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A survey on Smarandache notions in number theory II: pseudo-Smarandache function

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Abstract In this paper we give a survey on recent results on pseudo-Smarandache function.

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§1. Definition and simple properties

According to [11], the pseudo-Smarandache function $Z(n)$ is defined by

$$Z(n) = \min\left\{m:\ n\left|\frac{m(m+1)}{2}\right.\right\}.$$

Some elementary properties can be found in [11] and [1].

R. Pinch [20]. *For any given $L > 0$ there are infinitely many values of n such that $\frac{Z(n+1)}{Z(n)} > L$, and there are infinitely many values of n such that $\frac{Z(n-1)}{Z(n)} > L$.*

For any integer $k \geq 2$, the equation $\frac{n}{Z(n)} = k$ has infinitely many solutions n.

The ration $\frac{Z(2n)}{Z(n)}$ is not bounded.

Fix $\frac{1}{2} < \beta < 1$ and integer $t \geq 5$. The number of integers n with $e^{t-1} < n < e^t$ such that $Z(n) < n^\beta$ is at most $196t^2e^{\beta t}$.

The series $\sum_{n=1}^{\infty}\frac{1}{Z(n)^\alpha}$ is convergent for any $\alpha > 1$.

Some explicit expressions of $Z(n)$ for some particular cases of n were given by Abdullah-Al-Kafi Majumdar.

A. A. K. Majumdar [18]. *If $p \geq 5$ is a prime, then*

$$Z(2p) = \begin{cases} p-1, & \text{if } 4 \mid p-1, \\ p, & \text{if } 4 \mid p+1, \end{cases}$$

$$Z(3p) = \begin{cases} p-1, & \text{if } 3 \mid p-1, \\ p, & \text{if } 3 \mid p+1, \end{cases}$$

$$Z(4p) = \begin{cases} p-1, & \textit{if } 8 \mid p-1, \\ p, & \textit{if } 8 \mid p+1, \\ 3p-1, & \textit{if } 8 \mid 3p+1, \\ 3p, & \textit{if } 8 \mid 3p+1, \end{cases}$$

$$Z(6p) = \begin{cases} p-1, & \textit{if } 12 \mid p-1, \\ p, & \textit{if } 12 \mid p+1, \\ 2p-1, & \textit{if } 4 \mid 3p+1, \\ 2p, & \textit{if } 4 \mid 3p-1. \end{cases}$$

A. A. K. Majumdar [18]. *If $p \geq 7$ is a prime, then*

$$Z(5p) = \begin{cases} p-1, & \textit{if } 10 \mid p-1, \\ p, & \textit{if } 10 \mid p+1, \\ 2p-1, & \textit{if } 5 \mid 2p-1, \\ 2p, & \textit{if } 5 \mid 2p+1. \end{cases}$$

If $p \geq 11$ is a prime, then

$$Z(7p) = \begin{cases} p-1, & \textit{if } 7 \mid p-1, \\ p, & \textit{if } 7 \mid p+1, \\ 2p-1, & \textit{if } 7 \mid 2p-1, \\ 2p, & \textit{if } 5 \mid 2p+1, \\ 3p-1, & \textit{if } 7 \mid 3p-1, \\ 3p, & \textit{if } 7 \mid 3p+1. \end{cases}$$

If $p \geq 13$ is a prime, then

$$Z(11p) = \begin{cases} p-1, & \textit{if } 11 \mid p-1, \\ p, & \textit{if } 11 \mid p+1, \\ 2p-1, & \textit{if } 11 \mid 2p-1, \\ 2p, & \textit{if } 11 \mid 2p+1, \\ 3p-1, & \textit{if } 11 \mid 3p-1, \\ 3p, & \textit{if } 11 \mid 3p+1, \\ 4p-1, & \textit{if } 11 \mid 4p-1, \\ 4p, & \textit{if } 11 \mid 4p+1, \\ 5p-1, & \textit{if } 11 \mid 5p-1, \\ 5p, & \textit{if } 11 \mid 5p+1. \end{cases}$$

A. A. K. Majumdar [18]. *Let p and q be two primes with $q > p \geq 5$. Then*

$$Z(pq) = \min\{qy_0 - 1, px_0 - 1\},$$

where

$$\begin{aligned} y_0 &= \min\{y : \ x, y \in \mathbb{N}, qy - px = 1\}, \\ x_0 &= \min\{x : \ x, y \in \mathbb{N}, px - qy = 1\}. \end{aligned}$$

A. A. K. Majumdar [18]. *If $p \geq 3$ is a prime, then $Z(2p^2) = p^2 - 1$. If $p \geq 5$ is a prime, then $Z(3p^2) = p^2 - 1$.*

If $p \geq 3$ is a prime and $k \geq 3$ is an integer, then

$$\begin{aligned} Z(2p^k) &= \begin{cases} p^k, & \text{if } 4 \mid p-1 \text{ and } k \text{ is odd}, \\ p^k - 1, & \text{otherwise}, \end{cases} \\ Z(3p^k) &= \begin{cases} p^k, & \text{if } 3 \mid p+1 \text{ and } k \text{ is odd}, \\ p^k - 1, & \text{otherwise}. \end{cases} \end{aligned}$$

S. Gou and J. Li [2]. *The equation $Z(n) = Z(n+1)$ has no positive integer solutions.*
For any given positive integer M, there exists a positive integer s such that

$$|Z(s) - Z(s+1)| > M.$$

Y. Zheng [29]. *For any given positive integer M, there are infinitely many positive integers n such that*

$$|Z(n+1) - Z(n)| > M.$$

M. Yang [27]. *Suppose that n has primitive roots. Then $Z(n)$ is a primitive root modulo n if and only if $n = 2, 3, 4$.*

W. Lu, L. Gao, H. Hao and X. Wang [17]. *Let $p \geq 17$ be a prime. Then we have*

$$Z\left(2^p + 1\right) \geq 10p, \qquad Z\left(2^p - 1\right) \geq 10p.$$

L. Gao, H. Hao and W. Lu [?]. *Let $p \geq 17$ be a prime, and let a, b be distinct positive integers. Then we have*

$$Z\left(a^p + b^p\right) \geq 10p.$$

Y. Ji [10]. *Let r be a positive integer. Suppose that $r \neq 1, 2, 3, 5$. Then*

$$Z\left(2^r + 1\right) \geq \frac{1}{2}\left(-1 + \sqrt{2^{r+3} \cdot 5 + 41}\right).$$

Assume that $r \neq 1, 2, 4, 12$. *Then*

$$Z\left(2^r - 1\right) \geq \frac{1}{2}\left(-1 + \sqrt{2^{r+3} \cdot 3 - 23}\right).$$

§2. Mean values of the pseudo-Smarandache function

Y. Lou [16]. *For any real* $x > 1$ *we have*

$$\sum_{n \leq x} \ln Z(n) = x \ln x + O(x).$$

W. Huang [9]. *For any integer* $n > 1$ *we have*

$$\frac{\sum_{k=2}^{n} \frac{\ln Z(k)}{\ln k}}{n} = 1 + O\left(\frac{1}{\ln n}\right), \qquad \frac{Z(n)}{\sum_{k \leq n} \ln Z(k)} = O\left(\frac{1}{\ln n}\right).$$

L. Cheng [4]. *Let* $p(n)$ *denote the smallest prime divisor of* n, *and let* k *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n \leq x} \frac{p(n)}{Z(n)} = \frac{x}{\ln x} + \sum_{i=2}^{k} \frac{a_i x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

X. Wang, L. Gao and W. Lu [23]. *Define*

$$\overline{\Omega}(n) = \begin{cases} 0, & \text{if } n = 1, \\ \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_r p_r, & \text{if } n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}. \end{cases}$$

Let $k \geq 2$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n \leq x} Z(n)\overline{\Omega}(n) = \frac{\zeta(3) x^3}{3 \ln x} + \sum_{i=2}^{k} \frac{a_i x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where a_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

H. Hao, L. Gao and W. Lu [8]. *Let* $d(n)$ *denote the divisor function, and let* $k \geq 2$ *be any fixed positive integer. For any real* $x > 1$ *we have*

$$\sum_{n \leq x} Z(n) d(n) = \frac{\pi^4}{36} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{a_i x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where a_i $(i = 2, 3, \cdots, k)$ *are computable constants.*

X. Wang, L. Gao and W. Lu [24]. *Define*

$$D(n) = \min\left\{ m : \ m \in \mathbb{N}, n \mid \prod_{i=1}^{m} d(i) \right\}.$$

Let $k \geq 2$ be any fixed positive integer. For any real $x > 1$ we have

$$\sum_{n \leq x} Z(n) \ln D(n) = \frac{\zeta(3) \ln 2}{3} \cdot \frac{x^3}{\ln x} + \sum_{i=2}^{k} \frac{a_i x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where a_i $(i = 2, 3, \cdots, k)$ are computable constants.

§3. The dual of the pseudo-Smarandache function, the near pseudo-Smarandache function, and other generalizations

According to [21], the dual of the pseudo-Smarandache function is defined by

$$Z_*(n) = \max\left\{m \in \mathbb{N} : \frac{m(m+1)}{2} \mid n\right\}.$$

D. Liu and C. Yang [15]. *Let $\mathcal{A}$ denote the set of simple numbers. For any real $x \geq 1$ we have*

$$\sum_{\substack{n \leq x \\ n \in \mathcal{A}}} Z_*(n) = C_1 \frac{x^2}{\ln x} + C_2 \frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right),$$

where C_1, C_2 are computable constants.

X. Zhu and L. Gao [30]. *We have*

$$\sum_{n=1}^{\infty} \frac{Z_*(n)}{n^\alpha} = \zeta(\alpha) \sum_{m=1}^{\infty} \frac{2m}{m^\alpha (m+1)^{2\alpha}}.$$

The near pseudo Smarandache function $K(n)$ is defined as

$$K(n) = \sum_{i=1}^{n} i + k(n),$$

where $k(n) = \min\left\{k : \ k \in \mathbb{N}, n \mid \sum_{i=1}^{n} i + k\right\}$. Some recurrence formulas satisfied by $K(n)$ were derived in [19].

H. Yang and R. Fu [26]. *For any real $x \geq 1$ we have*

$$\begin{aligned}
\sum_{n \leq x} d\left(K(n) - \frac{n(n+1)}{2}\right) &= \frac{3}{4} x \log x + Ax + O\left(x^{\frac{1}{2}} \log^2 x\right), \\
\sum_{n \leq x} \phi\left(K(n) - \frac{n(n+1)}{2}\right) &= \frac{93}{28\pi^2} x^2 + O\left(x^{\frac{3}{2}+\epsilon}\right),
\end{aligned}$$

where $\phi(n)$ denotes the Euler function, A is a computable constant, and $\epsilon > 0$ is any real number.

Y. Zhang [28]. *For any real number $s > \frac{1}{2}$, the series*

$$\sum_{n=1}^{\infty} \frac{1}{K^s(n)}$$

is convergent, and

$$\sum_{n=1}^{\infty}\frac{1}{K(n)}=\frac{2}{3}\ln 2+\frac{5}{6},\qquad \sum_{n=1}^{\infty}\frac{1}{K^2(n)}=\frac{11}{108}\pi^2-\frac{22+2\ln 2}{27}.$$

Y. Li, R. Fu and X. Li [14]. *We have*

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in\mathcal{A}}}K(n) &= \frac{x^2\ln\ln x}{3\ln x}+B\frac{x^2}{\ln x}+\frac{2x^2\ln\ln x}{9\ln^2 x}+O\left(\frac{x^2}{\ln^2 x}\right),\\ \sum_{\substack{n\le x\\ n\in\mathcal{A}}}\frac{1}{K(n)} &= \frac{2}{3}(\ln\ln x)^2+D\ln\ln x+E+O\left(\frac{\ln\ln x}{\ln x}\right).\end{aligned}$$

L. Gao, R. Xie and Q. Zhao [5]. *Define*

$$p_d(n)=\prod_{d|n}d,\qquad q_d(n)=\prod_{\substack{d|n\\ d<n}}d.$$

For any real $x>1$ we have

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in\mathcal{A}}}K\left(p_d(n)\right) &= \frac{x^5}{5\ln x}\ln\ln x+A_1\frac{x^5}{\ln x}+\frac{x^5}{25\ln^2 x}\ln\ln x+O\left(\frac{x^5}{\ln^2 x}\right),\\ \sum_{\substack{n\le x\\ n\in\mathcal{A}}}K\left(q_d(n)\right) &= \frac{x^3}{3\ln x}\ln\ln x+A_2\frac{x^3}{\ln x}+\frac{x^3}{9\ln^2 x}\ln\ln x+O\left(\frac{x^3}{\ln^2 x}\right),\end{aligned}$$

where A_1, A_2 are computable constants.

Other generalizations on the near pseudo-Smarandache function have been given. For example, define

$$Z_3(n)=\min\left\{m:\ m\in\mathbb{N}, n\mid\frac{m(m+1)(m+2)}{6}\right\}.$$

The elementary properties were studied in [6] and [7].

Y. Wang [25]. *Define*

$$U_t(n)=\min\left\{k:\ 1^t+2^t+\cdots+n^t+k=m,n\mid m,k,t,m\in\mathbb{N}\right\}.$$

For any real number $s>1$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{U_1^s(n)} &= \zeta(s)\left(2-\frac{1}{2^s}\right),\\ \sum_{n=1}^{\infty}\frac{1}{U_2^s(n)} &= \zeta(s)\left(1+\frac{1}{5^s}-\frac{1}{6^s}+2\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\right),\\ \sum_{n=1}^{\infty}\frac{1}{U_3^s(n)} &= \zeta(s)\left(1+\left(1-\frac{1}{2^s}\right)^2\right).\end{aligned}$$

M. Tong [22]. *Define*

$$Z_0(n) = \begin{cases} \min\{m: \ m \in \mathbb{N}, n \mid m(m+1)\}, & \textit{if } 2 \mid n, \\ \min\{m: \ m \in \mathbb{N}, n \mid m^2\}, & \textit{if } 2 \nmid n. \end{cases}$$

For any real $x > 1$, we have

$$\sum_{n \le x} Z_0(2n-1) = \frac{3\zeta(3)}{\pi^2} x^2 + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

X. Li [12]. *Define*

$$C(n) = \min\left\{a + b: \ a, b \in \mathbb{N}, n \mid \frac{a(a+1)}{2} + b\right\}.$$

For any real $x > 1$, we have

$$\begin{aligned} \sum_{n \le x} C(n) &= \sqrt{2} x^{\frac{3}{2}} + O(x), \\ \sum_{n \le x} \frac{1}{C(n)} &= \ln 2 \cdot \sqrt{2x} + O(\ln x), \\ \sum_{n \le x} d(C(n)) &= \frac{1}{2} x \ln x + x\left(2\gamma + \frac{5}{2}\ln 2 - \frac{3}{2}\right) + O\left(x^{\frac{3}{4}}\right), \end{aligned}$$

where γ is the Euler constant.

Y. Li [13]. *Define*

$$D(n) = \max\left\{ab: \ a, b \in \mathbb{N}, n = \frac{a(a+1)}{2} + b\right\}.$$

For any real $x > 1$, we have

$$\begin{aligned} \sum_{n \le x} D(n) &= \frac{4\sqrt{6}}{45} x^{\frac{5}{2}} + O\left(x^2\right), \\ \sum_{n \le x} \frac{C(n)}{D(n)} &= \frac{9\sqrt{3}}{4} \ln x + O(1). \end{aligned}$$

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