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On the Distance Eccentricity Zagreb Indeices of Graphs

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Abstract: Let $G = (V, E)$ be a connected graph. The distance eccentricity neighborhood of $u \in V(G)$ denoted by $N_{De}(u)$ is defined as $N_{De}(u) = \{v \in V(G) : d(u, v) = e(u)\}$, where $e(u)$ is the eccentricity of u. The cardinality of $N_{De}(u)$ is called the distance eccentricity degree of the vertex u in G and denoted by $deg^{De}(u)$. In this paper, we introduce the first and second distance eccentricity Zagreb indices of a connected graph G as the sum of the squares of the distance eccentricity degrees of the vertices, and the sum of the products of the distance eccentricity degrees of pairs of adjacent vertices, respectively. Exact values for some families of graphs and graph operations are obtained.

Key Words: First distance eccentricity Zagreb index, Second distance eccentricity Zagreb index, Smarandachely distance eccentricity.

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§1. Introduction

In this research work, we concerned about connected, simple graphs which are finite, undirected with no loops and multiple edges. Throughout this paper, for a graph $G = (V, E)$, we denote $p = |V(G)|$ and $q = |E(G)|$. The complement of G, denoted by $\overline{G}$, is a simple graph on the same set of vertices $V(G)$ in which two vertices u and v are adjacent if and only if they are not adjacent in G. The open neighborhood and the closed neighborhood of u are denoted by $N(u) = \{v \in V : uv \in E\}$ and $N[u] = N(u) \cup \{u\}$, respectively. The degree of a vertex u in G, is denoted by $deg(u)$, and is defined to be the number of edges incident with u, shortly $deg(u) = |N(u)|$. The maximum and minimum degrees of G are defined by $\Delta(G) = max\{deg(u) : u \in V(G)\}$ and $\delta(G) = min\{deg(u) : u \in V(G)\}$, respectively. If $\delta = \Delta = k$ for any graph G, we say G is a regular graph of degree k. The distance between any two vertices u and v in G denoted by $d(u, v)$ is the number of edges of the shortest path joining u and v. The eccentricity $e(u)$ of a vertex u in G is the maximum distance between u and any other vertex v in G, that is $e(u) = \max\{d(u, v), v \in V(G)\}$.

The path, wheel, cycle, star and complete graphs with p vertices are denoted by P_p, W_p, C_p, S_p and K_p, respectively, and $K_{r,m}$ is the complete bipartite graph on $r + m$ vertices. All the definitions and terminologies about graph in this paper available in [6].

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The Zagreb indices have been introduced by Gutman and Trinajestic [5].

$$M_1(G) = \sum_{u\in V(G)} \big[deg(u)\big]^2 = \sum_{u\in V(G)} \sum_{v\in N(u)} deg(v) = \sum_{uv\in E(G)} \big[deg(u) + deg(v)\big].$$

$$M_2(G) = \sum_{uv\in E(G)} deg(u)deg(v) = \frac{1}{2} \sum_{u\in V(G)} deg(u) \sum_{v\in N(u)} deg(v).$$

Here, $M_1(G)$ and $M_2(G)$ denote the first and the second Zagreb indices, respectively. For more details about Zagreb indices, we refer to [2, 4, 9, 13, 11, 12, 7, 10, 8].

Let $u \in V(G)$. The distance eccentricity neighborhood of u denoted by $N_{De}(u)$ is defined as $N_{De}(u) = \{v \in V(G) :\ d(u,v) = e(u)\}$. The cardinality of $N_{De}(u)$ is called the distance eccentricity degree of the vertex u in G and denoted by $deg^{De}(u)$, and $N_{De}[u] = N_{De}(u) \cup \{u\}$, note that if u has a full degree in G, then $deg(u) = deg^{De}(u)$. And generally, a Smarandachely distance eccentricity neighborhood $N^S_{De}(u)$ of u on subset $S \subset V(G)$ is defined to be $N^S_{De}(u) = \{v \in V(G)\backslash S :\ d_{G\backslash S}(u,v) = e(u)\}$ with Smarandachely distance eccentricity $\left|N^S_{De}(u)\right|$. Clearly, $\left|N^{\emptyset}_{De}(u)\right| = deg^{De}(u)$. The maximum and minimum distance eccentricity degree of a vertex in G are denoted respectively by $\Delta^{De}(G)$ and $\delta^{De}(G)$, that is $\Delta^{De}(G) = \max_{u\in V} |N_{De}(u)|$, $\delta^{De}(G) = \min_{u\in V} |N_{De}(u)|$. Also, we denote to the set of vertices of G which have eccentricity equal to α by $V_e^{\alpha}(G) \subseteq V(G)$, where $\alpha = 1, 2, \cdots, diam(G)$. In this paper, we introduce the distance eccentricity Zagreb indices of graphs. Exact values for some families of graphs and some graph operations are obtained.

§2. Distance Eccentricity Zagreb Indices of Graphs

In this section, we define the first and second distance eccentricity Zagreb indices of connected graphs and study some standard graphs.

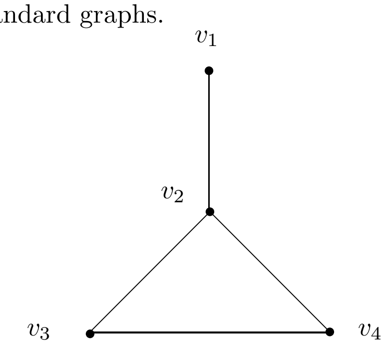

Fig.1

Definition 2.1 *Let $G = (V, E)$ be a connected graph. Then the first and second distance eccentricity Zagreb indices of G are defined by*

$$M_1^{De}(G) = \sum_{u\in V(G)} \big[deg^{De}(u)\big]^2,$$

$$M_2^{De}(G) = \sum_{uv\in E(G)} deg^{De}(u)deg^{De}(v).$$

Example 2.2 Let G be a graph as in Fig.1. Then

$$(i)\quad M_1^{De}(G) = \sum_{u\in V(G)} \big[deg^{De}(u)\big]^2 = \sum_{i=1}^{4} \big(deg^{De}(v_i)\big)^2$$
$$= \big(deg^{De}(v_1)\big)^2 + \big(deg^{De}(v_2)\big)^2 + \big(deg^{De}(v_3)\big)^2 + \big(deg^{De}(v_4)\big)^2$$
$$= (2)^2 + (3)^2 + (1)^2 + (1)^2 = 15.$$

$$(ii)\ M_2^{De}(G) = \sum_{uv\in E(G)} deg^{De}(u)deg^{De}(v)$$
$$= deg^{De}(v_1)deg^{De}(v_2) + deg^{De}(v_2)deg^{De}(v_3) + deg^{De}(v_2)deg^{De}(v_4) + deg^{De}(v_3)deg^{De}(v_4) = 13.$$

Calculation immediately shows results following.

Proposition 2.3 *(i) For any path P_p with $p \geq 2$,* $M_1^{De}(P_p) = \begin{cases} p+3, & p \text{ is odd,} \\ p, & p \text{ is even;} \end{cases}$

(ii) For $p \geq 3$, $M_1^{De}(C_p) = \begin{cases} 4p, & p \text{ is odd,} \\ p, & p \text{ is even;} \end{cases}$

(iii) $M_1^{De}(K_p) = M_1(K_p) = p(p-1)^2$;

(iv) For $r, m \geq 2$, $M_1^{De}(K_{r,m}) = r(r-1)^2 + m(m-1)^2$;

(v) For $p \geq 3$, $M_1^{De}(S_p) = (p-1)(p-2)^2 + (p-1)^2$;

(vi) For $p \geq 5$, $M_1^{De}(W_p) = (p-1)(p-4)^2 + (p-1)^2$.

Proposition 2.4 *(i) For $p \geq 2$,* $M_2^{De}(P_p) = \begin{cases} p+1, & p \text{ is odd,} \\ p-1, & p \text{ is even;} \end{cases}$

(ii) For $p \geq 3$, $M_2^{De}(C_p) = \begin{cases} 4p, & p \text{ is odd,} \\ p, & p \text{ is even;} \end{cases}$

(iii) $M_2^{De}(K_p) = M_2(K_p) = \frac{p(p-1)}{2}(p-1)^2$;

(iv) For $r, m \geq 2$, $M_2^{De}(K_{r,m}) = rm(r-1)(m-1)$;

(v) For $p \geq 3$, $M_2^{De}(S_p) = (p-1)^2(p-2)$;

(vi) For $p \geq 5$, $M_2^{De}(W_p) = (p-1)(p-4)(2p-5)$.

Proposition 2.5 *For any graph G with $e(v) = 2$, $\forall\, v \in V(G)$,*

(i) $M_1^{De}(G) = M_1(\overline{G})$;

(ii) $M_2^{De}(G) = q(p-1)^2 - (p-1)M_1(G) + M_2(G)$.

Proof Since $e(v) = 2$, $\forall v \in V(G)$, then $deg_G^{De}(v) = deg_{\overline{G}}(v)$. Hence the result. □

Corollary 2.6 *For any k-regular (p,q)-graph G with diameter two,*

(i) $M_1^{De}(G) = p(p-k-1)^2$;

(ii) $M_2^{De}(G) = \frac{1}{2}pk(p-k-1)^2$.

§3. Distance Eccentricity Zagreb Indices for Some Graph Operations

In this section, we compute the first and second distance eccentricity Zagreb indices for some graph operations.

Cartesian Product. The Cartesian product of two graphs G_1 and G_2, where $|V(G_1)| = p_1$, $|V(G_2)| = p_2$ and $|E(G_1)| = q_1$, $|E(G_2)| = q_2$ is denoted by $G_1 \Box G_2$ has the vertex set $V(G_1) \times V(G_2)$ and two vertices (u, u') and (v, v') are connected by an edge if and only if either $([u = v$ and $u'v' \in E(G_2)])$ or $([u' = v'$ and $uv \in E(G_1)])$. By other words, $|E(G_1 \Box G_2)| = q_1 p_2 + q_2 p_1$. The degree of a vertex (u, u') of $G_1 \Box G_2$ is as follows:

$$deg_{G_1 \Box G_2}(u, u') = deg_{G_1}(u) + deg_{G_2}(u').$$

The Cartesian product of more than two graphs is denoted by $\prod_{i=1}^{n} G_i$ $\big(\prod_{i=1}^{n} G_i = G_1 \Box G_2 \Box \ldots \Box G_n = (G_1 \Box G_2 \Box \ldots \Box G_{n-1}) \Box G_n\big)$, in which any two vertices $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are adjacent in $\prod_{i=1}^{n} G_i$ if and only if $u_i = v_i$, $\forall i \neq j$ and $u_j v_j \in E(G_j)$, where $i, j = 1, 2, \ldots, n$. If $G_1 = G_2 = \cdots = G_n = G$, we have the n-th Cartesian power of G, which is denoted by G^n.

Lemma 3.1([8]) *Let $G = \prod_{i=1}^{n} G_i$ and let $u = (u_1, u_2, \cdots, u_n)$ be a vertex in $V(G)$. Then*

$$e(u) = \sum_{i=1}^{n} e(u_i).$$

Lemma 3.2 *Let $G = \prod_{i=1}^{n} G_i$ and let $u = (u_1, u_2, \ldots, u_n)$ be a vertex in G. Then*

$$deg_G^{De}(u) = \prod_{i=1}^{n} deg_{G_i}^{De}(u_i).$$

Proof Since $e(u) = \sum_{i=1}^{n} e(u_i)$ (Lemma 3.1), then each distance eccentricity neighbor of u_1 in G_1 corresponds $deg_{G_2}^{De}(u_2)$ vertices in G_2 and each distance eccentricity neighbor of u_2 in G_2 corresponds $deg_{G_3}^{De}(u_3)$ vertices in G_3 and so on. Thus by using the Principle of Account

$$deg_G^{De}(u) = deg_{G_1}^{De}(u_1) deg_{G_2}^{De}(u_2) \cdots deg_{G_n}^{De}(u_n). \qquad \Box$$

Theorem 3.3 *Let $G = \prod_{i=1}^{n} G_i$. Then*

(i) $M_1^{De}(G) = \prod_{i=1}^{n} M_1^{De}(G_i)$;

(ii) $M_2^{De}(G) = \sum_{j=1}^{n} \prod_{\substack{i=1 \\ i \neq j}}^{n} M_1^{De}(G_i) M_2^{De}(G_j)$.

Proof Let $u = (u_1, u_2, \cdots, u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ be any two vertices in $V(G)$. Then

$$(i)\ M_1^{De}(G) = \sum_{u\in V(G)} \left(deg_G^{De}(u)\right)^2 = \sum_{u\in V(G)} \left(deg_{G_1}^{De}(u_1)deg_{G_2}^{De}(u_2)\dots deg_{G_n}^{De}(u_n)\right)^2$$
$$= \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\dots\sum_{u_n\in V(G_n)} \left(deg_{G_1}^{De}(u_1)\right)^2\left(deg_{G_2}^{De}(u_2)\right)^2\dots\left(deg_{G_n}^{De}(u_n)\right)^2$$
$$= \prod_{i=1}^{n} M_1^{De}(G_i).$$

(ii) To prove the second distance eccentricity Zagreb index we will use the mathematical induction. First, if $n = 2$, then

$$M_2^{De}(G_1\Box G_2) = \sum_{(u_1,u_2)(v_1,v_2)\in E(G_1\Box G_2)} deg_{G_1}^{De}(u_1)deg_{G_1}^{De}(v_1)deg_{G_2}^{De}(u_2)deg_{G_2}^{De}(v_2)$$
$$= \sum_{u_1\in V(G_1)}\sum_{(u_1,u_2)(u_1,v_2)\in E(G_1\Box G_2)} \left(deg_{G_1}^{De}(u_1)\right)^2 deg_{G_2}^{De}(u_2)deg_{G_2}^{De}(v_2)$$
$$+ \sum_{u_2\in V(G_2)}\sum_{(u_1,u_2)(v_1,u_2)\in E(G_1\Box G_2)} \left(deg_{G_2}^{De}(u_2)\right)^2 deg_{G_1}^{De}(u_1)deg_{G_1}^{De}(v_1)$$
$$= M_1^{De}(G_1)M_2^{De}(G_2) + M_1^{De}(G_2)M_2^{De}(G_1)$$
$$= \sum_{j=1}^{2}\prod_{\substack{i=1\\ i\neq j}}^{2} M_1^{De}(G_i)M_2^{De}(G_j).$$

Now, suppose the claim is true for $n-1$. Then

$$M_2^{De}\left(\Box_{i=1}^{n-1}G_i\Box G_n\right) = M_1^{De}\left(\Box_{i=1}^{n-1}G_i\right)M_2^{De}(G_n) + M_1^{De}(G_n)M_2^{De}\left(\Box_{i=1}^{n-1}G_i\right)$$
$$= \prod_{i=1}^{n-1} M_1^{De}(G_i)M_2^{De}(G_n) + M_1^{De}(G_n)\sum_{j=1}^{n-1}\prod_{\substack{i=1\\ i\neq j}}^{n-1} M_1^{De}(G_i)M_2^{De}(G_j)$$
$$= \sum_{j=1}^{n}\prod_{\substack{i=1\\ i\neq j}}^{n} M_1^{De}(G_i)M_2^{De}(G_j).$$

□

Composition. The composition $G = G_1[G_2]$ of two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, where $|V(G_1)| = p_1$, $|E(G_1)| = q_1$ and $|V(G_2)| = p_2$, $|E(G_2)| = q_2$ is the graph with vertex set $V(G_1)\times V(G_2)$ and any two vertices (u,u') and (v,v') are adjacent whenever u is adjacent to v in G_1 or $u = v$ and u' is adjacent to v' in G_2. Thus, $|E(G_1[G_2])| = q_1p_2^2 + q_2p_1$. The degree of a vertex (u,u') of $G_1[G_2]$ is as follows:

$$deg_{G_1[G_2]}(u,u') = p_2deg_{G_1}(u) + deg_{G_2}(u').$$

Lemma 3.4([8]) *Let $G = G_1[G_2]$ and $e(v)\neq 1$, $\forall\, v\in V(G_1)$. Then $e_G((u,u')) = e_{G_1}(u)$.*

Lemma 3.5 *Let $G = G_1[G_2]$ and $e(v)\neq 1$, $\forall\, v\in V(G_1)$. Then*

$$deg_G^{De}(u,u') = \begin{cases} p_2deg_{G_1}^{De}(u) + deg_{\overline{G_2}}(u'), & \text{if } u\in V_e^2(G_1);\\ p_2deg_{G_1}^{De}(u), & \text{otherwise.}\end{cases}$$

Proof From Lemma 3.4, we have $e_G(u,u') = e_{G_1}(u)$. Therefore, $N_G^{De}(u,u') = \{(x,x') \in V(G) : d\big((u,u'),(x,x')\big) = e_{G_1}(u)\}$. Now, if $u \notin V_e^2(G_1)$, then $N_G^{De}(u,u') = \{(x,x') \in V(G) : x \in N_{G_1}^{De}(u)\}$ and hence, $deg_G^{De}(u,u') = p_2 deg_{G_1}^{De}(u)$ and if $u \in V_e^2(G_1)$, then $deg_G^{De}(u,u') = p_2 deg_{G_1}^{De}(u) + deg_{\overline{G_2}}(u')$ (note that all the vertices of the copy of G_2 with the projection $u \in V(G_1)$ which are not adjacent to (u,u') have distance two from (u,u')). □

Theorem 3.6 *Let $G = G_1[G_2]$ and $e(v) \neq 1$, $\forall\, v \in V(G_1)$. Then*

$$M_1^{De}(G) = p_2^3 M_1^{De}(G_1) + |V_e^2(G_1)| M_1(\overline{G_2}) + 4p_2\overline{q_2} \sum_{u \in V_e^2(G_1)} deg_{G_1}^{De}(u).$$

Proof By definition, we know that

$$\begin{aligned}
M_1^{De}(G) = & \sum_{(u,u')\in V(G)} \big(deg_G^{De}(u,u')\big)^2 = \sum_{u\in V(G_1)} \sum_{u'\in V(G_2)} \big(deg_G^{De}(u,u')\big)^2 \\
= & \sum_{u\in V_e^2(G_1)} \sum_{u'\in V(G_2)} \big(p_2 deg_{G_1}^{De}(u) + deg_{\overline{G_2}}(u')\big)^2 \\
& + \sum_{u\in V(G_1)-V_e^2(G_1)} \sum_{u'\in V(G_2)} \big(p_2 deg_{G_1}^{De}(u)\big)^2 \\
= & \sum_{u\in V(G_1)} \sum_{u'\in V(G_2)} \big(p_2 deg_{G_1}^{De}(u)\big)^2 + \sum_{u\in V_e^2(G_1)} M_1(\overline{G_2}) \\
& + \sum_{u\in V_e^2(G_1)} \sum_{u'\in V(G_2)} 2p_2 deg_{\overline{G_2}}(u') deg_{G_1}^{De}(u) \\
= & p_2^3 M_1^{De}(G_1) + |V_e^2(G_1)| M_1(\overline{G_2}) + 4p_2\overline{q_2} \sum_{u\in V_e^2(G_1)} deg_{G_1}^{De}(u).
\end{aligned}$$
□

Theorem 3.7 *Let $G = G_1[G_2]$ and $e(v) \neq 1$ or 2, $\forall\, v \in V(G_1)$. Then*

$$M_2^{De}(G) = p_2^4 M_2^{De}(G_1) + p_2^2 q_2 M_1^{De}(G_1).$$

Proof By deifnition, we know that

$$\begin{aligned}
M_2^{De}(G) = & \frac{1}{2} \sum_{(u,u')\in V(G)} deg_G^{De}(u,u') \sum_{(v,v')\in N_G(u,u')} deg_G^{De}(v,v') \\
= & \frac{1}{2} \sum_{u\in V(G_1)} \sum_{u'\in V(G_2)} deg_G^{De}(u,u') \Big[\sum_{v\in N_{G_1}(u)} \sum_{v'\in V(G_2)} deg_G^{De}(v,v') + \sum_{v'\in N_{G_2}(u')} deg_G^{De}(u,v') \Big] \\
= & \frac{1}{2} \sum_{u\in V(G_1)} \sum_{u'\in V(G_2)} p_2 deg_{G_1}^{De}(u) \Big[\sum_{v\in N_{G_1}(u)} \sum_{v'\in V(G_2)} p_2 deg_{G_1}^{De}(v) + \sum_{v'\in N_{G_2}(u')} p_2 deg_{G_1}^{De}(u) \Big] \\
= & p_2^4 M_2^{De}(G_1) + p_2^2 q_2 M_1^{De}(G_1).
\end{aligned}$$

This completes the proof. □

Disjunction and Symmetric Difference. The disjunction $G_1 \vee G_2$ of two graphs G_1 and G_2 with $|V(G_1)| = p_1$, $|E(G_1)| = q_1$ and $|V(G_2)| = p_2$, $|E(G_2)| = q_2$ is the graph with vertex set $V(G_1)\times V(G_2)$ in which (u,u') is adjacent to (v,v') whenever u is adjacent to v in G_1 or u' is adjacent to v' in G_2. So, $|E(G_1\vee G_2)|=q_1p_2^2+q_2p_1^2-2q_1q_2$. The degree of a vertex (u,u') of $G_1\vee G_2$ is as follows:

$$deg_{G_1\vee G_2}(u,u')=p_2deg_{G_1}(u)+p_1deg_{G_2}(u')-deg_{G_1}(u)deg_{G_2}(u').$$

Also, the symmetric difference $G_1\oplus G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1)\times V(G_2)$ in which (u,u') is adjacent to (v,v') whenever u is adjacent to v in G_1 or u' is adjacent to v' in G_2, but not both. From definition one can see that, $|E(G_1\oplus G_2)|=q_1p_2^2+q_2p_1^2-4q_1q_2$. The degree of a vertex (u,u') of $G_1\oplus G_2$ is as follows:

$$deg_{G_1\oplus G_2}(u,u')=p_2deg_{G_1}(u)+p_1deg_{G_2}(u')-2deg_{G_1}(u)deg_{G_2}(u').$$

The distance between any two vertices of a disjunction or a symmetric difference cannot exceed two. Thus, if $e(v)\neq 1$, $\forall\, v\in V(G_1)\cup V(G_2)$, the eccentricity of all vertices is constant and equal to two. We know the following lemma.

Lemma 3.8 *Let G_1 and G_2 be two graphs with $e(v)\neq 1$, $\forall\, v\in V(G_1)\cup V(G_2)$. Then*

(i) $deg^{De}_{G_1\vee G_2}(u,u')=deg_{\overline{G_1\vee G_2}}(u,u')$;
(ii) $deg^{De}_{G_1\oplus G_2}(u,u')=deg_{\overline{G_1\oplus G_2}}(u,u')$.

Theorem 3.9 *Let G_1 and G_2 be two graphs with $e(v)\neq 1$, $\forall\, v\in V(G_1)\cup V(G_2)$. Then*

(i) $M_1^{De}(G_1\vee G_2)=M_1(\overline{G_1\vee G_2})$;
(ii) $M_2^{De}(G_1\vee G_2)=q_{G_1\vee G_2}(p_1p_2-1)^2-(p_1p_2-1)M_1(G_1\vee G_2)+M_2(G_1\vee G_2)$.

Proof The proof is straightforward by Proposition 2.5. □

Theorem 3.10 *Let G_1 and G_2 be any two graphs with $e(v)\neq 1$, $\forall\, v\in V(G_1)\cup V(G_2)$. Then*

(i) $M_1^{De}(G_1\oplus G_2)=M_1(\overline{G_1\oplus G_2})$;
(ii) $M_2^{De}(G_1\oplus G_2)=q_{G_1\oplus G_2}(p_1p_2-1)^2-(p_1p_2-1)M_1(G_1\oplus G_2)+M_2(G_1\oplus G_2)$.

Proof The proof is straightforward by Proposition 2.5. □

Join. The join G_1+G_2 of two graphs G_1 and G_2 with disjoint vertex sets $|V(G_1)|=p_1$, $|V(G_2)|=p_2$ and edge sets $|E(G_1)|=q_1$, $|E(G_2)|=q_2$ is the graph on the vertex set $V(G_1)\cup V(G_2)$ and the edge set $E(G_1)\cup E(G_2)\cup\{u_1u_2: u_1\in V(G_1); u_2\in V(G_2)\}$. Hence, the join of two graphs is obtained by connecting each vertex of one graph to each vertex of the other graph, while keeping all edges of both graphs. The degree of any vertex $u\in G_1+G_2$ is given by

$$deg_{G_1+G_2}(u)=\begin{cases} deg_{G_1}(u)+p_2, & \text{if } u\in V(G_1);\\ deg_{G_2}(u)+p_1, & \text{if } u\in V(G_2).\end{cases}$$

By using the definition of the join graph $G=\sum\limits_{i=1}^{n}G_i$, we get the following lemma.

Lemma 3.11 *Let* $G = \sum_{i=1}^{n} G_i$ *and* $u \in V(G)$. *Then*

$$deg_G^{De}(u) = \begin{cases} |V(G)| - 1, & u \in V_e^1(G_i); \\ p_i - 1 - deg_{G_i}(u), & u \in V(G_i) - V_e^1(G_i), \text{ for } i = 1, 2, \ldots, n. \end{cases}$$

Theorem 3.12 *Let* $G = \sum_{i=1}^{n} G_i$. *Then*

$$M_1^{De}(G) = \big(|V(G)| - 1\big)^2 \sum_{i=1}^{n} |V_e^1(G_i)| + \sum_{i=1}^{n} \Big[M_1(G_i) + p_i\big(p_i - 1\big)^2 - 4q_i(p_i - 1)\Big].$$

Proof By definition,

$$\begin{aligned} M_1^{De}(G) &= \sum_{u \in V(G)} \big[deg_G^{De}(u)\big]^2 = \sum_{i=1}^{n} \sum_{u \in V(G_i)} \big[deg_G^{De}(u)\big]^2 \\ &= \sum_{i=1}^{n} \sum_{u \in V_e^1(G_i)} \big[deg_G^{De}(u)\big]^2 + \sum_{i=1}^{n} \sum_{u \in V(G_i) - V_e^1(G_i)} \big[p_i - 1 - deg_{G_i}(u)\big]^2 \\ &= \big(|V(G)| - 1\big)^2 \sum_{i=1}^{n} |V_e^1(G_i)| + \sum_{i=1}^{n} M_1(\overline{G_i}). \end{aligned}$$

This completes the proof. □

Theorem 3.13 *Let* $G = \sum_{i=1}^{n} G_i$. *Then*

$$\begin{aligned} M_2^{De}(G) = \frac{1}{2}\big(|V(G)| - 1\big) \sum_{i=1}^{n} |V_e^1(G_i)| \Big[(|V(G)| - 1)\big(-1 + \sum_{j=1}^{n} |V_e^1(G_j)|\big) \\ + 2\sum_{j=1}^{n} \big(p_j^2 - p_j - 2q_j\big)\Big] + \sum_{i=1}^{n} \big[q_i(p_i - 1)^2 - (p_i - 1)M_1(G_i) + M_2(G_i)\big] \\ + \sum_{i=1}^{n-1} \big(p_i^2 - p_i - 2q_i\big) \sum_{j=i+1}^{n} \big(p_j^2 - p_j - 2q_j\big). \end{aligned}$$

Proof By definition, we get that

$$M_2^{De}(G) = \sum_{uv \in E(G)} deg_G^{De}(u) deg_G^{De}(v) = \frac{1}{2} \sum_{u \in V(G)} deg_G^{De}(u) \sum_{v \in N_G(u)} deg_G^{De}(v)$$

$$
\begin{aligned}
=&\frac{1}{2}\sum_{i=1}^{n}\sum_{u\in V(G_i)} deg_G^{De}(u)\Big[\sum_{v\in N_{G_i}(u)} deg_G^{De}(v)+\sum_{\substack{j=1\\j\neq i}}^{n}\sum_{v\in V(G_j)} deg_G^{De}(v)\Big]\\
=&\frac{1}{2}\sum_{i=1}^{n}\sum_{u\in V_e^1(G_i)}\big(|V(G)|-1\big)\Big[\big(|V(G)|-1\big)\big(|V_e^1(G_i)|-1\big)+\sum_{v\in V(G_i)-V_e^1(G_i)} deg_{\overline{G_i}}(v)\\
&+\sum_{\substack{j=1\\j\neq i}}^{n}\big[\big(|V(G)|-1\big)|V_e^1(G_j)|+\sum_{v\in V(G_j)-V_e^1(G_j)} deg_{\overline{G_j}}(v)\big]\Big]\\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{u\in V(G_i)-V_e^1(G_i)} deg_{\overline{G_i}}(u)\Big[\big(|V(G)|-1\big)|V_e^1(G_i)|+\sum_{v\in N_{G_i}(u)-V_e^1(G_i)} deg_{\overline{G_i}}(v)\\
&+\sum_{\substack{j=1\\j\neq i}}^{n}\big[\big(|V(G)|-1\big)|V_e^1(G_j)|+\sum_{v\in V(G_j)-V_e^1(G_j)} deg_{\overline{G_j}}(v)\big]\Big]\\
=&\frac{1}{2}\big(|V(G)|-1\big)\sum_{i=1}^{n}|V_e^1(G_i)|\Big[\big(|V(G)|-1\big)\big(-1+\sum_{j=1}^{n}|V_e^1(G_j)|\big)\\
&+\sum_{j=1}^{n}\big(p_j^2-p_j-2q_j\big)\Big]+\frac{1}{2}\sum_{i=1}^{n}\big(p_i^2-p_i-2q_i\big)\Big[\big(|V(G)|-1\big)\sum_{j=1}^{n}|V_e^1(G_j)|\\
&+\sum_{\substack{j=1\\j\neq i}}^{n}\big(p_j^2-p_j-2q_j\big)\Big]+\sum_{i=1}^{n}\big[q_i(p_i-1)^2-(p_i-1)M_1(G_i)+M_2(G_i)\big]\\
=&\frac{1}{2}\big(|V(G)|-1\big)\sum_{i=1}^{n}|V_e^1(G_i)|\Big[\big(|V(G)|-1\big)\big(-1+\sum_{j=1}^{n}|V_e^1(G_j)|\big)\\
&+2\sum_{j=1}^{n}\big(p_j^2-p_j-2q_j\big)\Big]+\sum_{i=1}^{n}\big[q_i(p_i-1)^2-(p_i-1)M_1(G_i)+M_2(G_i)\big]\\
&+\sum_{i=1}^{n-1}\big(p_i^2-p_i-2q_i\big)\sum_{j=i+1}^{n}\big(p_j^2-p_j-2q_j\big).
\end{aligned}
$$

Note that, the equality

$$\frac{1}{2}\sum_{i=1}^{n}\big(p_i^2-p_i-2q_i\big)\sum_{\substack{j=1\\j\neq i}}^{n}\big(p_j^2-p_j-2q_j\big)=\sum_{i=1}^{n-1}\big(p_i^2-p_i-2q_i\big)\sum_{j=i+1}^{n}\big(p_j^2-p_j-2q_j\big),$$

is applied in the previous calculation. □

Corollary 3.14 *If G_i $(i=1,2,\cdots,n)$ has no vertices of full degree ($V_e^1(G_i)=\phi$), then*

(i) $M_1^{De}\big(\sum_{i=1}^{n}G_i\big)=\sum_{i=1}^{n}M_1(\overline{G_i})$;

(ii)
$$
\begin{aligned}
M_2^{De}\big(\sum_{i=1}^{n}G_i\big)=&\sum_{i=1}^{n}\big[q_i(p_i-1)^2-(p_i-1)M_1(G_i)+M_2(G_i)\big]\\
&+\sum_{i=1}^{n-1}\big(p_i^2-p_i-2q_i\big)\sum_{j=i+1}^{n}\big(p_j^2-p_j-2q_j\big).
\end{aligned}
$$

Corona Product. The corona product $G_1 \circ G_2$ of two graphs G_1 and G_2, where $|V(G_1)| = p_1$, $|V(G_2)| = p_2$ and $|E(G_1)| = q_1$, $|E(G_2)| = q_2$ is the graph obtained by taking $|V(G_1)|$ copies of G_2 and joining each vertex of the i-th copy with vertex $u \in V(G_1)$. Obviously, $|V(G_1 \circ G_2)| = p_1(p_2+1)$ and $|E(G_1 \circ G_2)| = q_1 + p_1(q_2+p_2)$. It follows from the definition of the corona product $G_1 \circ G_2$, the degree of each vertex $u \in G_1 \circ G_2$ is given by

$$deg_{G_1 \circ G_2}(u) = \begin{cases} deg_{G_1}(u) + p_2, & \text{if } u \in V(G_1); \\ deg_{G_2}(u) + 1, & \text{if } u \in V(G_2). \end{cases}$$

We therefore know the next lemma.

Lemma 3.15 *Let $G = G_1 \circ G_2$ be a connected graph and let $u \in V(G)$. Then*

$$deg_G^{De}(u) = \begin{cases} p_2 deg_{G_1}^{De}(u), & u \in V(G_1); \\ p_2 deg_{G_1}^{De}(v), & u \in V(G) - V(G_1), \text{ where } v \in V(G_1) \text{ is adjacent to } u. \end{cases}$$

Theorem 3.16 *Let $G = G_1 \circ G_2$ be a connected graph. Then*

(*i*) $M_1^{De}(G) = p_2^2(p_2+1)M_1^{De}(G_1)$;
(*ii*) $M_2^{De}(G) = p_2^2 M_2^{De}(G_1) + p_2^2(q_2+p_2)M_1^{De}(G_1)$.

Proof By definition, calculation shows that

$$\begin{aligned}
(i)\ M_1^{De}(G) &= \sum_{u \in V(G)} \left[deg_G^{De}(u)\right]^2 \\
&= \sum_{u \in V(G_1)} \left[deg_G^{De}(u)\right]^2 + \sum_{v \in V(G_1)} \sum_{u \in V(G_2)} \left[deg_G^{De}(u)\right]^2 \\
&= \sum_{u \in V(G_1)} \left[p_2 deg_{G_1}^{De}(u)\right]^2 + \sum_{v \in V(G_1)} \sum_{u \in V(G_2)} \left[p_2 deg_{G_1}^{De}(v)\right]^2 \\
&= p_2^2 M_1^{De}(G_1) + p_2^3 M_1^{De}(G_1).
\end{aligned}$$

$$\begin{aligned}
(ii)\ M_2^{De}(G) &= \frac{1}{2} \sum_{u \in V(G)} deg^{De}(u) \sum_{v \in N(u)} deg^{De}(v) \\
&= \frac{1}{2} \sum_{u \in V(G_1)} deg_G^{De}(u) \left[\sum_{v \in N_{G_1}(u)} deg_G^{De}(v) + \sum_{v \in V(G_2)} deg_G^{De}(v) \right] \\
&\quad + \frac{1}{2} \sum_{v \in V(G_1)} \sum_{u \in V(G_2)} deg_G^{De}(u) \left[\sum_{w \in N_{G_2}(u)} deg_G^{De}(w) + deg_G^{De}(v) \right] \\
&= \frac{1}{2} \sum_{u \in V(G_1)} p_2 deg_{G_1}^{De}(u) \left[\sum_{v \in N_{G_1}(u)} p_2 deg_{G_1}^{De}(v) + p_2^2 deg_{G_1}^{De}(u) \right] \\
&\quad + \frac{1}{2} \sum_{v \in V(G_1)} \sum_{u \in V(G_2)} p_2 deg_{G_1}^{De}(v) \left[p_2 deg_{G_1}^{De}(v) deg_{G_2}(u) + p_2 deg_{G_1}^{De}(v) \right] \\
&= p_2^2 M_2^{De}(G_1) + p_2^2(q_2+p_2)M_1^{De}(G_1).
\end{aligned}$$

This completes the proof. □

Example 3.17 For any cycle C_{p_1} and any path P_{p_2},

$$(i)\ \ M_1^{De}(C_{p_1} \circ P_{p_2}) = \begin{cases} 4p_1p_2^2(p_2+1), & p_1 \text{ is odd;} \\ p_1p_2^2(p_2+1), & p_1 \text{ is even.} \end{cases}$$

$$(ii)\ M_2^{De}(C_{p_1} \circ P_{p_2}) = \begin{cases} 8p_1p_2^3, & p_1 \text{ is odd;} \\ 2p_1p_2^3, & p_1 \text{ is even.} \end{cases}$$

Example 3.18 For any two cycles C_{p_1} and C_{p_2},

$$(i)\ M_1^{De}(C_{p_1} \circ C_{p_2}) = \begin{cases} 4p_1p_2^2(p_2+1), & p_1 \text{ is odd;} \\ p_1p_2^2(p_2+1), & p_1 \text{ is even.} \end{cases}$$

$$(ii)\ M_2^{De}(C_{p_1} \circ C_{p_2}) = \begin{cases} 4p_1p_2^2(2p_2+1), & p_1 \text{ is odd;} \\ p_1p_2^2(2p_2+1), & p_1 \text{ is even.} \end{cases}$$

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