Smarandache-Zero Divisors in Group Rings

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The study of zero-divisors in group rings had become interesting problem since 1940 with the famous zero-divisor conjecture proposed by G.Higman [2]. Since then several researchers [1, 2, 3] have given partial solutions to this conjecture. Till date the problem remains unsolved. Now we introduce the notions of Smarandache zero divisors (S-zero divisors) and Smarandache week zero divisors (S-weak zero divisors) in group rings and study them. Both S-zero divisors and S-weak zero divisors are zero divisors but all zero divisors are not S-zero divisors or S-weak zero divisors. Even here we can modify the zero divisor conjecture and suggest the S-zero divisor conjecture and S-weak zero divisor conjecture for group rings. Thus the study has its own importance. Unlike in case of group rings of finite groups over field of characteristic zero where one is always gurranteed the zero divisors, we can not establish the same in case of S-zero divisors or S-weak zero divisors.

In this paper we study the conditions on Z_n, the ring of integer modulo n to have S-zero divisors and S-weak zero divisors. We have proved if n is a composite number of the form $n = p_1p_2p_3$ or ($n = p^{\alpha}$) where p_1, p_2 and p_3 are distinct primes or (p a prime, $\alpha \geq 3$), then Z_n has S-zero divisors. We further obtain conditions for Z_n to have S-weak zero divisors. We prove all group rings Z_2 G where G a cyclic group of non prime order and $| G | \geq 6$ have S-zero divisors. The result is extended to any prime p i.e. we generalize the result for Z_pG where p any prime. Conditions for group rings ZG and KG (Z the ring of integer and K any field of characteristic zero) to have S-zero divisors is derived.

This paper is organized into 5 sections. In section 1, we recall the basic definitions of S-zero divisors and S-weak divisors and some of its properties. In section 2, we find conditions for the finite ring Z_n to have S-zero divisors and S-weak zero divisors. We give examples to this effect. In section 3, we analize the group rings Z_2G where G is a finite group of order greater than or equal to 6 and G a cyclic group of non-prime order. In section 4 we find conditions for ZG to have S-zero divisors. We prove ZS_n and ZD_{2n} have S-zero divisors. Further if K is a field of characteristic zero then KS_n and KD_{2n} have S-zero divisors. When G is any group of order n and n a composite number, then ZG and KG have S-zero divisors. In section 5 we give the conclusion based on our study.

1. Some Basic Definitions:

Here we just recall some basic results about S-zero divisors and S-weak zero divisors.

Definition 1.1 [5]: Let R be a ring. An element $a \in R \setminus \{0\}$ is said to be a S-zero divisor if $a.b = 0$ for some $b \neq 0$ in R and there exists $x, y \in R \setminus \{0, a, b\}$ such that

i.	$a.x = 0$	or	$x.a = 0$
ii.	$b.y = 0$	or	$y.b = 0$
iii.	$x.y \neq 0$	or	$y.x \neq 0$

Note that in case of commutative rings we just need

i. $a.x = 0$, ii. $b.y = 0$ and iii. $x. y \neq 0$

Theorem 1.1 [5]: Let R be a ring. Every S-zero divisor is a zero divisor but a zero divisor in general is not a S-zero divisor.

Example 1.1: In $Z_4 = \{0, 1, 2, 3\}$ the ring of integers modulo 4, 2 is a zero divisor but it is not a S-zero divisor.

Definition 1.2 [5]: Let R be a ring. An element $a \in R \setminus \{0\}$ is a S-weak zero divisor if there exists $b \in R \setminus \{0, a\}$ such that $a.b = 0$ satisfying the following conditions:

There exists $x, y \in R \setminus \{0, a, b\}$ such that

i.	$a.x = 0$	or	$x.a = 0$
ii.	$b.y = 0$	or	$y.b = 0$
iii.	$x.y = 0$	or	$y.x = 0$

Example 1.2: Let $Z_{12} = \{0, 1, 2,\dots,11\}$ be the ring of integer modulo 12.

$3.4 \equiv 0 \pmod{12}$, $4.9 \equiv 0 \pmod{12}$,

and $3.8 \equiv 0 \pmod{12}$ also $8.9 \equiv 0 \pmod{12}$

So 3 and 4 are S-weak zero divisors in Z_{12}. We can check that Z_{12} contains S-zero divisor also. For

$6.8 \equiv 0 \pmod{12}$ and $3.8 \equiv 0 \pmod{12}$

$6.2 \not\equiv 0 \pmod{12}$ but $3.2 \equiv 0 \pmod{(12)}$

Thus Z_{12} has both S-zero divisor and S-weak zero divisors.

2. S-zero divisors and S-weak zero divisors in Z_n

In this section we find conditions for Z_n to have S-zero divisors and S-weak zero divisors and prove when n is of the form $n = p^\alpha$ ($\alpha \geq 3$) or $n = p_1p_2p_3$ where p_1, p_2, p_3 are distinct primes, then Z_n has S-zero divisor.

Proposition 2.1: $Z_p = \{0, 1, 2,\ldots, p-1\}$, the prime field of characteristic p where p is a prime has no S-zero divisors or S-weak zero divisors.

Proof: Let $Z_p = \{0, 1, 2,.., p-1\}$, where p is a prime. Z_p has no zero divisors. Hence Z_p can not have S-zero divisor as every S-zero divisor is a zero divisor.
Further Z_p cannot have S-weak zero divisor as every S-weak zero divisor is also a zero divisor.
Hence the claim

Example 2.1: Let $Z_6 = \{0, 1, 2, 3, 4, 5\}$ be the ring of integers modulo 6. Here
$\equiv 0 \pmod 6$, $3.4 \equiv 0 \pmod 6$
are the only zero divisors of Z_6. So Z_6 has no S-zero divisor and S-weak zero divisors.

Example 2.2: $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, the ring of integers modulo 8. Here
$4.4 \equiv 0$ (mod) and $2.4 \equiv 0 \pmod 8$, $4.6 \equiv 0 \pmod 8$
but $2.6 \not\equiv 0$ (mod8).
So Z has 4 as S-zero divisor, but has no S-weak zero divisors.

Theorem 2.1: Z_n has no S-weak zero divisors when n = 2p, p a prime.

Proof: To get any zero divisors $a, b \in Z_n \setminus \{0\}$ such that $a.b \equiv 0 \pmod n$ one of a or b must be p and the other an even number. So we cannot get $x, y \in Z_n \setminus \{p\}$ such that

$$a.x \equiv 0 \pmod n,\ b.y \equiv 0 \pmod n$$

and

$$x.y \equiv 0 \text{ (modn)}.$$

Hence the claim.

Theorem 2.2: Z_n has S-weak zero divisors when $n = p_1p_2$ (orp^2) where p_1 and p_2 are distinct odd primes with both of them greater than 3 (or p an odd prime greater than 3).

Proof: Take $a = p_1$, $b = p_2$, then $a.b \equiv 0 \pmod n$.
Again take $x = 3p_2$ and y $2p_1$ then $a.x \equiv 0 \pmod n$ and $b.y \equiv 0 \pmod n$, also $x.y \equiv 0 \pmod n$.

In case of = p^2 a similar proof holds good.
Hence the claim.

Theorem 2.3: Z_n has no S-zero divisors if $n = p_1p_2$ where p_1, p_2 are primes.

Proof: Let $n = p_1p_2$. Suppose $a.b \equiv 0 \pmod{n}$, $a, b \in Z_n \setminus \{0\}$ then p_1 is factor of a and p_2 is a factor of b or vice-versa.
Suppose p_1 is a factor of a and p_2 is a factor of b. Now to find $x, y \in Z_n \setminus \{0, a, b\}$ such that $a.x \equiv 0 \pmod{n} \Rightarrow p_2$ is a factor of x, and $b.y \equiv 0 \pmod{n} \Rightarrow p_1$ is a factor of y.
This forcing $x.y \equiv 0 \pmod{n}$.
Thus if $n = p_1p_2$, Z_n has no S-zero divisors.

Corollary 2.1: Z_n has no S-zero divisors when $n = p^2$.

Theorem 2.4: Z_n has S-zero divisors if $n = p_1^{\alpha}$ where p_1 is a prime and $\alpha \geq 3$.

Proof: Take

$$a = p_1^{\alpha-1},\ b = p_1^{\alpha-1} \text{ then } a.b \equiv 0 \text{ (mon n).}$$

Again take $x = p_1$ and $y = p_1\, p_2$ where p_2 is the prime next to p_1.
Then clearly

$$a.x \equiv 0 \pmod{n},\ b.y \equiv 0 \pmod{n} \text{ but } x.y \not\equiv 0 \pmod{n}.$$

Hence the claim.

Example 2.3: In $Z_{27} = \{0, 1, 2,\ldots, 26\}$; ring of integers modulo 27, we have

$$9.9 \equiv 0 \pmod{27},\qquad 9.3 \equiv 0 \pmod{27} \text{ also } 9.15 \equiv 0 \pmod{27}$$

but $3.15 \not\equiv 0 \pmod{27}$. So $3^2 = 9$ is a S-zero divisor in Z_{27}.

Theorem 2.5: Z_n has S-zero divisor when n is of the form $n = p_1p_2p_3$, where p_1, p_2, p_3 are primes.

Proof: Take $a = p_1p_2$, $b = p_2p_3$ then $a.b \equiv 0 \pmod{n}$. again take $x = p_1p_3$ and $y = p_1p_4$ where p_4 is the prime next to p_3. Then $a.x \equiv 0 \pmod{n}$ and $b.y \equiv 0 \pmod{n}$. But $x.y \not\equiv 0 \pmod{n}$.
Hence the claim

Example 2.4: Let $Z_{30} = \{0, 1, 2,\ldots, 29\}$ be the ring of integers modulo 30.
Here

$$6.15 \equiv 0 \pmod{30},\qquad 6.10 \equiv 0 \pmod{30},\qquad 15.14 \equiv 0 \pmod{30}$$

But $14.10 \not\equiv 0 \pmod{30}$, so 6 and 15 are S-zero divisor in Z_{30}.
We can generalize the Theorem 2.5 as follows:

Theorem 2.6: If $n = p_1p_2p_3..p_n$ then Z_n has S-zero divisors.

Proof: Take $a = p_1p_2\ldots p_{n-1}$, $b = p_2p_3\ldots p_{n-1}$. and $x = p_1p_3\ldots p_n$ $y = p_1 p_{n+1}$ whre p_{n+1} is the prime next to p_n.
Then it is easy to see that

$$a.b \equiv 0 \pmod n, \quad a.x\,y \equiv 0 \pmod n,\ b.y \equiv 0 \pmod n,$$

$$\text{But } x.y \not\equiv 0 \pmod n.$$

So a and b are S-zero divisors in Z_n.

Finally we can characterize Z_n for having S-zero divisors as follows.

Theorem 2.7: Z_n has S-zero divisors if and only if n is the product of atleast three primes.

3. S-zero divisors in the group ring Z_2G

Here we prove that the group ring Z_2G, where G is a finite cyclic group of non prime order and $|G| \geq 6$ has S-zero divisors. We illustrate by certain examples the non-existence of S-zero divisors before we prove our claim.

Example 3.1: Consider the group ring Z_2G of the group $G = \{g/g^2 = 1\}$ over Z_2. Clearly $(1 + g)^2 = 0$ is the only zero divisor, so it can not have S-zero divisors or S-weak zero divisors.

Example 3.2: Let $G = \{g / g^3 = 1\}$ be the cyclic group of order 3. Consider the group ring Z_2 G of the group G over Z_2.
Clearly

$$(1 + g + g^2)(g + g^2) = 0$$
$$(1 + g + g^2)(1 + g) = 0$$
$$(1 + g + g^2)(1 + g^2) = 0$$

are the only zero divisors in Z_2G. We see none of these are S-zero divisors or S-weak zero divisors.

Example 3.3: Consider the group ring Z_2G. where $G = \{ g / g^4 = 1\}$ is the cyclic group of order 4. Then

$$(1 + g)(1 + g + g^2 + g^3) = 0$$
$$(1 + g^2)(1 + g + g^2 + g^3) = 0$$
$$(1 + g^3)(1 + g + g^2 + g^3) = 0$$
$$(g + g^2)(1 + g + g^2 + g^3) = 0$$
$$(g + g^3)(1 + g + g^2 + g^3) = 0$$
$$(g^2 + g^3)(1 + g + g^2 + g^3) = 0$$

are some of the zero divisors in Z_2G. So it has S-zero divisors and no S-weak zero divisors.

Theorem 3.1: Let Z_2G be the group ring where G is a cyclic group of prime order p. Then the group ring Z_2G has no S-zero divisors or S-weak zero divisors.

Proof: The only possible zero divisors in Z_2G are

$$(1 + g + \ldots + g^{p-1})\, Y = 0$$

where $Y \in Z_2G$ and support of T is even
But we can not find $A \in Z_2G \setminus (1 + g + \ldots + g^{p-1})$
Such that $AY = 0$, $|\text{supp } A| < p$.
Hence Z_2G can not have S-zero divisors or S-weak zero divisors.

Theorem 3.2: Let Z_2G be the group ring of a finite cyclic group of composite order $n \geq 6$ ($n = p_1^{\alpha 1} p_1^{\alpha 2}$) then Z_2G has S-zero divisors.

Proof: Let G be a cyclic group of order n where G has atleast one subgroup H. Let H be generated by g^t, $t > 1$. Now

$$(1 + (g^t)^r)\,(1 + g + g^2 + \ldots + g^{n-1}) = 0;\ (g^t)^r \neq 1$$

again

$$(1 + g)\,(1 + g + g^2 + \ldots + g^{n-1}) = 0$$

also

$$(1 + (g^t)^r)\,(1 + g^t + (g^t)^2 + \ldots + (g^t)^{s+1}) = 0, \quad (\because\ (g^t)^{s+1} = 1)$$

clearly

$$(1 + g)\,(1 + g^t + (g^t)^2 + \ldots + (g^t)^{s+1}) \neq 0.$$

Thus Z_2G has S-zero divisors.

Theorem 3.3: Let Z_2S_n be the group ring of the symmetric group S_n over Z_2. then Z_2S_n has S-zero divisors.

Proof: Let $A = 1 + p_1 + p_2 + p_3 + p_4 + p_5$ and $B = p_4 + p_5$

$$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 1 & 3 & 2 & 4 & 5 & \cdots & n \end{pmatrix}, \qquad p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 3 & 2 & 1 & 4 & 5 & \cdots & n \end{pmatrix}$$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 2 & 1 & 3 & 4 & 5 & \cdots & n \end{pmatrix}, \qquad p_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 2 & 3 & 1 & 4 & 5 & \cdots & n \end{pmatrix}$$

$$p_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 3 & 1 & 2 & 4 & 5 & \cdots & n \end{pmatrix}$$

And 1 is the identity permutation.
Clearly

$$AB = 0$$

Take

$$X = 1 + p_1, \quad \text{and} \quad Y = 1 + p_4 + p_5$$

Then

$$AX = 0 \qquad \text{and} \qquad BY = 0$$

But

$$XY \neq 0.$$

Hence Z_2S_n has S-zero divisor.

Example 3.4: The group ring Z_2S_4 where S_4 is the symmetric group of order 4 has s-zero divisors.

Let $a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $\quad b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$, $\quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \in S_4$.

Put

$$A = (1 + a + b + c)$$

Let $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, then $g^4 = 1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$:

Now. Let $B = 1 + g + g^2 + g^3$,

Clearly

$$(1 + g) \sum_{s_i \in S_4} s_i = 0, \qquad (1 + g).\ B = 0$$

also

$$A \sum_{s_i \in S_4} s_i = 0,$$

$$A.B \neq = 0.$$

Hence $(1 + g)$ and $\sum_{s_i \in S_4} s_i$ are S-zero divisors in $Z_2\, S_4$.

Theorem 3.4: Let $Z_2\, D_{2n}$ be the group ring of the Dihedral group D_{2n}, over Z_2 where n is a composite number, then $Z_2\, D_{2n}$ has S-zero divisors.

Proof: Since n is a composite number, by the theorem 3.2, we see $Z_2\, D_{2n}$ has S-zero divisors.

Example 3.5: Let $Z_2\, D_8$ be the group ring, where D_8 is the Dihedral group of order 8. then $Z_2\, D_8$ has s-zero divisors.

Here

$$D_8 = \{a, b \,/\, a^2 = b^4 = 1, bab = a\}.$$

Let

$$x = 1 + b + b^2 + b^3,\ y = \sum_{g_i \in D_8} g_i = (1 + a + ab + ab^2 + ab^3 + b + b^2 + b^3)$$

then

$$x.y = 0$$

Suppose $A = (1 + b^2)$ and $B = (1 + ab)$.

Then clearly

$$A.x = 0 \text{ also } B.y = 0$$

But

$$A.B \neq 0.$$

Hence x and y are S-zero divisors in Z_2 D.

Example 3.6: Z_2G where $G = \{g \,/\, g^6 = 1\}$ be the group ring. Z_2G has S-zero divisors.

Let

$$X = (1 + g + g^2 + g^3 + g^4 + g^5), \text{ and } y = (1 + g^2 + g^4)$$

Then

$$x.y = 0.$$

Let $a = 1 + g^3$ and $b = 1 + g^2$

Clearly

$$a.x = 0 \qquad \text{and} \qquad b.y = 0$$

$$\text{but } a.\, b \neq 0.$$

Hence Z_2 G gas S-zero divisors.

4. S-zero divisors of group rings over rings of characteristic zero.

In this section we prove ZG, the group ring of a finite group G over Z,. the ring of integers, has non-trivial S-zero divisors. Further we prove ZS_n and ZD_{2n} has non-trivial S-zero divisors. Since $Z \subset K$ for any field K of characteristic zero, we can say KG has non-trivial S-zero divisors.

Example 4.1: ZG the group ring of the group G over Z; where $G = \{g \,/\, g^8 = 1\}$, we have for $(1-g^4)$, $(1 + g + g^2 + \ldots + g^7) \in ZG$

$(1-g^4).\ (1 + g + g^2 + \ldots + g^7) = 0$

also $(1-g).\ (1 + g + g^2 + \ldots + g^7) = 0$

$(1-g^4).\ (1 + g^2 + g^4 + g^6) = 0$

but $(1-g).\ (1 + g^2 + g^4 + g^6) \neq 0.$

Hence the claim

Theorem 4.1: Let G be a finite group where $|G| = n$, n is a composite number ; then the group ring ZG has S-zero divisors.

Proof: Let $a \in G$ such that $a^m = 1$, $m < n$. Now for $(1-a)$, $(1 + g_1 + \ldots + g_{n-1}) \in ZG$

We have

$$(1\text{-}a)\ (1 + g_1 + g_2 + \ldots + g_{n-1}) = 0, \quad g_i \in G$$

$$(1 - a)\ (1 + a + a^2 + \ldots + a^{m-1}) = 0,\ a \in G \text{ and } a^m = 1$$

$$(1\text{-}h)\ (1 + g_1 + g_2 + \ldots. + g_{n-1}) = 0,\ h \in G \text{ and } h \neq a^i,\ I = 1, 2, \ldots, m$$

$$\text{but } (1\text{-}h)\ (1 + a + a^2 + \ldots + a^{m-1}) \neq 0,\ m < n.$$

The complete the proof.

Theorem 4.2: The group ring ZS_n has S-zero divisors, where S_n is the symmetric group of degree n.

Proof: For $(1-p_1)$, $(1 + p_1 + p_2 + p_3 + p_4 + p_5) \in ZS_n$
We have

$$(1 - p_1)(1 + p_1 + p_2 + p_3 + p_4 + p_5) = 0$$

where p_i are as in theorem 3.3

$$\text{again } (1-p_1)(1 + p_1) = 0$$

$$(1 + p_2)(1 + p_1 + p_2 + p_3 + p_4 + p_5) = 0$$

$$\text{but } (1+p_1)(1 + p_2) \neq 0.$$

Hence the claim

Theorem 4.3: The group ring ZD_{2n} has S-zero divisors, where D_{2n} is the Dihedral group of order 2n, n is a composite number.

Proof: As n is a composite number, result will follow from the theorem 4.1

Theorem 4.4: Let KG be the group ring of G over K, where K is a field of characteristic zero and G any group of composite order, then KG has S-zero divisors.

Proof: As $Z \subset K$, so $ZG \subset KG$ and as ZG has S-zero divisors so KG has non-trivial S-zero divisors.

Example 4.2: Let Z_3G be the group ring of G over Z_3 where $G = \{g / g^6 = \}$. For $(1 + g + g^2 + g^3 + g^4 + g^5)$, $(1 + g + g^4) \in Z_3G$, we have

$$(1 + g + g^2 + g^3 + g^4 + g^5)(1 + g^2 + g^4) = 0$$

$(g^2 + 2)(1 + g^2 + g^4) = 0$
also $(1 + g + g^3)(1 + g + g^2 + g^3 + g^4 + g^5) = 0$
but $(g^2 + 2)(1 + g + g^3) \neq 0$.
Hence Z_3 G has s-zero divisors.

Theorem 4.5: The group ring Z_p G, where $G = \{g / g^n = 1\}$ and p/n, has S-zero divisors.

Proof: Here

$$(1 + g + g^2 + \ldots + g^{n-1})(1 + g^p + g^{2p} + \ldots + g^{rn-1}) = 0 \text{ where } rp = n$$

$$\text{again } (1 + g + g^2 + \ldots + g^{n-1})((p-1) + g)) = 0, \text{ (where (s, p) = 1)}$$

$$((p-1) + g^{tp})(1 + g^p + g^{2p} + \ldots + g^{rp-1}) = 0, t < r$$

$$\text{but } ((p-1) + g^8)((p-1) + g^{tp}) \neq 0.$$

Hence the claim

Example 4.3: The group ring Z_3S_3 has S-zero divisors.
Let $(1 + p_4 + p_5), (1 + p_1 + p_2 + p_3 + p_4 + p_5) \in Z_3S_3$
Now

$$(1 + p_4 + p_5), (1 + p_1 + p_2 + p_3 + p_4 + p_5) = 0$$
$$(2 + p_1)(1 + p_1 + p_2 + p_3 + p_4 + p_5) = 0$$

also $$(2 + p_5)(1 + p_4 + p_5) = 0$$

but $$(2 + p_5)(2 + p_1) \neq 0.$$

Theorem 4.6: The group ring $Z_p S_n$ when $p \mid n!$ (i.e $p < n$) has S-zero divisors.

Proof: As $p|n!$ S_n has a cyclic subgroup H of order p. We have for $(1 + h + h^2 + \ldots + h^{p-1})$, $\sum_{s_i \in S_n} s_i \in Z_pS_n$,

$$(1 + h + h^2 + \ldots + h^{p-1}) \sum_{g_i \in D_8} s_i = 0, \quad h \in H$$
$$\text{and } ((p-1) + s) \sum_{g_i \in D_8} s_i = 0, s \notin H, s \in S_n$$
$$\text{and } ((p-1) + h^r)(1 + h + h^2 + \ldots + h^{p-1}) = 0, r < p - 1$$
$$\text{but } ((p-1) + s)((p-1) + h^r) \neq 0.$$

So $Z_p S_n$ has S-zero divisors.

Theorem 4.7: Let $Z_p D_{2n}$ be the group ring of the Dihedral group D_{2n} over Z_p, p a prime such that p/n then $Z_p D_{2n}$ has S-zero divisors.

Proof: Given $Z_p D_{2n}$ is the group ring of D_{2n} over Z_p such that p is a prime and p/n. Now $p/n \Rightarrow D_{2n}$ has a cyclic subgroup of order p, say $H = < t >$.
For $(1 + t + t^2 + \ldots + t^{p-1})$, $\sum_{g_i \in D_{2n}} g_i \in Z_pD_{2n}$

We have

$$t^p = 1, t \in D_{2n}$$
$$(1 + t + t^2 + \ldots + t^{p-1}). \sum_{g_i \in D_{2n}} g_i = 0$$
$$((p-1) - t^r)(1 + t + t^2 + \ldots + t^{p-1}) = 0 \qquad r < p - 1$$
$$((p-1) - g_r) \sum_{g_i \in D_{2n}} g_i = 0 \ \ gr \neq t^r \text{ for } r = 1, 2, 3, \ldots, p-1.$$
$$\text{But } ((p-1)-t^r)((p-1)-g^r) \neq 0.$$

Hence the claim.

Conclusion

In the first place we proved that $Z_p S_n$ has S-zero divisors provided $p|n!$, p is a prime such that $p < n$. Further we proved ZS_n and ZD_{2n} has S-zero divisors only when n is a composite number.

Z_p D_{2n} has S-zero divisors for p|n. We are not in a position to obtain any nice algebraic structure for the collection of S-zero divisors or S-weak zero divisors in group rings. Also another interesting problem would be to find the number of S-zero divisors and s-weak zero divisors in case of Z_p S_n (p|n!, p < n) and Z_p D_{2n} (p|n). The solution even in case of Z_p G where G is a cyclic group of order n such that p/n, p a prime is not complete.

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