On single valued neutrosophic relations

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Abstract— Smarandache initiated neutrosophic sets (NSs) which can be used as a mathematical tool for dealing with indeterminate and inconsistent information. In order to apply NSs conveniently, single valued neutrosophic sets (SVNSs) were proposed by Wang et al. In this paper, we propose single valued neutrosophic relations (SVNRs) and study their properties. The notions of anti-reflexive kernel, symmetric kernel, reflexive closure, and symmetric closure of a SVNR are introduced, respectively. Their accurate calculate formulas and some properties are explored. Finally, single valued neutrosophic relation mappings and inverse single valued neutrosophic relation mappings are introduced, and some interesting properties are also obtained.

Keywords— Single valued neutrosophic sets; Single valued neutrosophic relations; Kernels; Closures; Single valued neutrosophic relation mappings; Inverse single valued neutrosophic relation mappings

1 Introduction

Smarandache [7] proposed neutrosophic sets (NSs) by combining the non-standard analysis, a tri-component logic/set/probability theory and philosophy. "It is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra" [7]. A NS has three membership functions: truth membership function, indeterminacy membership function and falsity membership function, in which each membership degree is a real standard or non-standard subset of the nonstandard unit interval $]0^-, 1^+[$ [6, 7].

In order to practice NSs in real-life applications conveniently, Wang et al. [10] introduced single valued neutrosophic sets (SVNSs) which were also called simplified neutro-

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sophic sets in [13]. SVNSs is a generalization of intuitionistic fuzzy sets [1], in which three membership functions are independent and their values belong to the unit interval $[0, 1]$. SVNSs have been a new hot research topic. Many researchers have addressed this issue. Majumdar and Samanta [4] studied similarity and entropy of SVNSs. Ye [11, 12] proposed correlation coefficients of SVNSs, and applied it to single valued neutrosophic decision-making problems. Şahin and Küçük [8] introduced a system of axioms for subsethood measure of SVNSs and presented a subsethood measure for SVNSs. Based on interval neutrosophic sets [9], Chi et al. [2] extended the TOPSIS method to the multiple attribute decision making problems based on interval neutrosophic sets. Peng [5] developed some novel operations of SVNSs and proposed an outranking approach for multi-criteria decision making problems with simplified neutrosophic numbers.

It is worthy pointing out that the research about the theoretic aspect of SVNSs is not quite a few. In this paper, we attempt to broad the theoretic aspect of SVNSs. In the literatures, the study of single valued neutrosophic relations (SVNRs) is still blank. We shall focus on the study of SVNRs in this paper. Concretely, the notions of SVNRs are introduced based on SVNSs. Several kinds of kernels and closures of a SVNR are developed. Furthermore, some results on SVNR mappings and inverse SVNR mappings are also obtained.

The rest of this paper is structured as follows. In Section 2, some notions and operations of NSs and SVNSs are provided. Section 3 introduces the notions of SVNRs and presents basic properties of SVNRs. Section 4 and Section 5 discuss kernels and closures of a SVNR, respectively. Their computational formulas and some properties are obtained. SVNR mappings and inverse SVNR mapping are investigated in Section 6. The last section summarizes the conclusions.

2 Preliminaries

In this section, we provide some basic notions and operations about NSs and SVNSs.

Definition 2.1([7]). Let U be a space of points (objects), with a generic element in U denoted by u. A NS A in U is characterized by three membership functions, a truth-membership function T_A, an indeterminacy membership function I_A and a falsity-membership function F_A, where $\forall u \in U$, $T_A(u)$, $I_A(u)$ and $F_A(u)$ are real standard or non-standard subsets of $]0^-, 1^+[$. There is no restriction on the sum of $T_A(u)$, $I_A(u)$ and $F_A(u)$, thus $0^- \leq \sup T_A(u) + \sup I_A(u) + \sup F_A(u) \leq 3^+$.

Definition 2.2([7]). Let A and B be two NSs in U. If for any $u \in U$, $\inf T_A(u) \leq \inf T_B(u)$, $\sup T_A(u) \leq \sup T_B(u)$, $\inf I_A(u) \geq \inf I_B(u)$, $\sup I_A(u) \geq \sup I_B(u)$, $\inf F_A(u) \geq \inf F_B(u)$ and $\sup F_A(u) \geq \sup F_B(u)$, then we called A is contained in B, denoted by $A \subseteq B$.

To apply NSs conveniently, Wang et al. proposed SVNSs as follows.

Definition 2.3([10]). Let U be a space of points (objects), with a generic element in U denoted by u. A SVNS A in U is characterized by three membership functions, a truth-membership function T_A, an indeterminacy membership function I_A and a falsity-membership function F_A, where $\forall u \in U$, $T_A(u), I_A(u), F_A(u) \in [0,1]$. That is $T_A : U \longrightarrow [0,1]$, $I_A : U \longrightarrow [0,1]$ and $F_A : U \longrightarrow [0,1]$. There is no restriction on the sum of $T_A(u)$, $I_A(u)$ and $F_A(u)$, thus $0 \leq T_A(u) + I_A(u) + F_A(u) \leq 3$.

Here A can also be denoted by $A = \{\langle u, T_A(u), I_A(u), F_A(u)\rangle \mid u \in U\}$. $\forall u \in U$, $(T_A(u), I_A(u), F_A(u))$ is called a single valued neutrosophic number (SVNN).

Definition 2.4([13]). Let A and B be two SVNSs in U. If for any $u \in U$, $T_A(u) \leq T_B(u)$, $I_A(u) \geq I_B(u)$ and $F_A(u) \geq F_B(u)$, then we called A is contained in B, i.e. $A \subseteq B$.
If $A \subseteq B$ and $B \subseteq A$, then we called A is equal to B, denoted by $A = B$.

It is easy to see that Definition 2.4 is consistent to Definition 2.2, and Definition 2.4 can be regard as a special case of Definition 2.2.

Definition 2.5([10]). Let A be a SVNSs in U. The complement of A is denoted by A^c, where $\forall u \in U$, $T_{A^c}(u) = F_A(u)$, $I_{A^c}(u) = 1 - I_A(u)$ and $F_{A^c}(u) = T_A(u)$.

Definition 2.6. Let A and B be two SVNSs in U.

(1) The union of A and B is a SVNS C, denoted by $C = A \cup B$, where $\forall u \in U$,
$T_C(u) = \max\{T_A(u), T_B(u)\}$, $I_C(u) = \min\{I_A(u), I_B(u)\}$ and $F_C(u) = \min\{F_A(u), F_B(u)\}$.

(2) The intersection of A and B is a SVNS D, denoted by $D = A \cap B$, where $\forall u \in U$,
$T_D(u) = \min\{T_A(u), T_B(u)\}$, $I_D(u) = \max\{I_A(u), I_B(u)\}$ and $F_D(u) = \max\{F_A(u), F_B(u)\}$.

Proposition 2.1. Let A and B be two SVNSs in U. The following results hold:

(1) $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

(2) $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

(3) $(A^c)^c = A$ ([10]).

(4) $(A \cup B)^c = A^c \cap B^c$.

(5) $(A \cap B)^c = A^c \cup B^c$.

Proof. The proof is straightforward from Definitions 2.4-2.6.

Note that Definition 2.6 is different from correspondence definitions in [10]. The union and intersection in [10] donot satisfy Proposition 2.1 (1) and (2).

3 Single valued neutrosophic relations

In this section, we introduce the notions of single valued neutrosophic relations and several special single valued neutrosophic relations.

Definition 3.1. A SVNS R in $U \times U$ is called a single valued neutrosophic relation (SVNR) in U, denoted by $R = \{\langle (u,v), T_R(u,v), I_R(u,v), F_R(u,v)\rangle \mid (u,v) \in U \times U\}$, where $T_R : U \times U \longrightarrow [0,1]$, $I_R : U \times U \longrightarrow [0,1]$ and $F_R : U \times U \longrightarrow [0,1]$ denote the truth-membership function, indeterminacy membership function and falsity-membership function of R, respectively.

In what follows, SVNR(U) will denote the family of all single valued neutrosophic relations in U.

Definition 3.2. Let R be a SVNR in U, the complement and inverse of R are defined as follows, respectively

$R^c = \{\langle (u,v), T_{R^c}(u,v), I_{R^c}(u,v), F_{R^c}(u,v)\rangle \mid (u,v) \in U \times U\}$,

where $\forall (u,v) \in U \times U$, $T_{R^c}(u,v) = F_R(u,v)$, $I_{R^c}(u,v) = 1 - I_R(u,v)$ and $F_{R^c}(u,v) = T_R(u,v)$.

$R^{-1} = \{\langle (u,v), T_{R^{-1}}(u,v), I_{R^{-1}}(u,v), F_{R^{-1}}(u,v)\rangle \mid (u,v) \in U \times U\}$,

where $\forall (u,v) \in U \times U$, $T_{R^{-1}}(u,v) = T_R(v,u)$, $I_{R^{-1}}(u,v) = I_R(v,u)$ and $F_{R^{-1}}(u,v) = F_R(u,v)$.

Example 3.1. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$. A SVNR R in U is given in Table 1. By Definitions 3.2, we can compute R^c and R^{-1} which are given in Tables 2 and 3, respectively.

The union and intersection of two SVNRs are introduced as follows.

Definition 3.3. Let R, S be two SVNRs in U.

(1) The union $R \cup S$ of R and S is defined by $R \cup S = \{\langle (u,v)$, $\max\,\{T_R(u,v), T_S(u,v)\}$,

Table 1: A SVNR R

R	x_1	x_2	x_3	x_4	x_5
x_1	(0.2,0.6,0.4)	(0,0.3,0.7)	(0.9,0.2,0.4)	(0.3,0.9,1)	(1,0.2,0)
x_2	(0.4,0.5,0.1)	(0.1,0.7,0)	(1,1,1)	(1,0.3,0)	(0.5,0.6,1)
x_3	(0,1,1)	(1,0.5,0)	(0,0,0)	(0.2,0.8,0.1)	(1,0.8,1)
x_4	(1,0,0)	(0,0,1)	(0.5,0.7,0.1)	(0.1,0.4,1)	(1,0.8,0.8)
x_5	(0,1,0)	(0.9,0,0)	(0,0.1,0.7)	(0.8,0.9,1)	(0.6,1,0)

Table 2: The complement R^c of R

R^c	x_1	x_2	x_3	x_4	x_5
x_1	(0.4,0.4,0.2)	(0.7,0.7,0)	(0.4,0.8,0.9)	(1,0.1,0.3)	(0,0.8,1)
x_2	(0.1,0.5,0.4)	(0,0.3,0.1)	(1,0,1)	(0,0.7,1)	(1,0.4,0.5)
x_3	(1,0,0)	(0,0.5,1)	(0,1,0)	(0.1,0.2,0.2)	(1,0.2,1)
x_4	(0,1,1)	(1,1,0)	(0.1,0.3,0.5)	(1,0.6,0.1)	(0.8,0.2,1)
x_5	(0,0,0)	(0,1,0.9)	(0.7,0.9,0)	(1,0.1,0.8)	(0,0,0.6)

$\min\{I_R(u,v), I_S(u,v)\}$, $\min\{F_R(u,v), F_S(u,v)\}\rangle \mid (u,v) \in U \times U\}$.

(2) The intersection $R \cap S$ of R and S is defined by $R \cap S = \{\langle (u,v), \min\{T_R(u,v), T_S(u,v)\}$, $\max\{I_R(u,v), I_S(u,v)\}$, $\max\{F_R(u,v), F_S(u,v)\}\rangle \mid (u,v) \in U \times U\}$.

Next, we give several special SVNRs.

Definition 3.4. Let R be a SVNR in U.

(1) If $\forall u, v \in U$, $T_R(u,v) = 0$ and $I_R(u,v) = F_R(u,v) = 1$, then R is called a null SVNR, denoted by $\emptyset_N$.

(2) If $\forall u, v \in U$, $T_R(u,v) = 1$, and $I_R(u,v) = F_R(u,v) = 0$, then R is called an absolute SVNR, denoted by U_N.

(3) If $\forall u, v \in U$, $T_R(u,v) = \begin{cases} 1, & u = v \\ 0, & u \neq v \end{cases}$ and $I_R(u,v) = F_R(u,v) = \begin{cases} 0, & u = v \\ 1, & u \neq v \end{cases}$, then R is called an identity SVNR, denoted by Id_N.

Table 3: The inverse R^{-1} of R

R^{-1}	x_1	x_2	x_3	x_4	x_5
x_1	(0.2,0.6,0.4)	(0.4,0.5,0.1)	(0,1,1)	(1,0,0)	(0,1,0)
x_2	(0,0.3,0.7)	(0.1,0.7,0)	(1,0.5,0)	(0,0,1)	(0.9,0,0)
x_3	(0.9,0.2,0.4)	(1,1,1)	(0,0,0)	(0.5,0.7,0.1)	(0,0.1,0.7)
x_4	(0.3,0.9,1)	(1,0.3,0)	(0.2,0.8,0.1)	(0.1,0.4,1)	(0.8,0.9,1)
x_5	(1,0.2,0)	(0.5,0.6,1)	(1,0.8,1)	(1,0.8,0.8)	(0.6,1,0)

By use of Definitions 3.2 and 3.4, the complement $(Id_N)^c$ of Id_N is a SVNR satisfying: $\forall u, v \in U$, $T_{(Id_N)^c}(u,v) = \begin{cases} 0, & u = v \\ 1, & u \neq v \end{cases}$ and $I_{(Id_N)^c}(u,v) = F_{(Id_N)^c}(u,v) = \begin{cases} 1, & u = v \\ 0, & u \neq v \end{cases}$.

Definition 3.5. Let R be a SVNR in U.

(1) If $\forall u \in U$, $T_R(u,u) = 1$ and $I_R(u,u) = F_R(u,u) = 0$, then R is called a reflexive SVNR.

(2) If $\forall u, v \in U$, $T_R(u,v) = T_R(v,u)$, $I_R(u,v) = I_R(v,u)$ and $F_R(u,v) = F_R(v,u)$, then R is called a symmetric SVNR.

(3) If $\forall u \in U$, $T_R(u,u) = 0$ and $I_R(u,u) = F_R(u,u) = 1$, then R is called an anti-reflexive SVNR.

(4) If $\forall u, v, w \in U$, $\vee_{v \in U}(T_R(u,v) \wedge T_R(v,w)) \leq T_R(u,w)$, $\wedge_{v \in U}(I_R(u,v) \vee I_R(v,w)) \geq I_R(u,w)$ and $\wedge_{v \in U}(F_R(u,v) \vee F_R(v,w)) \geq F_R(u,w)$, then R is called a transitive SVNR, where " $\vee$ " and " $\wedge$ " denote maximum and minimum, respectively.

Definition 3.6. Let R, S be two SVNRs in U. If $\forall u, v \in U$, $T_R(u,v) \leq T_S(u,v)$, $I_R(u,v) \geq I_S(u,v)$ and $F_R(u,v) \geq F_S(u,v)$, then we call R is contained in S (or R is less than S), denoted by $R \subseteq S$ (or $R \leq S$).

It is easy to verify that the union and intersection of SVNRs satisfy commutative law, associative law and distributive law. $\emptyset_N$ is a symmetric and anti-reflexive SVNR. U_N and Id_N are two symmetric and reflexive SVNRs. $(Id_N)^c$ is an anti-reflexive SVNR. If R is not an anti-reflexive SVNR, then there is no an anti-reflexive SVNR containing R. If R is not a reflexive SVNR, then there is no a reflexive SVNR contained in R. Moreover, if R is a reflexive SVNR, then $R \supseteq Id_N$, and if R is an anti-reflexive SVNR, then $R \subseteq (Id_N)^c$.

Theorem 3.1. Let R, S, P be three SVNRs in U. Then

(1) R is symmetric iff $R = R^{-1}$.

(2) $(R^c)^{-1} = (R^{-1})^c$.

(3) $(R^{-1})^{-1} = R$, $(R^c)^c = R$.

(4) $R \cup S \supseteq R$, $R \cup S \supseteq S$.

(5) $R \cap S \subseteq R$, $R \cap S \subseteq S$.

(6) If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$.

(7) If $P \supseteq S$ and $P \supseteq R$, then $P \supseteq R \cup S$.

(8) If $P \subseteq S$ and $P \subseteq R$, then $P \subseteq R \cap S$.

(9) If $R \subseteq S$, then $R \cup S = S$ and $R \cap S = R$.

(10) $(R\cup S)^{-1}=R^{-1}\cup S^{-1}$, $(R\cap S)^{-1}=R^{-1}\cap S^{-1}$.

(11) $(R\cup S)^{c}=R^{c}\cap S^{c}$, $(R\cap S)^{c}=R^{c}\cup S^{c}$.

Proof. Clearly, (1) and (3)-(9) hold. We only show (2), (10) and (11).

(2) $\forall u,v\in U$, $T_{(R^c)^{-1}}(u,v)=T_{R^c}(v,u)=F_R(v,u)=F_{R^{-1}}(u,v)=T_{(R^{-1})^c}(u,v)$,

$I_{(R^c)^{-1}}(u,v)=I_{R^c}(v,u)=1-I_R(v,u)=1-I_{R^{-1}}(u,v)=I_{(R^{-1})^c}(u,v)$,

$F_{(R^c)^{-1}}(u,v)=F_{R^c}(v,u)=T_R(v,u)=T_{R^{-1}}(u,v)=F_{(R^{-1})^c}(u,v)$.

So $(R^c)^{-1}=(R^{-1})^c$.

(10) $\forall u,v\in U$,

$T_{(R\cup S)^{-1}}(u,v)=T_{(R\cup S)}(v,u)=\max\{T_R(v,u),T_S(v,u)\}=\max\{T_{R^{-1}}(u,v),T_{S^{-1}}(u,v)\}=T_{R^{-1}\cup S^{-1}}(u,v)$,

$I_{(R\cup S)^{-1}}(u,v)=I_{(R\cup S)}(v,u)=\min\{I_R(v,u),I_S(v,u)\}=\min\{I_{R^{-1}}(u,v),I_{S^{-1}}(u,v)\}=I_{R^{-1}\cup S^{-1}}(u,v)$,

$F_{(R\cup S)^{-1}}(u,v)=F_{(R\cup S)}(v,u)=\min\{F_R(v,u),F_S(v,u)\}=\min\{F_{R^{-1}}(u,v),F_{S^{-1}}(u,v)\}=F_{R^{-1}\cup S^{-1}}(u,v)$.

Hence $(R\cup S)^{-1}=R^{-1}\cup S^{-1}$. Similarly, we can show $(R\cap S)^{-1}=R^{-1}\cap S^{-1}$.

(11) $\forall u,v\in U$,

$T_{(R\cup S)^c}(u,v)=F_{(R\cup S)}(u,v)=\min\{F_R(u,v),F_S(u,v)\}=\min\{T_{R^c}(u,v),T_{S^c}(u,v)\}=T_{R^c\cap S^c}(u,v)$,

$I_{(R\cup S)^c}(u,v)=1-I_{(R\cup S)}(u,v)=1-\min\{I_R(u,v),I_S(u,v)\}=\max\{1-I_R(u,v),1-I_S(u,v)\}=\max\{I_{R^c}(u,v),I_{S^c}(u,v)\}=I_{R^c\cap S^c}(u,v)$,

$F_{(R\cup S)^c}(u,v)=T_{(R\cup S)}(u,v)=\max\{T_R(u,v),T_S(u,v)\}=\max\{F_{R^c}(u,v),F_{S^c}(u,v)\}=F_{R^c\cap S^c}(u,v)$.

So $(R\cup S)^c=R^c\cap S^c$. Similarly, we can show $(R\cap S)^c=R^c\cup S^c$.

Remark 3.1. According to Theorem 3.1 (1) and (2), the complement of a symmetric SVNR is also a symmetric SVNR.

4 Kernels of SVNRs

In this section, we will define anti-reflexive kernel and symmetric kernel of a SVNR, then investigate their properties.

Definition 4.1. Let R be a SVNR in U.

(1) The maximal anti-reflexive SVNR contained in R is called anti-reflexive kernel of R,

Table 4: The anti-reflexive kernel $ar(R)$ of R

$ar(R)$	x_1	x_2	x_3	x_4	x_5
x_1	(0,1,1)	(0,0.3,0.7)	(0.9,0.2,0.4)	(0.3,0.9,1)	(1,0.2,0)
x_2	(0.4,0.5,0.1)	(0,1,1)	(1,1,1)	(1,0.3,0)	(0.5,0.6,1)
x_3	(0,1,1)	(1,0.5,0)	(0,1,1)	(0.2,0.8,0.1)	(1,0.8,1)
x_4	(1,0,0)	(0,0,1)	(0.5,0.7,0.1)	(0,1,1)	(1,0.8,0.8)
x_5	(0,1,0)	(0.9,0,0)	(0,0.1,0.7)	(0.8,0.9,1)	(0,1,1)

denoted by $ar(R)$.

(2) The maximal symmetric SVNR contained in R is called symmetric kernel of R, denoted by $s(R)$.

Theorem 4.1. Let R be a SVNR in U. Then

(1) $ar(R) = R \cap (Id_N)^c$.

(2) $s(R) = R \cap R^{-1}$.

Proof. (1) By Theorem 3.1 (5), $R \cap (Id_N)^c \subseteq R$. By the definition of Id_N, $\forall u \in U$, we have $T_{Id_N}(u,u) = 1$ and $I_{Id_N}(u,u) = F_{Id_N}(u,u) = 0$, then $T_{(Id_N)^c}(u,u) = 0$ and $I_{(Id_N)^c}(u,u) = F_{(Id_N)^c}(u,u) = 1$. Hence $T_{R\cap(Id_N)^c}(u,u) = \min\{T_R(u,u), T_{(Id_N)^c}(u,u)\} = 0$, $I_{R\cap(Id_N)^c}(u,u) = \max\{I_R(u,u), I_{(Id_N)^c}(u,u)\} = 1$ and $F_{R\cap(Id_N)^c}(u,u) = \max\{F_R(u,u), F_{(Id_N)^c}(u,u)\} = 1$. By Definition 3.5 (3), $R \cap (Id_N)^c$ is an anti-reflexive SVNR in U.

If K is an anti-reflexive SVNR in U and $K \subseteq R$. Obviously, $K \subseteq (Id_N)^c$. Hence $K \subseteq R \cap (Id_N)^c$. So $ar(R) = R \cap (Id_N)^c$.

(2) By Theorem 3.1 (9) and (3), $(R\cap R^{-1})^{-1} = R^{-1} \cap (R^{-1})^{-1} = R^{-1} \cap R = R \cap R^{-1}$, which implies that $R \cap R^{-1}$ is a symmetric SVNR in U. According to Theorem 3.1 (5), $R \cap R^{-1} \subseteq R$.

If K is a symmetric SVNR in U and $K \subseteq R$. By Theorem 3.1 (6), $K^{-1} \subseteq R^{-1}$. Then by Theorem 3.1 (1) and (5), $K = K^{-1} \subseteq R \cap R^{-1}$. So $s(R) = R \cap R^{-1}$.

Example 4.1. Consider U and R in Example 3.1. By Theorem 4.1, we can obtain $ar(R)$ and $s(R)$ which are given in Table 4 and Table 5, respectively.

Next, we discuss the properties of the anti-reflexive kernel operator ar and symmetric kernel operator s.

Theorem 4.2. The anti-reflexive kernel operator ar of the SVNR has the following prop-

Table 5: The symmetric kernel $s(R)$ of R

$s(R)$	x_1	x_2	x_3	x_4	x_5
x_1	(0.2,0.6,0.4)	(0,0.5,0.7)	(0,1,1)	(0.3,0.9,1)	(0,1,0)
x_2	(0,0.5,0.7)	(0.1,0.7,0)	(1,1,1)	(0,0.3,1)	(0.5,0.6,1)
x_3	(0,1,1)	(1,1,1)	(0,0,0)	(0.2,0.8,0.1)	(0,0.8,1)
x_4	(0.3,0.9,1)	(0,0.3,1)	(0.2,0.8,0.1)	(0.1,0.4,1)	(0.8,0.9,1)
x_5	(0,1,0)	(0.5,0.6,1)	(0,0.8,1)	(0.8,0.9,1)	(0.6,1,0)

erties:

(1) $ar(\emptyset_N) = \emptyset_N$, $ar((Id_N)^c) = (Id_N)^c$.

(2) $\forall R \in \text{SVNR}(U)$, $ar(R) \subseteq R$.

(3) $\forall R, S \in \text{SVNR}(U)$, $ar(R \cup S) = ar(R) \cup ar(S)$, $ar(R \cap Q) = ar(R) \cap ar(S)$.

(4) $\forall R, S \in \text{SVNR}(U)$, if $R \subseteq S$, then $ar(R) \subseteq ar(S)$.

(5) $\forall R \in \text{SVNR}(U)$, $ar(ar(R)) = ar(R)$.

Proof. (1) By the anti-reflexivity of $\emptyset_N$ and $(Id_N)^c$, obviously, $ar(\emptyset_N) = \emptyset_N$ and $ar((Id_N)^c) = (Id_N)^c$.

(2) $\forall R \in \text{SVNR}(U)$, by Theorems 4.1 (1) and 3.1 (5), $ar(R) = R \cap (Id_N)^c \subseteq R$.

(3) $\forall R, S \in \text{SVNR}(U)$, by Theorem 4.1 (1),

$ar(R \cup S) = (R \cup S) \cap (Id_N)^c = (R \cap (Id_N)^c) \cup (S \cap (Id_N)^c) = ar(R) \cup ar(S)$,

$ar(R \cap S) = (R \cap S) \cap (Id_N)^c = (R \cap (Id_N)^c) \cap (S \cap (Id_N)^c) = ar(R) \cap ar(S)$.

(4) $\forall R, S \in \text{SVNR}(U)$, if $R \subseteq S$, by (3) and Theorem 3.1 (4) and (9),

$ar(S) = ar(R \cup S) = ar(R) \cup ar(S) \supseteq ar(R)$.

(5) $\forall R \in \text{SVNR}(U)$, by Theorem 4.1 (1), $ar(R) = R \cap (Id_N)^c$. Hence

$ar(ar(R)) = ar(R \cap (Id_N)^c) = (R \cap (Id_N)^c) \cap (Id_N)^c = R \cap (Id_N)^c = ar(R)$.

Theorem 4.3. The symmetric kernel operator s has the following properties:

(1) $s(\emptyset_N) = \emptyset_N$, $s(U_N) = U_N$, $s(Id_N) = Id_N$.

(2) $\forall R \in \text{SVNR}(U)$, $s(R) \subseteq R$.

(3) $\forall R, S \in \text{SVNR}(U)$, $s(R \cap S) = s(R) \cap s(S)$.

(4) $\forall R, S \in \text{SVNR}(U)$, if $R \subseteq S$, then $s(R) \subseteq s(S)$.

(5) $\forall R \in \text{SVNR}(U)$, $s(s(R)) = s(R)$.

Proof. (1) By the symmetry of $\emptyset_N$, U_N and Id_N, we have

$s(\emptyset_N) = \emptyset_N$, $s(U_N) = U_N$ and $s(Id_N) = Id_N$.

(2) $\forall R \in \mathrm{SVNR}(U)$, by Theorems 4.1 (2) and 3.1 (5), $s(R) = R \cap R^{-1} \subseteq R$.

(3) $\forall R, S \in \mathrm{SVNR}(U)$, by Theorems 4.1 (2) and 3.1 (10), we have

$s(R \cap S) = (R \cap S) \cap (R \cap S)^{-1} = (R \cap S) \cap (R^{-1} \cap S^{-1}) = (R \cap R^{-1}) \cap (S \cap S^{-1}) = s(R) \cap s(S)$.

(4) $\forall R, S \in \mathrm{SVNR}(U)$, if $R \subseteq S$, by (3) and Theorem 3.1 (5) and (9),

$s(R) = s(R \cap S) = s(R) \cap s(S) \subseteq s(S)$.

(5) $\forall R \in \mathrm{SVNR}(U)$, by Theorem 4.1 (2), $s(R) = R \cap R^{-1}$. Hence

$s(s(R)) = s(R \cap R^{-1}) = (R \cap R^{-1}) \cap (R \cap R^{-1})^{-1} = (R \cap R^{-1}) \cap (R^{-1} \cap R) = R \cap R^{-1} = s(R)$.

According to Theorem 4.3 (1)-(3) and (5), the symmetric kernel operator s is an interior operator in fuzzy topology [3].

5 Closures of SVNRs

In this section, we introduce the concepts of reflexive closure and symmetric closure of a SVNR, and investigate their properties.

Definition 5.1. Let R be a SVNR in U. The minimal reflexive SVNR containing R is called reflexive closure of R, denoted by $\overline{r}(R)$.

Definition 5.2. Let R be a SVNR in U. The minimal symmetric SVNR containing R is called symmetric closure of R, denoted by $\overline{s}(R)$.

Theorem 5.1. Let R be a SVNR in U. Then

(1) $\overline{r}(R) = R \cup Id_N$.

(2) $\overline{s}(R) = R \cup R^{-1}$.

Proof. (1) By Theorem 3.1 (4), $R \cup Id_N \supseteq R$ and $R \cup Id_N \supseteq Id_N$. Then $\forall u \in U$, we have $T_{R \cup Id_N}(u,u) \geq T_{Id_N}(u,u) = 1$, $I_{R \cup Id_N}(u,u) \leq I_{Id_N}(u,u) = 0$ and $F_{R \cup Id_N}(u,u) \leq F_{Id_N}(u,u) = 0$, so $R \cup Id_N$ is a reflexive SVNR.

If K is a reflexive SVNR in U and $K \supseteq R$. By the reflexivity of K, $K \supseteq Id_N$. Thus by Theorem 3.1 (7), we have $K \supseteq R \cup Id_N$. So $\overline{r}(R) = R \cup Id_N$.

(2) By Theorem 3.1 (10), $(R \cup R^{-1})^{-1} = R^{-1} \cup (R^{-1})^{-1} = R^{-1} \cup R = R \cup R^{-1}$, which implies that $R \cup R^{-1}$ is a symmetric SVNR in U. By Theorem 3.1 (4), $R \cup R^{-1} \supseteq R$.

If K is a symmetric SVNR in U and $K \supseteq R$. By Theorem 3.1 (6), $K^{-1} \supseteq R^{-1}$.

Table 6: The reflexive closure $\overline{r}(R)$ of R

$\overline{r}(R)$	x_1	x_2	x_3	x_4	x_5
x_1	(1,0,0)	(0,0.3,0.7)	(0.9,0.2,0.4)	(0.3,0.9,1)	(1,0.2,0)
x_2	(0.4,0.5,0.1)	(1,0,0)	(1,1,1)	(1,0.3,0)	(0.5,0.6,1)
x_3	(0,1,1)	(1,0.5,0)	(1,0,0)	(0.2,0.8,0.1)	(1,0.8,1)
x_4	(1,0,0)	(0,0,1)	(0.5,0.7,0.1)	(1,0,0)	(1,0.8,0.8)
x_5	(0,1,0)	(0.9,0,0)	(0,0.1,0.7)	(0.8,0.9,1)	(1,0,0)

Table 7: The symmetric closure $\overline{s}(R)$ of R

$\overline{s}(R)$	x_1	x_2	x_3	x_4	x_5
x_1	(0.2,0.6,0.4)	(0.4,0.3,0.1)	(0.9,0.2,0.4)	(1,0,0)	(1,0.2,0)
x_2	(0.4,0.3,0.1)	(0.1,0.7,0)	(1,0.5,0)	(1,0,0)	(0.9,0,0)
x_3	(0.9,0.2,0.4)	(1,0.5,0)	(0,0,0)	(0.5,0.7,0.1)	(1,0.1,0.7)
x_4	(1,0,0)	(1,0,0)	(0.5,0.7,0.1)	(0.1,0.4,1)	(1,0.8,0.8)
x_5	(1,0.2,0)	(0.9,0,0)	(1,0.1,0.7)	(1,0.8,0.8)	(0.6,1,0)

According to Theorem 3.1 (1) and (4), $K = K^{-1} \supseteq R \cup R^{-1}$.

Therefore $\overline{s}(R) = R \cup R^{-1}$.

Example 5.1. Consider U and R given in Example 3.1 again. By Theorem 5.1, we can compute $\overline{r}(R)$ and $\overline{s}(R)$ which are given in Table 6 and Table 7, respectively.

Theorem 5.2. The reflexive closure operator $\overline{r}$ has the following properties:

(1) $\overline{r}(U_N) = U_N$, $\overline{r}(Id_N) = Id_N$.

(2) $\forall R \in$ SVNR(U), $R \subseteq \overline{r}(R)$.

(3) $\forall R, S \in$ SVNR(U), $\overline{r}(R \cup S) = \overline{r}(R) \cup \overline{r}(S)$, $\overline{r}(R \cap S) = \overline{r}(R) \cap \overline{r}(S)$.

(4) $\forall R, S \in$ SVNR(U), if $R \subseteq S$, then $\overline{r}(R) \subseteq \overline{r}(S)$.

(5) $\forall R \in$ SVNR(U), $\overline{r}(\overline{r}(R)) = \overline{r}(R)$.

Proof. (1) By the reflexivity of U_N and Id_N, $\overline{r}(U_N) = U_N$, $\overline{r}(Id_N) = Id_N$.

(2) $\forall R \in$ SVNR(U), by Theorems 5.1 (1) and 3.1 (4), $\overline{r}(R) = R \cup Id_N \supseteq R$.

(3) $\forall R, S \in$ SVNR(U), by Theorem 5.1 (1),

$\overline{r}(R \cup S) = (R \cup S) \cup Id_N = (R \cup Id_N) \cup (S \cup Id_N) = \overline{r}(R) \cup \overline{r}(S)$,

$\overline{r}(R \cap S) = (R \cap S) \cup Id_N = (R \cup Id_N) \cap (S \cup Id_N) = \overline{r}(R) \cap \overline{r}(S)$.

(4) $\forall R, S \in$ SVNR(U), $R \subseteq S$, by (3) and Theorem 3.1 (4) and (9), we have

$\overline{r}(S) = \overline{r}(R \cup S) = \overline{r}(R) \cup \overline{r}(S) \supseteq \overline{r}(R)$.

(5) $\forall R \in \mathrm{SVNR}(U)$, by Theorem 5.1 (1), $\overline{r}(R) = R \cup Id_N$. It follows that $\overline{r}(\overline{r}(R)) = \overline{r}(R \cup Id_N) = (R \cup Id_N) \cup Id_N = R \cup Id_N = \overline{r}(R)$.

Theorem 5.3. The symmetric closure operator $\overline{s}$ has the following properties:

(1) $\overline{s}(\emptyset_N) = \emptyset_N$, $\overline{s}(U_N) = U_N$, $\overline{s}(Id_N) = Id_N$.

(2) $\forall R \in \mathrm{SVNR}(U)$, $\overline{s}(R) \supseteq R$.

(3) $\forall R, S \in \mathrm{SVNR}(U)$, $\overline{s}(R \cup S) = \overline{s}(R) \cup \overline{s}(S)$.

(4) $\forall R, S \in \mathrm{SVNR}(U)$, if $R \subseteq S$, then $\overline{s}(R) \subseteq \overline{s}(S)$.

(5) $\forall R \in \mathrm{SVNR}(U)$, $\overline{s}(\overline{s}(R)) = \overline{s}(R)$.

Proof. (1) By the symmetry of $\emptyset_N$, U_N and Id_N, we have $\overline{s}(\emptyset_N) = \emptyset_N$, $\overline{s}(U_N) = U_N$ and $\overline{s}(Id_N) = Id_N$.

(2) $\forall R \in \mathrm{SVNR}(U)$, by Theorem 5.1 (2), $\overline{s}(R) = R \cup R^{-1} \supseteq R$.

(3) $\forall R, S \in \mathrm{SVNR}(U)$, by Theorems 5.1 (2) and 3.1 (10), we have $\overline{s}(R \cup S) = (R \cup S) \cup (R \cup S)^{-1} = (R \cup S) \cup (R^{-1} \cup S^{-1}) = (R \cup R^{-1}) \cup (S \cup S^{-1}) = \overline{s}(R) \cup \overline{s}(S)$.

(4) $\forall R, S \in \mathrm{SVNR}(U)$, if $R \subseteq S$, by (3) and Theorem 3.1 (4) and (9), $\overline{s}(S) = \overline{s}(R \cup S) = \overline{s}(R) \cup \overline{s}(S) \supseteq \overline{s}(S)$.

(5) $\forall R \in \mathrm{SVNR}(U)$, by Theorem 5.1 (2), $\overline{s}(R) = R \cup R^{-1}$. Hence $\overline{s}(\overline{s}(R)) = \overline{s}(R \cup R^{-1}) = (R \cup R^{-1}) \cup (R \cup R^{-1})^{-1} = (R \cup R^{-1}) \cup (R^{-1} \cup R) = R \cup R^{-1} = \overline{s}(R)$.

According to Theorem 5.3 (1)-(3) and (5), the symmetric closure operator $\overline{s}$ is a closure operator in fuzzy topology [3].

Lemma 5.1. $\forall R \in \mathrm{SVNR}(U)$, we have

(1) $(\overline{r}(R^c))^c = ar(R)$.

(2) $\overline{r}(ar(R)) = \overline{r}(R)$.

(3) $ar(\overline{r}(R)) = ar(R)$.

Proof. (1) By Theorem 5.1 (1), $\overline{r}(R^c) = R^c \cup Id_N$. By Theorems 3.1 (11) and 4.1 (1), $(\overline{r}(R^c))^c = (R^c \cup Id_N)^c = (R^c)^c \cap (Id_N)^c = R \cap (Id_N)^c = ar(R)$.

(2) By Theorems 4.1 (1) and 5.1 (1), $\overline{r}(ar(R)) = \overline{r}(R \cap (Id_N)^c) = (R \cap (Id_N)^c) \cup Id_N = (R \cup Id_N) \cap ((Id_N)^c \cup Id_N) = (R \cup Id_N) \cap U_N = \overline{r}(R)$.

(3) By Theorems 4.1 (1) and 5.1 (1), $ar(\overline{r}(R)) = ar(R \cup Id_N) = (R \cup Id_N) \cap (Id_N)^c = (R \cap (Id_N)^c) \cup (Id_N \cap (Id_N)^c) = (R \cap (Id_N)^c) \cup \emptyset_N = ar(R)$.

Theorem 5.4. $\forall R \in \mathrm{SVNR}(U)$, at most six different SVNRs can be obtained by using anti-reflexive kernel operator, reflexive closure operator and complement operator.

Proof. $\forall R \in \mathrm{SVNR}(U)$, by Lemma 5.1 (1), $(\overline{r}(R^c))^c = ar(R)$. Thus, we can replace anti-reflexive kernel operator with complement operator and reflexive closure operator.

(1) Take complement operator first, then reflexive closure operator on R. One can obtain only the following five SVNRs:

R^c, $\overline{r}(R^c)$, $(\overline{r}(R^c))^c$, $\overline{r}((\overline{r}(R^c))^c)$, $(\overline{r}((\overline{r}(R^c))^c))^c$.

It is because that by Lemma 5.1 and Theorem 3.1 (3),

$\overline{r}((\overline{r}((\overline{r}(R^c))^c))^c) = \overline{r}((\overline{r}(ar(R)))^c) = \overline{r}((\overline{r}(R))^c) = \overline{r}(ar(R^c)) = \overline{r}(R^c)$,

which implies that the sixth is the same as the second. The results of the latter steps will be repeated.

(2) Take reflexive closure operator first, then complement operator on R. By Lemma 5.1 (1) and (2), $\overline{r}(R) = \overline{r}(ar(R)) = \overline{r}((\overline{r}(R^c))^c)$, which implies that the first is the same as the fourth in (1). Hence it will be repeated.

(3) Take reflexive closure operator successively or complement operator successively on R. By Theorems 3.1 (3) and 5.2 (5), $(R^c)^c = R$, $\overline{r}(\overline{r}(R)) = \overline{r}(R)$. This is repeated emergence. The proof is complete.

To illustrate the idea developed in Theorem 5.4, we give the following example.

Example 5.2. Let $U = \{u_1, u_2\}$. A SVNR R in U is given as follows.

$R = \{\langle (u_1, u_1), 0.3, 0.2, 0.9\rangle, \langle (u_1, u_2), 1, 0, 0.2\rangle, \langle (u_2, u_1), 0, 0.4, 0.3\rangle, \langle (u_2, u_2), 0.5, 0.3, 1\rangle\}$.

By using anti-reflexive kernel operator, reflexive closure operator and complement operator, the following six different SVNRs can be obtained:

$R^c = \{\langle (u_1, u_1), 0.9, 0.8, 0.3\rangle, \langle (u_1, u_2), 0.2, 1, 1\rangle, \langle (u_2, u_1), 0.3, 0.6, 0\rangle, \langle (u_2, u_2), 1, 0.7, 0.5\rangle\}$,

$\overline{r}(R^c) = \{\langle (u_1, u_1), 1, 0, 0\rangle, \langle (u_1, u_2), 0.2, 1, 1\rangle, \langle (u_2, u_1), 0.3, 0.6, 0\rangle, \langle (u_2, u_2), 1, 0, 0\rangle\}$,

$(\overline{r}(R^c))^c = \{\langle (u_1, u_1), 0, 1, 1\rangle, \langle (u_1, u_2), 1, 0, 0.2\rangle, \langle (u_2, u_1), 0, 0.4, 0.3\rangle, \langle (u_2, u_2), 0, 1, 1\rangle\}$,

$\overline{r}((\overline{r}(R^c))^c) = \{\langle (u_1, u_1), 1, 0, 0\rangle, \langle (u_1, u_2), 1, 0, 0.2\rangle, \langle (u_2, u_1), 0, 0.4, 0.3\rangle, \langle (u_2, u_2), 1, 0, 0\rangle\}$,

$(\overline{r}((\overline{r}(R^c))^c))^c = \{\langle (u_1, u_1), 0, 1, 1\rangle, \langle (u_1, u_2), 0.2, 1, 1\rangle, \langle (u_2, u_1), 0.3, 0.6, 0\rangle, \langle (u_2, u_2), 0, 1, 1\rangle\}$,

$R = (R^c)^c = \{\langle (u_1, u_1), 0.3, 0.2, 0.9\rangle, \langle (u_1, u_2), 1, 0, 0.2\rangle, \langle (u_2, u_1), 0, 0.4, 0.3\rangle, \langle (u_2, u_2), 0.5, 0.3, 1\rangle\}$.

Lemma 5.2. $\forall R \in \mathrm{SVNR}(U)$, we have

(1) $(\overline{s}(R^c))^c = s(R)$.

(2) $\overline{s}(s(R)) = s(R)$.

(3) $s(\overline{s}(R)) = \overline{s}(R)$.

Proof. (1) By Theorem 5.1 (2), $\overline{s}(R^c) = R^c \cup (R^c)^{-1}$. By Theorems 3.1 and 4.1 (2), $(\overline{s}(R^c))^c = (R^c \cup (R^c)^{-1})^c = (R^c)^c \cap ((R^{-1})^c)^c = R \cap R^{-1} = s(R)$.

(2) and (3) The proofs are straightforward and follow from the definitions of symmetric kernel and symmetric closure.

Theorem 5.5. $\forall R \in \mathrm{SVNR}(U)$, at most six different SVNRs can be obtained by using symmetric kernel operator, symmetric closure operator and complement operator.

Proof. $\forall R \in U$, by Lemma 5.2 (1), $(\overline{s}(R^c))^c = s(R)$. Then we can replace symmetric kernel operator with complement operator and symmetric closure operator.

(1) Take complement operator first, then symmetric closure operator on R. One can only obtain the following three SVNRs:

R^c, $\overline{s}(R^c)$, $(\overline{s}(R^c))^c$.

It is because that by Theorem 3.1 (3) and Lemma 5.2, we have

$\overline{s}((\overline{s}(R^c))^c) = \overline{s}(s(R)) = s(R) = (\overline{s}(R^c))^c$, i.e. the fourth is the same as the third. $(\overline{s}((\overline{s}(R^c))^c))^c = (s(R))^c = \overline{s}(R^c)$, i.e. the fifth is the same as the second. The results of the latter steps will be repeated.

(2) Take symmetric closure operator first, then complement operator on R. Only the following two SVNRs can be constructed:

$\overline{s}(R)$, $(\overline{s}(R))^c$.

It is because that by Theorem 3.1 (3) and Lemma 5.2, we have

$\overline{s}((\overline{s}(R))^c) = \overline{s}(s(R^c)) = s(R^c) = (\overline{s}(R))^c$, which means that the third is the same as the second.

$(\overline{s}((\overline{s}(R))^c))^c = ((\overline{s}(R))^c)^c = \overline{s}(R)$, which means that the fourth is the same as the first. It will be repeated.

(3) Take symmetric closure operator successively or complement operator successively on R. By Theorems 3.1 (3) and 5.3 (5), $(R^c)^c = R$ and $\overline{s}(\overline{s}(R)) = \overline{s}(R)$. This is repeated emergence. The proof is complete.

The following example is given to illustrate the idea developed in Theorem 5.5.

Example 5.3. Consider U and R in Example 5.2. By using symmetric kernel operator, symmetric closure operator and complement operator, the following six different SVNRs can be obtained:

$R^c = \{\langle(u_1,u_1),0.9,0.8,0.3\rangle,\langle(u_1,u_2),0.2,1,1\rangle,\langle(u_2,u_1),0.3,0.6,0\rangle,\langle(u_2,u_2),1,0.7,0.5\rangle\}$,

$\overline{s}(R^c) = \{\langle(u_1,u_1),0.9,0.8,0.3\rangle,\langle(u_1,u_2),0.3,0.6,0\rangle,\langle(u_2,u_1),0.3,0.6,0\rangle,\langle(u_2,u_2),1,0.7,0.5\rangle\}$,

$(\overline{s}(R^c))^c = \{\langle(u_1,u_1),0.3,0.2,0.9\rangle,\langle(u_1,u_2),0,0.4,0.3\rangle,\langle(u_2,u_1),0,0.4,0.3\rangle,\langle(u_2,u_2),0.5,0.3,1\rangle\}$,

$\overline{s}(R) = \{\langle(u_1,u_1),0.3,0.2,0.9\rangle,\langle(u_1,u_2),1,0,0.2\rangle,\langle(u_2,u_1),1,0,0.2\rangle,\langle(u_2,u_2),0.5,0.3,1\rangle\}$,

$(\overline{s}(R))^c = \{\langle(u_1,u_1),0.9,0.8,0.3\rangle,\langle(u_1,u_2),0.2,1,1\rangle,\langle(u_2,u_1),0.2,1,1\rangle,\langle(u_2,u_2),1,0.7,0.5\rangle\}$,

$R = (R^c)^c = \{\langle(u_1,u_1),0.3,0.2,0.9\rangle,\langle(u_1,u_2),1,0,0.2\rangle,\langle(u_2,u_1),0,0.4,0.3\rangle,\langle(u_2,u_2),0.5,0.3,1\rangle\}$.

6 SVNR mappings

In this section, we introduce the notions of single valued neutrosophic relation mappings and inverse single valued neutrosophic relation mappings, then study some related properties.

Definition 6.1. Let U,V be two spaces of points (objects). f is a mapping from U to V.
(1) $f^{\to}$ is called a SVNR mapping from SVNR(U) to SVNR(V) induced by f. Concretely, $\forall R\in$ SVNR(U), $f^{\to}(R) = \{\langle(v_1,v_2),T_{f^{\to}(R)}(v_1,v_2),I_{f^{\to}(R)}(v_1,v_2),F_{f^{\to}(R)}(v_1,v_2)\rangle \mid (v_1,v_2)\in V\times V\}$, where $T_{f^{\to}(R)}(v_1,v_2) = \vee\{T_R(u_1,u_2)\mid u_i\in U, f(u_i)=v_i, i=1,2\}$, $I_{f^{\to}(R)}(v_1,v_2) = \wedge\{I_R(u_1,u_2)\mid u_i\in U, f(u_i)=v_i, i=1,2\}$, $F_{f^{\to}(R)}(v_1,v_2) = \wedge\{F_R(u_1,u_2)\mid u_i\in U, f(u_i)=v_i, i=1,2\}$.
(2) $f^{\leftarrow}$ is called an inverse SVNR mapping from SVNR(V) to SVNR(U) induced by f. Concretely, $\forall Q\in$ SVNR(V), $f^{\leftarrow}(Q) = \{\langle(u_1,u_2),T_{f^{\leftarrow}(Q)}(u_1,u_2),I_{f^{\leftarrow}(Q)}(u_1,u_2),F_{f^{\leftarrow}(Q)}(u_1,u_2)\rangle \mid (u_1,u_2)\in U\times U\}$, where $T_{f^{\leftarrow}(Q)}(u_1,u_2) = T_Q(f(u_1),f(u_2))$, $I_{f^{\leftarrow}(Q)}(u_1,u_2) = I_Q(f(u_1),f(u_2))$, $F_{f^{\leftarrow}(Q)}(u_1,u_2) = F_Q(f(u_1),f(u_2))$.

Next, we discuss some properties of SVNR mappings and inverse SVNR mappings.

Theorem 6.1. Let f be a mapping from U to V, $\forall R\in$ SVNR(U), $\forall T\in$ SVNR(V). Then
(1) $f^{\leftarrow}(f^{\to}(R))\supseteq R$. If f is one-one, then $f^{\leftarrow}(f^{\to}(R)) = R$.
(2) $f^{\to}(f^{\leftarrow}(Q))\subseteq Q$. If f is surjective, then $f^{\to}(f^{\leftarrow}(Q)) = Q$.

Proof. (1) $\forall u_1,u_2\in U$, by Definition 6.1, we have

$$\begin{aligned}T_{f^{\leftarrow}(f^{\to}(R))}(u_1,u_2) &= T_{f^{\to}(R)}(f(u_1),f(u_2))\\ &= \vee\{T_R(u_1',u_2')\mid u_i'\in U, f(u_i')=f(u_i), i=1,2\}\geq T_R(u_1,u_2),\\ I_{f^{\leftarrow}(f^{\to}(R))}(u_1,u_2) &= I_{f^{\to}(R)}(f(u_1),f(u_2))\\ &= \wedge\{I_R(u_1',u_2')\mid u_i'\in U, f(u_i')=f(u_i), i=1,2\}\leq I_R(u_1,u_2),\end{aligned}$$

$$\begin{aligned}F_{f^{\leftarrow}(f^{\rightarrow}(R))}(u_1,u_2) &= F_{f^{\rightarrow}(R)}(f(u_1),f(u_2))\\ &= \wedge\{F_R(u_1',u_2') \mid u_i'\in U, f(u_i')=f(u_i), i=1,2\} \le F_R(u_1,u_2).\end{aligned}$$

So $f^{\leftarrow}(f^{\rightarrow}(R)) \supseteq R$. By the proof procedure, it is easy to see that if f is one-one, then $f^{\leftarrow}(f^{\rightarrow}(R)) = R$.

(2) $\forall v_1, v_2 \in V$,

(i) If $v_1 \times v_2 \in f(U)\times f(U)$, we have

$$\begin{aligned}T_{f^{\rightarrow}(f^{\leftarrow}(Q))}(v_1,v_2) &= \vee\{T_{f^{\leftarrow}(Q)}(u_1,u_2) \mid u_i\in U, f(u_i)=v_i, i=1,2\}\\ &= \vee\{T_Q(f(u_1),f(u_2)) \mid u_i\in U, f(u_i)=v_i, i=1,2\}\\ &= T_Q(v_1,v_2),\\ I_{f^{\rightarrow}(f^{\leftarrow}(Q))}(v_1,v_2) &= \wedge\{I_{f^{\leftarrow}(Q)}(u_1,u_2) \mid u_i\in U, f(u_i)=v_i, i=1,2\}\\ &= \wedge\{I_Q(f(u_1),f(u_2)) \mid u_i\in U, f(u_i)=v_i, i=1,2\}\\ &= I_Q(v_1,v_2),\\ F_{f^{\rightarrow}(f^{\leftarrow}(Q))}(v_1,v_2) &= \wedge\{F_{f^{\leftarrow}(Q)}(u_1,u_2) \mid u_i\in U, f(u_i)=v_i, i=1,2\}\\ &= \wedge\{F_Q(f(u_1),f(u_2)) \mid u_i\in U, f(u_i)=v_i, i=1,2\}\\ &= F_Q(v_1,v_2).\end{aligned}$$

(ii) If $v_1 \times v_2 \notin f(U)\times f(U)$, we have

$$\begin{aligned}T_{f^{\rightarrow}(f^{\leftarrow}(Q))}(v_1,v_2) &= \vee\{T_{f^{\leftarrow}(Q)}(u_1,u_2) \mid u_i\in U, f(u_i)=v_i, i=1,2\}\\ &= \vee\emptyset = 0 \le T_Q(v_1,v_2),\\ I_{f^{\rightarrow}(f^{\leftarrow}(Q))}(v_1,v_2) &= \wedge\{I_{f^{\leftarrow}(Q)}(u_1,u_2) \mid u_i\in U, f(u_i)=v_i, i=1,2\}\\ &= \wedge\emptyset = 1 \ge I_Q(v_1,v_2),\\ F_{f^{\rightarrow}(f^{\leftarrow}(Q))}(v_1,v_2) &= \wedge\{F_{f^{\leftarrow}(Q)}(u_1,u_2) \mid u_i\in U, f(u_i)=v_i, i=1,2\}\\ &= \wedge\emptyset = 1 \ge F_Q(v_1,v_2).\end{aligned}$$

So $f^{\rightarrow}(f^{\leftarrow}(Q)) \subseteq Q$. By the proof procedure, it is easy to see that if f is surjective, then $f^{\rightarrow}(f^{\leftarrow}(Q)) = Q$.

Theorem 6.2. Let f be a mapping from U to V, $\forall R \in \mathrm{SVNR}(U)$.

(1) If f is surjective and R is reflexive, then $f^{\rightarrow}(R)$ is a reflexive SVNR in V.

(2) If f is one-one and R is anti-reflexive, then $f^{\rightarrow}(R)$ is an anti-reflexive SVNR in V.

(3) If R is symmetric, then $f^{\rightarrow}(R)$ is a symmetric SVNR in V.

Proof. (1) If f is surjective, then $\forall v \in V$ there exists $u \in U$ such that $f(u) = v$. By the reflexivity of R, we have $T_R(u,u) = 1$ and $I_R(u,u) = F_R(u,u) = 0$. Hence $\forall v \in V$, we have

$$T_{f^{\rightarrow}(R)}(v,v) = \vee\{T_R(u_1,u_2) \mid u_i \in U, f(u_i) = v, i = 1,2\} \geq T_R(u,u) = 1,$$
$$I_{f^{\rightarrow}(R)}(v,v) = \wedge\{I_R(u_1,u_2) \mid u_i \in U, f(u_i) = v, i = 1,2\} \leq I_R(u,u) = 0,$$
$$F_{f^{\rightarrow}(R)}(v,v) = \wedge\{F_R(u_1,u_2) \mid u_i \in U, f(u_i) = v, i = 1,2\} \leq F_R(u,u) = 0.$$

Thus $f^{\rightarrow}(R)$ is a reflexive SVNR in V.

(2) If R is anti-reflexive, then $\forall u \in U$, $T_R(u,u) = 0$ and $I_R(u,u) = F_R(u,u) = 1$. Thus $\forall v \in V$, (i) If $v \notin f(U)$, then $T_{f^{\rightarrow}}(R)(v,v) = 0$ and $I_{f^{\rightarrow}}(R)(v,v) = F_{f^{\rightarrow}}(R)(v,v) = 1$. (ii) If $v \in f(U)$, then there exists unique $u \in U$ such that $f(u) = v$ since f is one-one. Hence $\forall v \in V$, we have

$$T_{f^{\rightarrow}(R)}(v,v) = \vee\{T_R(u_1,u_2) \mid u_i \in U, f(u_i) = v, i = 1,2\} = T_R(u,u) = 0,$$
$$I_{f^{\rightarrow}(R)}(v,v) = \wedge\{I_R(u_1,u_2) \mid u_i \in U, f(u_i) = v, i = 1,2\} = I_R(u,u) = 1,$$
$$F_{f^{\rightarrow}(R)}(v,v) = \wedge\{F_R(u_1,u_2) \mid u_i \in U, f(u_i) = v, i = 1,2\} = F_R(u,u) = 1.$$

Therefore $f^{\rightarrow}(R)$ is an anti-reflexive SVNR in V.

(3) If R is symmetric, then $R = R^{-1}$. Hence $\forall v_1, v_2 \in V$,

$$\begin{aligned}
T_{f^{\rightarrow}(R)}(v_1,v_2) &= \vee\{T_R(u_1,u_2) \mid u_i \in U, f(u_i) = v_i, i = 1,2\} \\
&= \vee\{T_{R^{-1}}(u_1,u_2) \mid u_i \in U, f(u_i) = v_i, i = 1,2\} \\
&= \vee\{T_R(u_2,u_1) \mid u_i \in U, f(u_i) = v_i, i = 1,2\} \\
&= T_{f^{\rightarrow}(R)}(v_2,v_1), \\
I_{f^{\rightarrow}(R)}(v_1,v_2) &= \wedge\{I_R(u_1,u_2) \mid u_i \in U, f(u_i) = v_i, i = 1,2\} \\
&= \wedge\{I_{R^{-1}}(u_1,u_2) \mid u_i \in U, f(u_i) = v_i, i = 1,2\} \\
&= \wedge\{I_R(u_2,u_1) \mid u_i \in U, f(u_i) = v_i, i = 1,2\} \\
&= I_{f^{\rightarrow}(R)}(v_2,v_1), \\
F_{f^{\rightarrow}(R)}(v_1,v_2) &= \wedge\{F_R(u_1,u_2) \mid u_i \in U, f(u_i) = v_i, i = 1,2\} \\
&= \wedge\{F_{R^{-1}}(u_1,u_2) \mid u_i \in U, f(u_i) = v_i, i = 1,2\} \\
&= \wedge\{F_R(u_2,u_1) \mid u_i \in U, f(u_i) = v_i, i = 1,2\} \\
&= F_{f^{\rightarrow}(R)}(v_2,v_1).
\end{aligned}$$

So $f^{\rightarrow}(R)$ is a symmetric SVNR in V.

Theorem 6.3. Let f be a mapping from U to V, $\forall Q \in \text{SVNR}(V)$.

(1) If Q is reflexive, then $f^{\leftarrow}(Q)$ is a reflexive SVNR in U.

(2) If Q is anti-reflexive, then $f^{\leftarrow}(Q)$ is an anti-reflexive SVNR in U.

(3) If Q is symmetric, then $f^{\leftarrow}(Q)$ is a symmetric SVNR in U.

Proof. (1) If Q is reflexive, then $\forall v \in V$, $T_Q(v,v) = 1$ and $I_Q(v,v) = F_Q(v,v) = 0$.

Then $\forall u \in U$, $T_{f^{\leftarrow}}(Q)(u,u) = T_Q(f(u), f(u)) = 1$, $I_{f^{\leftarrow}}(Q)(u,u) = I_Q(f(u), f(u)) = 0$ and $F_{f^{\leftarrow}}(Q)(u,u) = F_Q(f(u), f(u)) = 0$. So $f^{\leftarrow}(T)$ is a reflexive SVNR in U.

(2) The proof is similar to (1).

(3) If Q is symmetric, then $Q = Q^{-1}$. Thus $\forall u_1, u_2 \in U$,

$T_{f^{\leftarrow}(Q)}(u_1, u_2) = T_Q(f(u_1), f(u_2)) = T_{Q^{-1}}(f(u_1), f(u_2)) = T_Q(f(u_2), f(u_1)) = T_{f^{\leftarrow}(Q)}(u_2, u_1)$,

$I_{f^{\leftarrow}(Q)}(u_1, u_2) = I_Q(f(u_1), f(u_2)) = I_{Q^{-1}}(f(u_1), f(u_2)) = I_Q(f(u_2), f(u_1)) = I_{f^{\leftarrow}(Q)}(u_2, u_1)$,

$F_{f^{\leftarrow}(Q)}(u_1, u_2) = F_Q(f(u_1), f(u_2)) = F_{Q^{-1}}(f(u_1), f(u_2)) = F_Q(f(u_2), f(u_1)) = F_{f^{\leftarrow}(Q)}(u_2, u_1)$.

So $f^{\leftarrow}(Q)$ is a symmetric SVNR in U.

Lemma 6.1. Let f be a mapping from U to V.

(1) If $R, S \in \text{SVNR}(U)$ and $R \subseteq S$, then $f^{\rightarrow}(R) \subseteq f^{\rightarrow}(S)$.

(2) If $Q, P \in \text{SVNR}(V)$ and $Q \subseteq P$, then $f^{\leftarrow}(Q) \subseteq f^{\leftarrow}(P)$.

Proof. The proof is straightforward from Definition 6.1.

Next, we give two main results of this section.

Theorem 6.4. Let f be a mapping from U to V, $\forall R \in \text{SVNR}(U)$. Then

(1) If f is surjective, then $f^{\rightarrow}(\overline{r}(R)) = \overline{r}(f^{\rightarrow}(R))$.

(2) $f^{\rightarrow}(\overline{s}(R)) = \overline{s}(f^{\rightarrow}(R))$.

(3) If f is bijective, then $f^{\rightarrow}(s(R)) = s(f^{\rightarrow}(R))$.

(4) If f is bijective, then $f^{\rightarrow}(ar(R)) = ar(f^{\rightarrow}(R))$.

Proof. (1) By Definition 5.1 and Theorem 6.2, $f^{\rightarrow}(\overline{r}(R))$ is a reflexive SVNR in V. By Theorem 5.1, $\overline{r}(R) = R \cup Id_N \supseteq R$. According to Lemma 6.1, $f^{\rightarrow}(\overline{r}(R)) \supseteq f^{\rightarrow}(R)$. If H is a reflexive SVNR in V and $f^{\rightarrow}(R) \subseteq H$. By Lemma 6.1 and Theorem 6.1, $R \subseteq f^{\leftarrow}(f^{\rightarrow}(R)) \subseteq f^{\leftarrow}(H)$. By Theorem 6.3, $f^{\leftarrow}(H)$ is a reflexive SVNR in U. Then $\overline{r}(R) \subseteq f^{\leftarrow}(H)$. According to Lemma 6.1 and Theorem 6.1, $f^{\rightarrow}(\overline{r}(R)) \subseteq f^{\rightarrow}(f^{\leftarrow}(H)) \subseteq H$. Therefore $f^{\rightarrow}(\overline{r}(R)) = \overline{r}(f^{\rightarrow}(R))$.

(2) The proof is similar to (1).

(3) By Definition 4.1 and Theorem 6.2, $f^{\rightarrow}(s(R))$ is a symmetric SVNR in V. By Theorem 4.1, $s(R) = R \cap R^{-1} \subseteq R$. According to Lemma 6.1, $f^{\rightarrow}(s(R)) \subseteq f^{\rightarrow}(R)$. If H is a symmetric SVNR in V and $f^{\rightarrow}(R) \supseteq H$. By Theorem 6.1 and Lemma 6.1, $R = f^{\leftarrow}(f^{\rightarrow}(R)) \supseteq f^{\leftarrow}(H)$. On the other hand, by Theorem 6.3, $f^{\leftarrow}(H)$ is a symmetric SVNR in U. Then $s(R) \supseteq f^{\leftarrow}(H)$. According to Theorem 6.1 and Lemma 6.1, then $f^{\rightarrow}(s(R)) \supseteq f^{\rightarrow}(f^{\leftarrow}(H)) = H$. So $f^{\rightarrow}(s(R)) = s(f^{\rightarrow}(R))$.

(4) The proof is similar to (3).

Theorem 6.5. Let f be a mapping from U to V, $\forall Q \in \mathrm{SVNR}(V)$.

(1) If f is bijective, then $f^{\leftarrow}(\overline{r}(Q)) = \overline{r}(f^{\leftarrow}(Q))$.

(2) If f is bijective, then $f^{\leftarrow}(\overline{s}(Q)) = \overline{s}(f^{\leftarrow}(Q))$.

(3) If f is one-one, then $f^{\leftarrow}(ar(Q)) = ar(f^{\leftarrow}(Q))$.

(4) $f^{\leftarrow}(s(Q)) = s(f^{\leftarrow}(Q))$.

Proof. (1) By Definition 5.1 and Theorem 6.3, $f^{\leftarrow}(\overline{r}(Q))$ is a reflexive SVNR on U. By Theorem 5.1, $\overline{r}(Q) = Q \cup Id_N \supseteq Q$. According to Lemma 6.1, $f^{\leftarrow}(\overline{r}(Q)) \supseteq f^{\leftarrow}(Q)$. If K is a reflexive SVNR in U and $f^{\leftarrow}(Q) \subseteq K$. By Theorem 6.1 and Lemma 6.1, $Q = f^{\rightarrow}(f^{\leftarrow}(Q)) \subseteq f^{\rightarrow}(K)$. By Theorem 6.2, $f^{\rightarrow}(K)$ is a reflexive SVNR in V. Then $\overline{r}(Q) \subseteq f^{\rightarrow}(K)$. According to Theorem 6.1 and Lemma 6.1, $f^{\leftarrow}(\overline{r}(Q)) \subseteq f^{\leftarrow}(f^{\rightarrow}(K)) = K$. So $f^{\leftarrow}(\overline{r}(Q)) = \overline{r}(f^{\leftarrow}(Q))$.

(2) The proof is similar to (1).

(3) By Definition 4.1 and Theorem 6.3, $f^{\leftarrow}(ar(Q))$ is an anti-reflexive SVNR in U. By Theorem 4.1, $ar(Q) = Q \cap Q^c \subseteq Q$. According to Lemma 6.1, $f^{\leftarrow}(ar(Q)) \subseteq f^{\leftarrow}(Q)$. If K is an anti-reflexive SVNR in U and $f^{\leftarrow}(Q) \supseteq K$. By Theorem 6.1 and Lemma 6.1, $K \supseteq f^{\rightarrow}(f^{\leftarrow}(Q)) \supseteq f^{\rightarrow}(K)$. By Theorem 6.2, $f^{\rightarrow}(K)$ is an anti-reflexive SVNR in V. Then $ar(Q) \supseteq f^{\rightarrow}(K)$. According to Lemma 6.1 and Theorem 6.1, $f^{\leftarrow}(ar(Q)) \supseteq f^{\leftarrow}(f^{\rightarrow}(K)) \supseteq K$. So $f^{\leftarrow}(ar(Q)) = ar(f^{\leftarrow}(Q))$.

(4) The proof is similar to (3).

7 Conclusion

In this paper, the theoretical point of view of SVNRs is investigated. We systematically study SVNRs, kernels and closures of a SVNR, and SVNR mappings. Some interesting properties are discussed. Based on these results, one can further probe the applications in real life situations of SVNRs.

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