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On certain new inequalities and limits for the Smarandache function

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I. Inequalities

1) If $n \geq 4$ is an even number, then $S(n) \leq \frac{n}{2}$.

—Indeed, $\frac{n}{2}$ is integer, $\frac{n}{2} > 2$, so in $(\frac{n}{2})! = 1 \cdot 2 \cdot 3 \cdots \frac{n}{2}$ we can simplify with 2, so $n \,|\, (\frac{n}{2})!$.

This simplies clearly that $S(n) \leq \frac{n}{2}$.

2) If $n > 4$ is an even number, then $S(n^2) \leq n$

—By $n! = 1 \cdot 2 \cdot 3 \cdots \frac{n}{2} \cdots n$, since we can simplify with 2, for $n > 4$ we get that $n^2 | n!$. This clearly implies the above stated inequality. For factonials, the above inequality can be much improved, namely one has:

3) $S\left((m!)^2\right) \leq 2m$ and more generally, $S\left((m!)^n\right) \leq n \cdot m$ for all positive integers m and n.

—First remark that $\frac{(mn)!}{(m!)^n} = \frac{(mn)!}{m!(mn-m)!} \cdot \frac{(mn-m)!}{m!(mn-2m)!} \cdots \frac{(2m)!}{m! \cdot m!} =$

$= C_{2m}^{m} \cdot C_{3m}^{m} \cdots C_{nm}^{m}$, where $C_n^k = \binom{n}{k}$ denotes a binomial coefficient. Thus $(m!)^n$ divides $(mn)!$, implying the stated inequality. For $n = 2$ one obtains the first part.

4) Let $n > 1$. Then $S\left((n!)^{(n-1)!}\right) \leq n!$

—We will use the well-known result that the product of n consecutive integers is divisible by $n!$. By $(n!)! = 1 \cdot 2 \cdot 3 \cdots n \cdot \left((n+1)(n+2) \cdots 2n\right) \cdots \left((n-1)!-1\right) \cdots (n-1)!\, n$

each group is divisible by $n!$, and there are $(n-1)!$ groups, so $(n!)^{(n-1)!}$ divides $(n!)!$. This gives the stated inequality.

5) For all m and n one has $[S(m), S(n)] \leq S(m \cdot S(n) \leq [m, n]$. where $[a, b]$ denotes the

$\ell \cdot c \cdot m$ of a and b.

—If $m = \prod_{Pi}^{ai}$, $n = \prod q_j^{bj}$ are the canonical representations of m, resp. n, then it is well-known that $S(m) = S\left(_{Pi}^{ai}\right)$ and $S(n) = S(q_j^{bj})$, where $S\left(_{Pi}^{ai}\right) = \max\left\{S\left(_{Pi}^{ai}\right) : i = 1, \cdots, r\right\}$; $S(q_j^{bj}) = \max\left\{S(q_j^{bj}) : j = 1, \cdots, h\right\}$, with r and h the number of prime divisors of m, resp. n. Then clearly $[S(m), S(n)] \leq S(m) \cdot S(n) \leq {}_{Pi}{}^{ai} \cdot q_j^{bj} \leq [m, n]$

6) $\left(S(m), S(n)\right) \geq \frac{S(m) \cdot S(n)}{mn} \cdot (m, n)$ for all m and n

—Since $\left(S(m), S(m)\right) = \frac{S(m) \cdot S(n)}{[S(m), S(n)]} \geq \frac{S(m) \cdot S(n)}{[m, n]} = \frac{S(m) \cdot S(n)}{nm} \cdot (m, n)$

by 5) and the known formula $[m,n] = \frac{mn}{(m, n)}$.

7) $\frac{\left(S(m), S(n)\right)}{(m, n)} \geq \left(\frac{S(mn)}{mn}\right)^2$ for all m and n

—Since $S(mn) \leq m\, S(n)$ and $S(mn) \leq n\, S(m)$ (See [1]), we have $\left(\frac{S(mn)}{mn}\right)^2 \leq \frac{S(m)\, S(n)}{mn}$, and the result follows by 6).

8) We have $\left(\frac{S(mn)}{mn}\right)^2 \leq \frac{S(m)\, S(n)}{mn} \leq \frac{1}{(mn)}$

—This follows by 7) and the stronger inequality from 6), namely $S(m)\, S(n) \leq [m\, n] = \frac{mn}{(m, n)}$

Corollary $S(mn) \leq \frac{mn}{\sqrt{mn}}$

9) Max $\left\{S(m), S(n)\right\} \geq \frac{S(mn)}{(mn)}$ for all m, n; where (m,n) denotes the $g \cdot c \cdot d$ of m and n.

—We apply the known result: max $\left\{S(m), S(n)\right\} = S([m, n])$ On the other hand, since

[m, n] | m · n, by Corollary 1 from our paper [1] we get $\frac{S(mn)}{mn} \leq \frac{S([m, n])}{[m, n]}$.

Since $[m, n] = \frac{mn}{(m, n)}$,

The result follows:

Remark. Inequality g) compliments Theorem 3 from [1],

namely that max $\left\{S(m), S(n)\right\} \leq S(mn)$.

10) Let $d(n)$ be the number of divisors of n. Then $\frac{S(n!)}{n!} \leq \frac{S\left(n^{d(n)/2}\right)}{n^{d(n)/2}}$

—We will use the known relation $\prod_{k|n} k = n^{d(n)/2}$, where the product is extended over all divisors k of n. Since this product divides $\prod_{k \leq n} k = n!$, by Corollary 1 from [1] we can write

$\frac{S(n!)}{n!} \leq \frac{S(\prod_{k/n} k)}{\prod_{k/n} k}$, which gives the desired result.

Remark If n is of the form m^2, then $d(n)$ is odd, but otherwise $d(n)$ is even. So, in each case $n^{d(n)/2}$ is a positive integer.

11) For infinitely many n we have $S(n+1) < S(n)$, but for infinitely many m one has

$S(m+1) > S(m)$.

—This is a simple application of 1). Indeed, let $n = p - 1$, where $p \geq 5$ is a prime. Then, by 1) we have $S(n) = S(p-1) \leq \frac{P-1}{2} < p$. Since $p = S(p)$, we have $S(p-1) < S(p)$. Let in the same manner $n = p + 1$. Then, as above, $S(p+1) \leq \frac{P+1}{2} < p = S(p)$.

12) Let p be a prime. Then $S(p!+1) > S(p!)$ and $S(p!-1) > S(p!)$

—Clearly, $S(p!) = p$. Let $p!+1 = \prod q_j^{\partial j}$ be the prime factorization of $p!+1$. Here each $q_j > p$, thus $S(p!+1) = S(q_j^{\partial j})$ (for certain j) $\geq S(p^{\partial j}) \geq S(p) = p$. The same proof applies to the case $p!-1$.

Remark: This offers a new proof for M).

13) Let P_k be the kth prime number. Then $S(p_1 p_2 \ldots P_k + 1) > S(p_1 p_2 \cdots P_k)$ and

$S(p_1 p_2 \ldots P_k - 1) > S(p_1 p_2 \cdots P_k)$

—Almost the same proof as in 12) is valid, by remarking that $S(p_1 p_2 \cdots P_k) = P_k$ (since $p_1 < p_2 < \cdots < p_k$).

14) For infinitely many n one has $\left(S(n)^2\right) < S(n-1) \cdot S(n+1)$ and for infinitely many m, $\left(S(m)\right)^2 > S(m-1) \cdot S(m+1)$.

—By $S(p+1) < p$ and $S(p-1) < p$ (See the proof in 11) we have

$\frac{S(p+1)}{S(p)} < \frac{S(p)}{S(p)} < \frac{S(p)}{S(p-1)}$. Thus $\left(S(p)\right)^2 > S(p-1) \cdot S(p+1)$.

On the other hand, by putting $x_n = \frac{S(n+1)}{S(n)}$, we shall see in part II, that $\limsup_{n \to \infty} x_n = +\infty$. Thus $x_{n-1} < x_n$ for infinitely many n, giving

$\left(S(n)\right)^2 < S(n-1) \cdot S(n+1)$.

II. <u>Limits:</u>

1) $\liminf_{n \to \infty} \frac{S(n)}{n} = 0$ and $\limsup_{n \to \infty} \frac{S(n)}{n} = 1$

—Clearly, $\frac{S(n)}{n} > 0$. Let $n = 2^m$. Then, since $S(2^m) \leq 2m$, and $\lim_{m \to \infty} \frac{2m}{2m} = 0$, we have $\lim_{m \to \infty} \frac{S(2^m)}{2^m} = 0$, proving the first part. On the other hand, it is well known that $\frac{S(n)}{n} \leq 1$. For $n = p_k$ (the kth prime), one has $\frac{S(p_k)}{p_k} = 1 \to 1$ as $k \to \infty$, proving the second part.

<u>Remark:</u> With the same proof, we can derive that $\liminf_{n \to \infty} \frac{S(n^r)}{n} = 0$ for all integers r.

—As above $S(2^{kr}) \leq 2kr$, and $\frac{2kr}{2^k} \to 0$ as $k \to \infty$ (r fixed), which gives the result.

2) $\liminf_{n \to \infty} \frac{S(n+1)}{S(n)} = 0$ and $\limsup_{n \to \infty} \frac{S(n+1)}{S(n)} = +\infty$

—Let p_r denote the rth prime. Since $(p_\Lambda ... p_r, 1) = 1$, Dirichlet's theorem on arithmetical progressions assures the existence of a prime p of the form $p = a \cdot p_\Lambda ... p_r - 1$.

Then $S(p+1) = S(ap_\Lambda \cdots p_r) \leq a \cdot S(p_\Lambda \cdots p_r)$ *by* $S(mn) \leq mS(n)$ (see [1])

But $S(p_\Lambda \cdots p_r) = max\{p_\Lambda, \cdots, p_r\} = p_r$. Thus $\frac{S(p+1)}{S(p)} \leq \frac{ap_r}{ap_\Lambda \cdots p_r - 1} \leq \frac{p_r}{p_\Lambda \cdots p_r - 1} \to 0$ as $r \to \infty$. This gives the first part.

Let now p be a prime of the form $p = bp_\Lambda \cdots p_r + 1$.

Then $S(p-1) = S(bp_\Lambda \cdots p_r) \leq b\, S(p_\Lambda \cdots p_r) = b \cdot p_r$,

and $\frac{S(p-1)}{S(p)} \leq \frac{bp_r}{bp_1 \cdots p_r + 1} \leq \frac{p_r}{p_\Lambda \cdots p_r} \to 0$ as $r \to \infty$.

3) $\liminf_{n\to\infty} \left[S(n+1) - S(n)\right] = -\infty$ and $\limsup_{m\to\infty} \left[S(n+1) - S(n)\right] = +\infty$

—We have $S(p+1) - S/p) \leq \frac{p+1}{2} - p = \frac{-p+1}{2} \to -\infty$ for an odd prime

p (see 1) and 11)$\big)$. On the other hand, $S(p) - S(p-1) \geq p - \frac{p-1}{2} = \frac{p+1}{2} \to \infty$

(Here $S(p) = p$), where $p-1$ is odd for $p \geq 5$. This finishes the proof.

4) Let $\sigma(n)$ denotes the sum of divisors of n. Then $\liminf_{n\to\infty} \frac{S(\sigma(n))}{n} = 0$

—This follows by the argument of 2) for $n = p$. Then $\sigma(\varphi) = p + 1$ and $\frac{S(p+1)}{p} \to 0$, where

$\{p\}$ is the sequence constructed there.

5) Let $\varphi(n)$ be the Enter totient function. Then $\liminf_{n\to\infty} \frac{S(\varphi(n))}{n} = 0$

—Let the set of primes $\{p\}$ be defined as in 2). Since $\varphi(n) = p - 1$ and $\frac{S(p-1)}{p} = \frac{S(p-1)}{S(p)} \to 0$,

the assertion is proved. The same result could be obtained by taking $n = z^k$. Then, since

$\varphi(2^k) = 2^{k-1}$, and $\frac{S(2^{k-1})}{2^k} \leq \frac{2\cdot(k-1)}{2^k} \to o$ as $k \to \infty$, the assertion follows:

6) $\liminf_{n\to\infty} \frac{S(S(n))}{n} = 0$ and $\max_{n\in\aleph} \frac{S(S(n))}{n} = 1$.

—Let $n = p!$ (p prime). Then, since $S(p!) = p$ and $S(p) = p$, from $\frac{p}{p!} \to 0$ $(p \to \infty)$

we get the first result. Now, clearly $\frac{S(S(n))}{n} \leq \frac{S(n)}{n} \leq 1$. By letting $n = p$ (prime), clearly

one has $\frac{S(S(p))}{p} = 1$, which shows the second relation.

7) $\liminf_{n\to\infty} \frac{\sigma(S(n))}{S(n)} = 1$.

—Clearly, $\frac{\sigma(k)}{k} > 1$. On the other hand, for $n = p$ (prime), $\frac{\sigma\left(S(p)\right)}{S(p)} = \frac{p+1}{p} \rightarrow 1$ as $p \rightarrow \infty$.

8) Let Q(n) denote the greatest prime power divisor of n. Then $\liminf_{n\rightarrow\infty} \frac{\varphi\left(S(n)\right)}{\partial(n)} = 0$.

—Let $n = p_1^k \cdots p_r^k$ ($k > 1$, fixed). Then, clearly $\partial(n) = p_r^k$.

By $S(n) = S(p_r^k)$ $\left(\text{since } S(p_r^k) > S(p_i^k) \text{ for } i < k\right)$ and $S(p_r^k) = j \cdot p_r$, with $j \leq k$ (which is known) and by $\varphi\ (\ j\ p_k) \leq j \cdot \varphi(p_r)\ \leq k(p_r - 1)$, we get $\frac{\varphi\left(S(n)\right)}{\partial(n)} \leq \frac{k\cdot(p_r-1)}{p_r^{\ k}} \rightarrow 0$ as $r \rightarrow \infty$ (k fixed).

9) $\lim_{\substack{m\rightarrow\infty \\ m\ even}} \frac{S(m^2)}{m^2} = 0$

—By 2) we have $\frac{S(m^2)}{m^2} \leq \frac{1}{m}$ for $m > 4$, even. This clearly inplies the above remark.

Remark. It is known that $\frac{S(m)}{m} \leq \frac{2}{3}$ if $m \neq 4$ is composite. From $\frac{S(m^2)}{m^2} \leq \frac{1}{m} < \frac{2}{3}$ for $m > 4$, for the composite numbers of the perfect squares we have a very strong improvement.

10) $\liminf_{n\rightarrow\infty} \frac{\sigma\left(S(n)\right)}{n} = 0$

—By $\sigma(\mathrm{n}) = \overline{Z}_{d/n}\, d = n\overline{Z}_{d/n} \frac{1}{d} \leq n\overline{Z}_{d=1}^{n} \frac{1}{d} < n \cdot (2\ log\, n)$, we get $\sigma(n) < 2n \log n$ for $n > 1$. Thus $\frac{\sigma\left(S(n)\right)}{n} < \frac{2\, S(n)\, log\, S(n)}{n}$. For $n = 2^k$ we have $S(2^k) \leq 2k$, and since $\frac{4k\ log\, 2k}{2^k} \rightarrow 0$ $(k \rightarrow \infty)$, the result follows.

11) $\lim_{n\rightarrow\infty} \sqrt[n]{S(n)} = 1$

—This simple relation follows by $1 \leq S(n) \leq n$, so $1 \leq \sqrt[n]{S(n)}\ \leq \sqrt[n]{n}$; and by $\sqrt[n]{n}\ \rightarrow 1$ as $n \rightarrow \infty$. However, 11) is one of a (few) limits, which exists for the Smarandache function.

Finally, we shall prove that:

12) $\limsup_{n\rightarrow\infty} \frac{\sigma\left(n\, S(n)\right)}{n\, S(n)}\ =\ +\infty.$

—We will use the facts that $S(p!) = p$, $\frac{\sigma(p!)}{p!} = \sum_{d|p!} \frac{1}{d} \geq 1 + \frac{1}{2} + \cdots + \frac{1}{p} \to \infty$ as $p \to \infty$, and the inequality $\sigma(ab) \geq a\,\sigma(b)$ (see [2]).

Thus $\frac{\sigma(S(p!)p!}{p! \cdot S(p!)} \geq \frac{S(p!) \cdot \sigma(p!)}{p! \cdot p} = \frac{\sigma(p!)}{p!} \to \infty$. Thus, for the sequence $\{n\} = \{p!\}$, the results follows.

References

[1] J. Sándor. On certain inequalities involving the Smarandache function. Smarandache Notions J. F (1996), 3 - 6;

[2] J. Sándor. On the composition of some arithmetic functions. Studia Univ. Babes-Bolyai, 34 (1989), F - 14.