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A Study on Equitable Triple Connected Domination Number of a Graph

M. Subramanian

Department of Mathematics

Anna University Regional Campus - Tirunelveli, Tirunelveli - 627 007, India

T. Subramanian

Department of Mathematics, University Voc College of Engineering

Anna University:Thoothukudi Campus, Thoothukudi- 628008, India

E-mail: ms_akce@yahoo.com, tsubramanian001@gmail.com

Abstract: A graph G is said to be *triple connected* if any three vertices lie on a path in G. A dominating set S of a connected graph G is said to be a *triple connected dominating set* of G if the induced subgraph $\langle S\rangle$ is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the *triple connected domination number* and is denoted by γ_{tc}. A triple connected dominating set S of V in G is said to be an *equitable triple connected dominating set* if for every vertex u in $V-S$ there exists a vertex v in S such that uv is an edge of G and $|deg(v)-deg(u)|\leq 1$.The minimum cardinality taken over all equitable triple connected dominating sets is called the *equitable triple connected domination number* and is denoted by γ_{etc}. In this paper we initiate a study on this parameter. In addition, we discuss the related problem of finding the stability of γ_{etc} upon edge addition on some classes of graphs.

Key Words: Connected domination, triple connected domination, equitable triple connected dominating set, equitable triple connected domination number, Smarandachely equitable dominating set.

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§1. Introduction

By a *graph*, we mean a finite, simple, connected and undirected graph $G(V,E)$, where V denotes its vertex set and E its edge set. Unless otherwise stated, the graph G is connected and has p vertices and q edges. For graph theoretic terminology, we refer to Harary [1].

Definition 1.1([2]) *A subset S of V in G is called a dominating set of G if every vertex in $V-S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets in G.*

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Definition 1.2([6]) *A dominating set S of V in G is said to be an equitable dominating set if for every vertex u in $V - S$ there exists a vertex v in S such that uv is an edge of G and $|deg(v) - deg(u)| \leq 1$, and Smarandachely equitable dominating set if $|deg(v) - deg(u)| \geq 1$ for all such an edge. The minimum cardinality taken over all equitable dominating sets in G is the equitable domination number of G and is denoted by γ_e.*

Definition 1.3([2]) *A dominating set S of V in G is said to be a connected dominating set of G if the induced sub graph $\langle S \rangle$ is connected. The minimum cardinality taken over all connected dominating sets in G is the connected domination number of G and is denoted by γ_c.*

Definition 1.4([3]) *A connected dominating set S of V in G is said to be an equitable connected dominating set if for every vertex u in $V - S$ there exists a vertex v in S such that uv is an edge of G and $|deg(v) - deg(u)| \leq 1$. The minimum cardinality taken over all equitable connected dominating sets in G is the equitable connected domination number of G and is denoted by γ_{ec}.*

The concept of triple connected graphs has been introduced by Paulraj Joseph et. al. [5] by considering the existence of a path containing any three vertices of G. They have studied the properties of triple connected graphs and established many results on them. A graph G is said to be *triple connected* if any three vertices lie on a path in G. All paths, cycles, complete graphs and wheels are some standard examples of triple connected graphs. But the star $K_{1,p-1}, p \geq 4$ is not a triple connected graph.

Definition 1.5([4]) *A dominating set S of a connected graph G is said to be a triple connected dominating set of G if the induced sub graph $\langle S \rangle$ is triple connected. The minimum cardinality taken over all connected dominating sets is the triple connected domination number and is denoted by γ_{tc}.*

In this paper, we extend the concept of triple connected domination to an equitable triple connected domination and study its properties.

Notation 1.6 Let G be a connected graph on m vertices $v_1, v_2, \ldots, v_m$. The graph obtained from G by attaching n_1 times a pendant vertex of P_{l_1} on the vertex v_1, n_2 times a pendant vertex of P_{l_2} on the vertex v_2 and so on, is denoted by $G(n_1P_{l_1}, n_2P_{l_2}, n_3P_{l_3}, \cdots, n_mP_{l_m})$ where $n_i, l_i \geq 0$ and $1 \leq i \leq m$.

Notation 1.7 We have $C_p(nP_k, 0, 0, \cdots, 0) \cong C_p(0, nP_k, 0, \cdots, 0) \cong \cdots \cong C_p(nP_k)$.

Example 1.8 The graph G_1 in Figure 1 is isomorphic to $C_3(2P_2)$.

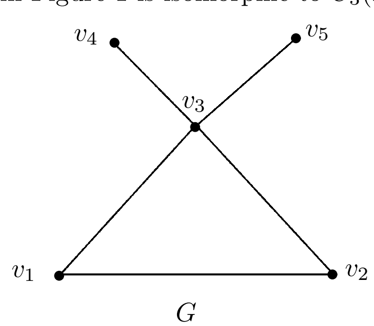

Figure 1 The graph $C_3(2P_2)$.

Proposition 1.9 *For any connected graph G with p vertices, $1 \leq \gamma_e(G) \leq p$.*

Proposition 1.10 *For any connected graph G with p vertices, $1 \leq \gamma_{ec}(G) \leq p$.*

§2. Equitable Triple Connected Domination Number of a Graph

In this section, we define the concept of equitable triple connected domination number of a graph.

Definition 2.1 *A subset S of V of a nontrivial graph G is said to be an equitable triple connected dominating set, if $\langle S\rangle$ is triple connected and for every vertex u in $V-S$ there exists a vertex v in S such that uv is an edge of G and $|deg(v)-deg(u)| \leq 1$. The minimum cardinality taken over all equitable triple connected dominating sets is called an equitable triple connected domination number of G and is denoted by $\gamma_{etc}(G)$. Any equitable triple connected dominating set with γ_{etc} vertices is called a γ_{etc}-set of G.*

Example 2.2 For the graph G_1 in Figure 2, $S = \{v_2, v_3, v_5, v_6\}$ forms a γ_{etc}-set of G_1. Hence $\gamma_{etc}(G_1) = 4$.

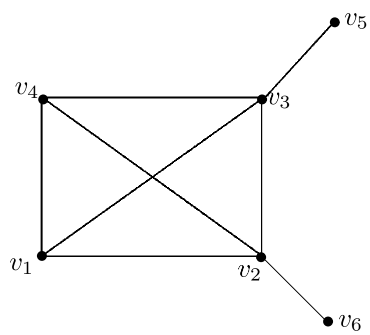

Figure 2 Graph with $\gamma_{etc} = 4$.

Remark 2.3 Any equitable triple connected dominating set is obviously equitable connected dominating set and any equitable connected dominating set is also an equitable dominating set and finally and any equitable dominating set is a dominating set. So it is permissible for the equitable triple connected dominating set S can have less than three vertices. If S has 1 (or 2) vertex (vertices) then S can be viewed as an equitable dominating set (or connected equitable dominating set).

Throughout this paper, we consider only connected graphs for which equitable triple connected dominating set exists.

Definition 2.4 *A bistar, denoted by $B(m,n)$ is the graph obtained by joining the centers of the stars $K_{1,m}$ and $K_{1,n}$. The center of a star $K_{1,p-1}$ with $p > 2$ vertices is its unique vertex of maximum degree.*

Definition 2.5 *A helm graph, denoted by H_n is the graph obtained from the wheel W_n by attaching a pendant vertex to each vertex in the outer cycle of W_n (the number of vertices of H_n is, $p = 2n-1$).*

Definition 2.6 *The friendship graph F_n is the graph constructed by identifying n copies of the cycle C_3 at a common vertex.*

Remark 2.7 It is to be noted that not every graph has a triple connected dominating set likewise not all graphs have an equitable triple connected dominating set. For example, the star graph $K_{1,3}$ does not have an equitable triple connected dominating set.

§3. Preliminary Results

We now proceed to determine the equitable triple connected domination number for some standard graphs.

(1) For any path of order $p, \gamma_{etc}(P_p) = \begin{cases} p & \text{if } p = 1 \\ p-1 & \text{if } p = 2 \\ p-2 & \text{if } p \geq 3. \end{cases}$

(2) For any cycle of order $p, \gamma_{etc}(C_p) = p - 2$.

(3) For any complete bipartite graph other than star of order $p = m + n$,

$$\gamma_{etc}(K_{m,n}) = \begin{cases} 2 & \text{if } |m-n| \leq 1 \text{ and } m, n \neq 1 \\ p & \text{if } |m-n| \geq 2 \text{ and } m, n \neq 1. \end{cases}$$

(4) For any complete graph of order $p, \gamma_{etc}(K_p) = 1$.

(5) For any wheel of order $p, \gamma_{etc}(W_p) = \begin{cases} 1 & \text{if } p = 4, 5 \\ 3 & \text{if } p = 6 \\ p-4 & \text{if } p \geq 7 \end{cases}$

Equitable triple connected dominating set does not exist for the following special graphs:

(6) For any star $K_{1,p-1}$ other than $K_{1,2}$.

(7) Helm graph H_n.

(8) Bistar $B(m, n)$.

Consider any star $K_{1,p-1}$ of order $p > 3$. Let v be its center and $v_1, v_2, \ldots, v_{p-1}$ be the pendant vertices which are adjacent to v. Since every minimum equitable dominating set S must contain all the pendant vertices $v_1, v_2, \cdots, v_{p-1}$ and we have $\langle S \rangle$ is not triple connected if $p - 1 > 2$. Hence $\gamma_{etc}(K_{1,p-1})$ does not exist if $p > 3$. Similarly we can prove all the other results.

Lemma 3.1 *If $\gamma_e(G) = 1$, then $\gamma_{etc}(G) = 1$.*

Lemma 3.2 *If $\gamma_{ec}(G) = 1$ (or 2 or 3), then $\gamma_{etc}(G) = 1$ (or 2 or 3).*

Lemma 3.3 *If $\gamma_{ec}(G) = 4$, then $\gamma_{etc}(G)$ need not be equal to 4.*

For $C_3(2P_2), \gamma_{ec}(C_3(2P_2)) = 4$, but $\gamma_{etc}(C_3(2P_2))$-set does not exist.

Theorem 3.4 *For any connected graph G with p vertices, we have $1 \leq \gamma_{etc}(G) \leq p$ and the bounds are sharp.*

Proof The lower bound follows from Definition 2.1 and the upper bound is obvious. For K_4 the lower bound is attained and for $K_{2,4}$ the upper bound is attained. □

Observation 3.5 For any connected graph G of order 1, $\gamma_{etc}(G) = p$ if and only if G is isomorphic to K_1.

Lemma 3.6 *There exists no connected graph G with $2 \leq p \leq 4$ vertices such that $\gamma_{etc}(G) = p$.*

Proof The proof is divided into cases following.

Case 1. The only connected graph with of order 2 is K_2 and for $K_2, \gamma_{etc}(K_2) = 1 = p - 1$ ([1]).

Case 2. There are only two connected graphs with three vertices which are P_3 or K_3 and for $G \cong P_3, K_3, \gamma_{etc}(G) = 1 = p - 2$ ([1]).

Case 3. The various possibilities of connected graphs on four vertices are: $K_{1,3}, P_4, C_3(P_2), C_4, K_4 - \{e\}, K_4$. If G is isomorphic to $P_4, C_4, C_3(P_2)$, $\gamma_{etc}(G) = 2 = p - 2$. If G is isomorphic to $K_4, K_4 - \{e\}, \gamma_{etc}(G) = 1 = p - 3$. If $G \cong K_{1,3}, \gamma_{etc}(G)$ does not exist. □

Theorem 3.7 *Let G be a connected graph with $p = 5$ vertices, then $\gamma_{etc}(G) = p$ if and only if G is isomorphic to $\overline{C_3 \cup 2K_1}$.*

Proof ([1]) For the various possibilities of connected graphs on five vertices are: $K_{1,4}$, $P_3(0, P_2, P_2), P_5$, $C_3(2P_2)$, $C_3(P_2, P_2, 0)$, $C_3(P_3)$, $C_4(P_2)$, C_5, F_2, $\overline{P}_5$, $K_{2,3}, K_4(P_2), \overline{C_3 \bigcup 2K_1}$, $\overline{P_2 \bigcup P_3}, \overline{P_3 \bigcup 2K_1}, W_5, K_5 - \{e\}, K_5$ and any one of the following graphs from G_1 to G_3 in Figure 3.

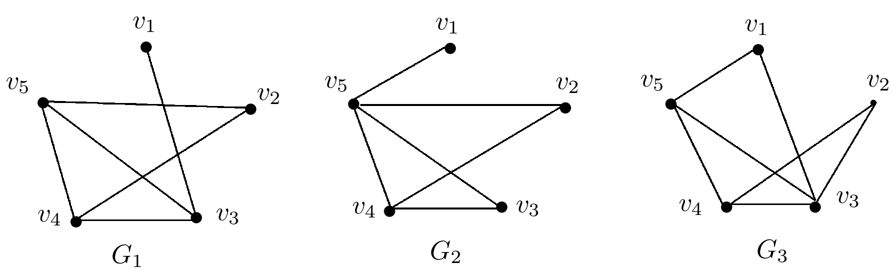

Figure 3 Graphs on 5 vertices.

If $G \cong K_5, W_5, K_5 - \{e\}$, then $\gamma_{etc}(G) = 1 = p - 4$. If $G \cong K_4(P_2)$, $C_3(P_3), K_{2,3}, \overline{P}_5$, $\overline{P_3 \cup 2K_1}$, $\overline{P_2 \cup P_3}, G_3$, then $\gamma_{etc}(G) = 2 = p - 3$. If $G \cong P_5, C_5, F_2$, $C_4(P_2), G_1, G_2$, then $\gamma_{etc}(G) = 3 = p - 2$. If $G \cong C_3(P_2, P_2, 0)$, then $\gamma_{etc}(G) = 4 = p - 1$. If $G \cong \overline{C_3 \cup 2K_1}$, then $\gamma_{etc}(G) = 5 = p$. If $G \cong K_{1,4}, C_3(2P_2)$, $P_3(0, P_2, P_2)$, then $\gamma_{etc}(G)$ does not exist. □

Theorem 3.8 *Let G be a connected graph with $p = 6$ vertices, then $\gamma_{etc}(G) = p$ if and only if*

G is isomorphic to $K_{2,4}$ or any one of the graphs: G_1, G_2, G_3 in Figure 4.

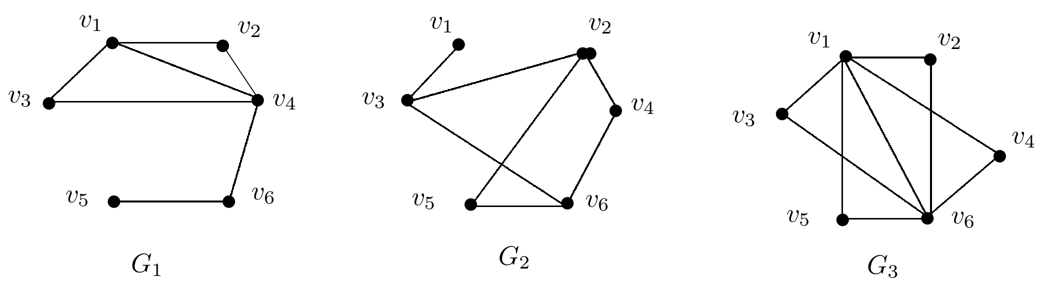

Figure 4 Graphs on 6 vertices with $\gamma_{etc}(G) = 6$.

Proof Let G be a connected graph with $p = 6$ vertices, and let $\gamma_{etc}(G) = 6$ ([1]). Among all of the connected graphs on 6 vertices, it can be easily verified that $G \cong K_{2,4}$ or any one of the graphs G_1, G_2, G_3 in Figure 4.The converse part is obvious. □

Lemma 3.9 *Let G be a connected graph of order 2 such that $\gamma(G) = \gamma_{etc}(G)$. Then $G \cong K_2$.*

Lemma 3.10 *Let G be a connected graph of order 3 such that $\gamma(G) = \gamma_{etc}(G)$. Then $G \cong K_3, P_3$.*

Lemma 3.11 *Let G be a connected graph of order 4 such that $\gamma(G) = \gamma_{etc}(G)$. Then G is isomorphic to one of the following graphs: $P_4, C_4, K_4, K_4 - \{e\}$.*

Proof For the various possibilities of connected graphs on four vertices are: $K_{1,3}, P_4, C_3(P_2)$, $C_4, K_4 - \{e\}, K_4$. If $G \cong P_4, C_4, \gamma(G) = \gamma_{etc}(G) = 2$. If $G \cong K_4, K_4 - \{e\}, \gamma(G) = \gamma_{etc}(G) = 1$. If $G \cong C_3(P_2), K_{1,3}, \gamma(C_3(P_2)) = 1$ but $\gamma_{etc}(C_3(P_2)) = 2$ and $\gamma(K_{1,3}) = 1$ but $\gamma_{etc}(K_{1,3})$ does not exist. Hence the lemma. □

Theorem 3.12 *Let G be a connected graph on order 5 such that $\gamma(G) = \gamma_{etc}(G)$. Then G is isomorphic to one of the following graphs: $C_3(P_3), \overline{P}_5,\ K_{2,3}, \overline{P_2 \cup P_3}, W_5, K_5 - \{e\}, K_5$.*

Proof For the various possibilities of connected graphs on five vertices are: $K_{1,4}, P_3(0, P_2, P_2)$, $P_5, C_3(2P_2), C_3(P_2, P_2, 0), C_3(P_3), C_4(P_2), C_5, F_2, \overline{P}_5, K_{2,3}, K_4(P_2)$, $\overline{C_3 \cup 2K_1}$, $\overline{P_2 \cup P_3}$, $\overline{P_3 \cup 2K_1}$, $W_5, K_5 - \{e\}, K_5$ and any one of the following graphs from G_1 to G_3 in Figure 3. Among all the above possibilities it can be easily verified that $\gamma(G) = \gamma_{etc}(G)$ only if $G \cong C_3(P_3), \overline{P}_5, K_{2,3}$, $\overline{P_2 \cup P_3}$, $W_5, K_5 - \{e\}, K_5$. □

Theorem 3.13 *Let G be a connected graph of order 6 such that $\gamma(G) = \gamma_{etc}(G)$. Then G is isomorphic to $K_{3,3}, K_6 - \{e\}, K_6$, or any one of the graphs: $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9, G_{10}$, G_{11}, G_{12} in Figure 5.*

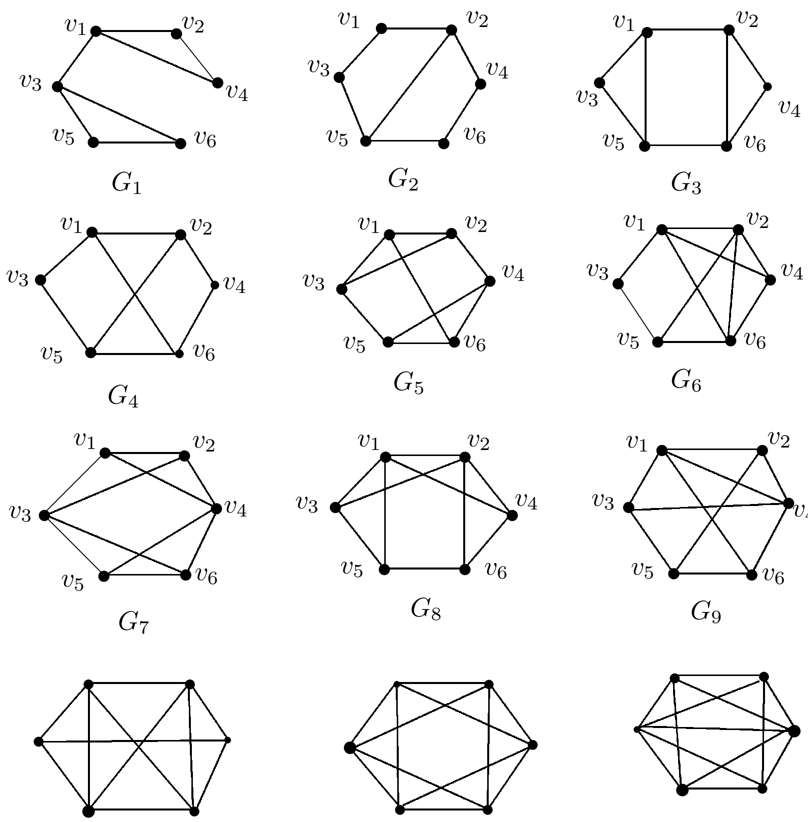

Figure 5 Graphs on 6 vertices such that $\gamma(G) = \gamma_{etc}(G)$.

Proof Let G be a connected graph of order 6 such that $\gamma(G) = \gamma_{etc}(G)$. It is straight forward to observe that $\gamma(G) = \gamma_{etc}(G)$ only if $G \cong K_{3,3}, K_6 - \{e\}, K_6$ or any one of the graphs $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9, G_{10}, G_{11}, G_{12}$ in Figure 5. $\square$

Observation 3.14 For any connected graph $G, \gamma_e(G) \leq \gamma_{ec}(G) \leq \gamma_{etc}(G)$ and the bounds can be strict as well as sharp for all possible cases.

(1) For the complete graph $K_5, \gamma_e(K_5) = \gamma_{ec}(K_5) = \gamma_{etc}(K_5) = 1$.

(2) For $K_4(P_3), \gamma_e(K_4(P_3)) = 2 < \gamma_{ec}(K_4(P_3)) = \gamma_{etc}(K_4(P_3)) = 3$.

(3) For the graph G_1 in Figure 6, $\gamma_e(G_1) = 3 < \gamma_{ec}(G_1) = 4 < \gamma_{etc}(G_1) = 5$.

(4) For the graph G_2 in Figure 6, $\gamma_e(G_2) = \gamma_{ec}(G_2) = 4 < \gamma_{etc}(G_2) = 5$.

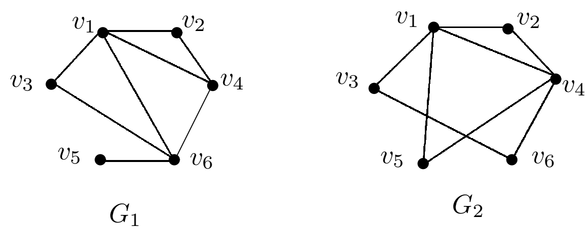

Figure 6

Lemma 3.15 *Let G be a connected graph of order 1 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Then $G \cong K_1$.*

Lemma 3.16 *Let G be a connected graph of order 2 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Then $G \cong K_2$.*

Lemma 3.17 *Let G be a connected graph of order 3 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Then $G \cong K_3, P_3$.*

Lemma 3.18 *Let G be a connected graph of order 4 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Then G is isomorphic to one of the following graphs: $P_4, C_4, K_4, C_3(P_2), K_4 - \{e\}$.*

Proof The various possibilities of connected graphs on four vertices are: $K_{1,3}, P_4, C_3(P_2)$, $C_4, K_4 - \{e\}, K_4$. If $G \cong P_4, C_4, C_3(P_2), \gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = 2$. If $G \cong K_4, K_4 - \{e\}$, $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = 1$. And if $G \cong K_{1,3}, \gamma_e(G) = \gamma_{ec}(G) = 4$ and $\gamma_{etc}(G)$ does not exist. Hence the lemma. □

Theorem 3.19 *Let G be a connected graph of order 5 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Then G is isomorphic to one of the following graphs: $C_3(P_3), C_4(P_2), \overline{P}_5, K_{2,3}, F_2, K_4(P_2)$, $\overline{P_2 \cup P_3}, \overline{P_3 \cup 2K_1}, W_5, K_5 - \{e\}, K_5$ or the graphs: G_1 to G_3 in Figure 3.*

Proof For the various possibilities of connected graphs on five vertices are: $K_{1,4}, P_3(0, P_2, P_2)$, $P_5, C_3(2P_2), C_3(P_2, P_2, 0), C_3(P_3), C_4(P_2), C_5, F_2, \overline{P}_5, K_{2,3}, K_4(P_2), \overline{C_3 \cup 2K_1}, \overline{P_2 \cup P_3}, \overline{P_3 \cup 2K_1}$, $W_5, K_5 - \{e\}, K_5$ and any one of the following graphs from G_1 to G_3 in Figure 3. If $G \cong K_5, W_5, K_5 - \{e\}$, then $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = 1$. If $G \cong K_4(P_2), C_3(P_3), K_{2,3}, \overline{P}_5, \overline{P_3 \cup 2K_1}$, $\overline{P_2 \cup P_3}, G_3$, then $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = 2$. If $G \cong F_2$ or $C_4(P_2)$ then $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = 3$. If $G \cong G_1$ or G_2 then $\gamma_e(G) = 2$, but $\gamma_{ec}(G) = \gamma_{etc}(G) = 3$. If $G \cong C_3(P_2, P_2, 0)$, then $\gamma_e(G) = 3$, but $\gamma_{ec}(G) = \gamma_{etc}(G) = 4$. If $G \cong \overline{C_3 \cup 2K_1}$, then$\gamma_e(G) = \gamma_{ec}(G) = 4$, but $\gamma_{etc}(G) = 5$. If $G \cong K_{1,4}$, then $\gamma_e(G) = \gamma_{ec}(G) = 5$, but $\gamma_{etc}(G)$ does not exist. If $G \cong P_3(0, P_2, P_2)$, then $\gamma_e(G) = 3, \gamma_{ec}(G) = 4$ and $\gamma_{etc}(G)$ does not exist. If $G \cong C_3(2P_2)$ then $\gamma_e(G) = \gamma_{ec}(G) = 4$, but $\gamma_{etc}(G)$ does not exist. If $G \cong P_5, C_5$, then $\gamma_e(G) = 2$,but $\gamma_{ec}(G) = \gamma_{etc}(G) = 3$. □

Theorem 3.20 *Let G be a connected graph of order 6 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Then $G \cong K_{2,4}$.*

Proof Let G be a connected graph of order 6 such that $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G)$. Among all of the connected graphs on 6 vertices, it can be easily seen that $K_{2,4}$ is the only graph with $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = 6$. □

Theorem 3.21 *If G is a connected graph of order $p = 2n$ for some positive integer $n \geq 2$ such that its vertex set and edge set are $V(G) = \{v_i : 1 \leq i \leq p\}$ and $E(G) = \{v_iv_{i+1} : 1 \leq i \leq p-1\} \cup \{v_iv_j :$ for $i = 1$ to $\frac{p}{2}$ and $j = \left(\frac{P}{2}+1\right)$ to $p\}$ respectively, then $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = n-1$.*

Example 3.22 For $p = 6(= 2n)$, By Theorem 3.21, the graph constructed is shown in Figure 7. Clearly any two adjacent vertices from the set $\{v_2, v_3, v_4, v_5\}$ forms a minimum equitable triple connected dominating set. Hence $\gamma_{etc}(G) = 2 = n-1$.

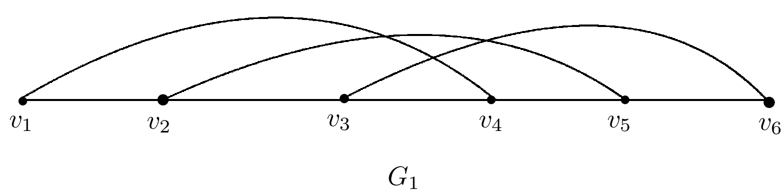

Figure 7 Graph illustrating the Theorem 3.21

Proposition 3.23 *Let G be a triple connected graph on order p. If its vertex set $V(G)$ can be partitioned into k sets $\{S_1, S_2, \cdots, S_k\}$ such that $S_1 = \{v : deg(v) = m_1\}$, $S_2 = \{v : deg(v) = m_2 \geq m_1 + 2\}$, $S_3 = \{v : deg(v) = m_3 \geq m_2 + 2\}, \cdots, S_k = \{v : deg(v) = m_k \geq m_{k-1} + 2\}$ where m_i's are increasing sequence of positive integers and $\langle S_i \rangle$ is $\overline{K_n}$ for some positive integer n, for $1 \leq i \leq k$, then $\gamma_e(G) = \gamma_{ce}(G) = \gamma_{etc}(G) = p$.*

Remark 3.24 The converse of Proposition 3.23 need not be true. Let G be a triple connected graph given in Figure 8. Clearly $\gamma_e(G) = \gamma_{ec}(G) = \gamma_{etc}(G) = p$, but there is no such partition of $V(G)$ as stated in Proposition 3.23. Since $V(G)$ can be partitioned in to $S_1 = \{v_{11}, v_{12}\}$ of degree $1, S_2 = \{v_3, v_4, v_5, v_7, v_8, v_9\}$ of degree $2, S_3 = \{v_2\}$ of degree $3, S_4 = \{v_6, v_{10}\}$ of degree 4 and $S_5 = \{v_1\}$ of degree 7 such that $\langle S_i \rangle$ is totally disconnected, for $1 \leq i \leq 5$.

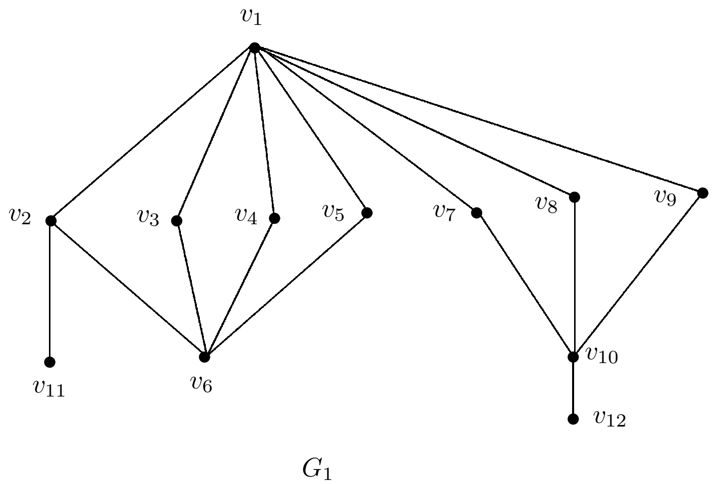

Figure 8 Counter example for Proposition 3.23

Lemma 3.25 *Let T be any tree, $\gamma_{etc}(T) = p$ if and only if $T \cong K_1$.*

Proof Let $T \cong K_1$, then clearly $\gamma_{etc}(T) = p$. Conversely, let T be a tree such that $\gamma_{etc}(T) = p$. Now $\langle T \rangle$ is triple connected, it follows that $T \cong P_p$ ([5] since a tree T is triple connected if and only if $T \cong P_p; p \geq 3$) and given that $\gamma_{etc}(T) = p$, we have $T \cong K_1$. □

Lemma 3.25 *Let T be any tree, $\gamma_{etc}(T) = p - 1$ if and only if $T \cong K_2$.*

Proof Let $T \cong K_2$, then clearly $\gamma_{etc}(T) = p - 1$. Conversely, let T be a tree such that $\gamma_{etc}(T) = p - 1$. Let v_p be the vertex not in $\gamma_{etc}(T)$-set. Suppose $deg(v_p) > 1$, then we can find a cycle in T, which is a contradiction. Hence $deg(v_p) = 1$. Since v_p is a pendant vertex we have $T - \{v_p\}$ is also a tree. Then $\langle T - \{v_p\} \rangle$ is triple connected, which follows that $T - \{v_p\} \cong P_{p-1}$ (from [5]) and hence $T \cong P_p$ and given that $\gamma_{etc}(T) = p - 1$, we have $T \cong K_2$. □

Theorem 3.27 *Let T be any tree on $p > 2$ vertices. Then either $\gamma_{etc}(T) = p - 2$ if $T \cong P_p$ or γ_{etc}-set does not exist.*

Proof The proof is divided into cases following.

Case 1. If T contains two pendant vertices. Then $T \cong P_p$ for which $\gamma_{etc}(T) = p - 2$, where $p > 2$.

Case 2. If T contains more than two pendant vertices.

Since any equitable triple connected dominating set must contain all the pendant vertices or its supports and also T is connected and acyclic it follows that γ_{etc}-set does not exist. □

§4. Equitable Triple Connected Domination Edge Addition Stable Graphs

In this section, we consider the problem of finding the stability of γ_{etc} upon edge addition of some classes of graphs such as cycles and complete bipartite graphs.

Definition 4.1 *A connected graph G is said to be an equitable triple connected domination edge addition stable (γ_{etc}-stable), if both G and $G+e$ have the same equitable triple connected domination number, where $G+e$ is a simple graph (i.e.) $\gamma_{etc}(G) = \gamma_{etc}(G+e)$.*

Definition 4.2 *A connected graph G is said to be an equitable triple connected domination edge addition positive critical (γ^{+}_{etc}- critical), if $G+e$ has greater equitable triple connected domination number than G, where $G+e$ is a simple graph (i.e.) $\gamma_{etc}(G) < \gamma_{etc}(G+e)$.*

Definition 4.3 *A connected graph G is said to be an equitable triple connected domination edge addition negative critical (γ^{-}_{etc}-critical), if G has greater equitable triple connected domination number than $G+e$, where $G+e$ is a simple graph (i.e.) $\gamma_{etc}(G) > \gamma_{etc}(G+e)$.*

Theorem 4.4 *The cycle $C_p(p > 3)$, is γ^{-}_{etc}-critical.*

Proof Let $C_p = v_1v_2\cdots v_pv_1$ be any cycle of length $p, p > 3$. Now $S = \{v_2, v_3, \cdots, v_{p-1}\}$ is the minimum equitable triple connected dominating set, we have $\gamma_{etc}(C_p) = p - 2$. Consider $C_p + e$, where $e = v_iv_j$.

Case 1. Let C_p+e contain $C_3 = v_1v_2v_3v_1$, where $e = v_iv_j = v_3v_1$. Since $S = \{v_3, v_4, \cdots, v_{p-1}\}$ forms a minimum equitable triple connected dominating set, we have $\gamma_{etc}(C_p + e) = p - 3$.

Case 2. Let C_p+e does not contain C_3. Let $e = v_iv_j$. Now $S = V(C_p+e)-\{v_{i+1}, v_{i+2}, v_{j+1}, v_{j+2}\}$, where $v_{i+2} = N(v_{i+1})-v_i$ and $v_{j+2} = N(v_{j+1})-v_j$ forms a minimum equitable triple connected dominating set. Hence $\gamma_{etc}(C_p + e) = p - 4$.

In both cases $\gamma_{etc}(C_p + e) < \gamma_{etc}(C_p)$. Hence $C_p(p > 3)$, is γ^{-}_{etc}-critical. □

Lemma 4.5 *The complete bipartite graph $K_{1,2}$ is γ_{etc}-stable.*

Lemma 4.6 *The complete bipartite graph $K_{2,2}$ is γ^{-}_{etc}-critical.*

Lemma 4.7 *The complete bipartite graph $K_{n,n}, (n > 2, p = 2n)$ is γ_{etc}-stable.*

Proof Let $K_{n,n}, (n > 2)$ be a complete bipartite graph and its vertex partition is given by $V = V_1 \cup V_2$ such that $V_1 = \{u_1, u_2, \cdots, u_n\}$ and $V_2 = \{v_1, v_2, \cdots, v_n\}$. Now $S = \{u_1, v_1\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n,n}) = 2$. If we add any edge to $K_{n,n}$ there is no change in the equitable triple connected domination number. Hence $K_{n,n}, (n > 2)$ is γ_{etc}-stable. □

Lemma 4.8 *If an edge e is added between the vertices of V_1. Then the complete bipartite graph $K_{3,2}$, is γ_{etc}-stable, where $V(K_{3,2}) = V_1 \cup V_2$ such that $V_1 = \{u_1, u_2, u_3\}$ and $V_2 = \{v_1, v_2\}$.*

Proof Here $S = \{u_1, v_1\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{3,2}) = 2$. If we add an edge e is added between the vertices of V_1 we see that there is no change in the equitable triple connected domination number. □

Lemma 4.9 *If an edge e is added between the vertices of V_2. Then the complete bipartite graph $K_{3,2}$ is γ^{+}_{etc}-critical, where $V(K_{3,2}) = V_1 \bigcup V_2$ such that $V_1 = \{u_1, u_2, u_3\}$ and $V_2 = \{v_1, v_2\}$.*

Proof Here $S = \{u_1, v_1\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{3,2}) = 2$. By adding an edge between the vertices of V_2, we see that $S = \{u_1, u_2, u_3, v_1, v_2\}$ is a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{3,2}) = 5$. $\square$

Lemma 4.10 *If an edge e is added between the vertices of* V_1. *Then the complete bipartite graph* $K_{n+1,n}, (n > 2)$ *is* γ_{etc}*-stable, where* $V(K_{n+1,n}) = V_1 \cup V_2$ *such that* $V_1 = \{u_1, u_2, \cdots, u_{n+1}\}$ *and* $V_2 = \{v_1, v_2, \cdots, v_n\}$.

Proof Here $S = \{u_1, v_1\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n+1,n}) = 2$. If we add an edge e is added between the vertices of V_1 we see that there is no change in the equitable triple connected domination number. $\square$

Lemma 4.11 *If an edge e is added between the vertices of* V_2. *Then the complete bipartite graph* $K_{n+1,n}, (n > 2)$ *is* γ^{+}_{etc}*-critical, where* $V(K_{n+1,n}) = V_1 \cup V_2$ *such that* $V_1 = \{u_1, u_2, \cdots, u_{n+1}\}$ *and* $V_2 = \{v_1, v_2, \cdots, v_n\}$.

Proof Here $S = \{u_1, v_1\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n+1,n}) = 2$. By adding an edge e between the vertices of V_2 say $e = v_1v_2$, we see that $S = \{v_1, u_1, v_i\}$ for $i \neq 2$ is a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n+1,n}) = 3$. $\square$

Lemma 4.12 *If an edge e is added between the vertices of* V_1. *Then the complete bipartite graph* $K_{n+2,n}, (n > 1)$ *is* γ^{-}_{etc}*-critical, where* $V(K_{n+2,n}) = V_1 \cup V_2$ *such that* $V_1 = \{u_1, u_2, \cdots, u_{n+1}, u_{n+2}\}$ *and* $V_2 = \{v_1, v_2, \cdots, v_n\}$.

Proof Here $S = \{u_1, u_2, \cdots, u_{n+1}, u_{n+2}, v_1, v_2, \cdots, v_n\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n+2,n}) = p$. By adding an edge e between the vertices of V_1 say $e = u_1u_2$, we see that $S = \{u_3, \cdots, u_{n+1}, u_{n+2}, v_1, v_2, \cdots, v_n\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n+2,n}) = p - 2$. $\square$

Lemma 4.13 *If an edge e is added between the vertices of* V_2. *Then the complete bipartite graph* $K_{n+2,n}, (n > 1)$ *is* γ_{etc}*-stable, where* $V(K_{n+2,n}) = V_1 \cup V_2$ *such that* $V_1 = \{u_1, u_2, \cdots, u_{n+1}, u_{n+2}\}$ *and* $V_2 = \{v_1, v_2, \cdots, v_n\}$.

Proof Here $S = \{u_1, u_2, \cdots, u_{n+1}, u_{n+2}, v_1, v_2, \cdots, v_n\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{n+2,n}) = p$. By adding an edge e between the vertices of V_2 say $e = v_1v_2$, we see that there is no change in the equitable triple connected domination number. $\square$

Theorem 4.14 *The complete bipartite graph* $K_{m,n}, (m - n > 2$ *and* $m + n = p)$ *is* γ_{etc}*-stable.*

Proof Let $K_{m,n}, (m - n > 2$ and $m + n = p)$ be a complete bipartite graph and its vertex partition is given by $V = V_1 \cup V_2$ such that $V_1 = \{u_1, u_2, \cdots, u_m\}$ and $V_2 = \{v_1, v_2, \cdots, v_n\}$. Now $S = \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$ forms a minimum equitable triple connected dominating set so that $\gamma_{etc}(K_{m,n}) = p$. If we add any edge to $K_{m,n}$ there is no change in the equitable triple connected domination number. $\square$

§5. Conclusion

We conclude this paper by giving the following interesting open problems for further study:

(1) Characterize connected graphs of order p for which $\gamma_{etc} = p$.

(2) For which graphs, $\gamma_e = \gamma_{ec} = \gamma_{etc} = p$.

References

[1] F. Harary, *Graph Theory*, Addison-Wesley, Reading Mass (1972).

[2] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York (1998).

[3] K. Kavitha, and N. G. David, Equitable Total Domination Edge Addition Stable Graphs, *J. Combin. Math. Combin. Comput.*, 92 (2015), 185–194.

[4] G. Mahadevan, A. Selvam, J. Paulraj Joseph and T. Subramanian, Triple connected domination number of a graph, *International Journal of Mathematical Combinatorics*, 3 (2012), 93–104.

[5] J. Paulraj Joseph J, M. K. Angel Jebitha, P. Chithra Devi and G. Sudhana, Triple connected graphs, *Indian Journal of Mathematics and Mathematical Sciences*, 8 (1) (2012), 61–75.

[6] E. Sampathkumar, Equitable Domination, *Kragujevac Journal of Mathematics*, 35(1) (2011), 191–197.