependent function as in Equation (5), and Table 8 shows those with nonlinear simplified dependent function as in Equation (6).

Table 7. Dependent degree with linear function of SNNs.

	Expert		
	d_1	d_2	d_3
$k(r_{1j}^k)$	0.7000, 0.5900, 0.7400, 0.6471	0.9100, 0.1500, 0.2250, 0.1250	0.7250, 0.3500, 0.7500, 0.5500
$k(r_{2j}^k)$	0.8900, 0.7250, 0.5500, 0.7600	0.8250, 0.8750, 0.5500, 0.7300	0.8750, 0.8750, 0.4200, 0.6500
$k(r_{3j}^k)$	0.7350, 0.5750, 0.7400, 0.6000	0.5250, 0.4750, 0.7400, 0.1750	0.7000, 0.7350, 0.7900, 0.3500
$k(r_{4j}^k)$	0.7300, 0.5800, 0.7500, 0.5750	0.5500, 0.5500, 0.6500, 0.5250	0.7600, 0.7900, 0.7400, 0.5500

Table 8. Dependent degree with nonlinear function of SNNs.

	Expert		
	d_1	d_2	d_3
$k(r_{1j}^k)$	$0.8234, 0.7415, 0.8503, 0.7855$	$0.9529, 0.2603, 0.3663, 0.2221$	$0.8402, 0.5182, 0.8565, 0.7095$
$k(r_{2j}^k)$	$0.9416, 0.8402, 0.7095, 0.8631$	$0.9038, 0.9333, 0.7095, 0.8438$	$0.9333, 0.9333, 0.5908, 0.7877$
$k(r_{3j}^k)$	$0.8470, 0.7297, 0.8503, 0.7492$	$0.6885, 0.6435, 0.8503, 0.2977$	$0.8230, 0.8470, 0.8820, 0.5182$
$k(r_{4j}^k)$	$0.8436, 0.7336, 0.8570, 0.7297$	$0.7095, 0.7095, 0.7877, 0.6885$	$0.8624, 0.8820, 0.8503, 0.7095$

Step 5. Calculate the comprehensive dependent degree of each alternative. According to Equation (19), experts weight vector λ and criteria weight vector w, there are comprehensive dependent degree $K(a_i)$ of each alternative a_i as in Table 9. Table 9 shows the values of $K(a_i)$ under different simplified dependent functions which include linear function and nonlinear functions with $\alpha = 3.0$, $\alpha = 2.0$, $\alpha = 1.5$, and $\alpha = 1.2$. As we can see, although the dependent degrees under different dependent functions are different from each other, the sorting result remains unchanged. In fact, the simplified dependent function can reflect the psychology status of decision makers. For example, Equation (6) describes risk aversion psychology, which means the curve slope will change with different evaluation values. The smaller the parameter α, the greater the extent of regret evasion from decision makers. Nevertheless, the result shows that it is not influenced by the risk aversion psychology changing in the decision makers. So it exhibits high stability.

Table 9. The comprehensive dependent degree of alternative set S.

	s_1	s_2	s_3	s_4	Sorting Result
$K(a_i)$ (linear)	0.5591	0.7276	0.6193	0.6549	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (nonlinear $\alpha = 3.0$)	0.6325	0.7928	0.6995	0.7367	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (nonlinear $\alpha = 2.0$)	0.6811	0.8322	0.7500	0.7868	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (nonlinear $\alpha = 1.5$)	0.7434	0.8778	0.8111	0.8453	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (nonlinear $\alpha = 1.2$)	0.8323	0.9321	0.8886	0.9148	$s_2 \succ s_4 \succ s_3 \succ s_1$

Step 6. Uncertainty analysis. Tables 10 and 11 show the sorting result under different distribution functions as in Figure 3. Although the dependent degree values are slightly different as the distribution function changes, the result remain unchanged, which illustrates the lower uncertainty and sensibility of the ranking result. Therefore, for decision makers in this case, the sorting result is sufficiently certain and stable.

Table 10. The sorting result with different distribution functions.

$\alpha = 2.0$	s_1	s_2	s_3	s_4	Sorting Result
$K(a_i)$ (uniform)	0.6806	0.8318	0.7496	0.7864	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (normal)	0.6793	0.8300	0.7481	0.7848	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (triangular)	0.6811	0.8322	0.7500	0.7868	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (appro-triangular)	0.6808	0.8320	0.7498	0.7866	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (trapezoid 1)	0.6864	0.8362	0.7553	0.7919	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (trapezoid 2)	0.6748	0.8276	0.7439	0.7809	$s_2 \succ s_4 \succ s_3 \succ s_1$

Table 11. The sorting result with different distribution functions.

$\alpha = 1.2$	s_1	s_2	s_3	s_4	Sorting Result
$K(a_i)$ (uniform)	0.8316	0.9319	0.8882	0.9144	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (normal)	0.8303	0.9297	0.8864	0.9124	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (triangular)	0.8323	0.9321	0.8886	0.9148	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (appro-triangular)	0.8318	0.9319	0.8884	0.9145	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (trapezoid 1)	0.8356	0.9339	0.8911	0.9169	$s_2 \succ s_4 \succ s_3 \succ s_1$
$K(a_i)$ (trapezoid 2)	0.8234	0.9283	0.8844	0.9136	$s_2 \succ s_4 \succ s_3 \succ s_1$

5.2. A Comparison Analysis

In order to verify the effectiveness of the proposed method based on the interval dependent degrees of SNNs, a comparison analysis was conducted. Several methods in Reference [42,45–48] were used on the above example and the same sorting results as in Table 12 were obtained, which is consistent with that derived from the proposed method. It indicates the effectiveness and feasibility of the proposed measure. However, this study presents a new method with maintaining uncertain and fuzzy information of SNNs in the algorithm process, while previous methods only consider three values of an SNN while ignoring its latent uncertain information. That makes the proposed method capable of uncertainty and stability analysis. From computational complexity, most previous methods need to take aggregation operations for three values of a SNN, respectively, in entire algorithm steps, while the proposed model only takes interval number operations throughout the process. Interestingly, although this study will transform SNN to an interval number, it avoids information loss and incompleteness by defining distribution functions. In summary, the advantages over the other methods are summarized below.

(1) In the proposed approach, the interval transformation operator is developed to convert SNNs into interval numbers, which avoids various complex aggregation operator processes for SNNs. The transformation operator is simple and convenient to perform. Then, the following process is built on the interval number which is relatively straightforward and understandable.

(2) As a result of the various kinds of aggregation operators, most previous methods considered will produce many intermediate results in neutrosophic MCGDM problems. However, in the proposed approach, the key parameters and main steps are directly integrated into the dependent function expression. Therefore, the proposed method takes less intermediate results and is more concise.

(3) In the proposed approach, the distribution function is conducted to describe latent uncertain information for SNNs. In this way, the fuzziness of the original information can be conserved and fully utilized which can be used to take some uncertainty analysis for the decision result. For decision makers, not only the sorting result is obtained, but the dynamic influences on the sorting result caused by any uncertainty in the decision environment will also be observed, which cannot be provided by the others methods. Therefore, the final ranking of the proposed approach is more conclusive and accurate.

Table 12. A comparison of different methods.

	Sorting Result	Best One	Worst One
SNNWA [42]	$s_2 \succ s_4 \succ s_3 \succ s_1$	s_2	s_1
Entropy of Euclidean [48]	$s_2 \succ s_4 \succ s_3 \succ s_1$	s_2	s_1
SNEE [46]	$s_2 \succ s_4 \succ s_3 \succ s_1$	s_2	s_1
SNNCI [47]	$s_2 \succ s_4 \succ s_3 \succ s_1$	s_2	s_1
GSNNWA [45]	$s_2 \succ s_4 \succ s_3 \succ s_1$	s_2	s_1
the proposed method	$s_2 \succ s_4 \succ s_3 \succ s_1$	s_2	s_1

6. Conclusions

In this paper, a novel method and framework based on the interval dependent degree of SNNs for MCGDM problems is proposed. Firstly, the interval dependent function is defined, in which the distribution function is used to describe inherent distribution information of a SNN. Subsequently, a transformation operator is constructed to convert SNNs into interval numbers, and then the interval dependent function for SNNs is build. Afterwards, the sorting result is obtained by computing and comparing the comprehensive dependent degree of each alternative. Finally, the uncertainty and stability analysis method for the result is given.

The proposed approach is convenient to perform, and is effective at decreasing original information loss. Its validity and feasibility also have been verified by an illustrative example and comparative analysis. The advantages over the other methods are demonstrated in the comparative analysis section. Through its uncertainty and stability analysis, the method can provide more reliable, persuasive, and accurate results. The proposed method not only provides a novel way of solving MCDM problems with simplified neutrosophic sets, but also enriches the theory of neutrosophic sets.

MCDM problems exist widely in many industrial and social application situations, such as medical diagnosis, investment decision, supplier selection, etc. To choose the appropriate solution, people often have to evaluate the effects of multiple criteria. Usually, the evaluation values are given not as a certain value but some degrees of truth, falsity, and indeterminacy which can be adequately described by SNNs. Moreover, the truth degree of a decision maker often covers a range and does not obey uniform distribution in the range. The proposed method, which can provide a more concise and comprehensive way of solving these problems, shows broad application prospects.

In the future, the proposed method will be extended to the other neutrosophic sets such as interval neutrosophic sets (INSs) [49], multi-valued neutrosophic sets (MVNSs) [50,51], and complex neutrosophic sets (CNSs) [52], etc. Further study as regards some complete uncertainty situations in which both the criteria and the weights are denoted as SNNs is also necessary. In addition, we consider exploring its possible applications in some non-traditional areas such as the game theory [53], which has become a new effective method for solving MCDM problems in recent years because of its non-linear dynamics description capability [54].

Author Contributions: L.X. conceived of the presented idea, verified the analytical methods and wrote the draft; X.L. started the literature findings, got data and discussed the findings of this work; C.P. conceived background study, discussed the findings and monitored the results; Y.G. checked and verified the mathematical models, the tables and figure.

Funding: This research is funded by Zhejiang Science Foundation Project of China under grant number LY16G010010, LY18F020001.

Acknowledgments: The authors would like to thank the editors and the reviewers for their comments on a draft of this article.

Conflicts of Interest: The authors declare that they do not have any conflict of interests.

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Neutrosophic Computability and Enumeration

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Received: 28 October 2018; Accepted: 13 November 2018; Published: 16 November 2018

Abstract: We introduce oracle Turing machines with neutrosophic values allowed in the oracle information and then give some results when one is permitted to use neutrosophic sets and logic in relative computation. We also introduce a method to enumerate the elements of a neutrosophic subset of natural numbers.

Keywords: computability; oracle Turing machines; neutrosophic sets; neutrosophic logic; recursive enumerability; oracle computation; criterion functions

1. Introduction

In classical computability theory, algorithmic computation is modeled by Turing machines, which were introduced by Alan M. Turing [1]. A *Turing machine* is an abstract model of computation defined by a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F, \{L, R\})$, where Q is a finite set of states, Σ is the alphabet, Γ is the tape alphabet, $q_0 \in Q$ is the starting state, $F \subset Q$ is a set of halting states, the set $\{L, R\}$ denotes the possible left (L) and right (R) move of the tape head, and δ is the transition function, defined as:

$$\delta : Q \times \Gamma \to Q \times \Sigma \times \{L, R\}.$$

Each transition is a step of the computation. Let w be a string over the alphabet Σ. We say that a Turing machine on input w *halts* if the computation ends with some state $q \in F$. The output of the machine, in this case, is whatever was written on the tape at the end of the computation. If a Turing machine M on input w halts, then we say that M is defined on w. Since there is a one-to-one correspondence between the set of all finite strings over Σ and the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, without loss of generality we may assume that Turing machines are defined from $\mathbb{N}$ to $\mathbb{N}$.

Standard Turing machines admit partial functions, i.e., functions that may not be defined on every input. The class of functions computable by Turing machines are called *partial recursive (computable)* functions. We shall not delve into the details about what is meant by a function or set that is computable by a Turing machine. We assume that the reader is familiar with the basic terminology. However, for a detailed account, the reader may refer to Reference [2–4]. Using a well known method called *Gödel numbering*, originated from Gödel's celebrated 1931 paper [5], it is possible to have an algorithmic enumeration of all partial recursive functions. We let Ψ_i denote the ith partial recursive function, i.e., the ith Turing machine.

If a partial recursive function is defined on every argument we say that it is *total*. Total recursive functions are simply called *recursive* or *computable*. Since there are countable infinitely many Turing machines, there are countable infinitely many computable functions. Computable sets and functions are widely used in mathematics and computer science. However, nearly all functions are non-computable.

Since there are $2^{\aleph_0}$ functions from $\mathbb{N}$ to $\mathbb{N}$ and only $\aleph_0$ many computable functions, there are uncountably many non-computable functions.

Oracle Turing machines, introduced by Alan Turing [6], are used for relativizing the computation with respect to a given set of natural numbers. An *oracle Turing machine* is a Turing machine with an extra *oracle tape* containing the characteristic function of a given set of natural numbers. The *characteristic function* of a set $S \subset \mathbb{N}$ is defined as:

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

We may also think of χ_S as an infinite binary sequence and call it the *characteristic sequence* of S. For a set $S \subset \mathbb{N}$, we let $S(i)$ denote $\chi_S(i)$. So the characteristic sequence of a set S simply gives the membership information about natural numbers regarding S. For a given an oracle Turing machine with the characteristic sequence of a set S provided in the oracle tape, functions are denoted as computable by the machine relative to the *oracle* S. If the oracle Turing machine with an oracle S computes a function f, then we say that f is *computable in* S or we say S *computes* f. We denote the ith oracle Turing machine with an oracle A by $\Psi_i(A)$. Then, it makes sense to write $\Psi_i(A) = B$ if A computes B.

Now we shall look at a non-standard Turing machine model based on neutrosophic sets. Neutrosophic logic, first introduced by Smarandache [7,8], is a generalization of classical, fuzzy and intuitionistic fuzzy logic. The key assumption of neutrosophy is that every idea not only has a certain degree of truth, as is generally taken in many-valued logic contexts, but also has degrees of falsity and indeterminacy, which need to be considered independently from each other. A neutrosophic set relies on the idea that there is a degree of probability that an element is a member of the given set, a degree that the very same element is *not* a member of the set, and a degree that the membership of the element is indeterminate for the set. For our purpose we take subsets of natural numbers. Roughly speaking, if n were a natural number and if A were a neutrosophic set, then there would be a probability distribution $p_{\in}(n) + p_{\notin}(n) + p_I(n) = 1$, where $p_{\in}(n)$ denotes the probability of n being a member of A, $p_{\notin}(n)$ denotes the probability of n not being a member of A, and $p_I(n)$ denotes the degree of probability that the membership of n is indeterminate in A. Since the probability distribution is expected to be normalized, the summation of all probabilities must be equal to unity. We should note however that the latter requirement can be modified depending on the application.

The above interpretation of a neutrosophic set can be in fact generalized to any multi-dimensional collection of attributes. That is, our attributes did not need to be merely about membership, non-membership, and indeterminacy, but it could range over any finite set of attributes $a_0, a_1, \ldots, a_k$ and $b_0, b_1, \ldots b_k$ so that the value of an element would range over (x, y, I) such that $x \in a_m$ and $y \in b_m$ for $0 \leq m \leq k$. The set of attributes can also be countably infinite or even uncountable. However, we are not concerned with these cases. We shall only consider the membership attribute discussed above.

We are particularly interested in subsets of natural numbers $A \subset \mathbb{N}$, in our study. Any neutrosophic subset A of natural numbers (we shall occasionally denote such a set by A^N) is defined in the form of ordered triplets:

$$\{\langle p_{\in}(0), p_{\notin}(0), p_I(0)\rangle, \langle p_{\in}(1), p_{\notin}(1), p_I(1)\rangle, \langle p_{\in}(2), p_{\notin}(2), p_I(2)\rangle, \ldots\},$$

where, for each $i \in \mathbb{N}$, $p_{\in}(i)$ denotes the degree of probability of i being an element of A, $p_{\notin}(i)$ denotes the probability of i being not an element of A, and $p_I(i)$ denotes the probability of i being indetermined. Since we assume a normalized probability distribution, we have that for every $i \in \mathbb{N}$:

$$p_{\in}(i) + p_{\notin}(i) + p_I(i) = 1.$$

2. Oracle Turing Machines with Neutrosophic Values

Now we can extend the notion of relativized computation based on neutrosophic sets and neutrosophic logic. For this we introduce oracle Turing machines with neutrosophic oracle tape. The general idea is as follows. Standard oracle tape contains the information of the characteristic sequence of a given set $A \subset \mathbb{N}$. We extend the definition of the characteristic function to neutrosophic sets as follows.

Definition 1. *Let $A \subset \mathbb{N}$ be a set. A neutrosophic oracle tape is a countably infinite sequence $t_0, t_1, \ldots$ where $t_i = \langle a, b, c \rangle$ is an ordered triplet and $a, b, c \in \mathbb{Q}$, so that a is the probability value of i such that $i \in A$, b is the probability value of i such that $i \notin A$, and c is the probability of i being indeterminate for A.*

The overall picture of a neutrosophic oracle tape can be seen in Figure 1. Now we need to modify the notion of the characteristic sequence accordingly.

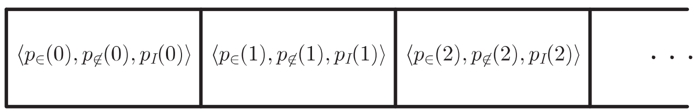

Figure 1. Neutrosophic oracle tape.

Definition 2. *Let $S \subset \mathbb{N}$ be a set and let B denote the blank symbol in the alphabet of the oracle tape. The neutrosophic characteristic function of S is defined by*

$$\chi_S^N(x) = \begin{cases} \langle 1, 0, I \rangle & \text{if } p_\in(x) > 0 \text{ and } p_\notin(x) > 0 \text{ and } p_I(x) > 0 \\ \langle B, 0, I \rangle & \text{if } p_\in(x) = 0 \text{ and } p_\notin(x) > 0 \text{ and } p_I(x) > 0 \\ \langle B, B, I \rangle & \text{if } p_\in(x) = 0 \text{ and } p_\notin(x) = 0 \text{ and } p_I(x) > 0 \\ \langle B, B, B \rangle & \text{if } p_\in(x) = 0 \text{ and } p_\notin(x) = 0 \text{ and } p_I(x) = 0 \\ \langle 1, B, I \rangle & \text{if } p_\in(x) > 0 \text{ and } p_\notin(x) = 0 \text{ and } p_I(x) > 0 \\ \langle 1, B, B \rangle & \text{if } p_\in(x) > 0 \text{ and } p_\notin(x) = 0 \text{ and } p_I(x) = 0 \\ \langle 1, 0, B \rangle & \text{if } p_\in(x) > 0 \text{ and } p_\notin(x) > 0 \text{ and } p_I(x) = 0 \\ \langle B, 0, B \rangle & \text{if } p_\in(x) = 0 \text{ and } p_\notin(x) > 0 \text{ and } p_I(x) = 0 \end{cases}$$

The idea behind this definition is to label the distributions which have significant probability value with respect to a pre-determined probability threshold value, in this case we assume this value to be 0 by default. Note that this threshold value could be defined for any $r \in \mathbb{Q}$ so that instead of being greater than 0, we would require the probability for that attribute to be greater than r in order to be labelled. We will talk about the properties of defining an arbitrary threshold value and its relation to neutrosophic computations in the next section.

Definition 3. *A neutrosophic oracle Turing machine is a Turing machine with an additional neutrosophic oracle tape $(Q, \Sigma, \Gamma, \Gamma', \delta, q_0, F, \{L, R\})$, where Q is a finite set of states, Σ is the alphabet, Γ is the tape alphabet, Γ' is the neutrosophic oracle tape alphabet containing the blank symbol B, $q_0 \in Q$ is the starting state, $F \subset Q$ is a set of halting states, the set $\{L, R\}$ denotes the possible left (L) and right (R) move of the tape head, and δ is the transition function defined as:*

$$\delta : Q \times \Gamma \times \Gamma' \to Q \times \Sigma \times \{L, R\}^2.$$

Theorem 1. *Any neutrosophic oracle Turing machine can be simulated by a standard Turing machine.*

Proof. Assuming the Church–Turing thesis in the proof, we only need to argue that standard oracle tapes can in theory represent neutrosophic oracle tapes. In fact any neutrosophic oracle tapes can be represented by three standard oracle tapes each of which contains one and only one attribute. The *i*th cell of the first oracle tape contains the probability value $p_{\in}(i)$. The *i*th cell of the second oracle tape contains the value $p_{\notin}(i)$. Similarly, the *i*th cell of the third oracle tape contains $p_I(i)$.

We also need to argue that a three-tape oracle standard Turing machine can be simulated by a single tape oracle Turing machine. Let Γ be the oracle alphabet. We define an extension Γ' of Γ by introducing a delimeter symbol # to separate each attribute for a given number *i*. We define another delimiter symbol $\perp$ to separate each $i \in \mathbb{N}$. Let $\Gamma' = \Gamma \cup \{\#, \perp\}$. Then, a neutrosophic oracle tape can be represented by a single oracle tape with the tape alphabet Γ'. The oracle tape will be in the form:

$$p_{\in}(0)\#p_{\notin}(0)\#p_I(0)\perp p_{\in}(1)\#p_{\notin}(1)\#p_I(1)\perp\ldots$$

The symbol $\perp$ determines a counter for *i*, whereas for each *i*, the symbol # determines a counter for the attribute. □

A neutrosophic set *A* *computes* another neutrosophic set *B* if using finitely many pieces of information of the characteristic sequence of *A* determines the *i*th entry of the characteristic sequence of *B* given any index $i \in \mathbb{N}$. Then, based on this definition, a set $B \subset \mathbb{N}$ is *neutrosophically computable* in *A* if $B = \Psi_e^N(A)$ for some $e \in \mathbb{N}$, where Ψ_e^N denotes the *e*-th neutrosophic oracle Turing machine. If $B = \Psi_e^N(A)$ for some $e \in \mathbb{N}$, we denote this by $B \leq_N A$. If $B \leq_N A$ and $A \leq_N B$, then we say that *A* and *B* are *neutrosophically equivalent* and denote this by $A \equiv_N B$. Intuitively, $A \equiv_N B$ means that *A* and *B* are neutrosophic subsets of natural numbers, and they have the same level of neutrosophic information complexity. We leave the discussion on the properties of the equivalence classes induced by $\equiv_N$ for another study as it is beyond the scope of this paper.

3. Neutrosophic Enumeration and Criterion Functions

We now introduce the concept of neutrosophic enumeration of the members of neutrosophic subsets of natural numbers. Since we talk about enumeration, we must only take countable sets into consideration. It is known from classical computability that, given a set $A \subset \mathbb{N}$, *A* is called *recursively enumerable* if there exists some $e \in \mathbb{N}$ such that *A* is the domain of Ψ_e. We want to define the neutrosophic counterpart of this notion, but we need to be careful about the indeterminate cases, an intrinsic property in neutrosophic logic.

Definition 4. *A set A is called neutrosophic Turing enumerable if there exists some $e \in \mathbb{N}$ such that A is the domain of Ψ_e^N restricted to elements whose probability degree of membership is greater than a given probability threshold. More precisely, if $r \in \mathbb{Q}$ is a given probability threshold, then A is neutrosophic Turing enumerable if A is the domain of $\Psi_e^N(\emptyset)$ restricted to those elements i such that $p_{\in}^A(i) \geq r$, where $p_{\in}^A(i)$ denotes the degree of probability of membership of i in A.*

If the *e*th Turing machine is defined on the argument *i*, we denote this by $\Psi_e(i)\downarrow$. The *halting set* in classical computability theory is defined as:

$$K = \{e : \Psi_e(e)\downarrow\}.$$

It is known that *K* is recursively enumerable but not recursive. Unlike in classical Turing computability, we show that neutrosophically computable sets allow us to neutrosophically compute the halting set. The way to do this goes as follows. A single neutrosophic subset of natural numbers is not enough to compute the halting set. Instead, we take the union of all neutrosophically computable subsets of natural numbers by taking an infinite join which will code the information of the halting set.

Let $\{A_i\}_{i\in\mathbb{N}}$ be a countable sequence of subsets of $\mathbb{N}$. The *infinite join* is defined by

$$\bigoplus\{A_i\} = \{\langle i, x\rangle : x \in A_i\},$$

where $\langle i, j\rangle$ is mapped to a natural number using a uniform pairing function $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$.

Theorem 2. *Let A^N be a neutrosophic subset of $\mathbb{N}$. Then, $\bigoplus\{A_i^N\} \equiv_N K$*

Proof. We first show that $\bigoplus\{A_i^N\} \geq_N K$. The infinite join of all neutrosophically computable sets computes the halting set. Let $\{\Psi_i^N\}_{i\in\mathbb{N}}$ be an effective enumeration of neutrosophic Turing functionals. Let $\{A_i^N\}$ be the corresponding neutrosophic sets, each of which is computable by Ψ_i^N. To compute K, we let $\bigoplus\{A_i^N\}$ be the infinite join of all A_i^N. To know whether $\Psi_i(i)$ is defined or not, we see if $p_\in(i) + p_\notin(i) > 0.5$. If so, then $\Psi_i(i)\downarrow$. Otherwise it must be that $p_I(i) > 0.5$. In this case, $\Psi_i(i)$ is undefined.

Next, we show $\bigoplus\{A_i^N\} \leq_N K$. To prove this, we assume that there exists an oracle for K. If $i \in K$, then there exist indices $x, y \in \mathbb{N}$ such that $\langle x, y\rangle = i$ and it must be that $p_\in(i) + p_\notin(i) > 0.5$ since $\Psi_i(i)$ is defined, but we may not know whether $i \in A_i$ or $i \notin A_i$. If $i \notin K$, then the same argument holds to prove this case as well. □

The use of the probability ratio 0.5 is for convenience. This notion will be generalized later on. Classically speaking, given a subset of natural numbers, we can easily convert it to a neutrosophic set preserving the membership information of the given classical set. Suppose that we are given a set $A \subset \mathbb{N}$ and we want to convert it to a neutrosophic set with the same characteristic sequence. The neutrosophic counterpart A^N is defined, for each $i \in \mathbb{N}$, as:

$$A^N(i) = \begin{cases} \langle i, 1-i, 0\rangle & \text{if } A(i) = 1 \\ \langle 1-i, i, 0\rangle & \text{otherwise.} \end{cases}$$

We now introduce the tree representation of neutrosophic sets and give a method, using trees, to approximate its classical counterpart. Suppose that we are given a neutrosophic subset of natural numbers in the form:

$$A^N = \{\langle p_\in(i), p_\notin(i), p_I(i)\rangle\}_{i\in\mathbb{N}}.$$

We use the probability distribution to decide which element will be included in the classical counterpart. If A^N is a neutrosophic subset of natural numbers, the *classical counterpart of A^N* is defined as:

$$A(i) = \begin{cases} 1 & \text{if } p_\in(i) > p_\notin(i) \\ 0 & \text{if } p_\in(i) < p_\notin(i). \end{cases}$$

Now we introduce a simple conversion using trees. The aim is to approximate to the classical counterpart of a given neutrosophic set A^N in a computable fashion. For this we start with a full ternary tree, as given in Figure 2, coding all possible combinations.

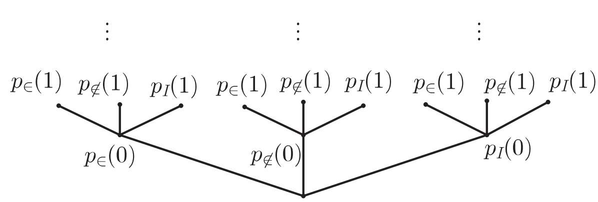

Figure 2. Approximating a neutrosophic set with a classical set through a ternary tree.

The correct interpretation of this tree is as follows. Each branch represent a possible element of the set we want to construct. For instance, if $p_{\in}(0)$ has the largest probability value among $p_{\in}(0)$, $p_{\notin}(0)$, $p_I(0)$, then we choose $p_{\in}(0)$ and define 0 to be an element of the classical set we construct. If either $p_{\notin}(0)$ or $p_I(0)$ is greater than $p_{\in}(0)$, then we know 0 is not an element of the constructed set. Since we are defining a classical set, the only time when some $i \in \mathbb{N}$ is in the constructed set is if $p_{\in}(i) > p_{\notin}(i)$ and $p_{\in}(i) > p_I(i)$. Continuing along this line, if $p_{\notin}(1)$, say, has the greatest probability value among $p_{\in}(1)$, $p_{\notin}(1)$, $p_I(1)$, then 1 will not be an element. So far, 0 is an element and 1 is not an element. So in the tree we choose the leftmost branch and then next we choose the middle branch. Repeating this procedure for every $i \in \mathbb{N}$, we end up defining a computable infinite path on this ternary tree which defines elements of the set being constructed. At each step, we simply take the maximum probability value and select that attribute. The infinite path defines a computable approximation to the classical counterpart of A^N using the tree method.

Earlier we defined Turing machines with a neutrosophic oracle tape. Suppose that the characteristic sequence of a neutrosophic set A can be considered as an oracle. Then, the e-th neutrosophic oracle Turing machine can compute a function of the same characteristic. That is, not only can Turing machines with neutrosophic oracles compute classical sets, but they can also compute neutrosophic sets. It is important to note that we need to modify the definition of standard oracle Turing machines in order to use neutrosophic sets. We add the symbol I to the alphabet of the oracle tape. The transition function δ^N is then defined as:

$$\delta^N : Q \times \Sigma \times \Gamma \to Q \times \Sigma \cup \{I\} \times \{L, R\}^2.$$

We say that the neutrosophic oracle Turing machine, say Ψ_e^N, *computes* a neutrosophic set B if $\Psi_e^N = B$.

We now turn to the problem of enumerating members of a neutrosophic subset A of natural numbers. Normally, general intuition suggests that we pick elements $i \in \mathbb{N}$ such that $p_{\in}(i) > 0.5$. It is important to note that, given $A = \{\langle p_{\in}(i), p_{\notin}(i), p_I(i)\rangle\}_{i\in\mathbb{N}}$, not every i will be enumerated if we use this probability criterion. However, changing the criterion depending on what aspect of the set we want to look at and depending on the application, would also change the enumerated set. Therefore, we would need a kind of criterion function to set a probability threshold regarding which elements of the neutrosophic set are to be enumerated.

In practice, one often encounters a situation where the given information is not directly used but rather analyzed under the criterion determined by a function. We examine how the computation behaves when we impose a function on the neutrosophic oracle tape. That is, suppose that $f : \mathbb{N} \to \{a, b, c\}$ is a function, where $a, b, c \in \mathbb{Q}$, which maps each cell of the neutrosophic oracle tape to a probability value. For example, f could be defined as a constant non-membership function which assigns every triplet in the cells to the non-membership $\notin$ attribute. In this case, the probability of any natural number not being an element of the considered oracle A is just 1. When these kinds of functions are used in the oracle information of A, we may be able to compute some useful information.

The intuition in using criterion functions is to select, under a previously determined probability threshold, a natural number from the probability distribution which is available in a given neutrosophic subset of natural numbers. As an example, let us imagine a neutrosophic subset A of natural numbers. Suppose for simplicity that A is finite and is defined as:

$$A = \{\langle 0.1, 0.4, 0.5\rangle, \langle 0.6, 0.3, 0.1\rangle, \langle 0, 0.9, 0.1\rangle\}.$$

First of all, we should read this as follows: A has neutrosophic information about the first three natural numbers $0, 1, 2$. In this example, $p_{\in}(0) = 0.1$, $p_{\notin}(0) = 0.4$, $p_I(0) = 0.5$. For the natural number 1, we have that $p_{\in}(1) = 0.6$, $p_{\notin}(1) = 0.3$, $p_I(1) = 0.1$. Finally, for the natural number 2, we have $p_{\in}(2) = 0$, $p_{\notin}(2) = 0.9$, $p_I(2) = 0.1$. Now if we want to know which natural numbers are in A, normally we would only pick the number 2 since $p_{\in}(2) > 0.5$. Our criterion of enumeration in this case is 0.5. In general, this probability value may not be always applicable. Moreover, this probability threshold value may not be constant. That is, we may want to have a different probability threshold for every natural number i. If our criterion were to select the ith element whose probability exceeds p_i, we would enumerate those numbers. For example, if the criterion is defined as

$$f(0) = 0, \quad f(1) = 0.8, \quad f(2) = 0.2,$$

then for the first triple, the probability threshold is 0, meaning that we enumerate the natural number 0 if $p_\in(0) > 0$. Obviously 0 will be enumerated in this case since $p_\in(0) = 0.1 > 0$. The probability threshold for enumerating the number 1 is 0.8, so it will not be enumerated since $p_\in(1) = 0.6 < 0.8$. Finally, the probability threshold for enumerating the number 2 is 0.2. In this case, 2 will not be enumerated since $p_\in(2) = 0 < 0.2$. So the enumeration set for A under the criterion f will be $\{0\}$.

We are now ready to give the formal definition of a criterion function.

Definition 5. *A criteria function is a mapping $f : \mathbb{N} \to \mathbb{Q}$ which, given a neutrosophic subset A of natural numbers, determines a probability threshold for each triple $\langle p_\in(i), p_\notin(i), p_I(i)\rangle$ in A.*

We first note a simple observation that if the criterion function is the constant function $f(n) = 0$ for any $n \in \mathbb{N}$, the enumeration set will be equal to $\mathbb{N}$ itself. However, this does not mean that the enumeration set will be empty if $f(n) = 1$. Given a neutrosophic set A, if $p_\in(i) = 1$ for all i, then the enumeration set for A will also be equal to $\mathbb{N}$.

We shall next give the following theorem. First we remind the reader that we call a function f *strictly decreasing* if $f(i+1) < f(i)$.

Theorem 3. *Let f be a strictly decreasing criterion function for a neutrosophic set A such that $p_\in(i) < p_\in(i+1)$ for every $i \in \mathbb{N}$, and let $\mathcal{E}_A$ be the enumeration set for A under the criterion f. Then, there exists some $k \in \mathbb{N}$ such that $|\mathcal{E}_A| < k$.*

Proof. Clearly, given A and that for each i, $p_\in(i) < p_\in(i+1)$, only those numbers i which satisfy $p_\in(i) > f(i)$ will be enumerated. Since the probability distribution of membership degrees of elements of A strictly increases and f is strictly decreasing, there will be some number $j \in \mathbb{N}$ such that $p_\in(i) \leq f(j)$. Moreover, for the same reason $p_\in(m) \leq f(m)$ for every $m > j$. Therefore, the number of elements enumerated is less than j. That is, $|\mathcal{E}_A| < j$. □

We denote the complement of a neutrosophic subset of natural numbers A by A^c and we define it as follows. Let $p^A_\in(i)$ denote the probability of i being an element of A and let $p^A_\notin(i)$ denote the probability of i being not an element of A. In addition, $p^A_I(i)$ denotes the probability of the membership of i being indeterminate. Then:

$$A^c(i) = \langle p^A_\notin(i), p^A_\in(i), p^A_I(i)\rangle.$$

So the complement of a neutrosophic set in consideration is formed by simply interchanging the probabilities of membership and non-membership for all $i \in \mathbb{N}$. Notice that the probability of indeterminacy remains the same. Our next observation is as follows. Suppose that A and A^c are neutrosophic subsets of natural numbers and f is a criterion function. If $\mathcal{E}_A \subset \mathcal{E}_{A^c}$, then clearly $p_\notin(i) \geq p_\in(i)$ for all $i \in \mathbb{N}$.

The probability distribution of members of a neutrosophic set can be also be given by a function $g(i,j)$ such that $i \in \mathbb{N}$ and $j \in \{1,2,3\}$ where j is the index for denoting the *membership* probability by 1, *non-membership* probability by 2, and *indeterminacy* probability by 3, respectively. For instance, for $i \in \mathbb{N}$, $g(i,2)$ denotes the probability of the non-membership of i generated by the function g. Now g being a computable function means, for any i, j, there is an algorithm to find the value of $g(i,j)$. We give the following theorem.

Theorem 4. *Let A be a neutrosophic set. If g is a computable function, then there exists a computable criterion function f such that $\mathcal{E}_A$ is the enumeration set of A under the criterion function f and, moreover, $\mathcal{E}_A = \mathbb{N}$.*

Proof. Suppose that we are given A. If g is a computable function that generates the probability distribution of members of A, then we can computably find $g(i, 1) = k$. We then simply let $f(i)$ be some $m \leq k$. Since $p_{\in}(i) = k \geq f(i)$, every i will be a member of $\mathcal{E}_A$. Since i is arbitrary, $\mathcal{E}_A = \mathbb{N}$. □

We say that a function f *majorizes* a function g if $f(x) > g(x)$ for all x. Suppose now that g is a quickly growing function in the sense that it majorizes every computable function. That is, assume that $g(i, j) \geq f(i)$ for every $i, j \in \mathbb{N}$ and every computable function f. Now in this case g is necessarily non-computable. Otherwise we would be able to construct a function h where $h(i)$ is chosen to be some $s > t$ such that $g(i, j)$ is defined at step t. So if g is not computable, we cannot apply the previous theorem on g. The only way to enumerate A is by using relative computability rather than giving a plain computable procedure. Suppose that we are given such a function g. Let $\Psi_i(A; i)$ denote the ith Turing machine with oracle A and input i. We define $g' = \{x : \Psi_x(g; x) \downarrow\}$ to be the *jump* of g, where $x = \langle i, j \rangle$ for a uniform pairing function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. The jump of g is basically the halting set relativized to g. If we want to enumerate members of A, we can then use g' as an oracle. Since, by definition, g' computes g, we enumerate members of A computably *in* g'.

We shall also note an observation regarding the relationship between A and A^c. Given a function f, unless $f(i)$ is strictly between $p^A_{\in}(i)$ and $p^A_{\notin}(i)$, we have that $\mathcal{E}_A = \mathcal{E}_{A^c}$. That is, the only case when $\mathcal{E}_A \neq \mathcal{E}_{A^c}$ is if $p^A_{\in}(i) < f(i) < p^A_{\notin}(i)$ or $p^A_{\notin}(i) < f(i) < p^A_{\in}(i)$. Let us examine each case. In the first case, since $f(i) > p^A_{\in}(i)$, i will not be enumerated into $\mathcal{E}_A$, but since $p^{A^c}_{\in}(i) > f(i)$, it will be enumerated into $\mathcal{E}_{A^c}$. The second case is just the opposite. That is, i will be enumerated into $\mathcal{E}_A$ but not into $\mathcal{E}_{A^c}$.

What about the cases where i is enumerated into both enumeration sets? It depends on how we allow our criteria function to operate over probability distributions. If we only want to enumerate those elements i such that $p_{\in}(i) \geq f(i)$, then we may have equal probability distribution among membership and non-membership attributes. We may have that $p_{\in}(i) = p_{\notin}(i) = 0.5$ and $p_I(i) = 0$. In this case, we get to enumerate i both into $\mathcal{E}_A$ and $\mathcal{E}_{A^c}$. However, if we allow the criterion function to operate in a way that i is enumerated if and only if $p_{\in}(i) > f(i)$, then it must be the case that $p_{\notin}(i) < f(i)$ so i will only be enumerated into $\mathcal{E}_A$.

The use of the criterion function may vary depending on the application and which aspect of the given neutrosophic set we want to analyze.

4. Conclusions

We introduced the neutrosophic counterpart of oracle Turing machines with neutrosophic values allowed in the oracle tape. For this we presented a new type of oracle tape where each cell contains a triplet of three probability values, namely for the membership, non-membership, and indeterminacy. The notion of neutrosophic oracle Turing machine is interesting in its own right since oracle information is used in relative computability of sets and enables us to investigate the computability theoretic properties of sets relative to one another. In this paper, we also introduced a method to enumerate the elements of a neutrosophic subset of natural numbers. For this we defined a criterion function to choose elements which satisfy a certain probability degree. This defines a method that can be used in many applications of neutrosophic sets, particularly in decision making problems, solution space searching, and many more. We proved some results about the relationship between the enumeration sets of a given neutrosophic subset of natural numbers and the criterion function. A future work of this study is to investigate the properties of equivalence classes induced by the operator $\equiv_N$. We may call this equivalence class, neutrosophic degree of computability. It would be interesting to study the relationship between neutrosophic degrees of computability and classical Turing degrees. The results also arise further developments in achieving of new generation of computing machines such as fuzzy cellular nonlinear networks paradigm or the memristor-based cellular nonlinear networks [9]. The latter of course has practical benefits.

Author Contributions: Conceptualization, A.Ç. and S.T.; Methodology, A.Ç.; Validation, A.Ç., S.T., and F.S.; Investigation, A.Ç. and S.T.; Resources, A.Ç., S.T., and F.S.; Writing-original draft preparation, A.Ç and S.T.; Writing-review & editing, A.Ç., S.T., and F.S.; Supervision, A.Ç., S.T. and F.S.; Funding Acquisition, F.S.

Funding: This research received no external funding

Conflicts of Interest: The authors declare no conflict of interest

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Neutrosophic Logic Based Quantum Computing

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Received: 18 October 2018; Accepted: 16 November 2018; Published: 20 November 2018

Abstract: We introduce refined concepts for neutrosophic quantum computing such as neutrosophic quantum states and transformation gates, neutrosophic Hadamard matrix, coherent and decoherent superposition states, entanglement and measurement notions based on neutrosophic quantum states. We also give some observations using these principles. We present a number of quantum computational matrix transformations based on neutrosophic logic and clarify quantum mechanical notions relying on neutrosophic states. The paper is intended to extend the work of Smarandache by introducing a mathematical framework for neutrosophic quantum computing and presenting some results.

Keywords: neutrosophic computation; neutrosophic logic; quantum computation; computation; logic

1. Introduction

1.1. Neutrosophy Theory

Neutrosophic set concept, introduced by Smarandache [1,2], is a more universal structure that extends the concepts of the classic set, fuzzy set [3] and intuitionistic fuzzy set [4]. Unlike intuitionistic fuzzy sets, the indeterminacy is explicitly defined in neutrosophic sets. A neutrosophic set has three basic components defined separately: Truth T, indeterminacy I and falsity F, regarding membership. Neutrosophy was proposed as an ambitious project by Smarandache as a new branch of philosophy as well, concerning "the origin, nature, and scope of neutralities, as well as their mutual effects with different intellectual spectra". The key assumption of neutrosophy is that every idea has not only a certain degree of truth, as is generally taken in many-valued logic contexts, but also degrees of falsity and indeterminacy need to be considered independently from each other. Neutrosophy has settled the baseline for a number of new mathematical theories generalizing both their classical and fuzzy counterparts, such as neutrosophic set theory, geometry, statistics, topology, analysis, probability, and logic. The neutrosophic framework has already been applied to practical applications in many different fields, such as decision-making, semantic web, and data analysis in medicine.

Now, let us look at the concepts of some subfields of neutrosophy. *Neutrosophic set* has a formal definition as follows: Let U be a universe of discourse or space, and M be a set in U. An element x from U is stated related to the set M as $x(T, I, F)$ and belongs to M in the following way: it is t % true in the set, i % indeterminate in the set, and f % false, where t varies in T, i varies in I, f varies in F. Statically T, I, F are subsets, but dynamically T, I, F are functions/operators depending on many known or unknown parameters. *Neutrosophic logic* is a general framework for the unification of many existing logics. The main idea of neutrosophic logic is to characterize each logical statement in a 3-dimensional neutrosophic space, where each dimension of the space represents respectively the truth (T), the falsehood (F), and the indeterminacy (I) of the statement under consideration, where T, I, F are standard or non-standard real subsets of $[0^-, 1^+]$. For instance, a statement can be between $[0.21, 0.55]$ true, 0.23 or between $(0.35, 0.45)$ indeterminate, and either 0.32 or 0.75 false. *Neutrosophic statistics* is the analysis of events characterized by the neutrosophic probability. The function that models the neutrosophic probability of a random variable x is called *neutrosophic distribution*: $NP(x) = (T(x), I(x), F(x))$, where $T(x)$ represents the probability that value x occurs, $F(x)$ represents the probability that value x does not occur, and $I(x)$ represents the indeterminate/unknown probability of value x. *Neutrosophic probability* is an extension of the classical probability and imprecise probability where a case, event or fact A occurs is t % true—where t varies in the subset T, i % indeterminate—where i varies in the subset I, and f % false—where f varies in the subset F. In classical probability $n_s up \leq 1$, while in neutrosophic probability $n_s up \leq 3^+$. In imprecise probability, the probability of an event is a subset T in $[0, 1]$, not a number p in $[0, 1]$, the rest was supposed to be the opposite, subset F (also from the unit interval $[0, 1]$); there is no indeterminate subset I in imprecise probability.

1.2. Quantum Mechanics and Computing

Quantum mechanics was started with Planck [5] and interpreted as real life problem by Einstein [6]. The mechanics was developed by Bohr, Heisenberg, Broglie, Schrödinger, Born, Dirac, Hilbert, Sommerfeld, Dyson, Wien, Pauli, Von Neumann and others [7–12] in the first 30 years of the 20th century. Computers are mechanisms that support transaction information by executing algorithms. An algorithm is a well-defined process to perform an information processing task. The task can always be translated into a realization. When creating complicated algorithms for a variety of tasks, working with some improved computational models is very useful, probably very important. However, when examining the actual limitations of a computation mechanism, it is key to remember the connection between computation and realization. Quantum computation explores how efficiently nature allows us to compute. The standard computational model is based on classical mechanics; the mechanics of the Turing machine relies on classical mechanics. Quantum information processing changes not only the physical paradigm used for computing and communication but also the concepts of knowledge and computation. Quantum computation is not synonymous with quantum effects to make calculations. Actual computing mechanisms of the quantum are based on a larger physical reality than is represented by the idealized computational model. Quantum information processing is the result of the use of the physical reality that quantum theory states to perform tasks that were previously thought to be infeasible or impossible. The mechanisms that perform quantum information processing are known as quantum computers. In the last few decades of the twentieth century, researchers tried to follow two of the most influential and revolutionary theories: information science and quantum mechanics. Their success provided an unfamiliar computation and information range of vision. This new insight has significantly changed how the relationship between quantum information theory, computation, knowledge, and physics is considered and has given rise to new applications and epoch-making algorithms. The theory of information, which contains the foundations of computer science and communication, made possible to address the important issues in computer science and communication. The Turing machine is a classical model that behaves entirely according to classical mechanical principles. Quantum mechanics has become an increasingly significant line in the progress of developing more efficient computing mechanisms. Until recently, the effect of quantum mechanics had been limited to low-level applications and it had no effect on how computation or communication was carried or worked. At the beginning of the 1980s, a number of scientists found that quantum mechanics had eye-opening effects that could be used in information processing. Richard Feynman [13], Yuri Manin [14], and other influential scientists realized that some quantum mechanical phenomena could not be efficiently simulated by a standard Turing machine. This observation has led to speculation that perhaps these quantum phenomena could be used to make computations more efficient in general. Such programme required re-thinking the underlying theoretical model of informatics and completely removed it from the classical circle. Quantum computing, a field that includes quantum information, quantum algorithms, quantum cryptography, quantum communication, and quantum games, explores the effects of using quantum mechanical phenomena for information modeling and processing instead of using the rules of classical mechanics in computations.

In the following sections, we will introduce a mathematical framework of the unification of neutrosophic theory and quantum theory, in a fully computational approach. In this context, we will reveal how one can have a computational approach to the solution of mathematical and algorithmic problems of a model that can be encountered in both the neutrosophic and quantum universes. In this sense, this paper presents a more computational approach to the neutrosophic quantum concept, i.e., neutrosophic quantum computation, whose groundwork was laid by the work of Smarandache [15].

2. Neutrosophic Quantum Computing

In this part, we define some fundamental notions of neutrosophic quantum computing. Some concepts will involve new interpretations and others will be straightforward generalizations. As also mentioned in Smarandache [15], we should note in the beginning of our paper that the reversibility condition of quantum computing has some challenging issues in the neutrosophic counterpart of this ambitious field. It is mainly due to the fact that neutrosophic states involve indeterminacy, so the inverse function of such states might not always be definable, hence the domain may not be uniquely recovered from the image. We propose an interesting open problem regarding a special case of this issue at the end of the paper.

We assume some basic familiarity with linear algebra and complex numbers including their basic properties like the norm of a complex vector, complex conjugation, complex number multiplication, etc. The reader may refer to Yanofsky and Mannucci's [16] or Nielsen and Chuang's [17] book for a detailed account on quantum computing and quantum information.

Definition 1. *A neutrosophic quantum bit (neutrobit) is a three-dimensional complex vector*

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \alpha|0\rangle + \beta|1\rangle + \gamma|I\rangle$$

such that $\alpha, \beta, \gamma \in \mathbb{C}$ *are called coefficients (or amplitudes) and* $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$*, where we define the basis vectors* $|0\rangle$*,* $|1\rangle$*,* $|I\rangle$ *in the canonical basis as*

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |I\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In comparison to classical quantum computation, the reader may have noticed a new basis vector $|I\rangle$ introduced above. We call this vector the *indeterminacy basis*.

A *coherent neutrosophic quantum state* $|\psi\rangle$ is a linear combination (superposition) of the basis vectors $|0\rangle$, $|1\rangle$ and $|I\rangle$ which is in the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|I\rangle$$

such that $\alpha, \beta, \gamma \in \mathbb{C}$ and that $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$.

Thus, a coherent neutrosophic quantum state is three-dimensional complex vector, which is of unit length.

Quantum systems evolve via special kind of matrix transformations. We define *neutrosophic Pauli gates* as given below:

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad W = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}.$$

The matrix X is actually the *NOT* gate which the reader might be familiar from classical quantum computation. That is, if $k \in \{0,1\}$, then $X|k\rangle = |1-k\rangle$. Notice that $X|I\rangle = |I\rangle$. Thus, we define the negation of the indeterminacy basis as itself. The next two gates are Y-rotation and Z-rotation (phase change). The new gate here is the W-transformation which can be simply thought of as a rotation around the $|I\rangle$ basis with an equal coefficient distribution of the bases between $|I\rangle$ and the basis on which the rotation is applied. The intuition behind these rotation gates will be understood better once we give the unit ball representation of neutrobits later on.

An important quantum gate in classical quantum computing is the *Hadamard transform*, which is defined as the matrix

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Standard Hadamard transform is defined on a single qubit since it is a 2×2 matrix. Hadamard matrix used in classical quantum computing is a unitary matrix. Thus, it is reversible, and is actually its own inverse. To introduce the neutrosophic counterpart of this transformation, we first need to define the notion of indeterminate (decoherent) superpositions to make sense of the use of the Hadamard transform in neutrosophic quantum computing. The terms *coherent* and *decoherent* superpositions of neutrobits were first introduced by Smarandache [15] for denoting quantum states with some indeterminacy. We modify these notions to make the Hadamard transform work on neutrobits.

Definition 2. *The reserved three-dimensional vector*

$$|0_I\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_I$$

is called the decoherent state of the $|0\rangle$ basis vector. We define $|1_I\rangle$ similarly. That is,

$$|1_I\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_I$$

is defined to be the decoherent state of $|1\rangle$. Any linear combination that includes either of these vectors is called a decoherent superposition.

The motivation behind this definition is to mix the *coherent* (stable) basis state $|0\rangle$ with the intrinsic property of neutrosophic logic, which is indeterminacy. A quantum system may still have a degree of indeterminancy even if the system appears to be in a pure basis state. A scalar α for any of these decoherent vectors is denoted by α_I. Thus, when we write α_I, for some number α, the reader should understand that we are refering to the coefficient of a decoherent state. For example, the vector

$$|\psi\rangle = \begin{bmatrix} \left(\frac{1}{\sqrt{2}}\right)_I \\ \left(\frac{1}{\sqrt{2}}\right)_I \\ 0 \end{bmatrix}$$

denotes the decoherent superposition state

$$\frac{1}{\sqrt{2}}|0_I\rangle + \frac{1}{\sqrt{2}}|1_I\rangle.$$

We could also define a decoherent state for $|I\rangle$, but, since the state $|I\rangle$ naturally involves an indeterminacy regarding which classical bit the state refers to, there is no need to repeat this decoherence. Thus, we adopt $|0_I\rangle$ and $|1_I\rangle$ as reserved basis vectors that will be used in decoherent superposition states. We should once again emphasize that $|0_I\rangle$ is different than the coherent basis state $|0\rangle$. It is also different than the coherent superposition state $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|I\rangle$. The latter says that the system is in a superposition of basis states $|0\rangle$ and $|I\rangle$, the former says that the system is in a possibly *indetermined* state $|0\rangle$. If $|\psi\rangle = |0\rangle$, this tells us that $|\psi\rangle$ is for certain in the basis state $|0\rangle$. The state $|0_I\rangle + |I\rangle$ says that the system is in a decoherent superposition of $|I\rangle$ and a possibly indetermined state $|0\rangle$. The distinction between coherent and decoherent states should now be clear. However, another way to imagine $|0_I\rangle$ as the state $|0\rangle$ with a bounded error $\epsilon > 0$.

Given the information above, we define the *neutrosophic Hadamard transform* as

$$H_N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \left(\frac{1}{\sqrt{2}}\right)_I \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \left(\frac{1}{\sqrt{2}}\right)_I \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{bmatrix}.$$

Then, it is easy to verify that

$$H_N|0\rangle = \tfrac{1}{\sqrt{3}}|0\rangle + \tfrac{1}{\sqrt{3}}|1\rangle + \tfrac{1}{\sqrt{3}}|I\rangle,$$

$$H_N|1\rangle = \tfrac{1}{\sqrt{3}}|0\rangle - \tfrac{1}{\sqrt{3}}|1\rangle - \tfrac{1}{\sqrt{3}}|I\rangle,$$

$$H_N|I\rangle = \tfrac{1}{\sqrt{2}}|0_I\rangle + \tfrac{1}{\sqrt{2}}|1_I\rangle.$$

3. Observables and Measurement

In classical mechanics, it is intuitively understood what is meant by an observable. An observable in classical mechanics is a quantity like velocity, momentum, position, temperature, etc. It is intuitively clear what these quantities are. In quantum mechanics, one needs to be more specific when talking about observables.

Definition 3. *Let A be an $n \times n$ matrix. We say that A is Hermitian if $A^\dagger A = AA^\dagger$, where $A^\dagger$ is called the Hermitian conjugate of A and is defined as the transpose of the complex conjugate matrix of A. An $n \times n$ matrix A is called unitary if $A^\dagger A = AA^\dagger = Id$, where Id is the identity matrix.*

We note that, in classical quantum computing, state evolution is obtained by applying unitary operators. There are two reasons for this. The first reason is that classical quantum computations are reversible. The second reason is that unitary transformations preserve inner products, hence they preserve the norm of the vectors. As we shall discuss later, this requirement is questionable in neutrosophic quantum computing.

In classical quantum computing, it is assumed that, for every observable, there corresponds a Hermitian operator. We use the same postulate for the neutrosophic case.

Measurement postulate. Observables in neutrosophic quantum computing are Hermitian operators.

Measurements are the outcomes of observables applied on the physical system in consideration. Classical quantum computing usually takes *projective measurements* in the sense that when we measure a state, the new state of the system becomes one of the basis states of the system. Thus, after the measurement, a general superposition state gets *projected* onto one of the basis vectors. We shall not adopt this requirement in neutrosophic quantum computing. The reason is the following. If the outcome were to be projected onto one of the basis states, the logic used here would be no different than the classical interpretation. Even if the state of the quantum system is projected onto a single basis state, we would still require a degree of probability of the same basis state being on other basis states. This is one reason why we should decoherent superposition states into account in neutrosophic quantum computing. It relies on the very nature of neutrosophic logic. For that matter, observables we take into consideration are non-projective.

Measuring an observable on a neutrosophic quantum bit yields not a single classical state, but a probability distribution of the basis states $|0\rangle, |1\rangle, |I\rangle$. This is perhaps one of the most important difference between classical quantum computation and neutrosophic quantum computation. Given a neutrobit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|I\rangle$, making a measurement on state $|\psi\rangle$ yields a triplet

$$\langle p_{|0\rangle}, p_{|1\rangle}, p_{|I\rangle} \rangle,$$

where $p_{|0\rangle}$ denoting the probability of $|\psi\rangle$ being in state $|0\rangle$, $p_{|1\rangle}$ denoting the probability of $|\psi\rangle$ being in state $|1\rangle$, and $p_{|I\rangle}$ denoting the probability of $|\psi\rangle$ being in the indetermined basis state $|I\rangle$. Thus, the outcome of observing a neutrobit gives a probability distribution of basis states. In classical quantum computing, the outcome of measurement on a qubit is a classical bit information.

Let us illustrate this idea. For example, given the neutrobit

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}|I\rangle,$$

in a coherent superposition, measuring some observable Ω on the state $|\psi\rangle$ should yield a neutrosophic quantum state $|\psi'\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|I\rangle$, of decoherent superposition.

It should be noted that the neutrosophic quantum state should not be confused with an ordinary superposition state of a classical quantum system. Thus, a pure state in a neutrosophic quantum system always looks like a superposition. A neutrosophic quantum state is in a coherent superposition of three basis states $|0\rangle, |1\rangle, |I\rangle$. However, as soon as we make a measurement on state $|\psi\rangle$, it yields a decoherent superposition, which is merely a triplet containing the information of probability distributions for each basis states. We state this as a theorem.

Theorem 1. *Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|I\rangle$ be a coherent neutrosophic quantum state. The outcome of a measurement on $|\psi\rangle$ is a three-dimensional real vector, particularly a decoherent neutrosophc quantum superposition.*

Proof. Suppose that we are given a coherent state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|I\rangle$. Without loss of generality, we may assume that the state is in a superposition rather than in a single coherent basis. Assume that we are given a Hermitian operator Ω which is not necessarily unitary and projective. Applying Ω on $|\psi\rangle$, since we assumed that Ω is non-projective, will still yield a linear combination of vectors, particularly a three-dimensional vector. Since the probability of seeing a single coherent basis state is a magnitude square of the coefficient corresponding to that basis vector, the probability of observing

$|0\rangle$ is some $p_{|0\rangle}$. Similarly, the probability of observing $|1\rangle$ is $p_{|1\rangle}$ and the probability of seeing $|I\rangle$ is some $p_{|I\rangle}$. Since Ω is non-projective, we observe a vector containing these probabilites as elements. However, since the outcome is decoherent, it should be that each probability value can be taken to be indetermined. That is, the outcome of the observation will be a vector

$$\begin{bmatrix} (p_{|0\rangle})_I \\ (p_{|1\rangle})_I \\ (p_{|I\rangle})_I \end{bmatrix}.$$

Since $|I\rangle_I = |I\rangle$, we have

$$\begin{bmatrix} (p_{|0\rangle})_I \\ (p_{|1\rangle})_I \\ (p_{|I\rangle}) \end{bmatrix}.$$

The vector above is a decoherent superposition state with numbers $p_{|0\rangle}$, $p_{|1\rangle}$, and $p_{|I\rangle}$. Since each number is the magnitude square of the coefficients of the state vector being measured, they cannot be complex valued. Thus, each of these numbers are real valued. □

4. Tensor Products and Entanglement

The usual tensor product of classical qubits generalizes to the neutrosophic case. Given two neutrobits

$$|\psi\rangle = \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix}, \quad |\phi\rangle = \begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix},$$

the tensor product is defined as

$$|\psi\rangle \otimes |\phi\rangle = \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix} \otimes \begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \alpha_1\gamma_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \\ \beta_1\gamma_2 \\ \gamma_1\alpha_2 \\ \gamma_1\beta_2 \\ \gamma_1\gamma_2 \end{bmatrix}.$$

The tensor product of measurement outcomes can also be defined. Assume that $|\psi'\rangle = \langle p^1_{|0\rangle}, p^1_{|1\rangle}, p^1_{|I\rangle}\rangle$ and $|\phi'\rangle = \langle p^2_{|0\rangle}, p^2_{|1\rangle}, p^2_{|I\rangle}\rangle$ are two probability distributions of two *decoherent* quantum states. Then, we define

$$p^{1\otimes 2}_{|0\rangle} = p^1_{|0\rangle} \cdot p^2_{|0\rangle},$$

$$p^{1\otimes 2}_{|1\rangle} = p^1_{|1\rangle} \cdot p^2_{|1\rangle},$$

$$p^{1\otimes 2}_{|I\rangle} = p^1_{|I\rangle} \cdot p^2_{|I\rangle}.$$

Then, we write the tensor product as $|\psi'\rangle \otimes |\phi'\rangle = \langle p^{1\otimes 2}_{|0\rangle}, p^{1\otimes 2}_{|1\rangle}, p^{1\otimes 2}_{|I\rangle}\rangle$.

The tensor product of measurement outcomes provides us with the ability to use compound outcome information of multiple neutrobit systems. We shall now look at the neutrosophic entanglement property. In classical quantum computation, a two qubit system is entangled if it is not the tensor product of two single-qubit systems. We adopt the same definition for neutrosophic coherent superposition states. However, entanglement is not defined on decoherent states. Suppose that we are given two neutrobits $|\psi\rangle = \alpha_1|0\rangle + \beta_1|1\rangle + \gamma_1|I\rangle$ and $|\phi\rangle = \alpha_2|0\rangle + \beta_2|1\rangle + \gamma_2|I\rangle$, the tensor product is defined exactly the same as in the classical case. That is,

$$|\psi\rangle \otimes |\phi\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \alpha_1\gamma_2|0I\rangle + \cdots + \gamma_2\gamma_2|II\rangle.$$

This is completely a coherent superposition. If we measure this two-neutrobit system, though, we get a 9-tuple containing probability distributions where each element of the 9-tuple denotes the probability of the compound system $|\psi\rangle \otimes |\phi\rangle$ being in the ith basis state for a two-neutrobit system. The reader should easily be able to verify that, for an n-neutrobit system, there are 3^n basis states.

5. More on Quantum Operators

As noted earlier, most quantum transformations are defined similarly as in the classical case. For a better understanding though, we shall discuss more about the action of the neutrosophic Hadamard transform. The neutrosophic Hadamard transform is defined as

$$H_N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \left(\frac{1}{\sqrt{2}}\right)_I \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \left(\frac{1}{\sqrt{2}}\right)_I \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{bmatrix}.$$

The indeterminate values $\frac{1}{\sqrt{2}}$ in the neutrosophic Hadamard transform denote the indeterminate decoherent counterpart of the basis states $|0\rangle$ and $|1\rangle$. Any state which involves any of these decoherent vectors is also decoherent. Despite that we leave $H_N|0_I\rangle$ and $H_N|1_I\rangle$ undefined, we define the logical NOT operator over the decoherent states as

$$NOT|0_I\rangle = |1_I\rangle,$$

$$NOT|1_I\rangle = |0_I\rangle.$$

We leave the action of H_N on two reserved decoherent vectors $|0_I\rangle$ and $|1_I\rangle$ undefined for the reason that creating a superposition from an already decoherent neutrosophic quantum state might prevent us to obtain the original input decoherence from the output decoherence. Thus, due to this reversibility problem, it is better if we leave the mentioned transformations undefined. Since $|I\rangle$ is a legitimate coherent state in neutrosophic quantum computation, we defined

$$H_N|I\rangle = \frac{1}{\sqrt{2}}|0_I\rangle + \frac{1}{\sqrt{2}}|1_I\rangle.$$

We may imagine a coherent neutrobit as a vector on a three-dimensional unit ball as given in Figure 1.

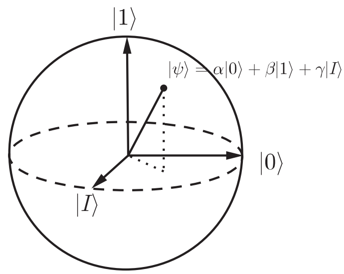

Figure 1. Representation of a neutrobit vector on a unit ball with real coefficients.

Of course, we assume in this image, for simplicity, that the amplitudes are real values. Allowing complex coefficients would require us to represent a neturobit on a four-dimensional geometry since an additional imaginary axis would need to be introduced. The basis vectors here are all mutually orthogonal. That is, the inner product of any of the two basis vectors is 0.

When we make a measurement on the state $|\psi\rangle$, we get a triplet $\langle p_{|0\rangle}, p_{|1\rangle}, p_{|I\rangle}\rangle$ which was defined earlier, where $p_{|0\rangle} = |\alpha|^2$, $p_{|1\rangle} = |\beta|^2$, $p_{|I\rangle} = |\gamma|^2$. The new state of the system in this case is a decoherent superposition of $|0\rangle$, $|1\rangle$ and $|I\rangle$ each with a degree of probability $p_{|0\rangle}$, $p_{|1\rangle}$, $p_{|I\rangle}$, respectively.

6. Results

We introduced a refined mathematical framework for neturosophic quantum computing based on the original work of Smarandache [15] and we gave a few standard transformations and notions that are to be used in neutrosophic quantum computations. Perhaps the most important difference from the classical quantum computation is the involvement of the indeterminacy basis and the separation between coherent and decoherent states. Treating the Hadamard transform as a function creating a superposition from a coherent state, we introduced the reserved decoherent vectors for this purpose. The measurement process is also slightly different in this case. The outcome of any measurement on a neutrobit gives a probability distribution, a decoherent state, of all possible basis states each with a certain degree of probability determined by the corresponding coefficients.

The computational complexity of the neutrosophic quantum gates, when applied to a quantum state, would be the same as their classical counterparts since the size of the transformation matrices in the neutrosophic counterpart does not change asymtotically. That is, for the neutrosophic Hadamard transform for instance, multiplying a 3×3 matrix with a three-dimensional vector does not give any difference in terms of computational complexity compared to its classical counterpart. The same observation can be easily seen with the other gates. The only complexity difference is with the tensor product that, since we are not working on a three-dimensional vector space, the size of the vector space grows by factors of 3 instead of 2 when taking tensor products of n many neutrobits. It should be noted that this is still a constant difference.

A practical application of neutrosophic quantum computing in the future would be used to solve hard problems involving indeterminate cases of multiple states when taken as a whole system. For example, it may not be known which one of the many possible channels that a quantum information is transferred through quantum communication channels. If we were to study the behavior of the transferred superposition quantum state, we would have to use neutrosophic quantum computing notions to describe the state of the transfer process that will involve the probability of the information being transferred on one particular channel, probability of the information not being transferred on the same channel, and a degree of indeterminacy of the information being transferred on that channel. This is required for a single channel. Thus, we would have a superposition of all possible probability distributions if we consider every channel taken together. The entire distribution will naturally define a decoherent quantum superposition state.

As stated in Smarandache [15], satisfying the reversibility condition of quantum computing is more problematic in the neutrosophic case due to the inclusion of indeterminate states. The first attempt to settle this problem is to try to make the neutrosophic Hadamard transform unitary, and hence reversible. We shall give the following open problem, for which we hope to encourage researchers in neutrosophic computation or quantum computing for finding a possible solution.

Open problem. Define a "reasonable" neutrosophic Hadamard transformation matrix, which is unitary.

By "reasonable", we mean preserving the original properties of the standard Hadamard transform such as creating a superposition of basis states, etc.

Another future work is to find a legitimate protocol for the teleportation of the state of a neutrobit from one location to another. This particularly has many applications in networks and communication. A typical quantum teleportation of a standard qubit is performed through classical bit channels. In order to send the state of a qubit, the first party sends two classical bits and the second part recovers the state of a qubit from the received classical bits. What kind of channels do we need to transport the state of a neutrobit? A classical channel may be a solution. A quantum channel, on the other hand, may not be sufficient to teleport a neutrobit due to the fact that the preservation of indeterminate states through the teleportation process becomes questionable. One idea is to separate the indeterminate state from the superposition and treat it as a classical quantum superposition state of all coherent basis states and then use the classical quantum teleportation protocol on this system.

Neutrosophic quantum computing is at its very early stage of development. We believe that this new field will attract many researchers in computer science, physics and mathematics for further advancement along with discovering many useful future applications.

Author Contributions: Conceptualization, A.Ç. and S.T.; Methodology, A.Ç.; Validation, A.Ç., S.T. and F.S.; Investigation, A.Ç. and S.T.; Resources, A.Ç., S.T. and F.S.; Writing—Original Draft Preparation, A.Ç. and S.T.; Writing—Review and Editing, A.Ç., S.T. and F.S.; Supervision, S.T. and F.S.; Funding Acquisition, F.S.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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ISBN 978-3-03897-476-5