An Exact Mapping from Navier-Stokes Equation to Schrödinger Equation via Riccati Equation

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In the present article we argue that it is possible to write down Schrödinger representation of Navier-Stokes equation via Riccati equation. The proposed approach, while differs appreciably from other method such as what is proposed by R. M. Kiehn, has an advantage, i.e. it enables us extend further to quaternionic and biquaternionic version of Navier-Stokes equation, for instance via Kravchenko's and Gibbon's route. Further observation is of course recommended in order to refute or verify this proposition.

1 Introduction

In recent years there were some attempts in literature to find out Schrödinger-like representation of Navier-Stokes equation using various approaches, for instance by R. M. Kiehn [1, 2]. Deriving exact mapping between Schrödinger equation and Navier-Stokes equation has clear advantage, because Schrodinger equation has known solutions, while exact solution of Navier-Stokes equation completely remains an open problem in mathematical-physics. Considering wide applications of Navier-Stokes equation, including for climatic modelling and prediction (albeit in simplified form called "geostrophic flow" [9]), one can expect that simpler expression of Navier-Stokes equation will be found useful.

In this article we presented an alternative route to derive Schrödinger representation of Navier-Stokes equation via Riccati equation. The proposed approach, while differs appreciably from other method such as what is proposed by R. M. Kiehn [1], has an advantage, i.e. it enables us to extend further to quaternionic and biquaternionic version of Navier-Stokes equation, in particular via Kravchenko's [3] and Gibbon's route [4, 5]. An alternative method to describe quaternionic representation in fluid dynamics has been presented by Sprössig [6]. Nonetheless, further observation is of course recommended in order to refute or verify this proposition.

2 From Navier-Stokes equation to Schrödinger equation via Riccati

Recently, Argentini [8] argues that it is possible to write down ODE form of 2D steady Navier-Stokes equations, and it will lead to second order equation of Riccati type.

Let ρ the density, μ the dynamic viscosity, and f the body force per unit volume of fluid. Then the Navier-Stokes equation for the steady flow is [8]:

$$\rho\,(v\cdot\nabla v) = -\nabla p + \rho\cdot f + \mu\cdot\Delta v\,. \tag{1}$$

After some necessary steps, he arrives to an ODE version of 2D Navier-Stokes equations along a streamline [8, p. 5] as follows:

$$u_1.\dot{u}_1 = f_1 - \frac{\dot{q}}{\rho} + v\cdot\dot{u}_1\,, \tag{2}$$

where $v = \frac{\mu}{\rho}$ is the kinematic viscosity. He [8, p. 5] also finds a general exact solution of equation (2) in Riccati form, which can be rewritten as follows:

$$\dot{u}_1 - \alpha\cdot u_1^2 + \beta = 0\,, \tag{3}$$

where:

$$\alpha = \frac{1}{2v}\,, \qquad \beta = -\frac{1}{v}\left(\frac{\dot{q}}{\rho} - f_1\right)s - \frac{c}{v}\,. \tag{4}$$

Interestingly, Kravchenko [3, p. 2] has argued that there is neat link between Schrödinger equation and Riccati equation via simple substitution. Consider a 1-dimensional static Schrödinger equation:

$$\ddot{u} + v\cdot u = 0 \tag{5}$$

and the associated Riccati equation:

$$\dot{y} + y^2 = -v\,. \tag{6}$$

Then it is clear that equation (5) is related to (6) by the inverted substitution [3]:

$$y = \frac{\dot{u}}{u}\,. \tag{7}$$

Therefore, one can expect to use the same method (7) to write down the Schrödinger representation of Navier-Stokes equation. First, we rewrite equation (3) in similar form of equation (6):

$$\dot{y}_1 - \alpha\cdot y_1^2 + \beta = 0\,. \tag{8}$$

By using substitution (7), then we get the Schrödinger equation for this Riccati equation (8):

$$\ddot{u} - \alpha\beta\cdot u = 0\,, \tag{9}$$

where variable α and β are the same with (4). This Schrödinger representation of Navier-Stokes equation is remarkably simple and it also has advantage that now it is possible to generalize it further to quaternionic (ODE) Navier-Stokes equation via quaternionic Schrödinger equation, for instance using the method described by Gibbon *et al.* [4, 5].

3 An extension to biquaternionic Navier-Stokes equation via biquaternion differential operator

In our preceding paper [10, 12], we use this definition for biquaternion differential operator:

$$\diamondsuit = \nabla^q + i\,\nabla^q = \left(-i\,\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}\right) + $$
$$+\ i\left(-i\,\frac{\partial}{\partial T} + e_1\frac{\partial}{\partial X} + e_2\frac{\partial}{\partial Y} + e_3\frac{\partial}{\partial Z}\right), \qquad (10)$$

where $e_1,\ e_2,\ e_3$ are *quaternion imaginary units* obeying (with ordinary quaternion symbols: $e_1 = i,\ e_2 = j,\ e_3 = k$): $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$ and quaternion *Nabla operator* is defined as [13]:

$$\nabla^q = -i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}\,. \qquad (11)$$

(Note that (10) and (11) include partial time-differentiation.)

Now it is possible to use the same method described above [10, 12] to generalize the Schrödinger representation of Navier-Stokes (9) to the biquaternionic Schrödinger equation, as follows.

In order to generalize equation (9) to quaternion version of Navier-Stokes equations (QNSE), we use first quaternion Nabla operator (11), and by noticing that $\Delta \equiv \nabla\nabla$, we get:

$$\left(\nabla^q\bar{\nabla}^q + \frac{\partial^2}{\partial t^2}\right)u - \alpha\beta\cdot u = 0\,. \qquad (12)$$

We note that the multiplying factor $\alpha\beta$ in (12) plays similar role just like $V(x) - E$ factor in the standard Schrödinger equation [12]:

$$-\frac{\hbar^2}{2m}\left(\nabla^q\bar{\nabla}^q + \frac{\partial^2}{\partial t^2}\right)u + \big(V(x) - E\big)\,u = 0\,. \qquad (13)$$

Note: we shall introduce the second term in order to "neutralize" the partial time-differentiation of $\nabla^q\bar{\nabla}^q$ operator.

To get *biquaternion* form of equation (12) we can use our definition in equation (10) rather than (11), so we get [12]:

$$\left(\diamondsuit\bar{\diamondsuit} + \frac{\partial^2}{\partial t^2} - i\,\frac{\partial^2}{\partial T^2}\right)u - \alpha\beta\cdot u = 0\,. \qquad (14)$$

This is an alternative version of *biquaternionic* Schrödinger representation of Navier-Stokes equations. Numerical solution of the new Navier-Stokes-Schrödinger equation (14) can be performed in the same way with [12] using Maxima software package [7], therefore it will not be discussed here.

We also note here that the route to *quaternionize* Schrödinger equation here is rather different from what is described by Gibbon *et al.* [4, 5], where the Schrödinger-equivalent to Euler fluid equation is described as [5, p. 4]:

$$\frac{D^2w}{Dt^2} - (\nabla Q)w = 0 \qquad (15)$$

and its quaternion representation is [5, p. 9]:

$$\frac{D^2w}{Dt^2} - q_b \otimes w = 0 \qquad (16)$$

with Riccati relation is given by:

$$\frac{D^q_a}{Dt + q_a \otimes q_a} = q_b \qquad (17)$$

Nonetheless, further observation is of course recommended in order to refute or verify this proposition (14).

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