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HISTORY OF THE SMARANDACHE FUNCTION

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1 Introduction

This function is originated from the Romanian professor Florentin Smarandache. It is defined as follows:

For any non-null integer n, $S(n) = \min\{m \mid m! \text{ is divisible by } n\}$.
So we have $S(1) = 0$, $S(2^5) = S(2^6) = S(2^7) = 8$.
If

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \cdots \cdot p_t^{\alpha_t} \tag{1}$$

is the decomposition of n into primes, then

$$S(n) = \max S\left(p_i^{\alpha_i}\right) \tag{2}$$

and moreover, if $[m, n]$ is the smallest common multiple of m and n then

$$S([m, n]) = \max\{S(m), S(n)\} \tag{3}$$

Let us observe that if $\wedge = \min$, $\vee = \max$, $\underset{d}{\wedge} =$ the greatest common divizor, $\overset{d}{\vee} =$ the smallest common multiple then S is a function from the lattice $\left(\mathbf{N}^*, \underset{d}{\wedge}, \overset{d}{\vee}\right)$ into the lattice $(\mathbf{N}, \wedge, \vee)$ for which

$$S\left(\overset{d}{\bigvee_{i=\overline{1,s}}} m_i\right) = \bigvee_{i=\overline{1,s}} S(m_i) \tag{4}$$

2 The calculus of S(n)

From (2) it results that to calculate $S(n)$ is necessary and sufficient to know $S\left(p_i^{\alpha_i}\right)$. For this let p be an arbitrary prime number and

$$a_n(p) = \frac{p^n-1}{p-1} \qquad b_n(p) = p^n \tag{5}$$

If we consider the usual numerical scale

$$(p):\ b_0(p), b_1(p), \ldots, b_k(p), \ldots$$

and the generalised numerical scale

$$[p]:\ a_1(p), a_2(p), \ldots, a_n(p), \ldots$$

then from the Legendre's formula

$$\alpha! = \prod_{p_i \leq \alpha} p_i^{E_{p_i}(\alpha)} \tag{6}$$

where $E_p(\alpha) = \sum\limits_{j\geq 1} \left[\frac{\alpha}{p^j}\right]$ it results that

$$S\left(p^{a_n(p)}\right) = b_n(p)$$

and even that: if

$$\alpha = k_\nu a_\nu(p) + k_{\nu-1}a_{\nu-1}(p) + \ldots + k_1 a_1(p) = \overline{k_\nu k_{\nu-1}\ldots k_1}_{[p]} \tag{7}$$

is the expression of α in the generalised scale $[p]$ then

$$S\left(p^\alpha\right) = k_\nu p^\nu + k_{\nu-1}p^{\nu-1} + \ldots + k_1 p \tag{8}$$

The right hand in (8) may be written as $p\left(\alpha_{[p]}\right)_{(p)}$. That is $S\left(p^\alpha\right)$ is the number obtained multiplying by p the exponent α written in the scale $[p]$ and "read" it in the scale (p). So, we have

$$S\left(p^\alpha\right) = p\left(\alpha_{[p]}\right)_{(p)} \tag{9}$$

For example to calculate $S\left(3^{100}\right)$ we write the exponent $\alpha = 100$ in the scale

$$[3]:\ 1,\ 4,\ 13,\ 40,\ 121,\ \ldots$$

We have $a_\nu(p) \le p \Leftrightarrow (p^\nu - 1)/(p-1) \le \alpha \Leftrightarrow \nu \le \log_p((p-1)\alpha + 1)$ and so ν is the integer part of $\log_p((p-1)\alpha+1)$,

$$\nu = [\log_p((p-1)\alpha + 1)]$$

For our example $\nu = [\log_3 201] = 4$. Then the first difit of $\alpha_{[3]}$ is $k_4 = [\alpha/a_4(3)] = 2$. So $100 = 2a_4(3) + 20$. For $\alpha_1 = 20$ it results $\nu_1 = [\log_3 41] = 3$ and $k_{\nu_1} = [20/a_3(3)] = 1$ so $20 = a_3(3) + 7$ and we obtain $100_{[3]} = 2a_4(3) + a_3(3) + a_2(3) + 3 = 2113_{[3]}$.
From (8) it results $S(3^{100}) = 3(2113)_{(3)} = 207$.
Indeed, from the Legendre's formula it results that the exponent of the prime p in the decomposition of $\alpha!$ is $\sum\limits_{j\ge1}\left[\frac{\alpha}{p^j}\right]$, so the exponent of 3 in the decomposition of $207!$ is $\sum\limits_{j\ge1}\left[\frac{207}{3^j}\right] = 69 + 23 + 7 + 2 = 101$ and the exponent of 3 in the decomposition of $206!$ is 99.
Let us observe that, as it is shown in [1], the calculus in the generalised scale $[p]$ is essentially different from the calculus in the standard scale (p), because

$$a_{n+1}(p) = pa_n(p) + 1 \text{ and } b_{n+1}(p) = pb_n(p)$$

Other formulae for the calculus of $S(p^\alpha)$ have been proved in [2] and [3].
If we note $S_p(\alpha) = S(p^\alpha)$ then it results [2] that

$$S_p(\alpha) = (p-1)\alpha + \sigma_{[p]}(\alpha) \tag{10}$$

where $\sigma_{[p]}(\alpha)$ is the sum of the digits of α written in the scale $[p]$

$$\sigma_{[p]}(\alpha) = k_\nu + k_{\nu-1} + \cdots + k_1$$

and also

$$S_p(\alpha) = \frac{(p-1)^2}{p}\left(E_p(\alpha) + \alpha\right) + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha)$$

where $\sigma_{(p)}(\alpha)$ is the sum of digits of α written in the scale (p), or

$$S_p(\alpha) = p\left(\alpha - \left[\frac{\alpha}{p}\right] + \left[\frac{\sigma_{[p]}(\alpha)}{p}\right]\right)$$

As a direct application of the equalities (2) and (8) in [16] is solved the following problem:

"Which are the numbers with the factorial ending in 1000 zeros ?"

The solution is
$S(10^{1000}) = S(2^{1000}5^{1000}) = \max\{S(2^{1000}), S(5^{1000})\} =$
$= \max\left\{2\left(1000_{[2]}\right)_{(2)}, 5\left(1000_{[5]}\right)_{(5)}\right\} = 4005.$ 4005 is the smallest natural number with the asked propriety.
4006, 4007, 4008, and 4009 verify the proprety but 4010 does not, because $4010! = 4009! \cdot 4010$ has 1001 zeros.
In [11] it presents an another calculus formula of $S(n)$:

$$S(n) = n + 1 - \left[\sum_{k=1}^{n} n^{-\left(n\sin\left(k!\frac{\pi}{n}\right)\right)^2}\right]$$

3 Solved and unsolved problems concerning the Smarandache Function

In [16] there are proposed many problems on the Smarandache Function.
M. Mudge in [12] discuses some of these problems. Many of them are unsolved until now. For example:
Problem (i) : Investigate those sets of consecutive integers $i, i+1, i+2, \ldots, i+x$ for which S generates a monotonic increasing (or indeed monotonic decreasing) sequence. (Note: For $1, 2, 3, 4, 5$, S generates the monotonic increasing sequence $0, 2, 3, 4, 5$).
Problem (ii) : Find the smallest integer k for which it is true that for all n less than some given n_0 at least one of $S(n)$, $S(n+1)$, $\ldots$, $S(n-k+1)$ is
(A) a perfect square
(B) a divisor of k^n
(C) a factorial of a positive integer
Conjecture what happens to k as n_0 tends to infinity.
Problem (iii) : Construct prime numbers of the form $\overline{S(n)S(n+1)\ldots S(n+k)}$.
For example $\overline{S(2)S(3)} = 23$ is prime, and $\overline{S(14)S(15)S(16)S(17)} = 75617$ also prime.
The first order forward finite differences of the Smarandache function are defined thus:
$D_s(x) = |S(x+1) - S(x)|$
$D_s^{(k)}(x) = D(D(\ldots$ k times $D_s(x)\ldots)$
Problem (iv) : Investigate the conjecture that $D_s^{(k)}(1) = 1$ or 0 for all k

greater than or equal to 2.

J. Duncan in [7] has proved that for the first 32000 natural numbers the conjecture is true.

J. Rodriguez in [14] poses the question than if it is possible to construct an increasing sequence of any (finite) length whose Smarandache values are strictly decreasing. P. Gronas in [9] and K. Khan in [10] give different solution to this question.

T. Yau in [17] ask the question that:

For any triplets of consecutive positive integers, do the values of S satisfy the Fibonacci relationship $S(n) + S(n+1) = S(n+2)$?

Checking the first 1200 positive integers the author founds just two triplets for which this holds:

$S(9) + S(10) = S(11)\,, \qquad S(119) + S(120) = S(121)\,.$

That is $S(11-2) + S(11-1) = S(11)$ and $S(11^2-2) + S(11^2-1) = S(11^2)$ but we observe that $S(11^3-2) + S(11^3-1) \neq S(11^3)$.

More recently Ch. Ashbacher has anounced that for n between 1200 and 1000000 there exists the following triplets satisfying the Fibonacci relationship:

$S(4900) + S(4901) = S(44902)$; $\quad S(26243) + S(26244) = S(26245)$;
$S(32110) + S(32111) = S(32112)$; $\quad S(64008) + S(64009) = S(64010)$;
$S(368138) + S(368139) = S(368140)$; $\quad S(415662) + S(415663) = S(415664)$;
but it is not known if there exists an infinity family of solutions.

The function $C_s\colon \mathbf{N}^* \mapsto \mathbf{Q}\,,\ C_s(n) = \frac{1}{n}\left(S(1) + S(2) + \cdots + S(n)\right)$ is the sum of Cesaro concerning the function S.

Problem (v) : Is there $\sum\limits_{n\geq 1} C_s^{-1}(n)$ a convergent series? Find the smallest k for which $\left(\underbrace{C_s \circ C_s \circ \cdots \circ C_s}_{k \text{ times}}\right)(m) \geq n\,.$

Problem (vi) : Study the function $S^{-1}_{\min}\colon \mathbf{N}\setminus\{1\} \mapsto \mathbf{N}\,,\ S^{-1}_{\min}(n) = \min S^{-1}(n)\,,$ where $S^{-1}(n) = \{m \in \mathbf{N} | S(m) = n\}\,.$

M. Costewitz in [6] has investigated the problem to find the cardinal of $S^{-1}(n)$.

In [2] it is shown that if for n we consider the standard decomposition (1) and $q_1 < q_2 < \cdots < q_s < n$ are the primes so that $p_i \neq q_j$, $i = \overline{1,t}$, $j = \overline{1,s}$, then if we note $e_i = E_{p_i}(n)$, $f_k = E_{q_k}(n)$ and $\hat{n} = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$, $\hat{n}_0 = \hat{n}/n$,

$q = q_1^{f_1} q_2^{f_2} \cdots q_s^{f_s}$, it result

$$\text{card } S^{-1}(n) = (d(\hat{n}) - d(\hat{n}_0))\, d(q) \tag{11}$$

where $d(r)$ is the number of divisors of r.
The generating function $F_S \colon \mathbf{N}^* \mapsto \mathbf{N}$ associated to S is defined by $F_S(n) = \sum_{d/n} S(d)$. For example $F_S(18) = S(1) + S(2) + S(3) + S(6) + S(9) + S(18) = 20$.
P. Gronas in [8] has proved that the solution of the diophantine equation $F_S(n) = n$ have the solution $n \in \{9, 16, 24\}$ or n prime.
In [11] is investigated the generating function for $n = p^\alpha$. It is shown that

$$F_S(p^\alpha) = (p-1)\frac{\alpha(\alpha+1)}{2} + \sum_{j=1}^{\alpha} \sigma_{[p]}(j) \tag{12}$$

and it is given an algorithm to calculate the sum in the right hand of (12). Also it is proved that $F_S(p_1 p_2 \cdots p_t) = \sum_{i=1}^{t} 2^{i-1} p_i$. Diophantine equations are given in [14] (see also [12]).
We mentione the followings:
(a) $S(x) = S(x+1)$ conjectured to have no solution
(b) $S(mx+n) = x$
(c) $S(mx+n) = m + nx$
(d) $S(mx+n) = x!$
(e) $S(x^m) = x^n$
(f) $S(x) + y = x + S(y)$, x and y not prime
(g) $S(x+y) = S(x) + S(y)$
(h) $S(x+y) = S(x)S(y)$
(i) $S(xy) = S(x)S(y)$
In [1] it is shown that the equation (f) has as solution every pair of composite numbers $x = p(1+q)$, $y = q(1+p)$, where p and q are consecutive primes, and that the equation (i) has no solutions $x, y > 1$.
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4 Generalizations of the Smarandache Function

In [4] are given three generalizations of the Smarandache Function, namely the Smarandache functions of the first kind are the functions $S_n : \mathbf{N}^* \mapsto \mathbf{N}^*$ defined as follows:
(i) if $n = u^i$ ($u = 1$ or $u = p$, prime number) then $S_n(a)$ is the smallest positive integer k with the property that $k!$ is a multiple of n^a.
(ii) if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ then $S_n(a) = \max_{1 \le j \le t} S_{p_j^{\alpha_j}}(a)$.
If $n = p$ then S_n is the function S_p defined by F. Smarandache in [15] ($S_p(a)$ is the smallest positive integer k such that $k!$ is divisible by p^n).
The Smarandache function of the second kind $S^k : \mathbf{N}^* \mapsto \mathbf{N}^*$ are defined by $S^k(n) = S_n(k)$, $k \in \mathbf{N}^*$.
For $k = 1$, the function S^k is the Smarandache function, with the modification that $S(1) = 1$.
If (a): $1 = a_1, a_2, \ldots, a_n, \ldots$
(b): $1 = b_1, b_2, \ldots, b_n, \ldots$
are two sequences with the property that

$$a_{kn} = a_k a_n \quad ; \quad b_{kn} = b_k b_n$$

Let $f_a^b : \mathbf{N}^* \mapsto \mathbf{N}^*$ be the function defined by $f_a^b(n) = S_{a_n}(b_n)$, (S_{a_n} is the Smarandache function of the first kind).
It is easy to see that:
(i) if $a_n = 1$ and $b_n = n$ for every $n \in \mathbf{N}^*$, then $f_a^b = S_1$.
(ii) if $a_n = n$ and $b_n = 1$ for every $n \in \mathbf{N}^*$, then $f_a^b = S^1$.
The Smarandache functions the third kind are functions $S_a^b = f_a^b$ in the case that the sequences (a) and (b) are different from those concerned in the situations (i) and (ii) from above.
In [4] it is proved that

$$S_n(a+b) \le S_n(a) + S_n(b) \le S_n(a) S_n(b) \text{ for } n > 1$$

$$\max\left\{S^k(a), S^k(b)\right\} \le S^k(ab) \le S^k(a) + S^k(b) \text{ for every } a, b \in \mathbf{N}^*$$

$$\max\left\{f_a^b(k), f_a^b(n)\right\} \le f_a^b(kn) \le b_n f_a^b(k) + b_k f_a^b(n)$$

so, for $a_n = b_n = n$ it results

$$\max\left\{S_k(k), S_n(n)\right\} \le S_{kn}(kn) \le n S_k(k) + k S_n(n) \text{ for every } k, n \in \mathbf{N}^*.$$

This relation is equivalent with the following relation written by means of the Smarandache function:

$$\max\left\{S(k^k), S(n^n)\right\} \le S\left((kn)^{kn}\right) \le nS(k^k) + kS(n^n)$$

In [5] it is presents an other generalization of the Smarandache function. Let $\mathcal{M} = \{S_m(n) | n, m \in \mathrm{N}^*\}$, let $A, B \in \mathcal{P}(\mathrm{N}^*) \setminus \emptyset$ and $a = \min A$, $b = \min B$, $a^* = \max A$, $b^* = \max B$. The set I is the set of the functions $I_A^B : \mathrm{N}^* \mapsto \mathcal{M}$ with

$$I_A^B(n) = \begin{cases} S_a(b) & , \quad \text{if } n < \max\{a, b\} \\ S_{a_k}(b_k) & , \quad \text{if } \max\{a, b\} \le n \le \max\{a^*, b^*\} \\ & \quad \text{where} \\ & \quad a_k = \max\limits_i \{a_i \in A | a_i \le n\} \\ & \quad b_k = \max\limits_j \{b_j \in B | b_j \le n\} \\ S_{a^*}(b^*) & , \quad \text{if } n > \max\{a^*, b^*\} \end{cases}$$

Let the rule $\top : I \times I \mapsto I$, $I_A^B \top I_C^D = I_{A \cup C}^{B \cup D}$ and the partial order relation $\rho \subset I \times I$, $I_A^B \rho I_C^D \Leftrightarrow A \subset C$ and $B \subset D$.
It is easy to see that $(I, \top, \rho)$ is a semilattice.
The elements u, $v \in I$ are ρ-strictly preceded by w if:
(i) $w \,\rho\, u$ and $w \,\rho\, v$
(ii) $\forall x \in I \setminus \{w\}$ so that $x \,\rho\, u$ and $x \,\rho\, v \Rightarrow x \,\rho\, w$.
Let $I^{\#} = \{(u, v) \in I \times I | u, v$ are ρ-strictly preceded $\}$, the rule $\perp : I^{\#} \mapsto I$, $I_A^B \perp I_C^D = I_{A \cap C}^{B \cap D}$ and the order partial relation r, $I_A^B r I_C^D \Leftrightarrow I_C^D \rho I_A^B$. Then the structure $(I^{\#}, \perp, r)$ is called the return of semilattice $(I, \top, \rho)$.

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