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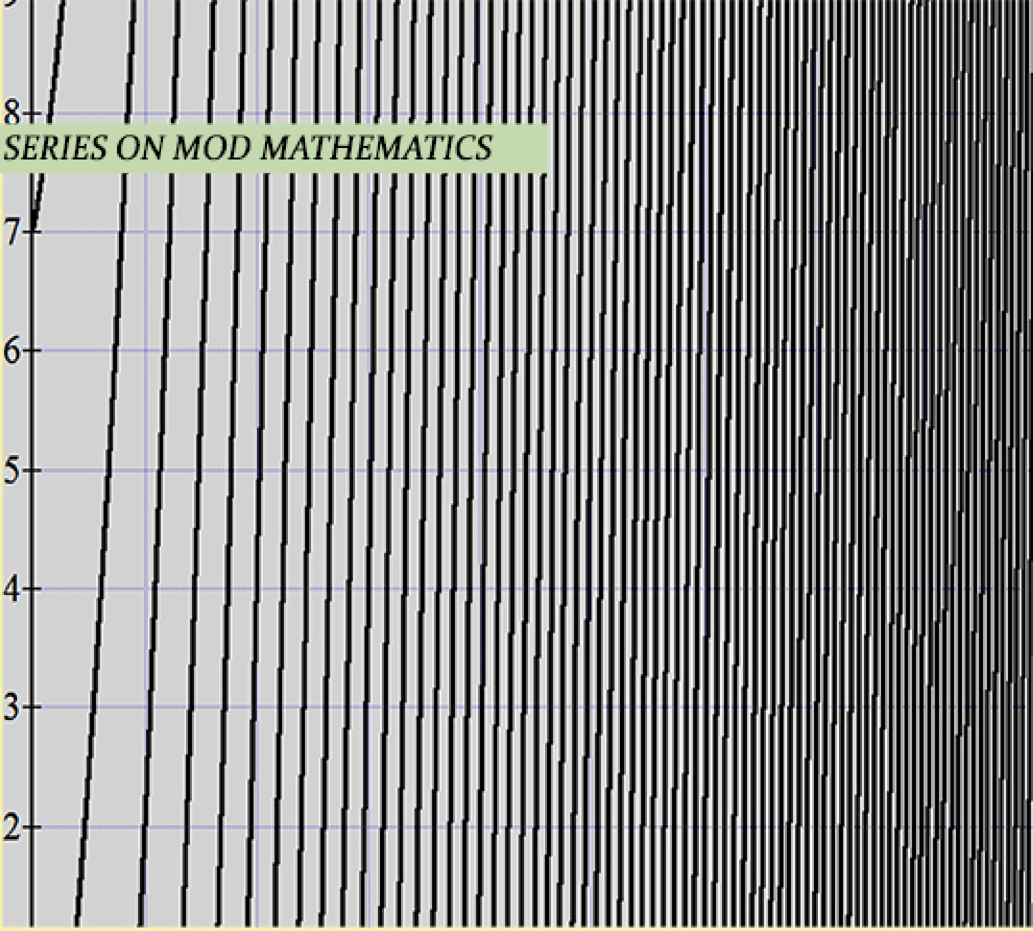


# PROBLEMS ON MOD STRUCTURES

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# PREFACE

In this book authors for the first time give several types of problems on MOD structures happens to be an interesting field of study as it makes the whole 4 quadrant plane into a single quadrant plane and the infinite line into a half closed open interval. So study in this direction will certainly yield several interesting results. The law of distributivity is not true. Further the MOD function in general do not obey all the laws of integration or differentiation. Likewise MOD polynomials in general do not satisfy the basic properties of polynomials like its roots etc.

Thus over all this study is not only innovative and interesting but challenging. So this book which is only full of problems based on MOD structures will be a boon to researchers. Further the MOD series books of the authors will certainly be an appropriate guide to solve these problems.

We wish to acknowledge Dr. K Kandasamy for his sustained support and encouragement in the writing of this book.

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## Chapter One

# PROBLEMS ON MOD INTERVALS

In this chapter we just recall all types of MOD intervals and the algebraic structure enjoyed by them. All problems related with these MOD intervals and the algebraic structures enjoyed by them are proposed here.

**DEFINITION 1.1:** *$S = \{[0, n); n \in Z^+ \setminus \{1\}$ is the MOD real interval mod n.*

*We see the infinite real line $(-\infty, \infty)$ can be mapped onto the MOD interval $[0, n); n \in Z^+ \setminus \{1\}$ in the following way.*

*Let $\eta: (-\infty, \infty) \to [0, n)$*

$$\eta(x) = \begin{cases} x & if \quad 0 < x < n \\ 0 & if \quad x = 0 \text{ or } tn\ (t \in (-\infty, \infty)) \\ n - x & if \quad -\infty < x < 0 \\ t & if \quad x > n > 0 \text{ and } \dfrac{x}{n} = s + \dfrac{t}{n} \\ n - t & if \quad -\infty < x < 0 \text{ and } \dfrac{x}{n} = s + \dfrac{t}{n} \end{cases}$$

*$\eta$ is the MOD real transformation.*

We will illustrate this by some examples.

***Example 1.1:*** Let S = {[0, 6)} be the real MOD interval modulo 6.

$$\begin{aligned} \eta: (-\infty, \infty) &\to [0, 6) \\ \eta(4.332) &= 4.332, \\ \eta(9.82) &= 3.82, \\ \eta(-7.68) &= 6 - 1.68 \\ &= 4.32. \end{aligned}$$

$$\begin{aligned} \eta(48) &= 0 \\ \eta(0) &= 0 \text{ and } \eta(-24) = 0 \\ \eta(0.331206) &= 0.331206 \\ \eta(-0.1150893) &= 5.8849107. \end{aligned}$$

$\eta$ is a MOD real transformation from $(-\infty, \infty)$ to $[0, 6)$.

***Example 1.2:*** Let S = {[0, 13)} be the real MOD interval.

$\eta: (-\infty, \infty) \to [0, 13)$ is defined by $\eta(0.84) = 0.84$

$$\begin{aligned} \eta(12.993) &= 12.993, \\ \eta(-1.234) &= 11.766, \\ \eta(65) &= 0, \\ \eta(-26) &= 0, \\ \eta(0) &= 0. \\ \eta(-13) &= 0 \text{ and so on.} \end{aligned}$$

This the way MOD real transformation $\eta$ from $(-\infty, \infty)$ to $[0, 13)$ is made.

We see $\eta$ is maps many points (or infinitely many points) onto a single point.

If we want to study the reverse the MOD transformation say $\eta^{-1} : [0, n) \to (-\infty, \infty)$.

We see $\eta^{-1}(0) = 0$, $\eta^{-1}(x) = tnx$ where $t \in Z^+$; so a single x can be or should be mapped onto infinitely many. This is true for every $x \in [0, n)$.

We will illustrate this situation by some examples.

***Example 1.3:*** Let S = {[0, 10)} be the MOD real interval.

Let $\eta^{-1} : [0, 10) \to (-\infty, \infty)$ defined by

$$\eta^{-1}(3) = \{10n + 3; n \in 10Z^+; 10n + 7 \in 10Z^-\}$$

Thus $\eta : (-\infty, \infty) \to [0, 10)$

$$\begin{aligned} \eta(3) &= 3 \\ \eta(-7) &= 3 \\ \eta(13) &= 3 \\ \eta(-17) &= 3 \end{aligned}$$

and so on.

Thus $\eta^{-1}$ the inverse 'map' is a pseudo map (as a single element is map onto) infinitely many elements, that is why the term pseudo map is used.

$\eta^{-1}(0.75) = \{0.75, 10.75, 20.75, 30.75, 40.75, \ldots, -9.25, -19.25, -29.25, \ldots\}$. Thus the single point 0.75 in [0, 10) is mapped onto infinitely many points in $(-\infty, \infty)$.

That is why we can $\eta^{-1}$ as the pseudo MOD real transformation. Likewise the mappings are carried out in a very systematic way.

We give one more example of this situation.

***Example 1.4:*** Let $\eta^{-1} : [0, 12) \to (-\infty, \infty)$ be the pseudo MOD map.

$$\eta^{-1}(0) = \{0, 12n, n \in Z^+\}$$
$$\eta^{-1}(1) = \{1, 13, 25, 37, \ldots \text{ and } -11, -23, -35, \ldots\}$$

$\eta^{-1}(2) = \{2, 14, 26, 38, 50, \ldots, -10, -22, -34, -46, \ldots\}$
$\eta^{-1}(3) = \{3, 15, 27, 39, 51, 63, \ldots, -9, -21, -33, \ldots\}$

Thus infinitely many points of $(-\infty, \infty)$ are associated with a single point of $[0, 12)$.

Consider $0.3189 \in [0, 12)$.

$\eta^{-1}(0.3189) = \{0.3189, 12.\ 3189, 24.3189, 36.3189, \ldots, 11.6811, 23.6811, 35.6811, 47.6811$ and so on$\}$.

Thus the single point 0.3189 is mapped on an infinite number of points by the pseudo MOD map

$\eta^{-1} : [0, 12) \to (-\infty, \infty)$. Further both MOD transformation map $\eta$ and pseudo inverse MOD maps are unique.

We have infinitely many real MOD intervals, however only one real interval $(-\infty, \infty)$.

Next we can define operations of $+$ and $\times$ on $[0, n)$.

$\{[0, n), +\}$ is a group of infinite order.

$\eta : (-\infty, \infty) \to [0, n)$ is compatible with respect to addition. For if we take the MOD interval $[0, 4)$.

Let $x = 5.732$ and $y = 0.193 \in (-\infty, \infty)$.

$$\begin{aligned}
x + y &= 5.925 \\
\eta(x) &= 1.732 \\
\eta(y) &= 0.193 \\
\eta(x) &= \eta(x) + \eta(y) \\
\eta(5.925) &= \eta(5.732) + \eta(0.193) \\
1.925 &= 1.732 + 0.193 \\
&= 1.925.
\end{aligned}$$

Hence we see $\eta$ can be compatible with respect to $+$.

We have only one group under '+' using $(-\infty, \infty)$ but there are infinitely many MOD interval real groups
$\{[0, m) \mid m \in Z^+ \setminus \{1\}, +\}$. This has subgroups of both finite and infinite order.

***Example 1.5:*** Let $G = \{[0, 24), +\}$ be the MOD interval real group of infinite order. $\{Z_{24}, +\} = H_1$ is a subgroup of finite order.

$H_2 = \{0, 6, 12, 18, +\}$ is a subgroup of finite order.
$H_3 = \{0, 8, 16\}$ is a subgroup of finite order.
$H_4 = \{0.1, 0.2, \ldots, 23, 23.1, 23.2, \ldots, 23.9, 0\} \subseteq G$ is a subgroup of G.

***Example 1.6:*** Let $G = \{[0, 17), +\}$ be a MOD interval group of infinite order.

Can we say G is a torsion group?

We can as in case of usual groups define MOD homomorphism from $G = \{[0, m), +\}$ to $H = \{[0, t), +\}$.

This will be illustrated by examples.

***Example 1.7:*** Let $G = \{[0, 6), +\}$ and $H = \{[0, 12), +\}$ be two MOD groups of infinite order.

Define $\phi : G \to H$

$\phi(0) = 0$
$\phi(x) = 2x$ if $x \in G$
$\phi(0.76) = 1.52$ and so on.

Clearly $\phi(x + y) = 2x + 2y$

Let $x = 3.84$ and $y = 4.57 \in G$.

$$\begin{aligned} \phi(x + y) &= \phi(3.84 + 4.57) \\ &= 2.41 \times 2 \\ &= 4.82 \qquad \ldots \quad \text{I} \end{aligned}$$

$$\begin{aligned} \phi(x) + \phi(y) &= \phi(3.84) + \phi(4.57) \\ &= 7.68 + 9.14 \\ &= 4.82 \qquad \ldots \quad \text{II} \end{aligned}$$

I and II are identical hence the claim.

This is the way operations or MOD homomorphisms are made.

Clearly one can define MOD isomorphisms of MOD interval group.

Next we proceed onto define the notion of MOD interval semigroups.

**DEFINITION 1.2:** *Let S = {[0, m), ×} be the MOD interval semigroup for a, b ∈ S. a ×b (mod m) ∈ S.*

We will illustrate this situation by some examples.

***Example 1.8:*** Let M = {[0, 12), ×} be the MOD interval semigroup.

Let $x = 3.2$ and $y = 6.9 \in M$

$x \times y = 3.2 \times 6.9 = 10.08 \pmod{12}$.

Let $x = 6$ and $y = 2$ so $x \times y = 6 \times 2 \equiv 0 \pmod{12}$.

Thus M has zero divisors.

Clearly $1 \in M$ is such that $1 \times x = x$ for all $x \in M$.

Thus M is a monoid. M has zero divisors. $11 \in M$ is such that $11 \times 11 = 1 \pmod{12}$.

Let $x = 2.4$ and $y = 5 \in [0, 12)$.

Clearly $x \times y = 2.4 \times 5 = 0 \pmod{12}$ so is a zero divisor. But $5 \in [0, 12)$ is a unit as $5 \times 5 \equiv 1 \pmod{12}$. Thus a unit contributes to a zero divisor.

This gives one a natural problem namely why in the MOD interval semigroups we have units contributing to zero divisors.

This is the one of the marked difference between the usual semigroups and MOD interval semigroups. Thus MOD interval semigroup behave in an odd way so we define such zero divisors contributed by units of the MOD interval semigroup as MOD pseudo zero divisors or just pseudo zero divisors.

***Example 1.9:*** Let $S = \{[0, 7), \times\}$ be the MOD interval semigroup of infinite order.

This has only pseudo zero divisors. For $x = 3.5$ and $y = 2 \in S$ is such that $x \times y = 0 \pmod 7$ is a pseudo zero divisor however $y = 2$ is a unit in S as $2 \times 4 \equiv 1 \pmod 7$.

Consider $x = 1.75$ and $y = 4 \in S$.

$x \times y = 0 \pmod 7$ is again pseudo zero divisor; however y is a unit of S.

Thus S has pseudo zero divisors which are not zero divisors for one of them is a unit.

Next we proceed onto study about the substructures in MOD interval monoids. MOD interval monoids have finite order subsemigroups but can it have ideals of finite order is a question mark.

***Example 1.10:*** Let $S = \{[0, 18), \times\}$ be the MOD interval monoid.

$$\begin{aligned} P_1 &= \{0, 9\}, \\ P_2 &= \{0, 2, 4, \ldots, 16\}, \\ P_3 &= \{0, 3, 6, \ldots, 15\}, \\ P_4 &= \{0, 6, 12\} \text{ and} \\ P_5 &= \{Z_{18}, \times\} \end{aligned}$$

are finite order MOD subsemigroups of S.

$H_1 = \{\langle 0.01\rangle, \times\}$ is an infinite order subsemigroup of S.
$H_2 = \{\langle 0.1\rangle\}$,
$H_3 = \{\langle 0.2\rangle\}$ are all some of the infinite order MOD subsemigroup of S.

None of them are ideals.

Finding ideals happens to be a very difficult problem.

***Example 1.11:*** Let S = $\{[0, 23), \times\}$ be the MOD interval semigroup (monoid).

S is of infinite order.

$P_1 = \{Z_{23}\}$ is a subsemigroup of finite order.

$P_2 = \{\langle 0.1\rangle\}$ is a subsemigroup of infinite order.

$P_3 = \{\langle 0.2\rangle\}$ is a subsemigroup of infinite order and so on.

Finding ideals of S happens to be a challenging problem.

Now MOD interval groups and semigroups are also built using matrices with entries from MOD intervals [21-2, 24].

Study in this direction is interesting.

Next we proceed onto describe the notion of MOD real interval pseudo rings for more [21].

***Example 1.12:*** Let R = {[0, 10), +, ×} be the MOD interval pseudo ring. R has zero divisors, units and idempotents. R has subrings as well as pseudo subrings.

However finding pseudo ideals of R happens to be a challenging problem.

***Example 1.13:*** Let W = {[0, 29), +, ×} be the MOD interval pseudo ring. This MOD pseudo ring has pseudo zero divisors, units and subrings of finite order.

Can P = {⟨0.1⟩, +, ×} be the MOD pseudo subring of W?

Finding MOD pseudo subrings of infinite order happens to be a very difficult problem.

$Z_{29}$ = P is a subring of order 29 which is not pseudo.

Infact P is a field so W is a S-real MOD pseudo ring.

Studying properties associated with them happens to be a very difficult problem.

***Example 1.14:*** Let S = {[0, 48), +, ×} be the pseudo MOD real interval ring of infinite order. This has several subrings of finite order which are not pseudo subrings.

Once again finding ideals of S happens to be a very difficult problem.

Next we introduce the MOD neutrosophic interval.

Let (–∞I, ∞I); $I^2 = I$ be the real neutrosophic interval.

[0, m)I = {aI | a ∈ [0, m), $I^2 = I$}, 1 < m < ∞ is defined as the MOD neutrosophic interval.

Clearly N = {[0, m)I} under + is an abelian group under product is a semigroup and under + and ×. N is only a MOD pseudo neutrosophic interval ring.

All properties associated with MOD real interval can be derived for N also. For more refer [24].

Next we proceed onto study the notion of MOD finite complex modulo integers $[0, m)i_F$; $i_F^2 = (m - 1)$, got from the complex infinite interval $(-\infty i, \infty i)$.

We can have MOD transformations both for MOD interval neutrosophic set [0, m)I as well as MOD interval finite complex modulo integers $[0, m)i_F$; $i_F^2 = m - 1$.

This is left as a matter routine for the reader however for more information refer [18, 21, 24].

Clearly on $[0, m)i_F$ for no value of m; $1 < m < \infty$ we can built MOD interval finite complex modulo integer semigroup under × or a MOD interval finite complex modulo integer pseudo ring as × is not a closed operation on $[0, m)i_F$ as $i_F \times 3i_F = 3i_F^2 = 3(m - 1) \notin [0, m)i_F$.

This is the marked difference between the other types of MOD intervals and the MOD finite complex modulo integer interval.

***Example 1.15:*** Let M = $\{[0, 8)i_F,\ i_F^2 = 7, +\}$ be a MOD interval finite complex modulo integer group of infinite order.

However M is not even closed under × as $2i_F \times 2.4i_F = 4.8i_F^2 = 4.8 \times 7 = 1.6 \notin$ M. Hence {M, +, ×} is also not a MOD complex modulo integer interval pseudo ring.

Only the algebraic structure viz. group under + can be defined on $[0, m)i_F$; $1 < m < \infty$ as $i_F^2 = m - 1$.

Now we proceed onto define the MOD dual number interval got from the real dual number interval $(-\infty g, \infty g)$ $g^2 = 0$; the transformation from $(-\infty g, \infty g)$ to $[0, m)g$ is the usual one discussed in [17].

We will illustrate this by some examples.

***Example 1.16:*** Let M = $\{[0, 18)g \mid g^2 = 0, +\}$ be the MOD interval dual number group. M is of infinite order. M has subgroups of finite order also.

***Example 1.17:*** Let P = $\{[0, 29)g \mid g^2 = 0, +\}$ be the MOD dual number group. M has subgroups of finite order.

Now we proceed onto build semigroup using $[0, m)g$; $g^2 = 0$. [17, 21, 24].

***Example 1.18:*** Let B = $\{[0, 12)g \mid g^2 = 0, \times\}$ be the MOD interval dual number semigroup. Clearly B is not a monoid. Further every proper subset of B is a subsemigroup of B. Infact B is a MOD interval dual number zero square semigroup of infinite order.

Further every subsemigroup is also an ideal of B. These happen for any $1 < m < \infty$.

Next we proceed to give examples of MOD interval dual number ring.

***Example 1.19:*** Let M = $\{[0, 25)g; +, \times\}$ be the MOD interval dual number zero square ring. Since distributive law is true it is not a pseudo ring only a ring.

Every subgroup of M under + is an ideal of M. M has both finite order as well as infinite order ideals.

For instance $P_1 = \{0, 5g, 10g, 15g, 20g\} \subseteq M$ is an ideal.

$P_2 = \{Z_{25}g \mid g^2 = 0\}$ is an ideal of M of order 25.

$P_3 = \{0, 0.1g, 0.2g, \ldots, g, \ldots, 24g, 24.1g, 24.2g, \ldots, 24.9g\} \subseteq M$ is a ideal of finite order.

Study of dual number MOD interval rings is important and interesting for two reasons it can contribute to zero square semigroups as well as subrings.

Next we consider the MOD interval special dual like number set $[0, m)h$, $h^2 = h$.

On $[0, m)h$ we proceed to give operations as $+$ or $\times$ or both $+$ and $\times$. [18, 21, 24].

All these situations will be represented by some examples.

***Example 1.20:*** Let $S = \{[0, 15)h, h^2 = h, +\}$ be an infinite MOD interval special dual like number group. S has subgroups of both finite and infinite order.

$P_1 = Z_{15}h$ is a subgroup of order 15.

$P_2 = \{0, 3h, 6h, 9h, 12h\}$ is a subgroup of order 5 and so on.

***Example 1.21:*** Let $P_1 = \{[0, 19)h, h^2 = h, +\}$ be the MOD interval special dual like number group of infinite order. $P_1$ has subgroups of both finite and infinite order.

For more refer [18, 21, 24].

Now we proceed onto give examples of MOD interval special dual like number semigroup.

***Example 1.22:*** Let $M = \{[0, 24)h, h^2 = h, \times\}$ be the MOD interval special dual like number semigroup.

M has a subsemigroup of finite order. M has zero divisors, idempotents but no units. M has subsemigroups of finite order. But M has no subgroups. That is M is not a S-subsemigroup.

***Example 1.23:*** Let B = $\{[0, 47)h \mid h^2 = h, \times\}$ be the MOD interval special dual like semigroup.

$P_1 = Z_{47}h$ is a finite subsemigroup. However B is not a S-semigroup.

***Example 1.24:*** Let M = $\{[0, 16)h, h^2 = h; \times\}$ be the MOD interval special dual like semigroup which has zero divisors and idempotents.

Thus we have several properties associated with MOD interval special dual like number semigroup of infinite order.

Next we proceed onto study the properties of MOD interval special dual like number pseudo ring.

All these rings are pseudo as distributive law is not true in general.

***Example 1.25:*** Let S = $\{[0, 82)h, h^2 = h, \times, +\}$ be the MOD interval pseudo special dual like number ring of infinite order.

S has subrings of finite order like $P_1 = \{0, 41h\}$,

$P_2 = \{Z_{82}h\}$ and

$P_3 = \{0, 2h, 4h, \ldots, 80h\}$ are subrings of finite order which are not pseudo. S has also subrings which are pseudo.

Finding subrings of infinite order and ideals of S happens to be a challenging problem.

For more about these structures please refer [19].

***Example 1.26:*** Let P = $\{[0, 31)h, h^2 = h, +, \times\}$ be the MOD interval special dual like number pseudo ring. P has finite subrings which are not pseudo. P has subrings of infinite order. Find ideals of P.

Thus P has pseudo zero divisors and so on.

Next we proceed onto study the notions associated with MOD interval special quasi dual numbers $[0, m)k$, $k^2 = (m - 1)k$, $1 < m < \infty$.

For more about these refer [19].

We will illustrate this by some example.

***Example 1.27:*** Let $M = \{[0, 24)k, k^2 = 23k, +\}$ be the MOD interval special quasi dual number group. M is an infinite group. M has subgroups of finite order as well as subgroups of infinite order.

***Example 1.28:*** Let $P = \{[0, 43)k, k^2 = 42k, +\}$ be the MOD interval special quasi dual number group. P has only few number of subgroups of finite order.

Next we proceed onto study MOD interval special quasi dual number semigroups.

***Example 1.29:*** Let $B = \{[0, 48)k, k^2 = 47k, \times\}$ be the MOD interval special quasi dual number semigroup. B has zero divisors.

Finding idempotents happen to be a difficult problem.

For more about these refer [19, 24].

***Example 1.30:*** Let $M = \{[0, 29)k, k^2 = 28k, \times\}$ be the MOD interval special quasi dual number semigroup. M has pseudo zero divisors.

Finding S-units or units is a challenging problem.

We propose several problems associated with this structure.

Next we proceed onto describe MOD interval specific quasi dual number pseudo rings.

***Example 1.31:*** Let B = $\{[0, 12)k, k^2 = 11k, +, \times\}$ be the MOD interval special quasi dual number pseudo rings. B is of infinite order. B has zero divisors. Finding idempotents is a difficult task.

However B has subrings of finite order which are not in general not pseudo. Finding ideals is yet another difficult task.

***Example 1.32:*** Let S = $\{[0, 19)k, k^2 = 18k, +, \times\}$ be the MOD interval special quasi dual number pseudo ring. P = $\{Z_{19}k\} \subseteq S$ is a subring of finite order.

Is S a S-ring? This question in general is difficult as we have to find a unit, but $k^2 = 18k$. [19].

So finding ideals, subrings etc; happens to be a challenging task.

This pseudo ring has pseudo zero divisors. Infact all MOD interval special quasi dual number pseudo rings in which m is a prime always contains pseudo zero divisors.

However even finding zero divisors is a challenging problem. Even if p is a prime finding units in those rings are very difficult.

But we have just recalled these concepts with illustrative examples mainly to make this book a self contained one.

Further these concepts are elaborately analysed in books [19-24] but the problems in this book are mainly given as the concept of MOD mathematics or more to be in a layman's language the notion of small mathematics is very new and adventures in the mathematical world. They behave in many places in a chaotic way as the real world.

They are not well organized as the real mathematical world. So this property of chaos has attracted the authors, so authors wish to share these concepts with those interested co-mathematicians.

The best way to do it is give or suggest a series of problems.

Some problems are simple and direct. Some of them are difficult problems. Some can be considered as open conjectures.

The main and sustained work on MOD mathematics is surprising us for when they are converted to small scale they do not possess all the properties when they are large.

This difficulty is not yet completely overcome by us.

In the following we suggest some problems for this chapter.

**Problems**

1. Can we say G = {[0, m), +, m $\in$ $Z^+ \setminus \{1\}$} have infinite number of subgroups finite order?

2. How many subgroups of infinite order can G = {[0, m), +}, (m a prime) contain?

3. Can G = {[0, m), +}, m $\in$ $Z^+ \setminus \{1\}$ be a torsion free group?

4. Define $\phi$ : G $\to$ H where G = {[0, 43), +} and H = {[0, 24), +} be two MOD interval groups be a MOD homomorphism.

   i) How is $\phi$ defined?

   ii) Is it ever possible to be have a one to one MOD homomorphism?

iii) Can we have ker$\phi$ to be a finite subgroups of G?

iv) Can we define a $\phi : G \to H$ so that ker$\phi$ is an infinite subgroup?

v) Let S = {Collection of all MOD homomorphism from G to H}.

What is the algebraic structure enjoyed by S?

vi) Find $\delta : H \to G$.
Let R = {Collection of all MOD homomorphism from H to G}.
What is the algebraic structure enjoyed by R?

vii) What is relation exist between R and S?

5. Let G = {[0, 48), +} and H = {[0, 35), +} be any two MOD interval group.

   Study questions (i) to (vii) of problem 4 for this G and H.

6. Characterize the property of MOD interval pseudo zero divisors in MOD interval semigroups (monoids) S = {[0, m), $\times$}.

7. Find special and distinct features enjoyed by MOD interval monoids S = {[0, n), $\times$}.

8. Can S = {[0, 19), $\times$} have only finite number of pseudo zero divisors?

9. Can a MOD interval monoid S = {[0, m), ×} be free from MOD pseudo zero divisors?

10. Prove all MOD interval monoids S = {[0, m), ×} are Smarandache semigroups.

11. Prove or disprove MOD interval monoids S = {[0, 24), ×} has more number of zero divisors than the MOD interval monoid R = {[0, 43), ×}.

12. Can P = {[0, p), ×} (p a prime) the MOD interval monoid have infinite number of pseudo zero divisor?

13. Can P in problem (12) have zero divisors which are not pseudo zero divisors?

14. Is it ever possible to find MOD interval real semigroups (monoids) which can have MOD interval ideals which is of finite order?

    Justify your claim.

15. Let S = {[0, 23), +, ×} be the MOD interval pseudo real ring.

    i) Find all zero divisors of S.
    ii) Can S have infinite number of zero divisors?
    iii) Can S have idempotents?
    iv) Can S have pseudo zero divisors?
    v) Can S have S-units, S-idempotents and S-zero divisors?
    vi) Can S have ideals of finite order?
    vii) How many of the subrings of S are of finite order?
    viii) Find all pseudo ideals of S.
    ix) Can S have S-pseudo ideals?

x) Can S have S-subrings of infinite order?
xi) Is S a S-pseudo interval ring?

16. Let P = {[0, 145), +, ×} be the MOD real interval pseudo ring.

    Study questions (i) to (xi) of problem (15) for this P.

17. Let S = {[0, m), +, ×, m a composite number} be the MOD real interval pseudo ring.

    Study questions (i) to (xi) of problem (15) for this S.

18. Let W = {[0, p), +, ×; p a prime} be the MOD real interval pseudo ring.

    Study questions (i) to (xi) of problem (15) for this W.

19. What are the distinct and special features enjoyed by N = {[0, m)I; $I^2$ = I}?

    i) Is N a group under +?
    ii) Can{N, ×} be a semigroup?
    iii) Can{N, +, ×} be a MOD pseudo interval neutrosophic ring?
    Study all properties of N, 1 < m < ∞.

20. What are the special features enjoyed by MOD interval finite complex modulo integers [0, m)$i_F$, $i_F^2$ = m – 1?

21. Prove this set [0, m)$i_F$ is not closed under product operation.

22. Show the MOD interval dual numbers [0, m)g, $g^2$ = 0 behave in a very different way from the other MOD intervals.

23. Obtain all the special features enjoyed by MOD special dual like numbers interval $[0, m)h$, $h^2 = h$; $1 < m < \infty$.

24. Prove $W = \{[0, 23)h, h^2 = h, +\}$ is a MOD interval special dual like number group of infinite order.

    $P_1 = Z_{23}h$ is a subgroup of W of order 23.
    $P_2 = \{0, 0.1h, 0.2h, \ldots, h, \ldots, 20h, 20.1h, \ldots, 22h, 22.4h, 22.5h, \ldots, 22.9h, +, h^2 = h\}$ is also subgroup of finite order in the MOD interval dual like number group $W = \{[0, 23)h, h^2 = h, +\}$.

    Find all finite number of MOD interval special dual like number subgroups.

25. Let $N = \{[0, 42)h, h^2 = h, +\}$ be the MOD interval special dual like number group.

    i) Find the number of subgroups of finite order.
    ii) Find the number of subgroups of infinite order.

26. Let $T = \{[0, 31)h, h^2 = h, +\}$ be the MOD interval special dual like number group.

    Study questions (i) and (ii) of problem (25) for this T.

27. Find all special properties satisfied by the MOD interval special dual like number semigroup $\{[0, m)h, h^2 = h, \times\}$ $(1 < m < \infty)$.

28. Let $B = \{[0, 20)h, h^2 = h, \times\}$ be the MOD interval special dual like number semigroup.

i) Find the number of finite order subsemigroups of B.
ii) Find all infinite order subsemigroups of B.
iii) Can B ever be a S-semigroup?
iv) Find all ideals of B.
v) Can ideals of B be of finite order? Justify your claim.
vi) Find any S-ideal of B.
vii) Find all idempotents and S-idempotents of B.
viii) Find all zero divisors and S-zero divisors of B.
ix) Find any other special feature enjoyed by B.

29. Let S = $\{[0, 23)h, h^2 = h, \times\}$ be the MOD interval special dual like number semigroup.

    Study questions (i) to (ix) of problem (28) for this S.

30. Let M = $\{[0, 45)h, h^2 = h, \times\}$ be the MOD interval special dual like number semigroup.

    Study questions (i) to (ix) of problem (28) for this M.

31. Let A = $\{[0, 26)h, h^2 = h, +, \times\}$ be the MOD interval special dual like number pseudo ring.

    i) Find the number of subrings which are not pseudo in A.
    ii) Can A be a S-ring?
    iii) Can A be a pseudo ring which has zero divisors as well as S-zero divisors?
    iv) Can A have S-idempotents?
    v) Can A have S-ideals?
    vi) Can A have ideals of finite order?
    vii) Can A have a subfield of infinite order?
    viii) Obtain any other special feature enjoyed by A.

32. Let M = $\{[0, 24)h, h^2 = h, +, \times\}$ be the MOD interval pseudo special dual like number ring.

    Study questions (i) to (viii) of problem (31) for this M.

33. Let N = $\{[0, 47)h, h^2 = h, +, \times\}$ be the MOD interval pseudo special dual like number ring.

    Study questions (i) to (viii) of problem (31) for this N.

34. Let Z = $\{[0, 248)h, h^2 = h, +, \times\}$ be the MOD interval special dual like number pseudo ring.

    Study questions (i) to (viii) of problem (31) for this Z.

35. Obtain all special features associated with MOD interval special dual like number algebraic structures.

36. Let S = $\{[0, 28)k, k^2 = 27k, +\}$ be the MOD interval special quasi dual number group.

    i) Prove $o(S) = \infty$.

    ii) How many subgroups of finite order does S have?

    iii) Find the number of subgroups of S of infinite order.

    iv) Obtain a transformation from group $(-\infty k, \infty k) \to S$. ($k^2 = -k$ in $(-\infty k, \infty k)$).

37. Let M = $\{[0, 15)k, k^2 = 14k, +\}$ be the MOD interval special quasi dual number group.

    Study questions (i) to (iv) of problem (36) for this M.

38. Let S = $\{[0, 47)k, k^2 = 46k, +\}$ be the MOD interval special quasi dual number group.

    Study questions (i) to (iv) of problem (36) for this S.

39. Let B = $\{[0, 45)k, k^2 = 44k, \times\}$ be the MOD interval special quasi dual number semigroup.

    i) Find all finite order subsemigroups of B.
    ii) Can B have ideals of finite order?
    iii) Find all infinite subsemigroups of B which are not ideals.
    iv) Can B be a S-semigroup?
    v) Find all pseudo zero divisors of B.
    vi) Find all zero divisors which are not pseudo zero divisors of B.
    vii) Can B have units?
    viii) Find all idempotents and S-idempotents of B.
    ix) Can B have S-zero divisors?
    x) Can B have S-ideals?
    xi) Obtain any other special or interesting property associated with B.

40. Let M = $\{[0, 23)k, k^2 = 22k, \times\}$ be the MOD interval special quasi dual number semigroup.

    Study questions (i) to (xi) of problem (39) for this M.

41. Let W = $\{[0, 24)k, k^2 = 23k, \times\}$ be the MOD interval special quasi dual number semigroup.

    Study questions (i) to (xi) of problem (39) for this W.

42. Let T = $\{[0, 42)k, k^2 = 41k, +, \times\}$ be the MOD interval special quasi dual number pseudo ring.

    Study all questions (i) to (xi) of problem (39) by appropriately changing from semigroups to pseudo rings.

43. Let S = $\{[0, 53)k, k^2 = 52k, +, \times\}$ be the MOD interval special quasi dual number pseudo ring.

    Study questions (i) to (xi) of problem (39) by appropriately changing from semigroup to pseudo ring.

44. Let P = $\{[0, m)k, k^2 = (m - 1)k, +, \times\}$ be the MOD interval special quasi dual number pseudo ring.

    Study questions (i) to (xi) of problem (39) by appropriately changing from semigroup to pseudo ring when

    i. m is a composite number.

    ii. m is a prime number.

45. Prove there exists a MOD transformation from the ring R = $\{(-\infty k, \infty k); k^2 = -k, +, \times\}$ to S = $\{[0, m)k, k^2 = (m - 1)k, +, \times\}$.

46. Obtain any other special features of MOD interval pseudo rings.

**Chapter Two**

# PROBLEMS ASSOCIATED WITH MOD PLANES

In this chapter we propose problems simple, difficult as well as open conjectures related to the seven types of MOD planes. To make this chapter a self contained one we just recall or describe these notions very briefly. However for more about these notions please refer [26-7]. Throughout this book the real plane

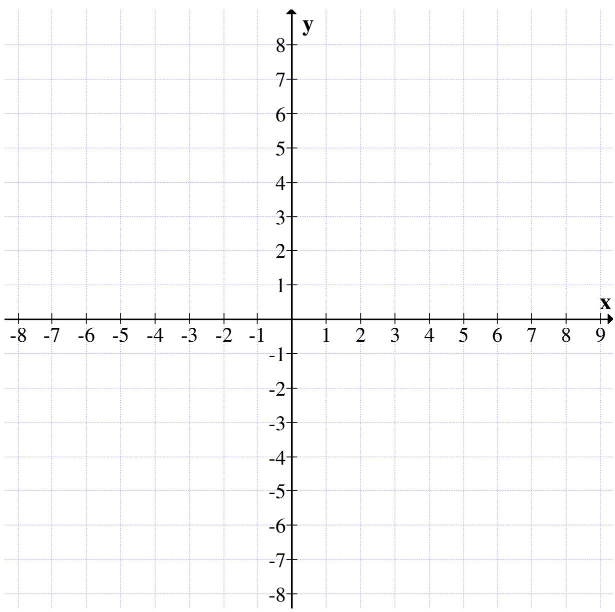


***Figure 2.1***

is a four quadrant plane as shown above.

The MOD real plane $R_n(m)$; $1 < m < \infty$ is a plane which has only one quadrant.

As it is the small size plane built on modulo integers it will be known as MOD-plane.

We will give some examples of MOD real planes.

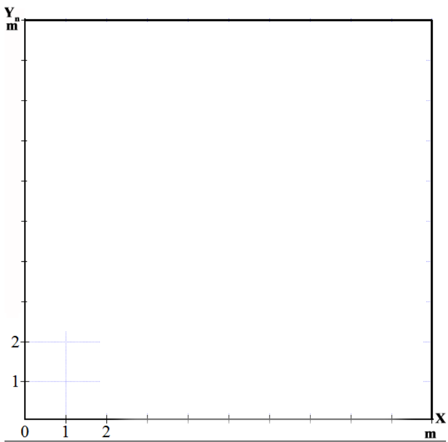


***Figure 2.2***

is a four quadrant plane as shown above.

The above diagram is the MOD real plane $R_n(m)$ $(1 < m < \infty)$ and elements of $R_n(m) = \{(a, b)$ where $a,b \in [0, m)$; that is a and b are elements of the MOD interval $[0,m)\}$

Thus $R_n(7)$, $R_n(148)$, $R_n(251)$, $R_n(10706)$, $R_n(20451)$ and so on are all MOD real planes and the main advantage of using MOD real planes is that MOD real planes are infinite in number whereas the real plane is only one.

Several problems in this direction are suggested at the end of this chapter.

Next we recall the description of the MOD complex modulo integer plane $C_n(m)$ ; $2 \leq m < \infty$.

$C_n(m) = \{a + b_{iF} \mid a, b \in [0, m),\ i_F^2 = (m - 1)\}$.

Just we give the diagram of this MOD complex modulo integer plane in the following.

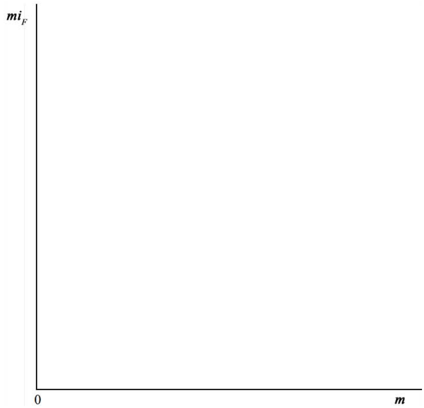


***Figure 2.3***

Any point a + $bi_F$ is represented as (a, b) in the MOD complex modulo integer plane $C_n(m)$. Thus in reality we have one and only one complex plane given b y the following figure.

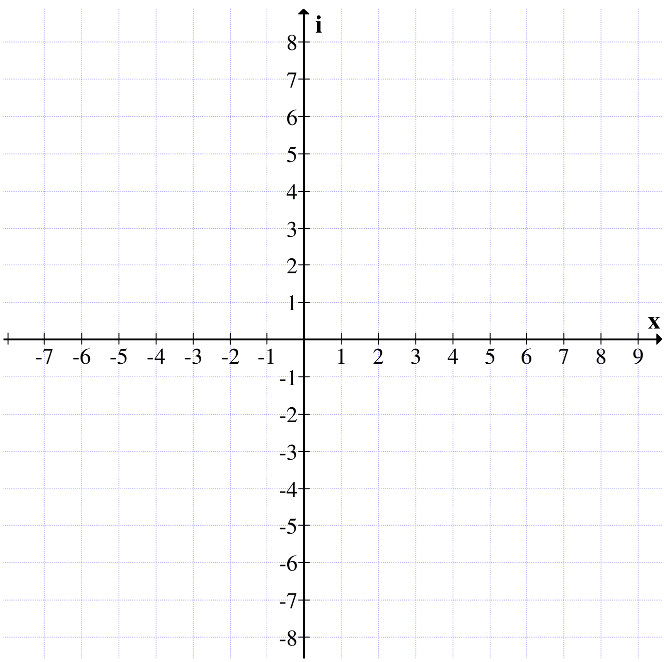


***Figure 2.4***

However there are infinitely many MOD complex modulo integer planes given by $C_n(m)$, $C_n(5)$, $C_n(2051)$, $C_n(999991)$, $C_n(7024)$ and so on.

Thus we can choose any appropriate complex modulo integer plane and work with the problem.

Infact we have infinite choice in contrast with one and only one complex plane which is an advantage.

We have suggested problems at the end of the chapter some of which are normal and a few can be taken for future research.

Next we proceed onto describe the MOD neutrosophic planes $R_n^I(m)$ ; $2 \leq m < \infty$ in the following.

For a fixed positive integer m the MOD neutrosophic plane $R_n^I(m) = \{a + bI \mid a, b \in [0, m)\ I^2 = I\}$ here the I is an indeterminacy which cannot be given a real value.

This cannot be misunderstood for idempotent, when we see $I \times I = I$ it means, it is an idempotent but an indeterminate idempotent which cannot be described as transformation or as a matrix; however many times we may multiply an indeterminate with itself it will only continue to be an indeterminate I only.

This mention is made for one should not get confused with special dual like number planes.
$R(g) = \{a + bg \mid g^2 = g$ where g is a special dual like number$\}$.

This concept is abundant in linear operators which are idempotents. Lattices which are abundant in idempotents.

We will first describe the neutrosophic plane before we illustrate the MOD neutrosophic plane $R_n^I(m)$; $2 < m < \infty$ by the following figure.

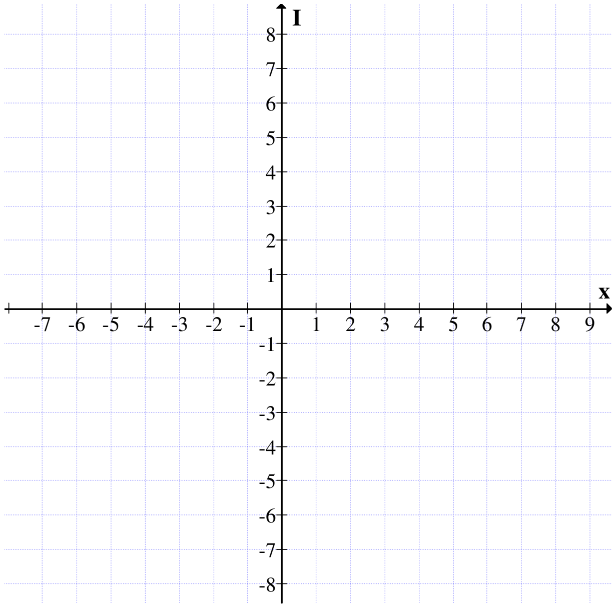


***Figure 2.5***

The above figure is a neutrosophic infinite plane.

We have one and only one neutrosophic infinite plane.

But we have infinitely many MOD neutrosophic planes $R_n^I(m)$; $2 \leq m < \infty$; we just give one illustrative example.

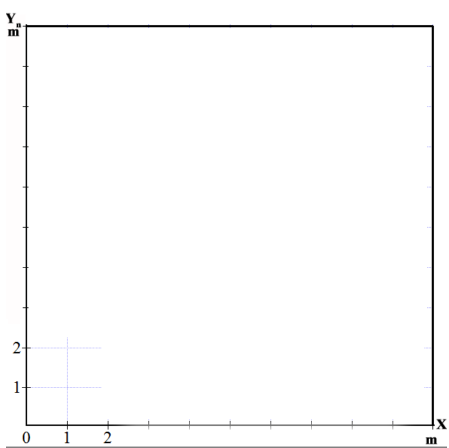


***Figure 2.6***

Any point a + bI in $R_n^I(m)$ is denoted in the MOD neutrosophic plane by (a,b).

Thus a + bI = (a, b) is the representation of its elements in the MOD neutrosophic planes.

Once again we have infinitely many MOD neutrosophic planes whereas there is one and only one neutrosophic plane. This is advantageous while working with real world problem for instead of working with one infinite structure we can appropriately use any one of the compact one quadrant MOD neutrosophic plane which has also the capacity to have infinite values but they can be realized as bounded infinities. For a precise transformation exists from the four quadrant plane to this one quadrant plane.

Several problems can be proposed in this direction which is done in at the end of this chapter.

Next we proceed onto describe the dual number plane and the MOD dual number plane in the following.

A dual number plane R(g) is an infinite four quadrant infinite plane where $R(g) = \{a + bg \mid a, b \in R \text{ and } g^2 = 0\}$ is as follows

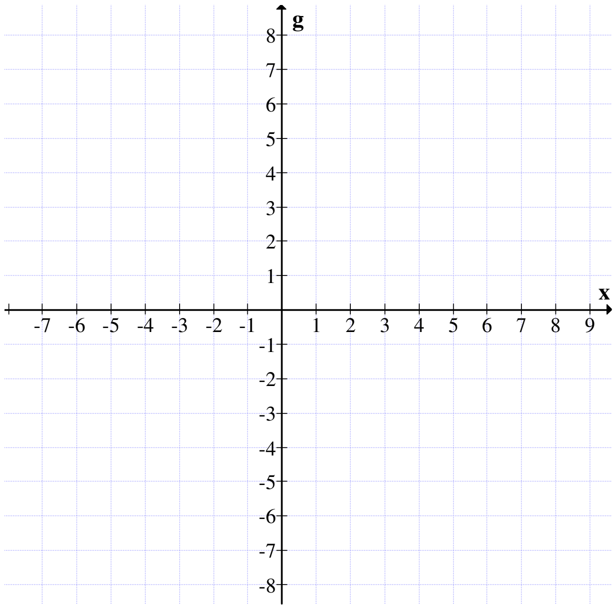


***Figure 2.7***

Any point a + bg is denoted by (a, b) in R(g).

This four quadrant dual number plane is unique and there is one and only one such plane.

However the MOD dual number planes are infinite in number and for any positive integer m it is denoted by

$R_n^g(m) = \{a + bg \mid a, b \in [0, m); g^2 = 0\}$ ; $2 \leq m < \infty$.

So there are infinitely many MOD dual number planes.

The dual number plane $R_n^g(m)$ is described by the following figure:

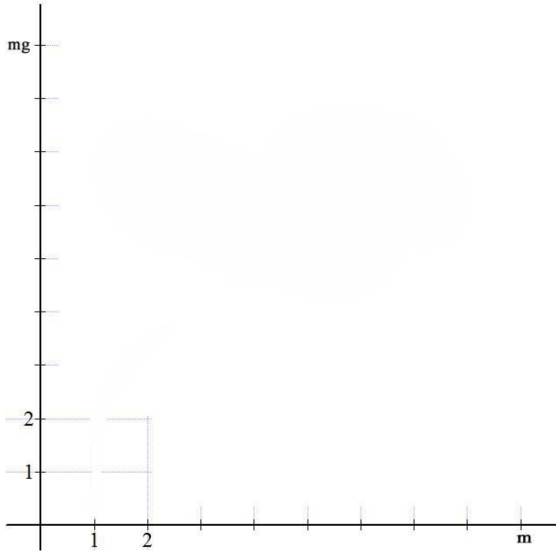


***Figure 2.8***

Thus as per need of the problem which needs the notion of dual numbers these planes can be used.

For $R_n^g(2)$, $R_n^g(3)$, $R_n^g(12)$, $R_n^g(18)$, $R_n^g(204899)$ and so on are some of the MOD dual number planes.

Problems related with them is given at the end of this chapter.

Infact one can built all algebraic structures, groups, semigroups and pseudo rings; but this is the only algebraic structure which can give infinite order zero square semigroups and rings (the rings are not pseudo).

Next we proceed onto describe special dual like number plane and MOD special dual like number planes in the following.

$R_n^g = \{(a + bg \mid g^2 = g, a, b \in R\}$ where $a + bg = (a, b)$ is the notation used.

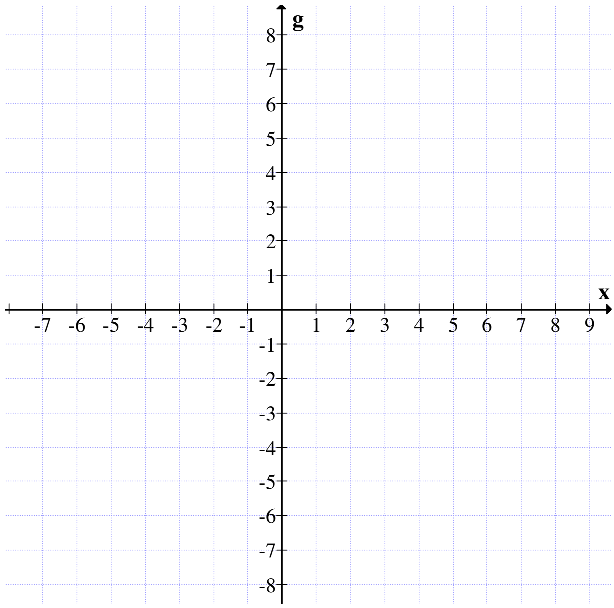


***Figure 2.9***

However in some places we have used $R_n(h)$ instead of $R_n(g)$ but from the very contest one can easily understand whether we are using dual number or special dual like number.

Any point a + bg in R(g) is denoted in the special dual like number plane by (a, b).

Next we proceed onto describe the notion of MOD special dual like number planes in the follows.

$R_n^g(m) = \{a + bg \mid a, b \in [0, m), g^2 = g\}$ denotes the MOD special dual like number plane.

The diagrammatic representation of $R_n^g(m)$ is as follows.

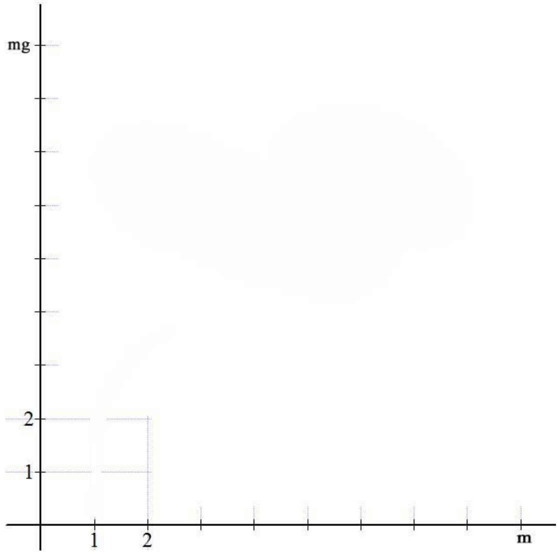


***Figure 2.10***

Any point a + bg in the MOD special dual like number plane is denoted by (a, b).

We have infinitely many special dual like number planes for m can vary in the infinite interval $2 \leq m < \infty$.

$R_n^g(10)$, $R_n^g(18)$, $R_n^g(18999)$, $R_n^g(250)$ and so on are some of the examples of MOD special dual like number planes.

As per need of the problem one can use any appropriate m; $2 \leq m < \infty$, these planes for storage space is comparatively less when compared with R(g).

Problems for the reader are given at the end of this chapter.

Some examples of special dual like numbers in the real world problems are square matrices A with $A \times A = A$, any $p \times q$ matrices B with $B \times_n B = B$ where $\times$ is the usual product and $\times_n$ is the natural product of matrices.

Also we have linear operators which can contribute to special dual like numbers.

Finally one can get several special dual like numbers from $Z_t$; the modulo integers where t is a composite number.

Also semi lattices and lattices are abundant with special dual like numbers.

Next we proceed onto describe the infinite plane of special quasi numbers and MOD special quasi dual numbers.

Let $R(g) = \{a + bg \mid g^2 = -g, a, b \in R\}$ be the infinite special quasi dual number plane.

We have only one such plane.

However there are infinitely many MOD special quasi dual number planes which will be described.

The special quasi dual number plane R(g) is as follows.

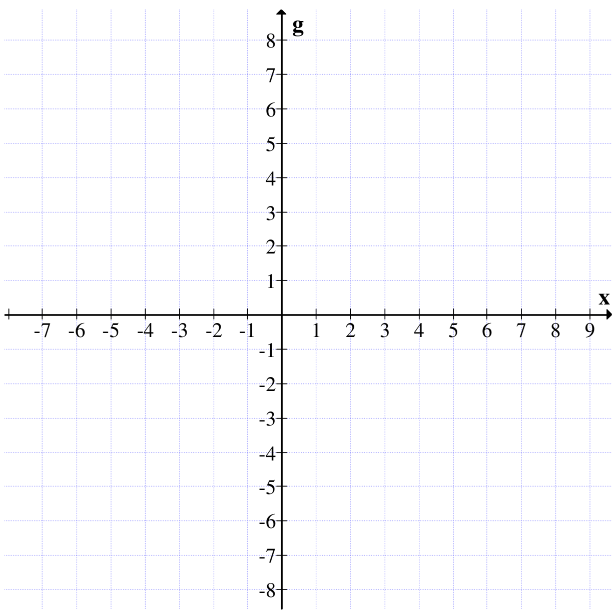


***Figure 2.11***

Any point a + bg ∈ R(g) is only represented as (a, b) in R(g), the real special quasi dual number plane.

Now the MOD special quasi dual number plane.

Now the MOD special quasi dual number plane $R_n^g(m) = \{a + bg \mid a, b \in [0, m)$ and $g^2 = (m-1)g\}$, $2 \leq m < \infty$. Here also any a + bg is represented as (a, b) in the MOD special quasi dual number plane $R_n^g(m)$.

The following diagram gives the MOD special quasi dual number plane $R_n^g(m)$.

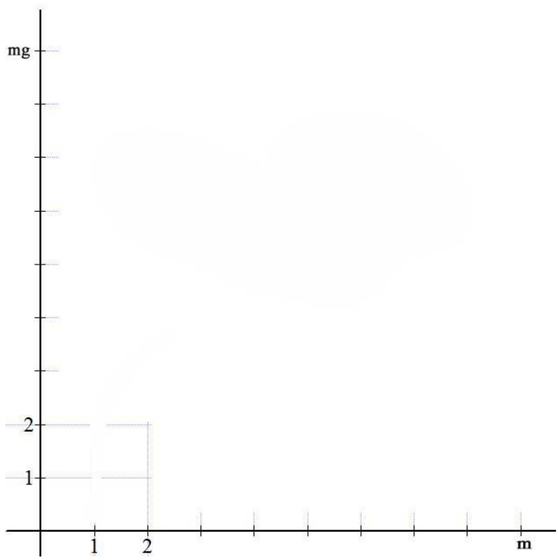


***Figure 2.12***

Now using six types of MOD planes we propose the following problems.

The authors wish to keep on record that all these definitions and notions are recalled to make this book self contained.

As the book is problems on MOD structures all the more we wanted to make it as self contained as possible.

**Problems**

1. What are the advantages of using MOD real planes instead of the real plane?
2. Can one say use of MOD real planes saves both time and economy?

3. What are the ways of defing the concept of distance between two points in the MOD real plane; $R_n(m)$, $2 \leq m < \infty$?

4. Illustrate the situation in problem (3) by some examples (that is used a fixed m).

5. Can the concept of analytical geometry be employed on MOD real planes?

6. Can the notion of Euclidean geometry be used on MOD real planes?

7. Give some nice problems in which MOD real planes would be better than usual real plane.

8. Can the concept of continuity be defined in MOD real planes?

9. Does the notion of continuity depends on the value of the m of the MOD real plane $R_n(m)$ and the function?

10. Give examples of functions which are continuous in the real plane and are not continuous in the MOD real plane.

11. Can a non continuous function in the real plane be continuous in the MOD real plane $R_n(m)$ depending on m?

12. Can you give examples of functions in the MOD real plane which are defined but not defined in the real plane?

13. Clearly $\frac{1}{x-3}$ = f(x) is not defined in the real plane at x = 3.

    i) Prove or disprove f(x) = $\frac{1}{x-3}$ is defined in the MOD real plane $R_n(m)$ (for some suitable m).

    ii) In which of the other MOD real planes f(x) = $\frac{1}{x-3}$ is defined?

14. Clearly $\frac{1}{(x-2)\,(x-7)\,(x+3)}$ = f(x) is not defined in the real plain at x = 2, x = 7 and x = –3.

    i) Give all MOD real planes in which

    $$f(x) = \frac{1}{(x-2)\,(x-7)\,(x+3)}$$ is defined at all points of the MOD real plane.

    ii) Can we have more than one real MOD plane for which the above statement (i) is true?

15. Can the notion of a circle be defined in the MOD real plane $R_n(m)$?

16. Prove in polynomial real planes $R_n(m)[x]$ as the polynomials in $R_n(m)[x]$ do not satisfy distributive law the notion of solving equations is an impossibility in $R_n(m)[x]$.

17. Illustrate the situation of problem in (16) for m = 5 and m = 12.

18. Prove $G = \{R_n(m), +\}$ is an infinite real MOD plane group under +.

19. Can G in problem 18 have decimals (that is elements $x \in R_n(m) \setminus \{(a,b) / a, b \in Z_m\}$ such that $\sum_t x = 0$ for some t?

20. Prove $S = \{R_n(m), \times\}$ is an abelian MOD real plane semigroup.

21. Prove or disprove S is not always a Smarandache semigroup.

22. Prove $S = \{R_n(m), \times\}$ have infinite number of zero divisors.

23. Prove or disprove $S = \{R_n(m), \times\}$ has only finite number of idempotents.

24. Can $S = \{R_n (m), \times\}$ have nilpotent elements of order greater than or equal to 2?

25. Can $S = \{R_n(m), \times\}$ have torsion free elements?

26. Can $S = \{R_n(m), \times\}$ have torsion elements which are infinite in number?

27. Will $S = \{R_n(m), \times\}$ have ideals which are of finite order?

28. Study problem (27) for $S = \{R_n(28), \times\}$.

29. Find all subsemigroups all of which are infinite order but are not ideals of $S = \{R_n(47), \times\}$.

30. Can $S = \{R_n(48), \times\}$ have infinite number of subsemigroups of finite order?

31. Let $M = \{R_n(11), \times\}$ be the semigroup of MOD real plane $R_n(11)$.

i) Find all zero divisors of M.
ii) Find all idempotents of M.
iii) Find all the ideals of M.
iv) Can M have ideals of finite order?
v) Can M have torsion free elements?
vi) Can M have infinite number of torsion elements?
vii) Prove M can have nilpotent elements of order greater than or equal to two.
viii) Can M have infinite number of units?
ix) Obtain all special properties associated with M.

32. Let R = $\{R_n(m), \times +\}$ be the MOD pseudo ring.

i) Can we have elements $x, y, z \in R_n(m) \setminus \{(a, b) \mid a, b \in Z_m\}$ which satisfy the distributive law $x \times (y + z) = x \times y + x \times z$ ?

ii) Is the non distributive nature in R an advantage to problems in general or disadvantage to those problems?

iii) Can one always accept that all natural problems for research really satisfy the distributive laws? Justify!

iv) Can we prove all practical problems in the world satisfy distributive law?

v) Can one just acknowledge that the non acceptance of distributive law does not drastically affect the solution of the problem.

vi) Under any of these circumstances can one claim that use of MOD real plane pseudo rings will give more accurate answer if appropriately adopted to certain problems.

vii) Find all idempotents of $\{R_n(m),+ \times\}$ the pseudo ring.

viii) Find all units of $\{R_n(m), +, \times\}$.

ix) Can one say the number of units in both $\{R_n(m), \times\}$ and that of $\{R_n(m), +, \times\}$ have same number of units?

x) Can we prove both $\{R_n(m), \times\}$ and $\{R_n(m), +, \times\}$ have the same number of idempotents?

xi) Can we say both $\{R_n(m), \times\}$ and $\{R_n(m), + \times\}$ have same number of zero divisors?

xii) Obtain any of the special features enjoyed by $\{R_n(m), +, \times\}$.

33. Let B = $\{R_n(23), +, \times\}$ be the MOD real plane pseudo ring.

Study questions (i) to (xii) of problem (32) for this B.

34. Study questions (i) to (xii) of problem (32) for this M = $\{R_n(48), +, \times\}$.

35. Let B = $\{R_n(m)[x]\}$ be the MOD real plane polynomials.

i) If p(x) $\in$ B is of degree n prove $\frac{dp(x)}{dx}$ is not a MOD polynomial of degree (n – 1).

ii) Prove $\int$ p(x) dx may or may not be defined in B.

iii) Prove if p(x) is a MOD polynomial of degree n then p(x) need not have n and only n roots.

iv) Prove p(x) can have finite number of units less than n.

v) Prove $p(x) \in B$ can have more than n roots even though $p(x)$ is a polynomial of degree n in B.

vi) Can one say as distributive law is not true we have $p(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) \neq x^n - (\alpha_1 + \alpha_2 + \ldots + \alpha_n) x^{n-1} + \sum_{i \neq j} \alpha_i \alpha_j x^{n-2} + \ldots \mp \alpha_1 \ldots \alpha_n = q(x)$ in general.

vii) Prove $p(\alpha_1) = 0$ then $q(\alpha_1) \neq 0$ in general.

viii) Does there exist a p(x) of degree n with coefficients from $R_n(m) \setminus \{(a,b) / a, b \in Z_n\}$ satisfy the equality.

$$(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) = x^n - (\alpha_1 + \alpha_2 + \ldots + \alpha_n) x^{n-1} + \sum_{i \neq j} \alpha_i \alpha_j x^{n-2} \ldots + \alpha_1 \ldots \alpha_n?$$

ix) Let $p(x) = (x + 0.312)(x + 2.5302)(x + 4.2) \in R_n(5)$; can equation in (viii) be true for this p(x).

36. Can we have polynomials in $R_n(m)[x]$ so that the derivatives are as in case of $R[x]$?

37. Can we have $p(x) \in R_n(m)[x]$ so that the integral exists as in case of $p(x) \in R[x]$?

38. Give examples of polynomials in $R_n(m)[x]$ in which both integral and derivatives does not exists.

39. Given a MOD polynomial p(x) in $R_n(m)[x]$ for which integral exist but derivatives does not exist.

40. Give an example of a MOD polynomial p(x) in $R_n[m][x]$ for which derivative exist but integral does not exist.

41. What are the special features enjoyed by $C_n(m)$?

42. Can $C_n(m)$ have units?

43. Can $C_n(m)$ have idempotents?

44. Prove $\{C_n(m), \times\}$ is only a semigroup of infinite order.

45. Obtain all the special features enjoyed by $\{C_n(m), +\}$.

46. Prove $C_n(45)$ have subgroups of finite order.

47. What are the special features enjoyed by $\{C_n(m), \times\}$?

48. Find all zero divisors of $\{C_n(m), \times\}$.

49. Will $\{C_n(m), \times\}$ be always a S-semigroup?

50. Can ideals of $\{C_n(m), \times\}$ be of finite order?

51. Obtain any other differences between $\{C_n(m), \times\}$ and $\{R_n(m), \times\}$.

52. Prove $\{C_n(m), +, \times\}$ is a MOD complex modulo integer pseudo ring.

53. Let P = $\{C_n(45), \times\}$ be the MOD complex modulo integer semigroup.

    Study questions (i) to (ix) of problem (31) for this P.

54. Let M = $\{C_n(47), \times\}$ be the MOD complex modulo integer semigroup.

    Study questions (i) to (ix) of problem (31) for this M.

55. Let W = $\{C_n(15), \times, +\}$ be the MOD complex modulo integer pseudo ring.

    i) Study questions (i) to (xii) of problem (32) for this W.

ii) Obtain any other special and interesting feature enjoyed by W.

56. Let V = $\{C_n(37), \times, +\}$ be the MOD complex modulo integer pseudo ring.

Study questions (i) to (xii) of problem (32) for this V.

57. Let M = $\{C_n(48), + \times\}$ be the MOD complex modulo integer pseudo ring.

Study questions (i) to (xii) of problem (32) for this M.

58. Characterize all properties that can be associated with the MOD complex modulo integer plane set $\{C_n(p)\}$, p a prime or a composite number.

59. Let = $\{ R_n^I (m)\}$ be the MOD neutrosophic plane.

Find all the special features enjoyed by the MOD neutrosophic plane.

60. Let = $\{ R_n^I (20), \times\}$ be the MOD neutrosophic semigroup.

i) Find all ideals of M.
ii) Prove ideals of M are of infinite order.
iii) Find all subsemigroups of finite order.
iv) Find all subsemigroups of infinite order which are not ideals.
v) Find all zero divisors and S-zero divisors of M.
vi) Find all idempotents of M.
vii) Find nilpotents of order higher than two in M.
viii) What are the special features enjoyed by M?
ix) Can M be a S-semigroup?
x) Can M have S-ideals?

61. Let P = $\{ R_n^I (37), \times\}$ be the MOD neutrosophic semigroup.

Study questions (i) to (x) of problem (60) for this P.

62. Let Q = { $R_n^I(42)$, +, ×} be the MOD neutrosophic pseudo ring.

    i) Find zero divisors of Q.

    ii) Can Q have S-zero divisors?

    iii) Prove Q has idempotents.

    iv) Can Q have S-idempotents?

    v) Prove Q has ideals of only infinite order.

    vi) Prove Q has S-ideals.

    vii) Prove Q has subrings of finite order.

    viii) Can Q have pseudo subrings of infinite order which are not ideals of Q?

    ix) Obtain any other special feature enjoyed by Q.

    x) Compare MOD neutrosophic pseudo rings with the MOD complex modulo integer pseudo ring and MOD real pseudo ring.

63. Let Q = { $R_n^I(53)$, +, ×} be the MOD neutrosophic pseudo ring.

    Study questions (i) to (x) of problem (62) for this Q.

64. Let T = { $R_n^I(50)$, +, ×} be the MOD neutrosophic pseudo ring.

    Study questions (i) to (x) of problem (62) for this T.

65. Let W = { $R_n^I(m)$; $g^2 = 0$; $q \leq m < \infty$} be the MOD dual number plane.

i) Obtain all the special features enjoyed by W.

ii) Can the distance concept be defined on this plane?

iii) Can a circle be defined on this plane?

iv) Prove {W, ×} = S is the MOD special dual number semigroup of infinite order.

v) Can S have infinite number of idempotents?

vi) Prove S has infinite number of zero divisors?

vii) Prove {W, ×} = S has zero square subsemigroups.

viii) S = {W, ×} has only ideals of infinite order prove.

ix) Prove S = {W, ×} have subsemigroups of infinite order which are not ideals.

x) Obtain any other special feature enjoyed by S = {W, ×}.

xi) Can S have infinite number of units?

xii) Can these MOD dual number semigroup find any applications to real world problems?

xiii) What are the advantages of using these MOD dual number planes in the place of dual number plane?

xiv) Study all the special features enjoyed by B = {W, +}, the MOD dual number group.

xv) Can B have subgroups of finite order?

xvi) Is every element in B a torsion element? Justify your claim!

xvii) Study the algebraic structure enjoyed by the MOD dual number pseudo ring where M = {W, +, ×}.

xviii) Prove M has subsemigroups of infinite order which are not pseudo.

xix) Prove all ideals of M are of infinite order.

xx) Prove or disprove all zero divisors of M and that of the MOD semigroup S are the same.

xxi) Prove or disprove all units in S and M are same.

xxii) Prove or disprove all idempotents of S and M are the same.

xxiii) Find all subrings of M of finite order.

xxiv) Obtain any other special feature enjoyed by M.

66. Let V = { $R_n^g$ (58), $g^2 = 0$} be the MOD dual number plane.

i) Study questions (i) to (xxiv) of problem (65) for this V.

ii) Obtain any other feature which is unique about V.

67. Let T = { $R_n^h$ (m), $h^2 = h$} be the MOD special dual like number plane.

i) Derive all the special features enjoyed by T.

ii) Can T find nice applications of the real world problem?

iii) {T, ×} = W be the MOD special dual like number plane semiring, prove W is a commutative semigroups of infinite order.

iv) Prove all ideals of W are of infinite order.

v) Find the number of subsemigroups of finite order.

vi) Show W can have subsemigroups of infinite order which are not ideals.

vii) Find all zero divisors of W.

viii) Can W have S-zero divisors?

ix) Characterize all idempotents of W.

x) Can W have S-units?

xi) Prove or disprove W can have only finite number of units.

xii) Prove {T, +} = A be the MOD special dual like number group, A has infinite order subsemigroups.

xiii) How many subgroups of A are of finite order for a given M?

xiv) Prove S = {T, +, ×} be the MOD special dual like number pseudo ring does not satisfy the distributive law.

xv) How many subrings of S are of finite order?

xvi) Find all zero divisors which are not S-zero divisors of S (at least characterize).

xvii) Can S have infinite number of units?

xviii) Does S contain infinite number of idempotents?

xix) Find all idempotents of S which are not S-idempotents.

xx) Give any other special feature enjoyed by these MOD special dual like number of planes?

68. Let W = { $R_n^h(53)$, $h^2 = h$} be the MOD special dual like number plane.

Study questions (i) to (xx) of problem (67) for this W.

69. Let M = { $R_n^h(48)$, $h^2 = h$} be the MOD special dual like number plane.

Study questions (i) to (xx) of problem (67) for this M.

70. Let P = { $R_n^h$ (256), $h^2 = h$} be the MOD special dual like number plane.

Study questions (i) to (xx) of problem (67) for this M.

71. Let P = { $R_n^h(49)$ | $h^2 = h$} be the MOD special dual like number plane.

Study questions (i) to (xx) of problem (67) for this P.

72. Let S = { $R_n^k(m)$, $k^2 = (m – 1)k$} be the MOD special quasi dual number plane.

i) Study the special and distinct features associated with S.

ii) What are advantages of using the MOD special quasi dual number planes?

iii) Give some applications of this MOD special quasi dual number planes to real world problems.

iv) Prove the MOD special quasi dual number plane semigroup (S, ×} is commutative and is of infinite order.

v) Prove all ideals of {S, ×} are of infinite order.

vi) Find all subsemigroups of {S, ×} which are of finite order.

vii) What are the advantages of using this semigroup {S, ×} instead of the MOD real semigroup $\{R_n(m),\times\}$ or $\{C_n(m), \times\}$ or $\{R_n^g(m), \times\}$ or $\{R_n^I(m), \times\}$?

viii) Find all units and S-units of {S, ×}.

ix) Can we claim {S, ×} has infinite number of zero divisors?

x) Find all idempotents of {S, ×} which are not S-idempotents of {S, ×}.

xi) Study the special properties enjoyed by the MOD special quasi dual number group {S, +}.

xii) Find all subgroups of finite order in {S, +}.

xiii) Can {S, +} have subgroups of infinite order, infinite in number or only finite in number?

xiv) Prove {S, + , ×} is only a pseudo MOD special quasi dual number ring of infinite order.

xv) Find all subrings of {S, +, ×} of finite order.

xvi) Can {S, +, ×} have infinite number of units?

xvii) Does $\{S, +, \times\}$ contain S-units?

xviii) Can $\{S, +, \times\}$ have infinite number of idempotents?

xix) Find all S-idempotents of $\{S, +, \times\}$.

xx) Can $\{S, +, \times\}$ have infinite number of zero divisors?

xxi) Characterize all S-zero divisors of $\{S, +, \times\}$.

xxii) Prove all ideals of $\{S, +, \times\}$ are only of infinite order.

xxiii) Obtain any other interesting special feature enjoyed by $\{S, +, \times\}$.

73. Let $S_1 = \{ R_n^k(19), k^2 = 18k\}$ be the MOD special quasi dual number plane.

i) Study questions (i) to (xxii) of problem (72) for this $S_1$.

ii) Does the product reflect on of the properties as $m = 19$ is a prime?

74. Let $B = \{ R_n^k(24), k^2 = 23k\}$ be the MOD special quasi dual number plane.

i) Study questions (i) to (xx) of problem (72) for this B.

ii) Does the fact $m = 24$ a composite number have any impact on number of subrings, number of zero divisors and number of units?

iii) Compare $S_1$ of problem (73) with this B.

75. Let M = { $R_n^k(3^7)$, $k^2 = (3^7 - 1)k$} be the MOD special quasi dual number plane.

    i) Study questions (i) to (xx) of problem (72) for this M.

    ii) As $m = 3^7$ does it have any impact on the number of idempotents of M or on S-idempotents (if any) on M.

    iii) Compare M of this problem with B and $S_1$ of problem (74) and (73) respectively.

    iv) Can we say if m is a power of a prime number some of the properties enjoyed by these MOD special quasi dual number plane is distinctly different from when m is a composite number?

76. Compare the algebraic structures enjoyed by the planes $R_n(m)$, $R_n^I(m)$, $C_n(m)$, $R_n^g(m)$, $R_n^h(m)$ and $R_n^k(m)$ when they are defined on them.

    i) Which of the 6 planes is a very powerful as a group?

    ii) Which of the 6 planes will find more applications?

    iii) Which of the six planes behaves more like the real plane?

## Chapter Three

# PROBLEMS ON SPECIAL ELEMENTS IN MOD STRUCTURES

In this chapter for the first time we propose the following problems some of which are difficult and others are at research level.

We have already recalled the notion of MOD interval and MOD planes of different types [26, 30].

In the first place the concept of special pseudo zero divisors was defined that is in [0,5); 4 is a unit but $2.5 \in [0, 5)$ is such that $2.5 \times 4 \equiv 0 \pmod 5$ as well as $2.5 \times 2 \equiv 0 \pmod 5$ [21,24].

Thus 2 and 4 are units as $2 \times 3 \equiv 1 \pmod 5$ and $4 \times 4 \equiv 1 \pmod 5$; but they also lead to zero divisors so we define them as special pseudo divisors.

For in usual algebraic structures this sort of situation will never occur.

Only due to MOD intervals such sort of special pseudo divisors in possible.

Let [0, 8) be the MOD interval. Let $5 \in [0,8)$; clearly $5 \times 5 = 1 \pmod 8$.

Now for $1.6 \in [0,8)$ we have $1.6 \times 5 = 8 \pmod 8 = 0$. Thus 1.6 is a special pseudo MOD zero divisor but 5 is a unit.

Will $1.6 \in [0,8)$ be a unit is another open conjecture?

That is can for some t; $(1.6)^t = 1$ for some $t > 2$.

Yet $5 \times 1.6 = 0$.

Can this sort of mathematical magic occur?

This is a open conjecture which is given in the problems suggested at the end of the chapter.

Now baring the study of these MOD interval special pseudo zero divisors we can have these types of special MOD pseudo zero divisors in the plane.

They happen to be different from the special MOD interval pseudo zero divisors.

We will illustrate these by the following examples.

Let $R = R_n(7) = \{(x, y) \mid x, y \in [0.7)\}$.

Take $(3.5, 1.75) \in R$; we have $(2, 4) \in R$.

$(3.5, 1.75) \times (2, 4) = (0, 0)$ is again a MOD pseudo zero divisors of the plane.

Consider $(1.75, 0), (4, 3.116) \in R$.

Clearly $(1.75, 0) \times (4, 3.116) = (0, 0)$ is a MOD special pseudo zero divisor.

So finding all pseudo zero divisors, finding idempotents and S-idempotents of R is a challenging problem.

Similarly finding MOD nilpotents and units from $(x, y) \in R_n(7) \setminus (Z_7 \times Z_7)$ happens to be a open conjecture.

Even finding the number of elements which are units and nilpotents happens to be a very difficult problem.

We suggest the following problems.

**Problems**

1. Let $B = \{[0, m), \times\}$ be the MOD real interval under product.

   i) Can $[0,m)$ have infinite number of units?
   ii) Can $[0, m)$ have MOD pseudo special zero divisors?
   iii) Can the elements of $[0, m)$ which is a special pseudo zero divisor be both the units?
   iv) Characterize all MOD pseudo special pseudo zero divisors?
   v) Characterize all torsion elements $x \in [0, m) \setminus Z_m$.
   vi) Can $x \in [0,m) \setminus Z_m$ be idempotents?
   vii) Can $x \in [0,m) \setminus Z_m$ be S-units?
   viii) Can $x \in [0,m) \setminus Z_m$ be S-zero divisors?
   ix) Obtain any other special feature associated with it.

2. Let $P = \{[0, m)k, \times\}$ be the MOD special quasi dual number interval semigroup ($k^2 = (m - 1) k$).

   i) Can P have the identity with respect to $\times$.
   ii) Study questions (i) to (ix) of problem (1) for this P.

3. Let M = {[0,m) h, ×} be the MOD special dual like number ($h^2 = h$) semigroup.

   i) What is the multiplicative identity of this semigroup?
   ii) Study questions (i) to (ix) of problem (1) for this M.

4. Let W = {[0, m)I, ×; $I^2 = I$} be the MOD neutrosophic interval semigroup.

   i) Find the multiplicative identity of W.
   ii) Study questions (i) to (ix) of problem (1) for this W.
   iii) Obtain any applications of W as a MOD neutrosophic set.

5. Let S = {[0, 25)I, $I^2 = I$, ×} be the MOD neutrosophic semigroup.

   Study questions (i) to (ix) of problem (1) for this S.

6. Let T = {[0, 47)I, $I^2 = I$, ×} be the MOD neutrosophic semigroup.

   Study questions (i) to (ix) of problem (1) for this T.

7. Let V = {[0, 24)I, $I^2 = I$, ×} be the MOD neutrosophic semigroup.

   Study questions (i) to (ix) of problem (1) for this V.

8. Compare the MOD neutrosophic semigroups in problems 5, 6 and 7.

9. Let N = {[0,27), ×} be the MOD real interval semigroup.

   Study questions (i) to (ix) of problem (1) for this N.

10. Let Z = {[0, 48), ×} be the MOD interval real semigroup.

    Study questions (i) to (ix) of problem (1) for this Z.

11. Let W = {[0, 53), ×} be the MOD interval real semigroup.

    Study questions (i) to (ix) of problem (1) for this W.

12. Compare the MOD interval real semigroups N, Z and W in problems 9, 10 and 11 respectively.

13. Let S = {[0, 19) k, $k^2 = 18k$, ×} be the MOD interval special quasi dual number semigroup.

    Study questions (i) to (ix) of problem (1) for this S.

14. Let D = {[0, 28)k, $k^2 = 27k$, ×} be the MOD interval special quasi dual number semigroup.

    Study questions (i) to (ix) of problem (1) for this D.

15. Let F = {[0, 243)k, $k^2 = 242k$, ×} be the MOD interval special quasi dual number semigroup.

    Study questions (i) to (ix) of problem (1) for this F.

16. Compare the MOD interval special quasi dual number semigroups S, D and F in problems 13, 14 and 15 respectively.

17. Let Y = {[0, 96)h, $h^2 = h$, ×} be the MOD interval special dual like number semigroup.

    i) Study questions (i) to (ix) of problem (1) for this Y.
    ii) Derive any other distinct feature enjoyed by Y.

18. Let R = {[0, 47)h, $h^2 = h$, ×} be the MOD interval special dual like number semigroup.

    Study questions (i) to (ix) of problem (1) for this R.

19. Let P = {[0, 625) h, $h^2 = h$, ×} be the MOD interval special dual like number semigroup.

    Study questions (i) to (ix) of problem (1) for this P.

20. Compare the MOD interval special dual like number semigroups; Y, R and P of problems 17, 18 and 19 respectively.

21. Let N = {[0,m)g | $g^2 = 0$, ×} be the MOD interval special dual number semigroup.

    i) N is a zero square semigroup.
    ii) Prove N has no identity.
    iii) Prove N has no idempotents.
    iv) Prove N has all order subsemigroups {2, 3, 4, …, ∞}.
    v) Can N have S-zero divisor?
    vi) Prove N cannot be a S-semigroup.

22. Let M = {[0, 42)g, $g^2 = 0$, ×} be the MOD interval dual number semigroup.

    Study questions (i) to (vi) of problem (21) for this M.

23. Let R = {[0, m), + ×} be the MOD real interval pseudo ring.

    i) Prove or disprove the number of zero divisors of R is the same as the number of zero divisors of S = {[0, m), ×}, the MOD real interval semigroup.
    ii) Prove all S-zero divisors of S and R are the same.
    iii) Will the number of units in S and R be the same?
    iv) Prove or disprove all S-units of S and R are the same.
    v) Prove S and R have the same S-idempotents and idempotents.
    vi) Prove ideals of R and S are not the same in general.
    vii) Can R have MOD pseudo zero divisors which are identical with that of S and vice versa?

24. Let R = {[0, 48), +, ×} be the MOD real interval pseudo ring.

    Study questions (i) to (vii) of problem (23) for this R.

25. Let M = {[0, 37), +, ×} be the MOD real interval pseudo ring.

    Study questions (i) to (vii) of problem (23) for this M.

26. Let Z = {[0, 256), +, ×} be the MOD real interval pseudo ring.

    Study questions (i) to (vii) of problem (23) for this Z.

27. Let W = {[0, 47) g, $g^2 = 0$, +, ×} be the MOD interval dual number pseudo ring.

    Study questions (i) to (vii) of problem (23) for this W.

28. Let T = {0, 49), k, $k^2 = 48k$, +, ×} be the MOD interval special quasi dual number pseudo ring.

    Study questions (i) to (vii) of problem (23) for this T.

29. Let N = {[0, 120) h, $h^2 = h$, +, ×} be the MOD interval special dual like number pseudo ring.

    Study questions (i) to (vii) of problem (23) for this N.

30. Let S = {[0, 55)I, $I^2 = I$, +, ×} be the MOD interval neutrosophic pseudo ring.

    Study questions (i) to (vii) of problem (23) for this S.

31. Can W = {[0, 120)$i_F$, $i_F^2 = 119$; +, ×} be the MOD interval complex modulo integer pseudo ring. Justify?

32. Find all special and distinct features enjoyed by $R = \{[0, m)i_F;\ i_F^2 = (m-1); +\}$ the MOD interval complex modulo integer group.

33. Let $R = \{R_n(m), \times\}$ be the MOD real plane semigroup.

    i) Find all MOD pseudo zero divisors of R.
    ii) Is the number of such MOD pseudo zero divisors of R infinite or finite?
    iii) Can we say the number of units of $R \setminus \{(Z_m \times Z_m)\}$?
    iv) Is the collection finite or infinite?
    v) Does $T = R \setminus \{(Z_m \times Z_m)\}$ have idempotents?
    vi) Find all nontrivial idempotents of T.
    vii) Can $(x, y) \in R \setminus \{(x, y) / x, y \in Z_m\}$ be pseudo zero divisor as well as zero divisor?
    viii) Can the number of nilpotents in $R \setminus \{(x, y) / x, y \in Z_m\}$ be infinite?

34. Let $P = \{R_n(20), \times\}$ be the MOD real plane semigroup.

    Study questions (i) to (viii) of problem 33 for this P.

35. Let $M = \{R_n(47), \times\}$ be the MOD real plane semigroup.

    Study questions (i) to (viii) of problem (33) for this M.

36. Let $T = \{R_n(2^{14}), \times\}$ be the MOD real plane semigroup.

    Study questions (i) to (viii) of problem (33) for this T.

37. Let $S = \{R_n^I(m);\ I^2 = I, \times\}$ be the MOD neutrosophic plane semigroup.

    Study questions (i) to (viii) of problem (33) for this S.

38. Let L = { $R_n^I(42)$; $I^2 = I$, $\times$} be the MOD neutrosophic plane semigroup.

    Study questions (i) to (viii) of problem (33) for this L.

39. Let Z = { $R_n^I(23)$; $I^2 = I$, $\times$} be the MOD neutrosophic plane semigroup.

    Study questions (i) to (viii) of problem (33) for this Z.

40. Let S = { $R_n^I(3^{20})$, $I^2 = I$, $\times$} be the MOD neutrosophic plane semigroup.

    Study questions (i) to (viii) of problem (33) for this S.

41. Let V = { $R_n^g(14)$, $g^2 = 0$, $\times$} be the MOD special dual number plane semigroup.

    Study questions (i) to (viii) of problem 33 for this V.

42. Let Z = { $R_n^g(53)$, $g^2 = 0$, $\times$} be the MOD special dual number plane semigroup.

    Study questions (i) to (viii) of problem (33) for this Z.

43. Let N = { $R_n^g(7^{12})$, $g^2 = 0$, $\times$} be the MOD special dual number plane semigroup.

    Study questions (i) to (viii) of problem (33) for this N.

44. Let M = { $R_n^g(m)$, $g^2 = 0$, $\times$} be the MOD special dual number plane semigroup.

    i) Study questions (i) to (viii) of problem (33) for this M.
    ii) Obtain all special features enjoyed by M.

45. Let V = $\{C_n(m), \times;\ i_F^2 = (m-1)\}$ be the MOD special complex modulo integer plane semigroup.

    i) Study questions (i) to (viii) of problem (33) for this V.
    ii) Obtain all special features enjoyed by V.
    iii) Find the difference between V and $R_n(m)$, $R_n^I(m)$ and $R_n^g(m)$.

46. Let S = $\{C_n(29);\ i_F^2 = 28, \times\}$ be the MOD special complex modulo integer semigroup.
    Study questions (i) to (viii) of problem (33) for this S.

47. Let Z = $\{C_n(48),\ i_F^2 = 47, \times\}$ be the MOD complex modulo integer semigroup.

    Study questions (i) to (viii) of problem (33) for this Z.

48. Let T = $\{C_n(5^{40}),\ i_F^2 = 5^{40} - 1, \times\}$ be the MOD complex modulo integer semigroup.

    Study questions (i) to (viii) of problem (33) for this T.

49. Let D = $\{R_n^h(m), h^2 = h, \times\}$ be the MOD special dual like number semigroup.

    i) Study questions (i) to (viii) of problem 33 for this D.
    ii) Find the special features enjoyed by D.
    iii) Compare D with $R_n^g(m)$, $R_n^I(m)$, $R_n(m)$ and $C_n(m)$.

50. Let P = $\{R_n^h(128), h^2 = h, \times\}$ be the MOD special dual like number semigroup.

    Study questions (i) to (viii) of problem (33) for this P.

51. Let B = { $R_n^h(47)$, $h^2 = h$, ×} be the MOD special dual like number semigroup.

    Study questions (i) to (viii) of problem (33) for this B.

52. Let V = { $R_n^h(11^8)$, $h^2 = h$, ×} be the MOD special dual like number semigroup.

    Study questions (i) to (viii) of problem (33) for this V.

53. Let S = { $R_n^k(m)$, $k^2 = (m – 1)k$, ×} be the MOD special quasi dual number plane semigroup.
    Study questions (i) to (viii) of problem (33) for this S.

54. Let Z = { $R_n^k(24)$; $k^2 = (m – 1)k$, ×} be the MOD special quasi dual number plane semigroup.

    Study questions (i) to (viii) of problem (33) for this Z.

55. Let R = { $R_n^k(59)$, $k^2 = 58k$, ×} be the MOD quasi dual number plane semigroup.

    Study questions (i) to (viii) of problem (33) for this R.

56. Let Z = { $R_n^k(13^7)$, $k^2 = (13^7 – 1)k$, ×} be the MOD special quasi dual number plane semigroup.

    Study questions (i) to (viii) of problem (33) for this Z.

57. Let M = {$R_n(m)$ , +, ×} be the MOD real plane pseudo ring.

    i) Find all zero divisors of M.
    ii) Can we say both M and S = {$R_n(m)$, ×} have same number of zero divisors?
    iii) Find all zero divisors of $R_n(m) \setminus \{(x, y) / x, y \in Z_m\}$.
    iv) Find all special MOD pseudo zero divisors of $R_n(m) \setminus \{(x, y) / x, y \in Z_m\}$.

v) Is the number of zero divisors and MOD special pseudo divisors different?
vi) Can we say all MOD special pseudo zero divisors in $R_n(m) \setminus \{(x, y) / x, y \in Z_m\}$ are only units of $R_n(m) \setminus \{(x, y) / x, y \in Z_m\}$?
vii) Does $R_n(m) \setminus \{(x, y) / x, y \in Z_m\}$ have torsion elements?
viii) Can $R_n(m) \setminus \{(x, y) / x, y \in Z_m\}$ have torsion free elements?
ix) Find all nilpotents of $R_n(m) \setminus \{(x, y) / x, y \in Z_m\}$.
x) Obtain any other special features associated with the pseudo ring $R_n(m)$.
xi) Find all idempotents of $R_n(m) \setminus \{(x, y) / x, y \in Z_m\}$.
xii) Find all ideals of $R_n(m)$.
xiii) Prove all ideals of $R_n(m)$ are of infinite order.
xiv) Prove $R_n(m)$ has subrings of finite order.
xv) Can there be finite order subring of order greater than m in $R_n(m)$?
xvi) Obtain all special features enjoyed by $\{R_n(m), +, \times\}$.

58. Let $S = \{R_n(47), +, \times\}$ be the MOD real plane pseudo ring.

    Study questions (i) to (xvi) of problem (57) for this S.

59. Let $M = \{R_n(12), + \times\}$ be the MOD real plane pseudo ring.

    Study questions (i) to (xvi) of problem (57) for this M.

60. Let $N = \{R_n(3^8), + \times\}$ be the MOD real plane pseudo ring.

    Study questions (i) to (xvi) of problem (57) for this N.

61. Let $M = \{R_n^g(m); g^2 = 0, +, \times\}$ be the MOD dual number plane pseudo ring.

    i) Study questions (i) to (xvi) of problem (57) for this M.
    ii) Prove M has subrings which are zero square subrings.

iii) Prove M has infinite number of zero divisors than the real plane pseudo ring $R_n(m)$.

62. Let V = { $R_n^g(24^3)$, $g^2 = 0$, +, ×} be the MOD dual number plane pseudo ring.

    i) Study questions (i) to (xvi) of problem (57) for this V.
    ii) Study questions (ii) and (iii) of problem (61) for this V.

63. Let S = { $R_n^g(2^{21})$, $g^2 = 0$, +, ×} be the MOD dual number plane pseudo ring.

    i) Study questions (i) to (xvi) of problem (57) for this S.
    ii) Study questions (ii) and (iii) of problem (61) for this S.

64. Let A = { $R_n^g(41)$, $g^2 = 0$, +, ×} be the MOD dual number plane pseudo ring.

    i) Study questions (i) to (xvi) of problem (57) for this A.
    ii) Study questions (ii) and (iii) of problem (61) for this A.

65. Let B = { $R_n^I(m)$, $I^2 = I$, +, ×} be the MOD neutrosophic plane pseudo ring.

    i) Study questions (i) to (xvi) of problem (57) for this B.
    ii) Obtain any other special features enjoyed by B.

66. Let W = { $R_n^I(97)$, $I^2 = I$, +, ×} be the MOD neutrosophic plane pseudo ring.

    i) Study questions (i) to (xvi) of problem 57 for this W.

67. Let F = { $R_n^I(420)$, $I^2 = I$, +, ×} be the MOD neutrosophic plane pseudo ring.

    i) Study questions (i) to (xvi) of problem (57) for this F.

68. Let G = $\{R_n^I(7^{10}), I^2 = I, +, \times\}$ be the MOD neutrosophic plane pseudo ring.

    i) Study questions (i) to (xvi) of problem (67) for this G.

69. Compare the MOD neutrosophic pseudo rings W, F and G in problems (66), (67) and (68) respectively.

70. Let T = $\{C_n(m), i_F^2 = m - 1, +, \times\}$ be the MOD complex modulo integer pseudo ring.

    i) Study questions (i) to (xvi) of problem (57) for this T.
    ii) Obtain any other nice feature enjoyed by T.

71. Let R = $\{C_n(96), i_F^2 = 95, +, \times\}$ be the MOD complex modulo integer pseudo ring.

    Study questions (i) to (xvi) of problem (57) for this R.

72. Let S = $\{C_n(47), i_F^2 = 46, +, \times\}$ be the MOD complex modulo integer plane pseudo ring.

    Study questions (i) to (xvi) of problem (57) for this S.

73. Let V = $\{C_n(3^{12}); i_F^2 = 3^{12} - 1, +, \times\}$ be the MOD complex modulo integer plane pseudo ring.

    Study questions (i) to (xvi) of problem (57) for this V.

74. Compare the MOD complex modulo integer plane pseudo rings R, S and V in problems 71, 72 and 73 respectively.

75. Let P = $\{R_n^k(m), k^2\ (m - 1)k, +, \times\}$ be the MOD special quasi dual number pseudo ring.

    i) Study questions (i) to (xvi) of problem (57) for this P.

ii) Obtain any other special feature associated with these pseudo ring.

76. Let B = { $R_n^k(42)$, $k^2 = 41k$, +, ×} be the MOD special quasi dual number pseudo ring.

    Study questions (i) to (xvi) of problem (57) for this B.

77. Let D = { $R_n^k(37)$, $k^2 = 36k$, +, ×} be the MOD special quasi dual number pseudo ring.

    Study questions (i) to (xvi) of problem (57) for this D.

78. Let E = { $R_n^k(7^{21})$, $k^2 = (7^{21} - 1)k$, +, ×} be the MOD special quasi dual number plane pseudo ring.

    Study questions (i) to (xvi) of problem (57) for this E.

79. Compare the MOD special quasi dual number planes pseudo rings in problems (76), (77) and (78) with each other.

80. Let H = { $R_n^h(m)$; $h^2 = h$, +, ×} be the MOD special dual like number plane pseudo ring.

    i) Study questions (i) to (xvi) of problem (57) for this H.
    ii) Obtain any other special striking feature associated with H.

81. Let J = { $R_n^h(23)$, $h^2 = h$, +, ×} be the MOD special dual like number plane pseudo ring.

    Study questions (i) to (xvi) of problem (57) for this J.

82. Let K = $\{R_m^h(48), h^2 = h, +, \times\}$ be the MOD special dual like number plane pseudo ring.

    Study questions (i) to (xvi) of problem (57) for this K.

83. Let L = $\{R_m^h(13^4), h^2 = h, +, \times\}$ be the MOD special dual like number plane pseudo ring.

    Study questions (i) to (xvi) of problem (57) for this L.

84. Compare the MOD special dual like number plane pseudo rings, J, K and L in problems 81, 82 and 83 respectively with each other.

85. Obtain any special features enjoyed by these pseudo rings built using MOD dual number planes.

86. Can S-zero divisors be present in $\{R_m^g(m); g^2 = 0, +, \times\}$?

87. Prove the notion of special pseudo zero divisors is impossible in $\{[0, m)g, g^2 = 0, +, \times\}$.

88. Is it possible to define special pseudo zero divisors in case of $\{R_n^g(m), g^2 = 0, +, \times\}$?

**Chapter Four**

# PROBLEMS ON MOD NATURAL NEUTROSOPHIC ELEMENTS

In this chapter we propose some problems on MOD neutrosophic numbers in [0, m), [0, m)I, [0, m)g, [0, m)h, [0, m)k, $R_n(m)$, $R_n^I(m)$, $R_n^g(m)$, $R_n^h(m)$ and $R_n^k(m)$; the MOD intervals and MOD planes respectively.

For the first time the natural neutrosophic numbers were introduced in [29]. The problem of finding natural neutrosophic numbers was proposed by Florentin Smarandache on several occasions. The MOD mathematics series [29] has answered this question completely.

Here a brief description of this concept is given for more one can refer [29].

Let $Z_9$ be the set of modulo integers. Now we try to introduce the operation of division / on $Z_9$ by / (division). $\{Z_9, /\}$ leads to natural neutrosophic numbers $\{Z_9, /\} = \{0, 1, 2,$

…, 8, 0/0, 1/0, 2/0, …, 8/0, 1/3, 2/3, …, 8/3, 0/6, 1/6, 2/6, …, 8/6}.

These are denoted by $I_0^9$ = {1/0, 2/0, …, 8/0, 0/0}, $I_3^9$ = {1/3, 2/3, 0/3, …, 8/3}, $I_6^9$ = {1/6, 2/6, 0/6, …, 8/6} that is division by 0 or 3 or 6 are indeterminates called as the natural neutrosophic numbers.

Further these numbers behave in a different way for; $I_3^9 \times I_3^9 = I_0^9$, $I_6^9 \times I_6^9 = I_0^9$ and $I_6^9 \times I_3^9 = I_0^9$.

Thus there are natural neutrosophic numbers which are neutrosophic zero divisors.

Consider the modulo integers $Z_{12}$ = {0, 1, 2, …, 11}. $I_2^{12}, I_0^{12}, I_3^{12}\ I_6^{12}\ I_4^{12}, I_8^{12}, I_{10}^{12}, I_9^{12}$ are the collection of all natural neutrosophic numbers of $Z_{12}$ or related with $Z_{12}$; where

$I_2^{12}$ = {0/2, 1/2, 2/2, …, 12/2},
$I_0^{12}$ = {0/0, 1/0, 2/0, …, 12/0} and so on.
$I_9^{12}$ = {0/0, 1/9, 2/9, …, 12/9}.

$$
\begin{array}{ll}
I_2^{12} \times I_6^{12} = I_0^{12}, & I_4^{12} \times I_4^{12} = I_4^{12}, \\
I_6^{12} \times I_4^{12} = I_0^{12} & I_6^{12} \times I_6^{12} = I_0^{12}, \\
I_4^{12} \times I_2^{12} = I_8^{12} & I_9^{12} \times I_9^{12} = I_9^{12} \\
I_4^{12} \times I_9^{12} = I_0^{12} & I_8^{12} \times I_8^{12} = I_4^{12} \\
I_2^{12} \times I_8^{12} = I_4^{12} & I_2^{12} \times I_9^{12} = I_6^{12} \\
I_9^{12} \times I_6^{12} = I_8^{12} & \text{and so on.}
\end{array}
$$

Thus natural neutrosophic zero divisors and natural neutrosophic nilpotents are got by this method.

Further there are some neutrosophic idempotents for instance $I_9^{12}$ and $I_4^{12}$ are neutrosophic idempotents related with $Z_{12}$.

Next we study the natural neutrosophic elements of $Z_{16}$.

$I_0^{16}, I_2^{16}, I_4^{16}, I_6^{16}, I_8^{16}, I_{10}^{16}, I_{12}^{16}$ and $I_{14}^{16}$, $I_6^{16} \times I_6^{16} = I_4^{16}$,

$I_4^{16} \times I_4^{16} = I_0^{16}$. $I_6^{16} \times I_6^{16} \times I_6^{16} \times I_6^{16} = I_0^{16}$ is a nilpotent neutrosophic element of order four.

$I_8^{16} \times I_8^{16} = I_0^{16}$ is a natural neutrosophic nilpotent of order two.

$I_2^{16} \times I_2^{16} \times I_2^{16} \times I_2^{16} = I_0^{16}$ is again a neutrosophic nilpotent of order four.

$I_{10}^{16} \times I_{10}^{16} \times I_{10}^{16} \times I_{10}^{16} = I_0^{16}$ is again a neutrosophic nilpotent of order four.

$I_{12}^{16} \times I_{12}^{16} = I_0^{16}$ is a neutrosophic nilpotent of order two.

$I_{10}^{16} \times I_{10}^{16} \times I_{10}^{16} \times I_{10}^{16} = I_0^{16}$ is again a neutrosophic nilpotent of order four.

$I_{14}^{16} \times I_{14}^{16} = I_4^{16}$ so $I_{14}^{16}$ is again a natural neutrosophic nilpotent of order four.

We see the natural neutrosophic elements associated with $Z_{16}$ are either neutrosophic nilpotents of order two or of order four only.

Clearly there are no neutrosophic natural idempotents associated with $Z_{16}$.

Let $Z_{15}$ be the modulo integer. $I_0^{15}, I_3^{15}, I_6^{15}, I_9^{15}, I_5^{15}, I_{10}^{15}$ and $I_{12}^{15}$ are the associated natural neutrosophic elements of $Z_{15}$.

$$I_3^{15} \times I_3^{15} = I_9^{15},$$

$$I_3^{15} \times I_3^{15} \times I_3^{15} = I_{12}^{15},$$

$$I_3^{15} \times I_3^{15} \times I_3^{15} \times I_3^{15} = I_6^{15},$$

$$I_6^{15} \times I_3^{15} = I_3^{15}.$$

So $I_3^{15}$ is not a neutrosophic nilpotent of order two or an idempotent.

However $I_6^{15} \times I_6^{15} = I_6^{15}$ is a neutrosophic natural idempotent.

$$I_9^{15} \times I_9^{15} = I_6^{15};$$

$I_9^{15}$ is not a neutrosophic nilpotent or neutrosophic idempotent.

$$I_{12}^{15} \times I_5^{15} = I_0^{15}$$

$I_{10}^{15} \times I_{10}^{15} = I_{10}^{15}$ is a natural neutrosophic idempotent.

But $I_3^{15} \times I_5^{15} = I_0^{15}$,

$$I_3^{15} \times I_{10}^{15} = I_0^{15},$$

$$I_6^{15} \times I_{10}^{15} = I_0^{15},$$

$$I_6^{15} \times I_5^{15} = I_0^{15},$$

$I_{12}^{15} \times I_{10}^{15} = I_0^{15}$ and

$$I_5^{15} \times I_9^{15} = I_0^{15}.$$

All elements are natural neutrosophic zero divisors.

Next we study the complex finite modulo integers $C(Z_{10})$ and its associated natural neutrosophic elements got by division which are as follows.

$I_0^C, I_5^C. I_{5i_F}^C, I_2^C, I_{2i_F}^C, I_4^C, I_{4i_F}^C, I_8^C, I_{8i_F}^C, \ldots, I_{8+2i_F}^C, I_{8+5i_F}^C, I_{4i_F+8}^C, I_{8+6i_F}^C, I_{8+8i_F}^C$
where

$I_0^C = \{0/0, 1/0, 2/0, \ldots, 9+9i_F/0\}$
$I_5^C = \{0/5, 1/5, 2/5, \ldots, 9+8i_F/5, 9+9i_F/5\}$
$I_{2i_F}^C = \{0/2i_F, 1/2i_F, 2/2i_F, \ldots, 9+9i_F/2i_F\}$ and so on.

It is left as an open conjecture to find elements which are natural neutrosophic elements in $C(Z_m)$ for a given m.

Let $C(Z_7)$ be the finite complex modulo integers.

The natural neutrosophic elements associated with $C(Z_7)$ is $I_0^C$.

Infact is a difficult problem to find other natural neutrosophic elements associated with $C(Z_7)$.

Next we recall the notion of natural neutrosophic elements of the neutrosophic set $\langle Z_m \cup I\rangle = \{a + bI / I^2 = I \text{ and } a, b \in Z_m\}$.

For more about these notions refer [13,14, 29].

Now we study the natural neutrosophic elements of $\langle Z_m \cup I\rangle$ for m = 4.

Clearly the natural neutrosophic elements associated with $\langle Z_4 \cup I\rangle$ are

$I_0^I, I_2^I, I_{2I}^I, I_{2+2I}^I, I_{1+3I}^I, I_I^I$, and so on where

$I_0^I = \{0/0, 1/0, 2/0, 3/0, 4/0, I/0, 1+I/0, \ldots, 3+3I/0\}$,

$I_2^I$ = {0/2, 1/2, 2/2, …, I/2, …, 3+3I/2} and so on.

$I_I^I$ = {0/I, 1/I, 2/I, …., 3+3I/I}.

$I_{3+I}^I$ is also a natural neutrosophic zero divisor as $I_{3+I}^I \times I_{2I}^I = I_0^I$ and $I_{1+3I}^I \times I_{2I}^I = I_0^I$.

Study in this direction is both interesting and innovative. Consider $\langle Z_3 \cup I\rangle$. To find the natural neutrosophic elements associated with $\langle Z_3 \cup I\rangle$.

The natural neutrosophic elements are $I_0^I$ and $I_{1+2I}^I$ we see $\left(I_{1+2I}^I\right)^2 = I_{1+2I}^I$ so $I_{1+2I}^I$ is a natural neutrosophic idempotent of $\langle Z_4 \cup I\rangle$. $I_I^I$ and $I_{1+2I}^I$ are such that

$I_I^I \times I_{1+2I}^I = I_0^I$ is a natural neutrosophic zero divisor.

$I_{2+I}^I \times I_I^I = I_0^I$ is again a natural neutrosophic zero divisor.

$I_{2I}^I \times I_{1+2I}^I = I_0^I$ and $I_{2I}^I \times I_{2+I}^I + I = I_0^I$ are all natural neutrosophic zero divisors.

Thus some set of associated natural neutrosophic elements of $\langle Z_3 \cup I\rangle$ are $\{I_0^I, I_I^I, I_{2I}^I, I_{1+2I}^I, I_{2+I}^I\}$.

The problem of finding all natural neutrosophic elements is a difficult problem. However $Z_3$ has only one natural neutrosophic element viz. $I_0^I$ and nothing more.

Further $\langle Z_3 \cup I\rangle$ has more number of natural neutrosophic elements than $Z_3$ which has only one neutrosophic element $I_0^3$.

Thus $\langle Z_5 \cup I\rangle$ has more than one natural neutrosophic element. For $I_0^I, I_I^I, I_{2I}^I, I_{3I,}^I I_{4I}^I, I_{1+4I}^I, I_{4+I}^I, I_{4+I}^I, I_{2+3I}^I, I_{3I+2}^I$ are some of the natural neutrosophic elements associated with $\langle Z_5 \cup I\rangle$.

Hence finding the number of natural neutrosophic elements in $\langle Z_m \cup I\rangle$ is a challenging problem.

Consider $\langle Z_6 \cup I\rangle$, the number of natural neutrosophic elements are given by $I_0^I, I_2^I, I_3^I, I_4^I$, $I_I^I$, $I_{2I}^I$, $I_{3I}^I$, $I_{4I}^I, I_{5I}^I$, $I_{1+5I}^I, I_{5+I}^I, I_{2+4I}^I, I_{4+2I}^I$ $I_{3+3I}^I, I_{2+2I}^I, I_{4+4I}^I$ and so on.

$I_3^I \times I_3^I = I_3^I$, $I_4^I \times I_4^I = I_4^I$ are some of the natural neutrosophic idempotents of $\langle Z_6 \cup I\rangle$.

We see

$$I_{1+5I}^I \times I_I^I = I_0^I,\ I_{3+3I}^I \times I_2^I = I_0^I,\ \ I_{4+2I}^I \times I_{3I}^I = I_0^I,\ \ \ I_{2+4I}^I \times I_{5I}^I = I_0^I.$$

$I_{2+4I}^I \times I_{3+3I}^I = I_0^I$ and so on are some of the natural neutrosophic zero divisors of $\langle Z_6 \cup I\rangle$. Thus $\langle Z_6 \cup I\rangle$ has several natural neutrosophic elements.

However it is difficult to find natural neutrosophic nilpotents in $\langle Z_m \cup I\rangle$ for m a prime or a composite number and m not of the from $p^t$ where p is a prime and $t \geq 2$.

Further we have several problems some of which are open conjectures that are proposed in the end of this chapter about $\langle Z_m \cup I\rangle$, $2 < m < \infty$.

Next we proceed onto analyse the natural neutrosophic elements of $\langle Z_m \cup g\rangle$ the modulo dual numbers. $\langle Z_m \cup g\rangle = \{a + bg / a, b \in Z_m, g^2 = 0\}$.

The natural neutrosophic elements of $\langle Z_9 \cup g\rangle$ are $\{ I_0^9$, $I_3^9$, $I_6^9$, $I_g^9$, $I_{2g}^9$, $I_{3g}^9$, $I_{4g}^9$, $I_{5g}^9$, $I_{6g}^9$, $I_{7g}^9$, $I_{8g}^9$, $I_{3+3g}^9$, $I_{6+6g}^9$, $I_{3+6g}^9$,

$I^9_{6+3g}$, $I^9_{3+4g}$, $I^9_{6+4g}$, $I^9_{6+g}$, $I^9_{3+g}$, $I^9_{3+2g}$, $I^9_{6+2g}$, $I^9_{3+3g}$, $I^9_{6+3g}$, $I^9_{3+5g}$, $I^9_{3+7g}$, $I^9_{6+5g}$, $I^9_{6+7g}$, $I^9_{3+8g}$, $I^9_{6+9g}$ }, where $I^9_g$ = {0/g, 1/g, 2/g, …, 1+g/g, …, 8+8g/g} and so on.

For $I^9_{6+8g} \times I^9_{3g} = I^9_0$, $I^9_{3+4g} \times I^9_{6g} = I^9_0$ and so on.

$I^9_{3g} \times I^9_{6g} = I^9_0$ and $I^9_3 \times I^9_3 = I^9_0$. This has nilpotents and zero divisors.

Thus the natural neutrosophic elements of the modulo dual numbers $\langle Z_m \cup g\rangle$ behave in a distinct way [17].

Next the notion of natural neutrosophic numbers in $\langle Z_m \cup h\rangle$; $h^2 = h$. The special dual like number modulo integers is analysed. $\langle Z_m \cup h\rangle = \{a + bh / a, b \in Z_m, h^2 = h\}$ [18].

We now describe the natural neutrosophic elements of $\langle Z_{12} \cup h\rangle$ in the following.

The natural neutrosophic elements of $\langle Z_{12} \cup h\rangle$ are $I^h_{01}$, $I^h_h$, $I^h_{2+10h}$, $I^h_{2h+10}$, $I^h_2$, $I^h_4$ $I^h_6$ $I^h_8$, $I^h_{10}$, $I^h_3$, $I^h_2$, $I^h_4$ $I^h_6$ $I^h_8$, $I^h_{10}$, $I^h_3$, $I^h_9$, $I^h_6$, $I^h_{3h}$, $I^h_{2h}$ $I^h_{4h}$, $I^h_{5h}$, $I^h_{6h}$, $I^h_{7h}$, $I^h_{8h}$, $I^h_{9h}$, $I^h_{10h}$, $I^h_{11h}$, $I^h_{1+11h}$, $I^h_{h+11}$, $I^h_{3+9h}$, $I^h_{3+9h}$, $I^h_{4+8h}$, $I^h_{75+7h}$, $I^h_{7+5h}$, $I^h_{6+6h}$, $I^h_{3+6h}$, $I^h_{6+9h}$, $I^h_{6+3h}$, $I^h_{9+6h}$ and so on where $I^{12}_{2+h}$ = {0/2+h, 1/2+h, …, h/2+h, …, 11h+11/2+h} and so on.

$I^h_h \times I^h_{9+6h} = I^h_0$, $I^h_3 \times I^h_{4h} = I^h_0$, $I^h_{4h} \times I^h_{4h} = I^h_{4h}$ and $I^h_9 \times I^h_9 = I^h_9$. Thus the natural neutrosophic elements associated with $\langle Z_{12} \cup h\rangle$ can be natural neutrosophic idempotents, natural neutrosophic nilpotents and natural neutrosophic zero divisors.

$I^h_{7h} \times I^h_{5+7h} = I^I_0$ is a natural neutrosophic zero divisors.

Next we study the special dual like number $\langle Z_7 \cup h\rangle$.

The natural neutrosophic elements associated with $\langle Z_7 \cup h\rangle$ are $I_0^h$, $I_h^h$, $I_{2h}^h$, $I_{3h}^h$, $I_{4h}^h$, $I_{5h}^h$, $I_{6h}^h$, $I_{1+6h}^h$, $I_{6+h}^h$, $I_{2+5h}^h$, $I_{5+2h}^h$, $I_{3+4h}^h$, $I_{4+3h}^h$, …

We see $I_{3+4h}^h \times I_{2h}^h = I_0^h$, $I_{1+6h}^h \times I_{6h}^h = I_0^h$ thus there are several natural neutrosophic zero divisors.

Further $I_h^h \times I_h^h = I_h^h$ is an natural neutrosophic idempotent of $\langle Z_7 \cup h\rangle$.

It is important to note all $\langle Z_m \cup h\rangle$ has atleast one idempotent given by $I_h^h$ and $\langle Z_m \cup h\rangle$ has atleast m number of zero divisors if m is even and (m – 1) number of zero divisors if m is odd.

Next we study the natural neutrosophic elements associated with $\langle Z_m \cup k\rangle$, $k^2 = (m - 1)k$ the special quasi dual number modulo integers. Clearly $\langle Z_m \cup k\rangle = \{a + bk \mid a, b \in Z_m, k^2 = (m-1)k\}$. For more refer [19].

Now we give the natural neutrosophic elements associated with $\langle Z_5 \cup k\rangle$ are $I_0^k$, $I_k^k$, $I_{1+k}^k$, $I_{2+2k}^k$, $I_{3+3k}^k$, $I_{4+4k}^k$ and so on, where $I_0^k$ = {0/0, 1/0, …, 4/0, k/0, …, 4+4k/0},

$I_{2+2k}^k$ = {0/2+2k, ½+2k, …, 4+4k/2+2k} and so on.

We see $I_k^k \times I_{1+k}^k = I_0^k$ $I_k^k \times I_{2+2k}^k = I_{2+2k}^k$ $I_{3+3k}^k$, $I_{4+4k}^k$ and so on.

We see $I_k^k \times I_{1+k}^k = I_0^k$ $I_k^k \times I_{2+2k}^k = I_0^k$ are natural neutrosophic zero divisors.

Next we study the natural neutrosophic elements of $\langle Z_4 \cup k\rangle$.

The natural neutrosophic elements of $\langle Z_4 \cup k\rangle$ are $I_0^k$, $I_k^k$, $I_{1+k}^k$ $I_{2+2k}^k, I_2^k, I_{2k}^k, I_{3+3k}^k$.

Clearly $\langle Z_4 \cup k\rangle$ has natural neutrosophic idempotents natural neutrosophic nilpotents and natural neutrosophic zero divisors.

$I_k^k \times I_{1+k}^k = I_0^k$, $I_k^k \times I_{3+3k}^k = I_0^k$ and $I_k^k \times I_{2+2k}^k = I_0^k$ are natural neutrosophic zero divisors associated with natural neutrosophic elements of $\langle Z_4 \cup k\rangle$.

$I_2^k \times I_{2k}^k = I_0^k$, $I_{2k}^k \times I_{2k}^k = I_0^k$ and $I_{2+2k}^k \times I_{2+2k}^k = I_0^k$ are the natural neutrosophic nilpotents of order two.

Consider $I_{1+k}^k \times I_{1+k}^k = I_{1+k}^k$ are natural neutrosophic idempotent. For more refer [29].

In view of all these we have the following result.

There is always a natural neutrosophic idempotent associated with $\langle Z_m \cup k\rangle$ given by $I_{1+2k}^k$; for $I_{1+2k}^k \times I_{1+k}^k = I_{1+k}^k$ as $(1 + k) \times (1 + k) = 1 + 2k + k^2$

$$= 1 + 2k + (m - 1)\,k \qquad = 1 + (m + 1)k$$
$$= 1 + k \pmod m.$$

Hence the claim.

Further all natural neutrosophic elements associated with $\langle Z_m \cup k\rangle$ has atleast $(m - 1)$ number of neutrosophic natural zero divisors.

We see $I_k^k \times I_{t+tk}^k = I_0^k$, $t = 1, 2, \ldots, m - 1$.

Hence the claim.

Now having seen examples of all natural neutrosophic elements of modulo integers, we now proceed onto study MOD real intervals, MOD complex modulo interval integers, MOD dual number intervals and so on.

We see [0, m) the real MOD interval has natural neutrosophic elements which are pseudo zero divisors also. Consider the MOD real interval [0, 5).

Clearly $I_0^{[0,5)}$, $I_{1.25}^{[0,5)}$, $I_{2.5}^{[0,5)}$, $I_{3.125}^{[0,5)}$ where $I_0^{[0,5)}$ = {0/0, 0.0 … 5/0, …, 4.999/0 …} and so on are all pseudo MOD zero divisors of [0,5).

Thus it is left as a open conjecture to find the pseudo MOD neutrosophic numbers.

For $I_4^{[0,5)} \times I_{1.25}^{[0,5)} = I_0^{[0.5)}$ so a unit acts as a zero divisor also in the MOD real neutrosophic interval [0, 5).

That is why we call these type of zero divisors as special pseudo zero divisors and their associated neutrosophic element as pseudo neutrosophic zero divisors.

Consider the MOD real interval [0, 6); $1.2 \times 5 = 0 \pmod 6$.

But 5 is a unit in [0,6) so 1.2 is a special pseudo zero divisor and $I_{1.2}^{[0,6)}$ is a natural neutrosophic special pseudo zero divisor and we are forced to include $I_5^{[0,6)}$ as the natural neutrosophic unit as it is the factor which contributes to the special pseudo neutrosophic zero divisors.

Consider the MOD dual number interval [0,m)g, $g^2 = 0$. Every element in [0,m)g is a zero divisor so $I_x^{[0,m)g}$ is a neutrosophic zero divisor for all $x \in [0,m)g$.

So they are infinite in number so not much of development is possible for the MOD dual number intervals.

Consider the MOD neutrosophic interval $[0,m)I$; this behaves more like a real MOD interval.

Next consider the complex MOD interval $[0,m)i_F$, $i_F^2 = (m - 1)$ we see under product $[0, m)i_F$ is not even closed so nothing can be said about MOD complex intervals $[0,m)i_F$; $2 \leq m < \infty$.

Now consider the MOD special dual like number interval $[0, m)h$, $h^2 = h$.

This MOD interval also behaves more like the MOD real interval.

Next consider the MOD special quasi dual number interval $[0,m)k$; $k^2 = (m - 1) k$, $2 \leq m < \infty$.

Clearly this MOD interval also behaves like the MOD real interval. So the study of the MOD intervals $[0,m)I$, $[0, m)h$ and $[0,m)k$ is akin to the study of MOD real interval.

Hence only a few problems are proposed at the end of this chapter.

Next we proceed onto study about MOD neutrosophic elements of the MOD real plane, MOD finite complex number plane, MOD neutrosophic plane and so on.

Let $R_n(m)$ be the real MOD plane.

This has several MOD natural neutrosophic elements which are MOD neutrosophic idempotents, MOD neutrosophic nilpotents and MOD neutrosophic zero divisors [29].

We will illustrate this situation by some examples.

Let $R_n(10)$ be the MOD real plane.

$I^R_{(0,0)}, I^R_{(5,5)}, I^R_{(0,5)}, I^R_{(0,0)}, I^R_{(6,5)}, I^R_{(5,6)}, I^R_{(6,6)}, I^R_{(6,0)}, I^R_{(0,6)}$, $I^R_{(2,0)}, I^R_{(0,2)}, I^R_{(2,2)}$, $I^R_{(4,0)}, I^R_{(0,4)}, I^R_{(4,4)}$ are some of the natural neutrosophic elements of $Z_m \times Z_m$.

Here $I^R_{(0,0)}$ = {(0,0)/(0,0), (0,a)/(0,0), …, (a,b)/(0,0) where a, b $\in Z_m$}.

It is pertinent to note $Z_m \times Z_m$ is only the cross product and not the MOD plane.

Infact $I^R_{(2.5,0)}, I^R_{(0,2.5)}$, $I^R_{(1.25,0)}, I^R_{(0,1.25)}$, $I^R_{(2.5,0)}, I^R_{(1.25,2.5)}$ and so on are some of the MOD neutrosophic elements of the MOD plane.

For $I^R_{(1,25,2.5)} \times I^R_{(8,8)} = I^R_{(0,0)}$ is a MOD zero neutrosophic divisor of $R_n(10)$ and $I^R_{(6,0)} \times I^R_{(6,0)} = I^R_{(6,0)}$ is the MOD neutrosophic idempotents and so on.

Infact all MOD neutrosophic elements of the MOD interval [0, 10) will contribute to MOD neutrosophic elements in the MOD real plane $R_n(10)$.

Thus all MOD neutrosophic elements of [0, m) will contribute to MOD neutrosophic elements of the MOD plane $R_n(m)$.

Hence we only propose some problems in the direction.

Similarly $R^I_n(m)$ = {a + bI / a, b $\in$ [0, m), $I^2$ = I} will contribute to MOD neutrosophic of $R^I_n(m)$. These MOD neutrosophic elements are contributed by the intervals [0, m), [0, m)I and more.

Apart from other types of MOD neutrosophic elements got from a + bI.

So study of MOD neutrosophic elements will remain mostly akin to MOD neutrosophic elements from [0, m) and [0, m)I; hence left as an exercise to the reader.

Now we study $R_n^I(m) = \{a + bg \mid g^2 = 0, a, b \in [0, m)\}$. The MOD neutrosophic elements of the MOD dual number plane will be MOD neutrosophic elements from [0, m), [0,m)g and more.

However this study is not that similar to the earlier case but in a way only a little similar to $R_n^I(m)$.

We can find the MOD neutrosophic elements from the MOD complex plane $C_n(m) = \{a + bi_F \; i_F^2 = (m - 1); a, b \in [0,m)\}$.

One part of MOD neutrosophic elements will be contributed by [0, m) however for other part one has to work thoroughly.

Infact it can be left as an open conjecture.

Next for the special dual like number MOD plane

$R_n^h(m) = \{a + bh \mid a, b \in [0, m), h^2 = h\}$ also works in a similar fashion as that of $R_n^I(m)$.

Finally the MOD special quasi dual number plane works entirely different from all the other MOD planes for

$R_n^k(m) = \{a + bk \mid k^2 = (m - 1)k, a, b \in [0, m)\} \; 2 \leq m < \infty$.

Study is innovative and interesting but one has work a lot in the direction. In this regards several problems are suggest.

Consider $R_n(6)$, $R_n^g(6)$, $R_n^h(6)$, $R_n^k(6)$ $C_n(6)$ and $R_n^I(6)$.

We take same sets of elements and show how differently the product behaves.

Let $x = (2.35, 1.5)$ and $y = (3.5, 2.7) \in R_n(6)$.

$x \times y = (2.225, 4.05).$ … I

Let $x = 2.35 + 1.5\,I$ and $y = 3.5 + 2.7\,I \in R_n^I(6)$.

$x \times y = 2.225 + 3.645I$ … II

Let $x = 2.35 + 1.5g$ and

$y = 3.5 + 2.7\,g \in R_n^g(6)$

$x \times y = (2.225 + 5.595\,g)$ … III

Let $x = 2.35 + 1.5i_F$ and

$y = 3.5 + 2.7i_F \in C_n(6)$.

$x \times y = 2.225 + 5.595\,i_F + 2.25$
$= 4.475 + 5.595\,i_F \in C_n(6).$ … IV

Let $x = 2.35 + 1.5$ and

$y = 3.5 + 2.7h \in R_n^h(6)$

$x \times y = 2.225 + 3.645\,h$ … V

Lastly $x = 2.35 + 1.5\,k$ and

$y = 3.5 + 2.7\,k \in R_n^k(6)$

$x \times y = 2.225 + 1.845\,k$ … VI

We see all the six values are different except $R_n^I(m)$ and $R_n^h(m)$ but I is an indeterminate where as h is a determinate.

We propose the following problems some of which are at research level and a few are open conjectures.

## Problems

1. For a given m find all the natural neutrosophic elements associated with $Z_m$.

2. Find all the natural neutrosophic elements associated with $Z_{48}$.

3. Find all natural neutrosophic elements associated with $Z_{256}$.

   i) Can we say that none of the natural neutrosophic elements associated with $Z_{256}$ will be neutrosophic idempotents except $I_0^{256} \times I_0^{256} = I_0^{256}$ ?

   ii) Can we claim every natural neutrosophic element of $Z_{256}$ will be natural nilpotents?

4. Obtain any other special features associated with the natural neutrosophic elements related with $Z_m$; $2 \leq m < \infty$.

5. Let $C(Z_m)$ be the finite complex modulo integers.

   i) Find the number of natural neutrosophic elements associated with $C(Z_m)$ for a fixed m.

   ii) Can we say when m is a prime number the number of natural neutrosophic elements would be the least?

   iii) Can we claim the more m has divisors; $C(Z_m)$ will have more natural neutrosophic elements?

   iv) Compare the number of natural neutrosophic elements m $C(Z_{210})$ and $C(Z_{505})$.

6. Let $C(Z_{101})$ be the finite complex modulo integers.

   i) Find the number of natural neutrosophic elements associated with $C(Z_{101})$.

   ii) How many natural neutrosophic idempotents are related with $C(Z_{101})$?

iii) Can there be natural neutrosophic nilpotents related with $C(Z_{101})$?

7. Let $C(Z_{96})$ be the finite complex modulo integers.

   Study questions (i) to (iii) of problem (6) for this $C(Z_{96})$.

8. Can we say $C(Z_{96})$ will have more number of natural neutrosophic elements associated with it than $C(Z_{101})$?

9. Compare the number of natural neutrosophic elements associated with $C(Z_m)$ and $Z_n$ for a fixed m.

10. Characterize all m which are such that both $C(Z_m)$ and $Z_m$ have same number of natural neutrosophic elements.

11. Can we say in general $C(Z_m)$ will have more number of natural neutrosophic elements than $Z_m$ when m is a composite number?

12. Obtain any other special feature associated with the natural neutrosophic elements of $C(Z_m)$.

13. Find the number of natural neutrosophic numbers associated with $\langle Z_m \cup I\rangle$; $2 \leq m < \infty$.

14. Let $\langle Z_{15} \cup I\rangle$ be the modulo neutrosophic integers.

    i) Find all the associated natural neutrosophic elements of $\langle Z_{15} \cup I\rangle$.
    ii) Prove there are natural neutrosophic zero divisors.
    iii) Prove there are natural neutrosophic elements.
    iv) Does the associated natural neutrosophic elements has nilpotents?
    v) Find any other special features enjoyed by these natural neutrosophic elements.
    vi) Prove the collection of all natural neutrosophic elements is a semigroup under product.

15. Let $\langle Z_{19} \cup I\rangle$ be the modulo neutrosophic integers.

    Study questions (i) to (vi) of problem (14) for this $\langle Z_{19} \cup I\rangle$.

16. Let $\langle Z_{3^{12}} \cup I\rangle = S$ be the neutrosophic modulo integers.

    Study questions (i) to (vi) of problem (14) for this S.

17. Let $T = \langle Z_m \cup g\rangle$ be the dual number modulo integers.

    i) Find all the natural neutrosophic elements associated with $\langle Z_m \cup g\rangle$ for a fixed m.
    ii) Prove there are many natural neutrosophic zero divisors.
    iii) Study questions (i) to (vi) of problem (14) for this T.

18. Let $W = \langle Z_{19} \cup g\rangle$ be the dual number modulo integers.

    Study questions (i) to (vi) of problem (14) for this W.

19. Let $M = \langle Z_{18} \cup g\rangle$ be the dual number modulo integers.

    Study questions (i) to (vi) of problem (14) for this M.

20. Let $P = \langle Z_{2^{20}} \cup g\rangle$ be the dual number modulo integers.

    Study questions (i) to (vi) of problem (14) for this P.

21. Let $T = \langle Z_m \cup h\rangle$; $h^2 = h$ be the special dual like number modulo integers.

    Study questions (i) to (vi) of problem (14) for this T.

22. Let $S = \langle Z_{10} \cup h\rangle$; $h^2 = h$ be the special dual like number modulo integers.

    Study questions (i) to (vi) of problem (14) for this S.

23. Let B = $\langle Z_{5^6} \cup h \rangle$; $h^2 = h$ be the special dual like number modulo integer.

    Study questions (i) to (vi) of problem (14) for this B.

24. Let D = $\langle Z_{47} \cup h \rangle$; $h^2 = h$ be the special dual like number modulo integers.

    Study questions (i) to (vi) of problem (14) for this D.

25. Obtain all the special features of the natural neutrosophic elements associated with $\langle Z_m \cup k \rangle$; the special quasi dual number modulo integers.

26. Let W = $\langle Z_m \cup k \rangle$, $k^2 = (m - 1)k$ the special quasi dual number modulo integers.

    i) Find all the natural neutrosophic elements associated with $\langle Z_m \cup k \rangle$.
    ii) How many are natural neutrosophic zero divisors?
    iii) Find all natural neutrosophic nilpotents.
    iv) How many are natural neutrosophic idempotents?

    (Study for a fixed m, m a prime and m a composite number).

27. Let T = $\langle Z_{23} \cup k \rangle$; $k^2 = 22k$ be the special quasi dual number modulo integers.

    Study questions (i) to (iv) of problem (26) for this T.

28. Let P = $\langle Z_{7^6} \cup k \rangle$; $k^2 = (7^6 - 1)k$ be the special quasi dual number modulo integers.

    Study questions (i) to (iv) of problem (26) for this P.

29. Let B = $\langle Z_{24} \cup k \rangle$; $k^2 = 23k$ be the special quasi dual number modulo integers.

Study questions (i) to (iv) of problem (26) for this B.

30. Compare the natural neutrosophic elements of T, P and B given in problems 27, 28 and 29 respectively.

31. Can there be a real MOD interval [0, m) without special pseudo zero divisors?

32. Find all special pseudo zero divisors of [0, 16).

33. Find all special pseudo zero divisors of the MOD real interval [0,43).

34. Obtain all special pseudo zero divisors of the MOD real interval [0,45).

35. Finding the total number of MOD natural neutrosophic elements of the MOD real interval [0,m) $2 \leq m < \infty$ happens to be a open conjecture.

36. It is a open conjecture to find the total number of MOD special pseudo neutrosophic zero divisors of the MOD real interval [0,m).

37. Find all MOD neutrosophic elements of P = [0, 13), the MOD real interval.

38. Find all MOD neutrosophic special pseudo zero divisors of P in problem 37.

39. Let M = [0, 48) be the MOD real interval;

    i) Find all MOD neutrosophic elements of M.
    ii) Find all MOD special pseudo zero divisors of [0,48).

40. Let W = $[0, 3^{24})$ be the MOD real interval.

    Study questions (i) and (ii) of problem (39) for this W.

41. Compare the nature of the MOD neutrosophic elements and MOD special pseudo neutrosophic zero divisors in problems (37), (39) and (40) for P, M and W respectively.

42. Prove [0,m)g, $g^2 = 0$ the MOD dual number interval has infinite number of MOD neutrosophic zero divisors and MOD neutrosophic nilpotents of order two and has no MOD natural neutrosophic idempotents.

43. Prove or disprove [0,m)I and [0,m) have the same type of MOD neutrosophic zero divisors, idempotents, and special pseudo zero divisors.

44. Compare the MOD neutrosophic elements of [0,m)I and [0,m).

45. Study for P = [0,24)I the MOD neutrosophic zero divisors, MOD neutrosophic idempotents and MOD neutrosophic pseudo zero divisors.

46. Study M = $[0, 3^{10})$I, the MOD neutrosophic interval; for MOD neutrosophic elements.

47. Compare P and M of problems (45) and (46) with W = [0,29)I.

48. Study the MOD special dual like interval [0,m)h $h^2 = h$ for natural MOD neutrosophic elements.

    i) Can we say [0,m)I and [0,m)h have same type of natural MOD neutrosophic elements?
    ii) Find the difference between [0,m)h and [0,m).
    iii) Find the difference between [0,m)I and [0,m).

49. Let P = [0,m),k $k^2 = (m – 1)k$ be the MOD special quasi dual number interval.

    i) Study questions (i) and (ii) of problem (39) for this P.

50. Let S = [0,96)k, $k^2$ = 985k be the MOD special quasi dual number interval.

    Study questions (i) to (ii) of problem (39) for this S.

51. Compare the MOD special quasi dual number interval [0,m)k, MOD special dual like number interval, [0,m)h MOD special quasi dual number interval; [0,m)g, and the neutrosophic MOD interval; [0,m)I and so on.

52. Find the MOD neutrosophic elements of the real MOD plane $R_n(m)$.

53. Find the collection of all MOD neutrosophic elements of the MOD real plane $R_n(27)$.

54. Find the collection of all MOD neutrosophic elements of the MOD real plane $R_n(47)$.

55. Find the collection of all MOD neutrosophic elements of the MOD real plane $R_n(24)$.

    (i) Does $R_n(24)$ contain pseudo zero divisors?
    (ii) Does $R_n(24)$ contain pseudo natural MOD neutrosophic zero divisors?

56. Find the collection of all MOD neutrosophic elements of the MOD complex modulo integer plane $C_n(m)$; $i_F^2 = m - 1$.

57. Find the collection of all MOD neutrosophic elements of the MOD neutrosophic plane $C_n(45)$; $i_F^2 = 44$.

58. Let S = $C_n(29)$, $i_F^2 = 28$ be the MOD complex plane. Find all the MOD neutrosophic elements of S.

59. Let M = $C_n(48)$, $i_F^2 = 47$ be the MOD complex modulo integer plane. Find all MOD neutrosophic elements of M.

60. Compare the MOD neutrosophic elements of S and M in problems 58 and 59 respectively.

61. Find the collection of all MOD neutrosophic elements of $R_n^I(m)$ the MOD neutrosophic plane.

62. Let M = $R_n^I(48)$ be the MOD neutrosophic plane find the MOD neutrosophic elements of M.

63. Let $M_1$ = $R_n^I(13)$ be the MOD neutrosophic plane find the MOD neutrosophic elements of $M_1$.

64. Compare M and $M_1$ in problems (62) and (63) respectively.

65. Let N = $R_n^g(m)$ be the MOD dual number plane.

    Find all MOD neutrosophic elements of N.

66. Let P = $R_n^g(28)$ be the MOD dual number plane.

    i) Compare P with N of (65).
    ii) Obtain all MOD neutrosophic elements of P.

67. Let S = $R_n^g(47)$ be the MOD dual number plane.

    i) Study questions (i) and (ii) of problem (66) for this S.

68. Let R = $R_n^h(m)$ be the MOD special dual like number plane.

    (i) Study questions (i) and (ii) of problem (66) for this R.

    (ii) Compare R of this problem with S of problem (67)

69. Let T = $R_n^k(m)$, $k^2 = (m - 1)k$ be the MOD special quasi dual number plane.

   i) Study questions (i) to (ii) of problem (66) for this T.
   ii) Compare T with R of problem in (67).

69. Let M = $\{R_n^k(24), k^2 = 23k\}$ be the MOD special quasi dual number plane.

   i) Study questions (i) to (ii) of problem (66) for this M.
   ii) Compare M with P = $\{R_n^h(42), h^2 = h\}$.

70. Let Y = $\{R_n^k(426), k^2 = 425k\}$ be the MOD special quasi dual number plane.

   i) Find all MOD neutrosophic elements of Y.
   ii) Prove Y has pseudo zero divisors.

**Chapter Five**

# PROBLEMS ON MOD INTERVAL POLYNOMIALS AND MOD PLANE POLYNOMIALS

In this chapter for the first time we introduce the notion of MOD interval polynomials and MOD plane polynomials. For more refer [26-30].

Clearly $[0, m)[x] = \left\{ \sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i \in [0, m) \right\}$ is the MOD real interval polynomials.

Finding roots of [0, m)[x] is an impossibly. As [0, m)[x] is only a pseudo ring we see it is difficult to get roots.

Several difficult open problems are suggested in this which are open conjecture.

Study $[0,96)[x] = \left\{ \sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i \in [0, 96) \right\}$, the MOD real interval polynomials.

Find all roots of $p(x) = x^3 + 90 \in [0, 96)[x]$.

Let $[0,m)I[x]$ be the neutrosophic polynomial interval pseudo ring.

We see $p(x) \in R[x]$ may be continuous in the plane R; however these polynomials in $[0,m)[x]$ behave very differently.

Next we can have MOD interval dual number polynomials,

$$[0, m)g[x] = \left\{ \sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i \in [0, m)g \right\}.$$

These polynomials $p(x) \in [0,m)g[x]$ has no solution in many cases as this polynomial ring is a zero square ring and is not pseudo.

We see $[0, m)I[x]$ is the MOD neutrosophic interval pseudo polynomial ring.

The problem of solving the equations arises from the fact I is an indeterminate and not a invertible element that is I has no inverse.

So solving these equations is also impossible for we can only say or find roots of Ix and not x.

Similar situation arise in case of $[0,m)h[x]$ and $[0, m)k[x]$, $h^2 = h$ and $k^2 = (m - 1)k$ respectively. Thus we get only values of Ix, gx, hx or kx and not for x.

The case where x gets the value is $[0, m)[x]$ the MOD real interval pseudo ring.

Next we study the polynomials in MOD planes.

$$R_n(m)[x] = \left\{ \sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i \in R_n(m) \right\}.$$

We see even the simple equation $y = x^2$ is not continuous has several zeros in the MOD (real plane $R_n(m)$).

Such illustrations and study is carried out in [Books 1 & 2]. However here we propose some problems.

Similarly the equation $y = sx + t \in R_n(m)$; $1 \leq s, t < m$ behaves in a very different way has several zeros.

So a linear equation in the MOD plane can have several zeros.

Next the study of a simple equation $y = nx$, $1 < n < m$ in $R_n(m)$ has many zeros.

Hence we have given several open problems in this chapter regarding roots of a polynomial.

Next if we consider MOD polynomials in the MOD complex modulo number plane $C_n(m)$, $m - 1 = i_F^2$ we see the similar situation arises. For more information refer (Books 1, 2).

Thus for instance even simple polynomials like $p(x) = (a + bi_F)x^2 + c + di_F$, a, b, d, c $\in$ [0,10) behave in a very unnatural way for varying values of a, b, c, d $\in$ [0, 10).

Study in this direction is also left open for interested reader.

Consider $5x + 8 + 2i_F = 0 \in C_n(10)$.

Solving this equation is near to an impossibility as $5^2 = 5 \pmod{10}$, so $5x = 2 + 8i_F$ is the solution.

Consider $(2 + 4i_F)\, x + 5 = 0 \in C_n(10)$.

Clearly $(2 + 4i_F)x = 5$ is the solution for 2 and 4 are zero divisors in $C_n(10)$.

$(2 + 4i_F) \times 5 \equiv 0 \pmod{10}$.

So one cannot lessen the coefficient.

Infact $(2 + 4i_F)\ x = 5$ multiply by 3 one gets $(6 + 2i_F)\ x = 5$ so are both same one is left to wonder.

Next we study MOD neutrosophic polynomials in $R_n^I(m)[x]$.

Since I is an indeterminate if the coefficients of x or its power contains I it is not solvable for x.

Consider $3Ix + 2 + 4I \in C_n^I(5)[x]$

$$3Ix = 3 + I$$
$$Ix = 1 + 2I$$

So we solve this not for x only for Ix.

Consider $\quad 2x + 0.7 + 2.43i_F \in C_n^I(5)[x]$

$$2x = 4.3 + 2.57i_F.$$

So $\quad x = 2.9 + 2.71i_F$ is the value of x.

Let $\quad 2.5\ x^2 + 4.31 + 2i_F \in C_n^I(5)[x]$

$$2.5\ x^2 = 0.69 + 3i_F \text{ since}$$
$$2.5 \times 2 \equiv 0 \pmod 5;$$

we see value of x cannot be got only the value of $x^2$ is got.

One can find $\sqrt{2.5x^2} = \sqrt{0.69+3i_F}$ so that $x(1.58113883) = \sqrt{0.59+3i_F}$ we do not know whether 1.58113883 is a unit or not in [0, 5).

So solving even a quadratic equation in $C_n(m)[x]$ is not that easy.

Further in $C_n(m)[x]$ or for that matter in all types of MOD planes we see

$(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) \neq x^n (\alpha_1 + \alpha_2 + \ldots + \alpha_n) x^{n-1} + \sum_{i \neq j} \alpha_i \alpha_j x^{n-2} \ldots \pm \alpha_1 \alpha_2 \ldots \alpha_n$ as the distributive law is not true, so after solving the p(x) one faces lots of difficulty all these are left as open conjecture to the reader.

Next we study the polynomial in the MOD neutrosophic pseudo polynomial rings.

$R_n^I(m)[x] = \left\{ \sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i = d_i + c_i I \in R_n^I(m) \right\}$ is the MOD neutrosophic polynomial pseudo ring.

As $2 \leq m < \infty$ we have infinitely many such rings for each value of m, however there is only one neutrosophic ring, $\langle R \cup I\rangle[x] = \left\{ \sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i \in \langle R \cup I\rangle \right\}$.

Solving MOD polynomial equations in $R_n^I(m)[x]$ is as difficult as in MOD complex polynomials and MOD real polynomials.

Consider $0.5Ix + 10.2I + 0.99 \in R_n^I(12)$.

Clearly $0.5Ix = 1.8I + 11.01$. However value of x cannot be found only the value of 0.5Ix is got.

Thus even linear equations in general are not solvable in $R_n^I(12)$.

So solving quadratic equations and equations of higher degree is a very difficult problem, hence left as a open conjecture or for future study.

Let $5x^2 + 3.7I \in R_n^I(12)$

$5x^2 = 8.3I$

$x^2 = 5.5$ I so x 2.34520788I so for this quadratic equation has a solution.

However

$5x^2 + 3.7 \text{ I} \neq 0$ if
x = 2.34520788I but its value is 7.2 only.

We see $x^2 = 5.5$ I is true but $5x^2 = 8.3$I is not true.

There is a fallacy in solving so left as a open conjecture.

In case of modulo addition or multiplication how to solve equations.

Thus there are several intricate problems involved while solving equations in MOD neutrosophic polynomials in $R_n^I(m)[x]$.

Next we proceed onto describe and discuss about the MOD dual number polynomials $R_n^g(m)[x] = \left\{\sum_{i=1}^{\infty} a_i x^i \,\middle|\, a_i \in R_n^g(m)\right.$ where $a_i = x + yg$, x, y $\in$ [0,m), $g^2 = 0\}$.

Solving these equations is a very difficult task for
$0.7g\, x^3 + 7.3 = 0$ is in $R_n^g(10)[x]$.

Clearly as $g^2 = 0$ we cannot cancel g using any inverse.

Now $0.7g\, x^3 = 2.67$.

We cannot take the cube root as g is a dual number and is it possible to take cube root of a dual number is still a open problem for the dual number can be matrix or a transformation or from $Z_n$ the ring of modulo integers n a composite number.

Under these circumstances only $0.7gx^3$ value is got is not further solvable. Consider

$3x^3 + 0.786 \in R_n^g(10)[x]$.
$3x^3 = 9.214$ (multiplying by 7 taking mod10 we get)
$x^3 = 4.498$.

So x = 1.650719001 is one of the roots of the given MOD polynomial.

Thus one cannot always say all the polynomials in MOD planes are not solvable.

Some of them may be solvable partially and a few fully and many of the polynomials never solvable.

Even solving
$p(x) = (6.78 + 9.33g)\, x^2 + (4.775 + 0.33g)x + (0.52 + 9.32g)$ in $R_n^g(10)[x]$ is near to an impossibility.

Readers can try this as a recreation or pursue this as a hobby.

Several open conjectures are laid before them.

Next the notion of MOD special quasi dual like number polynomials in

$$R_n^h(m)[x] = \left\{\sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i = x + yh \in R_n^h(m),\ h^2 = h \text{ and } x, y \in [0, m), 2 \leq m < \infty\right\}$$ are discussed in the following.

Let $p(x) = hx^3 + 0.335 + 6.7h \in R_n^h(7)[x]$. Solving this for x is a difficult problem.

For if $hx^3 = 6.665 + 0.3h$ then what is the cube root of 6.665 + 0.3h. This is not a easy work for square roots can be done as in case of complex numbers but finding cube root or $n^{th}$ root happens to be a challenging problem.

However if $hx^3 = 6.4h$ then we can solve for x as follows $hx = 1.856635533$ h is one of the roots. However as in case of other MOD plane some are solvable but certainly all polynomial equations are not solvable.

Next we proceed onto study the solving of MOD polynomials in

$R_n^k(m)[x] = \left\{\sum_{i=1}^{\infty} a_i x^i \,\middle|\, a_i = x + yh \in R_n^k(m),\ x, y \in [0, m),\right.$ $\left.k^2 = (m-1)k,\ 2 \leq m < \infty\right\}$.

Let $p(x) = 8x^3 + 6.24\,k \in R_n^k(11)[x]$.

$8x^3 = 4.75k$, so multiplying the equation by 7 and taking mod we get

$$x^3 = 0.25\,k$$

$x = 0.62999605249$ is one of the values of x for this p(x).

Solving equations is not an easy task but this research is interesting and innovative.

Further each MOD plane differently and infact one has infinite number of such MOD planes, be it real MOD plane or complex MOD plane or dual number MOD plane and so on.

In the following we suggest some problems which are open conjectures, some of them at research level and some of them are simple.

**Problems:**

1. Prove or disprove the fundamental theorem of algebra for roots for the MOD real interval polynomials [0,m)[x].

2. Can a polynomial of degree n have less than n number of roots?

   Justify your claim.

3. Can a polynomial of degree n in [0,m)[x] have more than n roots?

   Justify your claim.

4. As distributive law is not true can we say finding roots of [0,m)[x] is an impossibility?

5. Let [0,m)I[x] be the MOD neutrosophic interval polynomials.

   i) Does there exist p(x) ∈ [0,m)I[x] of degree n which have n and only n roots?

   ii) Give a polynomial p(x) ∈ [0,m)I[x] of degree n which has only less than n roots.

   iii) Prove a polynomial p(x) of degree n in [0, m)I[x] has more than n roots.

   iv) Show the distributive law is not true in [0,m)I[x] which affects the roots and the number of roots of p(x) in [0,m)I[x].

6. Let R = [0,45)I[x] be the MOD neutrosophic polynomial pseudo ring.

   Study questions (i) to (iv) of problem (5) for this R.

7. Let S = [0,37)I[x] be the MOD neutrosophic polynomial pseudo ring.

Study questions (i) to (iv) of problem (5) for this S.

8. Let W = $[0,m)g[x]$ be the MOD dual number interval pseudo ring.

   i) Solve $p(x) = 0.8g\,x^3 = 4.8gx^2 + 6.1gx + 4.21g \in W$.
      a) Can p(x) have more than 3 roots?
      b) Can p(x) have less than 3 roots?

   ii) Show roots of degree one alone is solvable provided coefficients of x is 1 so it is not possible.

   iii) Can $5.03gx^2 + 7.2g = 0$ have two roots or no root?

9. Prove or disprove dual number MOD interval polynomials are not solvable if the highest coefficient is not 1 or not possible.

10. Let P = $[0,m)h[x]$, $h^2 = h$ be the MOD interval special dual like number polynomial pseudo ring.

    i) Show $p(x) \in P$ cannot be solved in general.

    ii) Even linear polynomial in P cannot be solved prove or disprove.

    iii) Obtain any other special feature enjoyed by P.

    iv) Prove $p(x) \in P$ behaves in an odd way in all MOD interval polynomials baring $[0,m)[x]$.

11. Let M = $[0,45)h[x]$, $h^2 = h$ be the MOD interval special dual like number plane.

    Study questions (i) to (iv) of problem (10) for this M.

12. Study the MOD interval special quasi dual number polynomial pseudo ring B = $[0,m)I[x]$; $k^2 = (m-1)k$ their roots.

13. Study questions (i) to (iv) of problem (10) for this B

14. Let M = $[0,24)I[x]$, $k^2 = 23k$ be the MOD interval special quasi dual number pseudo ring.

    Study questions (i) to (iv) of problem (10) for this M.

15. For a polynomial p(x) fixed compare the roots in $[0,10)[x]$, $[0,10)I[x]$, $[0,10)g[x]$; $g^2 = 0$, $[0,10)\ h[x]$, $h^2 = h$ and $[0,10)k[x]$, $k^2 = 9\ k$.

16. Find any special feature associated with polynomials in $R_n(m)[x]$.

17. Prove the equation $y = x^2 + 1$ for varying m in $R_n(m)[x]$ has different sets of roots.

18. Give a polynomial of degree 5 in $R_n(6)[x]$ which has more than n roots.

19. Let $p(x) = x^3 + 0.375 + 4.7h \in R_n^h(5)[x]$. Find all roots of p(x).

20. Does there exist a polynomial p(x) of degree 7 in $R_n(5)[x]$ which has less than seven roots?

21. Give the graph of the polynomial $y = 4x + 2 \in R_n(8)[x]$.

    i) Is the curve a continuous one?

    ii) Find all the zeros of $y = 4x + 2$.

    iii) Compare $y = 4x + 2 \in R_n(8)[x]$ with $y = (4x + 2) \in R_n(6)[x]$.

22. Let $y = x^2 \in R_n(4)[x]$ be the function in the MOD plane.

    i) Find all zeros of $y = x^2$ in $R_n(4)[x]$.

    ii) Can this function be continuous?

    iii) Trace the graph of $y = x^2$ in $R_n(4)[x]$.

    iv) Can this function have more than two zeros?

    v) Is the function continuous in the real plane?

    vi) Can the function be continuous in any other MOD plane?

23. Let $y = 3x^2 + 4 \in R_n(5)[x]$.

    i) Study questions (i) to (vi) of problem (22) for this function.
    ii) Study this function in the planes $R_n(6)$, $R_n(7)$, $R_n(10)$, $R_n(12)$ and $R_n(23)$.

24. Let $y = x^3 + 2 \in R_n(3)[x]$.

    i) Study questions (i) to (vi) of problem (22) for this function.
    ii) Study the function in the MOD plane $R_n(8)$, $R_n(91)$, $R_n(24)$ and $R_n(19)$.

25. Let $y = 4x^2 + x + 1 \in R_n(6)[x]$.

    i) Study questions (i) to (vi) of problem (22) for this function.
    ii) Study this questions in the planes $R_n(9)$, $R_n(23)$, $R_n(48)$ and $R_n(121)$.

26. Solving polynomial equations be it linear or otherwise is a difficult task.

So this problem is left as an open conjecture.

27. Solve $p(x) \in R_n(20)[x] = \left\{ \sum_{i=1}^{\infty} a_i x^i \,\middle|\, a_i x^i / a_i \in R_n(20)\right.$ where $a_i = (t_i, v_i)$, $t_i, u_i \in [0,20)\}$ where $p(x) = (0,2)x + (10.31, 3.2)$.

28. Solve $p(x) = (4,9)\, x^3 + (2,1.09)x + (6.67, 4) \in R_n(12)[x]$.

    i) How many roots exist for p(x)?
    ii) Can p(x) have more than 3 roots?

29. Solve $p(x) = (4,0.7)x^5 + 0.7, 0.8)x^4 + (5.2, 0)x^3 + (0, 4.3) \in R_n(4)[x]$.

    i) Study questions (i) to (vi) of problem (21) for this p(x).
    ii) Study for this p(x) in $R_n(19)[x]$, $R_n(12)[x]$, $R_n(43)[x]$; $R_n(4)[x]$ and $R_n(10)[x]$.

30. Find polynomials other than y = x which is continuous in $R_n(m)\,[x]$, $2 \leq m < \infty$.

31. Let $p(x) \in C_n(m)\,[x]$ be a MOD polynomial with complex coefficients.

    Study for the same p(x) but for varying m; $2 \leq m < \infty$.

32. Compare polynomials in $C_n(m)[x]$ and $R_n(m)[x]$.

33. In which MOD plane a polynomial p(x) will have more zeros in $C_n(m)[x]$ or $R_n(m)[x]$?

34. Study the roots of the polynomial $f(x) = ax^3 + bx + 1 \in C_n(7)[x]$ where $a = 3.47 + 2.4i_F$, $b = 0.46 + 2.34i_F$ and $1 = 1 + 0i_F \in C_n(7)$.

i) Can f(x) have more than three roots? Justify your claim.

ii) If $a = 3 + 4i_F$ and $b = 4 + 3i_F$ will f(x) has only three roots?

iii) If a = 3.5, b = 1.75 and $1 = 1 + 0i_F$, can f(x) have only three roots?

35. Let $p(x) = 5x^3 + 7x + 34 \in C_n(10)[x]$ and also this $p(x) \in R_n(10)[x]$.

i) Will p(x) have different sets of roots in $R_n(10)[x]$ and $C_n(10)[x]$?
ii) Can we say p(x) will have 3 roots?
iii) Can p(x) have more than three roots?
iv) Will p(x) have less than three roots?
v) Is p(x) solvable in $R_n(10)[x]$?

36. Can we write program for solving roots of a polynomial equation in $R_n^{(m)}[x]$ and $C_n^{(m)}[x]$?

37. What will be the probable applications of these polynomials in [0,m)[x]?

38. Can we say this paradigm shift will give new dimension to solving polynomial equations?

39. Let $p(x) = 0.03x^5 + 0.01i_F \in C_n(2)[x]$.

i) Does the roots of p(x) exist?
ii) Will p(x) have atleast one root?
iii) Can p(x) have more than 5 roots?
iv) Derive a method to solve p(x) in $C_n(2)[x]$.
v) Is $0.03 \in [0,2)$ an invertible element?

40. It is left as an open conjecture to find

i) All zero divisors and nilpotents in [0,2) ([0,m) in general, $2 \leq m < \infty$).

ii) Finding idempotents in [0,2) ([0,m) in general, $2 \leq m < \infty$).

iii) Finding units of [0,2) ([0,m) in general, $2 \leq m < \infty$).

41 It is yet another open conjecture of finding the total number of MOD natural neutrosophic elements in [0,m), [0, m)h and [0,m)k for $2 \leq m < \infty$.

42. Prove [0,m)g and [0,m)I has infinite number of MOD neutrosophic elements.

i) Does they contain as many as the cardinality of [0,m) itself?

43. Prove once [0,m) is solved one can solve all the MOD plane $R_n(m)$, $2 \leq m < \infty$.

44. Solve the equation $p(x) = 2.7\ Ix^2 + 4 + 6.3I \in R_n^I(8)[x]$.

i) Does p(x) have more than one root?
ii) Can x be got only in terms of xI; be got as a solution?
iii) Can p(x) have more than 2 roots?
iv) Study this problem in $R_n^I(11)[x]$ and $R_n^I(12)[x]$.

45. Let $p(x) = 2.034\ x^3 + 0.75 + 0.745\ I \in R_n^I(3)[x]$.

Study questions (i) to (iv) of problem (44) for this p(x).

46. Let $q(x) = 0.770\ x^2 + 0.55x + (0.312\ I + 0.13) \in R_n^I(2)[x]$.

Study questions (i) to (iv) of problem (44) for this p(x).

47. Let $q(x) = 6x^3 + (3 + 4I) \in R_n^I(7)[x]$.

    Study questions (i) to (iv) of problem (44) for this q(x).

48. Let $q(x) = 0.78x + 4.5I \in R_n^I(5)[x]$.

    i) Study for roots in $R_n^I(5)[x]$.
    ii) Can q(x) have more than one root?
    iii) Is 0.78 a unit or a zero divisor in $R_n^I(5)$?

49. Let $p(x) = (0.2 + 0.5g)\, x^2 + (0.6 + 0.8) \in R_n^g(4)[x]$.

    i) Study questions (i) to (iv) of problem (44) for this p(x).
    ii) Can we say 0.2 + 0.5g is a zero divisor or an idempotent in $R_n^g(4)$?

50. Let $p(x) = 2x^2 + (4 + 6g) \in R_n^g(10)[x]$.

    i) Study questions (i) to (iv) of problem (44) for this p(x).

    ii) Does a root exist as 2 is a zero divisor in $R_n^g(10)$?

51. Let $p(x) = (3 + 2g)\, x^2 + (4 + 3g) \in R_n^g(6)[x]$.

    i) Study questions (i) to (iv) of problem (44) for this p(x).

    ii) Can we say (3 + 2g) is a zero divisor in $R_n^g(6)$?

iii) Can we say polynomials of the form p(x) is not solvable even in $\langle Z_6 \cup g\rangle [x] = \{\sum_{i=1}^{\infty} (a_i + b_i g)\, x^i / a_i + b_i g \in \langle Z_6 \cup g\rangle$ where $g^2 = 0$ and $a_i\, b_i \in Z_6\}$?

52. Let $p(x) = (0.6 + 0.2g)\, x^3 + 0.33g \in R_n^g(6)[x]$.

i) Study questions (i) to (iv) of problem (44) for this p(x).

53. Finding units, idempotents, zero divisors and nilpotents of the MOD dual number plane $R_n^g(m)$, $g^2 = 0$, $2 \leq m < \infty$ happens to be a very challenging open problem.

54. Find all units, zero divisors and idempotents of $R_n^g(53)$, $R_n^g(12)$ and $R_n^g(625)$, $g^2 = 0$.

55. Let $p(x) = (7 + h)x^2 + (4 + 3h) \in R_n^h(14)[x]$, $h^2 = h$.

Find the roots of p(x) and study questions (i) to (iv) of problem (44) for this p(x).

56. Let $f(x) = (0.7 + 1.4h)\, x^2 + (2 + 7h) \in R_n^h(14)[x]$.

Study questions (i) to (iv) of problem (44) for this f(x).

57. Finding all the units, zero divisors and idempotents of $R_n^h(m)$, $h^2 = h$, $2 \leq m < \infty$ happens to be a open conjecture.

58. Find all units, zero divisors, idempotents and nilpotent elements of the MOD special dual like number MOD planes $R_n^h(43)$, $R_n^h(24)$ and $R_n^h(81)$.

59. Find all innovative techniques that can be used in solving polynomials in $R_n^h(m)$, $h^2 = h$ and $2 \leq m < \infty$.

60. Let $p(x) = 3kx^3 (9 + 5k) \in R_n^k(15)[x]$ $k^2 = 14k$.

    i) Solve for x or kx.
    ii) Study questions (i) to (iv) of problem (44) for this p(x).
    iii) Will the solution p(x) be the same in $R_n^k(15)$ and $\langle Z_{15} \cup g\rangle$?

61. Let $p_1(x) \in 0.3kx^3 + (0.09 + 0.5k) \in R_n^k(15)[x]$, $k^2 = 14k$.

    Study questions (i) to (iv) of problem (44) for this $p_1(x)$. Compare p(x) in problem (60) with this $p_1(x)$.

62. It is left as an open conjecture to find the units, idempotents, zero divisors of the MOD special quasi dual number plane $R_n^k(m)$, $k^2 = (m - 1)k$; $2 \leq m < \infty$.

63. Find the zero divisors, units and idempotents of the MOD special quasi dual number plane $R_n^k(10)$, $R_n^k(47)$ and $R_n^k(16)$.

64. Solve the equation $q(x) = (4 + 0.4k)\, x^3 + 0.004\, kx^2 + (0.04\, k + 4.4)\, x + 0.0004 \in R_n^k(5)[x]$; $k^2 = 4k$.

    i) Study questions (i) to (iv) of problem (44) for this q(x).
    ii) Study q(x) in $R_n^h(5)[x]$, $h^2 = h$,
    iii) Study q(x) in $R_n(5)[x]$
    iv) Study q(x) in $C_n(5)[x]$.
    v) Study q(x) in $R_n^g(5)[x]$.
    vi) Compare the solutions in all the 5 MOD planes.

Chapter Six

# PROBLEMS ON DIFFERENT TYPES OF MOD FUNCTIONS

In this chapter we just recall different types of MOD functions viz. MOD trigonometric, MOD logarithmic, MOD exponential and their generalizations. However such study is carried out in [27].

Here our main aim is to suggest problems some of which are open conjectures and some of them are simple.

For more about these MOD functions refer [27].

**Problems:**

1. Define MODsine; nsinx function.

2. Does values / graph of nsinx vary for varying $R_n(m)$; $2 \leq m < \infty$.

3. Study problem (2) for ntanx, ncosx and nsecx.

4. Describe the trigonometric function MOD cosecx (ncosecx) in $R_n(7)$.

5. Can we say MOD trigonometric functions are in no way different from usual trigonometric functions?

   Justify your claim.

6. Give some special features about ntan3x in the MOD real plane $R_n(9)$.

7. What are the advantages of using MOD trigonometric function?

8. Obtain any special properties enjoyed by MOD trigonometric functions.

9. Can we see nsinmx behaves in the same way in all MOD planes m a fixed number?

10. For problem (9), study the situations in the MOD planes $R_n(t)$.

    i) if t / m
    ii) If $(t_1 m) = 1$
    iii) If $(t_1 m) = d$
    iv) m / t.

11. Compare the MOD trigonometric functions nsinx and ncosx in the MOD real plane $R_n(15)$.

12. Study ntanx in the MOD planes $R_n(t)$ where t is as in problem 10.

13. Compare ntanx with ncot x.

14. Can any of the MODtrignometric functions be continuous in $R_n(t)$; for any value of t; $2 \leq t$ m.

15. Let nsin5θ be the MOD trigonometric function in $R_n(5)$.

    i) Is this function continuous?
    ii) If $R_n(5)$ is replaced by $R_n(2)$ or $R_n(3)$ or $R_5(4)$, $R_n(10)$ and $R_n(6)$ study the nature of the curve.
    iii) Study ncosec7θ in the MOD plane $R_n(7)$.

16. Study nsec9θ in the MOD planes $R_n(3)$, $R_n(9)$ and $R_n(27)$.

17. Study nsec(mθ); $2 \leq t < \infty$ in the MOD planes $R_n(t)$ where $(t, \theta) = 1$, $t / \theta$ and $(t, \theta) = d$.

18. Obtain any special feature associated with MOD trigonometric functions.

19. Study MOD exponential functions in the MOD planes.

20. How does the study of MOD trigonometric functions in MOD planes $R_n(m)$ different from the trigonometric functions in the real plane?

21. Study the MOD trigonometric function $n\sin^{-1}x$ in the MOD plane $R_n(15)$.

22. Study $n\cot^{-1}5x$ in the MOD planes $R_n(5)$, $R_n(7)$, $R_n(4)$, $R_n(15)$ and $R_n(10)$.

    Are these functions graph different in different planes or is it the same?

23. Study the MOD inverse trigonometric function $n\sec^{-1}x$ in the MOD planes $R_n(2)$, $R_n(4)$, $R_n(3)$, $R_n(5)$, $R_n(6)$, $R_n(13)$ and $R_n(18)$.

    Draw the curves in these seven MOD planes and compare them.

24. Is this equation nsinθ + ntan2θ = 0 solvable in $R_n(5)$. Find the values of θ.

25. Solve for θ the following functions.

i) $\frac{n\cos 3\theta}{1+n\tan\theta}$ in $R_n(13)$

ii) $\frac{n\cot 2\theta}{n\sin\theta + n\cos\theta}$ in $R_n(10)$

iii) $\frac{n\sec 3\theta}{1+n\tan^2 3\theta}$ in $R_n(5)$.

26. Solve for θ the following MOD trigonometric inverse functions.

i) $\frac{n\sin^{-1} 2\theta}{1+n\tan^{-1}\theta}$ in $R_n(17)$.

ii) $\frac{n\cot^{-1} 5\theta}{ncis^{-1} 2\theta + 1}$ in $R_n(5)$

iii) $\frac{n\tan^{-1} 5\theta}{1+n\cot^{-1} 5\theta}$ in $R_n(15)$

iv) $\frac{n\cos^{-1} 3\theta}{1+n\sec^{-1} 3\theta}$ in $R_n(7)$

27. Trace the curve $ne^x$ in the MOD plane $R_n(2001589)$.

28. Obtain all the special features enjoyed by the MOD logarithmic functions.

29. Derive all special features in studying MOD exponential function.

30. Distinguish the logarithmic functions from the MOD logarithmic functions.

Chapter Seven

# PROBLEMS ON MOD DIFFERENTIATION AND MOD INTEGRATION OF FUNCTIONS IN THE MOD-PLANES

In this chapter we just give some relevant problems related with differentiation and integration of MOD functions defined on MOD planes. For more about these please refer [27].

It is important to keep on record that differentiation and integration of MOD functions in general behaves in a very odd way. They do not obey most rules associated with differentiation or integration.

Further a function which is continuous in the real plane or a complex plane need not in general be continuous in the MOD planes.

So at this stage itself most of the properties associated with MOD functions do not behave like usual functions

In fact we see polynomial functions behave very haphazardly.

For a $n^{th}$ degree polynomial in the MOD interval need not have any of the derivatives to exist in [0,6)[x].

Thus if $f(x) = x^{36} + 3.714x^{18} + 4.025\,x^6 + 3.7 \in [0,6)[x]$.

$$\text{Then } \frac{df}{dx} = 0.$$

There is a vast difference between MOD polynomials and usual polynomials in the study of differentiation.

This has been elaborately discussed in the books [26-30].

Likewise if

$f(x) = 0.7x^{23} + 4.02x^{17} + 2.5135\,x^5 + 0.33 \in [0,6)[x]$ then we see

$$\int f(x)\,dx = \int 0.74x^{23}\,dx + \int 4.02x^{17}dx + \int 2.5135\,x^5\,dx + \int 0.33\,dx$$

$= \dfrac{0.7x^{24}}{24} + \dfrac{4.02x^{18}}{18} + \dfrac{2.5135x^6}{6} + \dfrac{0.33}{1}x + c$ is not defined for in [0,6) the values of $\dfrac{1}{24}$, $\dfrac{1}{18}$ and $\dfrac{1}{6}$ are not defined.

Here we proceed onto give some more examples.

Consider the MOD interval polynomial pseudo ring.

$$[0,12)x = \left\{ \sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i \in [0,12) \right\}.$$

$p(x) = 3.7\,x^{24} + 1.2\,x^{12} + 3x^4 + 4x^3 + 6x^2 + 7.331 \in [0,12)x$.

$$\text{Clearly } \frac{dp(x)}{dx} = 0.$$

That is the derivative is zero though p(x) is of degree 24.

The integral of p(x) is also not defined for;

$\int p(x)\,dx = \frac{3.7x^{24}}{1} + 1.2x^{13} + \frac{3x^5}{5} + \frac{4x^4}{4} + \frac{6x^3}{3} + 7.331x +$ c, where c is a constant.

Since $\frac{1}{4}$ and $\frac{1}{3}$ are not defined in [0,12) as they are zero divisors in [0,12).

We see the integral of p(x) is also not defined.

Thus finding polynomials

$p(x) \in [0,m)x = \left\{\sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i \in [0,m)\right\}$ whose derivative and integrals are defined properly and those which satisfy the usual or classical rules of differentiation and integration also true happens to be a very challenging problem.

Consider

$p(x) = 2x^7 + 0.7x^{14} + 6.5x^{28} + 32x^7 \in [0,24)x = \left\{\sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i \in [0,14)\right\}$.

We see $\frac{dp(x)}{dx} = 0$.

Further $\int p(x)\,dx = \frac{2x^8}{8} + \frac{0.7x^{15}}{14} + \frac{6.5x^{29}}{29} + \frac{32x^7}{8} +$ c is not defined as $\frac{1}{8}$, $\frac{1}{14}$ are undefined in [0,14).

So in [0,m)x there exists several polynomials whose derivatives are zero and the integrals does not exists or are undefined [26-30],

Characterizing such polynomials happens to be a open conjecture.

Next we study polynomials in $R_n(m)[x] = \left\{ \sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i \in R_n(m) = \{(a, b) \mid a, b \in [0,m)\} \right.$.

It is important to note that even in case of polynomial in the MOD plane $R_n(m)[x]$ they in general do not satisfy the classical properties of differentiation or integration.

Consider $p(x) \in R_n(4)[x] = \left\{ \sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i \in R_n(4) = \{(a, b) \mid a, b \in [0, 4)\} \right.$ where

$p(x) = (2,1.2)x^8 + (3.1, 0.315)x^4 + (2, 2)x^2 + (0.1, 0)$.

Clearly $\frac{dp(x)}{dx} = 0$.

However

$$\int p(x)\, dx = \frac{(2,1.2)x^9}{1} + \frac{(3.1,0.315)x^5}{1} + \frac{(2,2)x^3}{3} + (0.1,0)x + c.$$

As $\frac{1}{3} = 3$ the integral exists in this case.

However the derivative is zero.

Thus even the polynomials in the MOD plane do not satisfy all the classical properties of derivatives and integrals in general.

Suppose we have the polynomials from the MOD plane $R_n(24)[x]$ then all polynomials whose power or degree is multiples of 24.

Also if the polynomials have coefficients from $\{(a,b \mid a, b \in \{0, 2, 4, 6, \ldots, 22\}\}$ then there are possibilities where the derivatives are zero.

$p(x) = (2,4)x^{12} + (16,8)x^6 + (12,6)x^4 +(12,12)\, x^2 + (8,6) \in R_n(24)[x]$.

Then the derivative is zero and the integrals do not exist.

Also all MOD polynomials whose powers are say $p_1, p_2, p_3, \ldots, p_n$ and these $p_i$'s take values from $\{1, 5, 7, 11, 13, 17, 19, 23\}$ are such that their integrals are not defined.

Let $p(x) = (0, (5.33) + (2, 7.21)\, x + (6.022, 5.001)x^2 + (5, 0.331)x^5 \in R_n(24)[x]$ be such that $\frac{dp(x)}{dx}$ exists but its integral is not defined.

Let $p(x) \in R_n(m)[x]$, we see even if p(x) is of degree n, then p(x) need not have all the n derivatives to exist.

Secondly even the second derivative of p(x) need not be a polynomial of degree (n – 1).

For instance if

$p(x) = (3, 6)x^4 + (2.01, 3.2)\, x^2 + (0.77, 6.2) \in R_n(12)[x]$. p(x) is of degree four.

Consider $\frac{dp(x)}{dx} = 0 + (4.02, 6.4)\, x + 0$.

Thus the first derivative results in a polynomial of degree one.

$$\frac{d^2p(x)}{dx^2} = (4.02, 6.4).$$

That is the second derivative is a constant polynomial and the third derivative $\frac{d^3p(x)}{dx^3} = (0,0)$.

Hence our claim that a nth degree polynomial in MOD planes need not be derivable n times.

Sometimes the first derivative itself can be zero.

Likewise one can define polynomials using the complex MOD integer plane $C_n(m)$.

$$C_n(m)[x] = \left\{\sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i = c + d_{i_F} \in C_n(m);\, c_1\, d \in [0,m)\right\}.$$

The degree of the polynomial roots of the polynomials are defined in an analogous way as that of $R_n(m)[x]$.

Suppose $p(x) = (5 + 3i_F)x^{10}\ (7.5 + 15i_F)\ x^4 = 15i_F\ x^2 + 4.32 + 0.75i_F \in C_n(30)[x]$.

We see even the first derivative is zero.

Further the integral of p(x) does not exist as the terms $\frac{1}{5}, \frac{1}{3}$ are not defined in $C_n(3)$.

Hence one cannot as in case of polynomials in C[x] define the derivatives or integrals of polynomials in $C_n(m)[x]$; $2 \leq m < \infty$.

Thus it is difficult to find conditions on polynomials in $C_n(m)[x]$ to have both the derivative and integrals to exist.

Let $p(x) = (2.5 + 1.25i_F)\ x^4 + 2.5\ i_F\ x^2 + 3.002 + 0.67i_F \in C_n(5)[x]$.

$$\frac{dp(x)}{dx} = 0 \text{ and } \int p(x)\,dx \text{ is not defined.}$$

Thus finding integrals and derivatives that follows all classical conditions happens to be a challenging problem.

Next we study polynomials in

$R_n^g(m)[x] = \left\{ \sum_{i=1}^{\infty} a_i x^i \;\middle|\; a_i \in R_n^g(m) = \{a + bg \mid a, b \in [0, m),\right.$

$g^2 = 0\}$, known as the dual number coefficient polynomials.

If $p(x) \in R_n^g(m)[x]$ we cannot say that p(x) is integrable or derivable in general.

Most of the polynomial do not satisfy the properties of classical derivatives.

For let $p(x) = (4 + 2.5g)x + (2.5 + 1.25\ g)x^4 + 3.12 + 4.706g \in R_n(5)[x]$.

Clearly $\frac{dp(x)}{dx} = 0$ which shows the classical property is not satisfied.

Further $\int p(x)\,dx$ is not even defined as $\frac{1}{5}$ is not defined in $[0,5)$.

Thus in general a polynomial $p(x) \in R_n(5)[x]$ is not integrable.

Thus characterizing those polynomials which are integrable (or (and) differentiable) in $R_n^g(m)[x]$ happens to be a open conjecture.

Next we can study MOD special dual like number polynomials in

$$R_n^h(m)[x] = \left\{ \sum_{i=1}^{\infty} a_i x^i \,\middle|\, a_i = c + bh \in R_n^h(m),\ c,\ b \in [0,m), \right.$$

$h^2 = h$, $2 \leq m < \infty\}$.

Clearly polynomials p(x) in $R_n^h(m)[x]$ will be known as MOD special dual like number polynomials.

In general as in case of other MOD polynomials these polynomials also do not in general and further do not satisfy the classical properties of differentiation or integration.

Apart from this these polynomials also do not satisfy the classical properties of roots of polynomials.

Study in these directions are just left as open conjectures.

We will present an example or two in this direction.

Let $p(x) = (1.50 + 3h)x^{12} + (0.75 + 1.50\ h)x^8 + 0.773 + 4.0075\ h \in R_n^h(6)[x]$.

Clearly $\frac{dp(x)}{dx} = 0$ and $\int p(x)\,(dx)$ is not defined.

Let $p(x) = hx^2 + (2.31 + 4.35\ h) \in R_n^h(6)[x]$.

Clearly this is a quadratic MOD polynomial equation in $R_n^h(6)[x]$ but is not solvable in $R_n^h(6)[x]$.

Similarly 3.2 hx + (4.21 + 5.2h) = 0 is not solvable in $R_n^h(6)[x]$.

Next we study MOD polynomials with coefficients from the special quasi dual numbers or equivalently the MOD special quasi dual number polynomials in $R_n^k(m)[x]$.

Let $R_n^k(m)[x] = \left\{ \sum_{i=1}^{\infty} a_i x^i \,\middle|\, a_i \in R_n^k(m) = \{a + bk \mid a, b \in [0, m)\ k^2 = (m-1)k, 2 \leq m < \infty\}. \right.$

Let
$p(x) = (5.3 + 0.3k)\ x^{12} + (4 + 3k)\ x^6 + 3.718 + 4.37\ k \in R_n^k(6)[x]$.

Clearly $\frac{dp(x)}{dx} = 0$. However the integral exist.

Let $(0.3k + 4.57)\ x^2 + (4.32 + 2.01\ k) \in R_n^k(6)[x]$.

Solve for x.

Further it is important to keep on record that it is not possible to solve for x even when it is a linear equation in x.

Consider $(0.7 + 4.2\ k)x + (0.75 + 2k) \in R_n^k(6)[x]$.

Solve for x.

Let $p(x) = (3.7 + 4.2k)x^5 + (2.5 + 1.25)x^4 + (2.5k)x^2 + (4.32 + 2.701\ k) \in R_n^k(5)$.

Prove the derivative is zero and the integral does not exist.

Characterizing those polynomials in $R_n^k(m)$ which are both integrable as well as derivable happens to be a challenging problem.

Next we consider about solving for roots for polynomials in $R_n^k(m)$. Consider $p(x) = kx^5 + 3.7 + 4.7k \in R_n^k(10)$.

Clearly p(x) cannot be solved for x however one may be at times in a position to get the value of kx and x.

Solve $3.52\ kx + 4.37 + 2.5k \in R_n^k(15)$. Here also we cannot solve for x only for kx.

Let $x^2 + 4 \in R_n^k(20)$ solving for x is a easy task for $x^2 = 16$ so $x = \pm 4$. But if $x^2 + 4k$ is considered instead then $x^2 = 16k$.

$x = 4\sqrt{k}$. Finding value of $\sqrt{k}$ is a very difficult task.

Now in the following we suggest some problems which are open conjectures some of them are difficult problems other easy exercise.

**Problems**

1. Find conditions on the MOD interval real polynomials $p(x) \in [0, m)[x]$ to have derivatives and to satisfy the classical properties.

2. Study the integrals of polynomials $p(x) \in [0,m)[x]$.

3. Characterize those $p(x) \in [0,12)[x]$ which follows the classical properties of derivatives.

4. Study question three for $p(x) \in [0,19)[x]$.

5. Let $p(x) \in R_n(m)[x]$. Find condition on p(x) to have derivatives.

6. Prove p(x) in general in $R_n(15)[x]$ are not integrable.

7. Find conditions on all those polynomials in $R_n(4)[x]$ which are both integrable and derivable.

8. Characterize those polynomials $p(x) \in R_n(5)[x]$ which are derivable but not integrable and vice versa.

9. Let $p(x) \in R_n(11)[x]$.

   i) Find all roots of p(x).
   ii) If $p(x) \in R_n(11)[x]$ is such that p(x) is both derivable and integrable can we say p(x) has same number of roots as its degree?
   iii) Characterize all polynomials p(x) in $R_n(11)[x]$ of degree n, but has less than n roots.
   iv) Characterize all polynomial p(x) in $R_n(11)[x]$ of degree n but has roots greater than n.

10. What are special and distinct features associated with the MOD polynomial interval $[0,m)[x]$? $(2 \leq m < \infty)$

11. Enumerate all the special features enjoyed by the MOD polynomials in $R_n(m)[x]$ $(2 \leq m < \infty)$.

12. Prove both $[0, m)[x]$ and $R_n(m)[x]$ are only pseudo rings and not rings.

13. Can we say as the distributive property is not true, is the only reason for polynomials not satisfying the classical properties?

14. What is the reason for the polynomials in $[0,m)[x]$ and $R_n(m)[x]$ to behave in a chaotic manner?

15. Study the MOD complex modulo integer polynomials in $C_n(m)[x]$.

16. Characterize all those polynomials in $C_n(m)[x]$ which follow both the classical laws of differentiation and integration.

17. Characterize all those polynomials in $C_n(m)[x]$, $2 \leq m < \infty$ which satisfy all the classical laws of differentiation.

18. Characterize all these polynomials in $C_n(m)[x]$, $2 \leq m < \infty$ which satisfy all the classical laws of integration.

19. Characterize all those polynomials in $C_n(m)[x]$ which satisfy the classical properties of roots.

20. Study questions (16) to (19) for the polynomials in $C_n(15)[x]$.

21. Study questions (16) to (19) for polynomials in $C_n(43)[x]$.

22. Study questions (16) to (19) for polynomials in $C_n(3^{12})[x]$.

23. Solve $p(x) = (.37 + 4.2\, i_F)\, x^2 + (7.3 + 5.4\, i_F) \in C_n(10)[x]$.

24. Show as the distributive laws are not true in $C_n(m)[x]$ finding roots of a polynomial in $C_n(m)[x]$ is a difficult task by some examples.

25. Let $p(x) = 27\, x^3 + (5 + 6\, i_F) \in C_n(7)[x]$.

    i) Solve for x.
    ii) Will p(x) have three roots in $C_n(7)[x]$?
    iii) If $\alpha$, $\beta$ and $\gamma$ are the roots of p(x) can we prove $(3x - \alpha)\,(3x - \gamma)\,(3x - \beta) = p(x)$.
    iv) If $p_1(x) = x^3 + (0.5 + 0.72\, i_F) \in C_n(7)[x]$ will (iii) be true for the roots $\alpha, \beta, \gamma$ of $p_1(x)$.

26. Prove in $C_n(m)[x]$; $(x - \alpha_1)\,(x - \alpha_2) \ldots (x - \alpha_n) \neq x^n - (\alpha_1 + \alpha_2 + \ldots + \alpha_n)\, x^{n-1} + \sum_{i \neq j} \alpha_i\alpha_j\, x^{n-2} \ldots \pm \alpha_1 \ldots \alpha_n$ where

$\alpha_1, \alpha_2, \ldots, \alpha_n$ are roots of a polynomial p(x) and are in $C_n(m)[x]$.

27. When will problem (25) be true in $R_n(m)[x]$?

28. How will this problem in (25) be over come?

29. Can we have this sort of problem in real world problem? (if so give some examples).

30. Is it advantageous to use MOD polynomials of MOD planes than usual real plane and complex plane?

31. Let $p(x) \in R_n^g(m)[x] = \left\{ \sum_{i=1}^{\infty} a_i x^i \,\middle|\, a_i \in R_n^g(m) = \{a + bg = (a, b);\ g^2 = 0;\ 2 \le m < \infty\} \right.$ be the MOD dual number polynomial with coefficients from the MOD dual number plane $R_n^g(m)$.

    i) When is p(x) totally solvable?
    ii) If p(x) is of degree n can we say p(x) will have n and only n roots.
    iii) When will p(x) satisfy all the classical properties of differentiation?
    iv) When will p(x) satisfy all the laws of differentiation?
    v) Obtain any other special or distinct feature associated with p(x).
    vi) Can we say because p(x) is a MOD dual number coefficient polynomial they enjoy some special features?
    vii) Characterize all p(x) in $R_n^g(m)[x]$ for which both the integral or derivative does not exist.
    viii) Characterize all those polynomials in $R_n^g(m)[x]$ which satisfies all properties of derivatives but are not integrable.

ix) Characterize those polynomials in $R_n^g(m)[x]$ which satisfy all properties of integrals but does not satisfy the properties of derivatives.

32. Let $R_n^g(12)[x]$ be the dual number polynomials with coefficients from the MOD dual number plane $R_n^g(12) = \{a + bg = (a, b) \mid g^2 = 0, a, b \in [0, 12)\}$.

    Study questions (i) to (ix) of problem (31) for this polynomials in $R_n^g(12)[x]$.

33. Let $R_n^g(43)[x]$ be the dual number polynomials with coefficients from the MOD dual number plane $R_n^g(43)$.

    Study questions (i) to (ix) of problem (31) of polynomials in $R_n^g(43)[x]$.

34. Let $R_n^g(5^7)[x]$ be the collection of all polynomials with coefficients from the MOD dual number plane $R_n^g(5^7)$.

    Study questions (i) to (ix) of problem (31) for polynomials in $R_n^g(5^7)[x]$.

35. Let $p(x) = (0.33 + 0.5g)x^3 + 7.32 + 4.52\,g \in R_n^g(12)[x]$.

    i) Find all roots of p(x).
    ii) Is p(x) integrable?
    iii) Does the derivative of p(x) satisfy the usual laws of derivation?

36. Let $p(x) = 4.3g\,x^4 + (5.2 + 3g)x^2 + 3.7g + 0.45 \in R_n^g(13)[x]$.

    Study questions (i) to (iii) of problem (35) for this p(x).

37. Let $3.7315g\ x^3 + (5 + 2g)x^2 + 4.32gx + 1.4 + 3g \in R_n^g(2^5)[x]$.

    Study questions (i) to (iii) of problem (35) for this p(x).

38. Obtain all special features associated with the MOD special dual like number polynomials in $R_n^h(m)[x]$ with coefficients from
    $R_n^h(m) = \{a + bh = (a,b) \mid h^2 = h, a,b \in [0, m)\}$.

39. Let $p(x) = (0.8 + 4.923\ k)\ (5.92 + 0.74\ k)x + (6.93 + 0.89k)x^2 \in R_n^h(6)[x]$; $k^2 = 5k$ special quasi dual MOD number polynomial.

    i) Find the roots of p(x).
    ii) Can p(x) have more than two roots?
    iii) Is x uniquely solvable for this p(x)?

40. Let $w(x) = (6.93 + 0.89\ h)\ x^3 + (4.32 + 5.75\ h) \in R_n^h(7)[x]$ where $h^2 = h$ be the MOD special dual like number polynomial.

    Study questions (i) to (iii) of problem (39) for this w(x).

41. Let $t(x) = (8.43 + 15.3h)\ x^6 + (10.3 + 0.7783\ h)x^3 + (17 + 2.315\ h) \in R_n^g(19)[x]$, $h^2 = h$, be the MOD special dual like number polynomial.

    Study questions (i) to (iii) of problem (38) for this t(x).

42. Let $m(x) = (4.331 + 0.389h)\ x^4 + (7.3 + 6.4h)\ x^3 + (0.37 + 4.3h)\ x^2 + (1.73 + 1.05\ h)\ x + 7.83 + 6.03\ h \in R_n^h(8)[x]$, $h^2 = h$ be the MOD special dual like number polynomial.

    Study questions (i) to (iii) of problem (38) for this m(x).

43. Let $p(x) = (4.38 + 0.7\,h)x^{16} + (2.312 + 6.449\,h)x^8 + (4.32 + 7.3\,h)x + (4.311 + 0.77\,h) \in R_n^h(8)[x]$ be the MOD special dual like number polynomial.

    i) Find $\frac{dp(x)}{dx}$ does the derivative satisfy the usual laws of derivation?
    ii) Is $\int p(x)\,dx$ defined? Justify your claim.
    iii) Characterize those MOD polynomials in $R_n^h(8)[x]$ for which the integrals follow the usual laws of integration.
    iv) Characterize all those MOD polynomials which follows all the classical rules of derivatives.
    v) Characterize those MOD polynomials in $R_n^h(8)[x]$ which satisfy both the classical law of integration as well as that of differentiation.

44. Let $R_n^h(47)[x]$ be the collection of all MOD special dual like number polynomials.

    Study questions (i) to (v) of problem 42 for this $R_n^h(47)[x]$.

45. Let $R_n^h(24)[x]$ be the collection of all MOD special dual like number polynomials.

    Study questions (i) to (v) of problem (42) for this $R_n^h(24)[x]$.

46. Enumerate all the special features enjoyed by the MOD special quasi dual number polynomials $R_n^k(m)[x]$, $m^2 = (m-1)k$.

47. Characterize all MOD special quasi dual number polynomials in $R_n^k(m)[x]$ which are integrable but whose derivative is zero.

48. Characterize all those MOD special quasi dual number polynomials in $R_n^k(m)[x]$ which follows the usual laws of differentiation but is not integrable.

49. Characterize all those MOD quasi dual number polynomials which does not satisfy both classical properties of differentiation and integration.

50. Let $p(x) = kx^4 + (6.2 + 4.5\,k) \in R_n^k(4)[x]$ be the MOD special quasi dual number polynomial.

    i) Does p(x) satisfy the classical laws of derivation?
    ii) Prove or disprove p(x) satisfy the classical laws of integration.
    iii) Can p(x) have four roots in $R_n^k(4)$, $k^2 = 3k$?
    iv) Is p(x) solvable?
    v) Let $p_1(x) = 5.32k\, x^3 + (2.5 + 7.5k) \in R_n^k(10)[x]$, $k^2 = 9k$ be the MOD special quasi dual number polynomial.

       Study questions (i) to (iv) of problem (49) for this $p_1(x)$?

51. Let $p(x) = (2.5 + 7.3\,k)\,x^4 + (1.5 + 0.72\,k)\,x^3 + (3 + 4.3k)x^2 + (0.311 + 4k)x + (0.72 + 5.1k) \in R_n^k(23)[x]$ be the MOD special quasi dual number polynomial.

    Study questions (i) to (iv) of problem (49) for this p(x).

52. Let $q(x) \in R_n^k(140)[x]$ where coefficients of the MOD special quasi dual number polynomials are from $\langle Z_{140} \cup k\rangle = \{a + bk \mid a,\ b \in Z_{140}, k^2 = 139\,k\}$.

    i) Can we say all classical laws of derivatives will be true for these polynomials in $\langle Z_{140} \cup k\rangle[x]$?

ii) Can the classical properties of integration be true for these polynomials in $\langle Z_{140} \cup k\rangle[x]$?

iii) Can we say a nth degree polynomial in $\langle Z_{140} \cup k\rangle[x]$ will have n and only n roots?

iv) Study questions (i) to (iii) of problems for $p(x) = 14k\, x^{10} + 20x^7 + 10k \in \langle Z_{140} \cup k\rangle[x]$.

53. Compare polynomials in $\langle Z_m \cup k\rangle[x]$ with the polynomials in $\langle Z_m \cup h\rangle[x]$ and $\langle Z_m \cup g\rangle[x]$.

54. Study MOD neutrosophic polynomials in $R_n^I(m)[x]$.

55. Derive all the special features enjoyed by $R_n^I(43)[x]$.

56. If $p(x) = 2.3\, Ix^7 + (5.3 + 2.16I) \in R_n^I(10)$ be the MOD neutrosophic polynomial.

i) Can we say p(x) has seven and only seven roots?

ii) Is p(x) solvable in $R_n^I(10)$?

57. Let $q(x) = (5 + 4I)\, x^3 + (2 + 7I)\, x + 4I \in R_n^I(8)[x]$.

i) Will q(x) have only 3 roots or more?

ii) Find $\frac{dq(x)}{dx}$.

iii) Find $\int q(x)\, dx$.

58. Enumerate all the special features enjoyed by $R_n^I(m)[x]$.

59. Characterize those MOD neutrosophic polynomials which has both derivatives and integrals which satisfy all the classical properties.

# FURTHER READING


1. Herstein., I.N., Topics in Algebra, Wiley Eastern Limited, (1975).

2. Lang, S., Algebra, Addison Wesley, (1967).

3. Smarandache, Florentin, Definitions Derived from Neutrosophics, In Proceedings of the First International Conference on Neutrosophy, Neutrosophic Logic, Neutrosophic Set, Neutrosophic Probability and Statistics, University of New Mexico, Gallup, 1-3 December (2001).

4. Smarandache, Florentin, Neutrosophic Logic—Generalization of the Intuitionistic Fuzzy Logic, Special Session on Intuitionistic Fuzzy Sets and Related Concepts, International EUSFLAT Conference, Zittau, Germany, 10-12 September 2003.

5. Vasantha Kandasamy, W.B., On Muti differentiation of decimal and Fractional Calculus, The Mathematics Education, 38 (1994) 220-225.

6. Vasantha Kandasamy, W.B., On decimal and Fractional Calculus, Ultra Sci. of Phy. Sci., 6 (1994) 283-285.

7. Vasantha Kandasamy, W.B., Muti Fuzzy differentiation in Fuzzy Polynomial rings, J. of Inst. Math and Comp. Sci., 9 (1996) 171-173.

8. Vasantha Kandasamy, W.B., Some Applications of the Decimal Derivatives, The Mathematics Education, 34, (2000) 128-129.

9. Vasantha Kandasamy, W.B., On Partial Differentiation of Decimal and Fractional calculus, J. of Inst. Math and Comp. Sci., 13 (2000) 287-290.

10. Vasantha Kandasamy, W.B., On Fuzzy Calculus over fuzzy Polynomial rings, Vikram Mathematical Journal, 22 (2002) 557-560.

11. Vasantha Kandasamy, W.B., Linear Algebra and Smarandache Linear Algebra, Bookman Publishing, (2003).

12. Vasantha Kandasamy, W. B. and Smarandache, F., N-algebraic structures and S-N-algebraic structures, Hexis, Phoenix, Arizona, (2005).

13. Vasantha Kandasamy, W. B. and Smarandache, F., Neutrosophic algebraic structures and neutrosophic N-algebraic structures, Hexis, Phoenix, Arizona, (2006).

14. Vasantha Kandasamy, W. B. and Smarandache, F., Smarandache Neutrosophic algebraic structures, Hexis, Phoenix, Arizona, (2006).

15. Vasantha Kandasamy, W.B., and Smarandache, F., Fuzzy Interval Matrices, Neutrosophic Interval Matrices and their Applications, Hexis, Phoenix, (2006).

16. Vasantha Kandasamy, W.B. and Smarandache, F., Finite Neutrosophic Complex Numbers, Zip Publishing, Ohio, (2011).

17. Vasantha Kandasamy, W.B. and Smarandache, F., Dual Numbers, Zip Publishing, Ohio, (2012).

18. Vasantha Kandasamy, W.B. and Smarandache, F., Special dual like numbers and lattices, Zip Publishing, Ohio, (2012).

19. Vasantha Kandasamy, W.B. and Smarandache, F., Special quasi dual numbers and Groupoids, Zip Publishing, Ohio, (2012).

20. Vasantha Kandasamy, W.B. and Smarandache, F., Algebraic Structures using Subsets, Educational Publisher Inc, Ohio, (2013).

21. Vasantha Kandasamy, W.B. and Smarandache, F., Algebraic Structures using [0, n), Educational Publisher Inc, Ohio, (2013).

22. Vasantha Kandasamy, W.B. and Smarandache, F., Algebraic Structures on the fuzzy interval [0, 1), Educational Publisher Inc, Ohio, (2014).

23. Vasantha Kandasamy, W.B. and Smarandache, F., Algebraic Structures on Fuzzy Unit squares and Neutrosophic unit square, Educational Publisher Inc, Ohio, (2014).

24. Vasantha Kandasamy, W.B. and Smarandache, F., Special pseudo linear algebras using [0, n), Educational Publisher Inc, Ohio, (2014).

25. Vasantha Kandasamy, W.B. and Smarandache, F., Algebraic Structures on Real and Neutrosophic square, Educational Publisher Inc, Ohio, (2014).

26. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., MOD planes, EuropaNova, (2015).

27. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., MOD Functions, EuropaNova, (2015).

28. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Multidimensional MOD planes, EuropaNova, (2015).

29. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Natural Neutrosophic numbers and MOD Neutrosophic numbers, EuropaNova, (2015).

30. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Algebraic Structures on MOD planes, EuropaNova, (2015).

31. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., MOD Pseudo Linear Algebras, EuropaNova, (2015).

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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The study of MOD structures
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The authors in this book
propose several problems
on MOD structures, some of
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