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On the F.Smarandache function and its mean value

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Abstract For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : n \mid m!, n \in N\}$. The main purpose of this paper is using the elementary methods to study a mean value problem involving the F.Smarandache function, and give a sharper asymptotic formula for it.

Keywords F.Smarandache function, mean value, asymptotic formula.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : n \mid m!, n \in N\}$. For example, the first few values of $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $\cdots$. About the elementary properties of $S(n)$, some authors had studied it, and obtained some interesting results, see reference [2], [3] and [4]. For example, Farris Mark and Mitchell Patrick [2] studied the elementary properties of $S(n)$, and gave an estimates for the upper and lower bound of $S(p^\alpha)$. That is, they showed that

$$(p-1)\alpha + 1 \leq S(p^\alpha) \leq (p-1)[\alpha + 1 + \log_p \alpha] + 1.$$

Murthy [3] proved that if n be a prime, then $SL(n) = S(n)$, where $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, and $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1, 2, \cdots, k$. Simultaneously, Murthy [3] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [4] completely solved this problem, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \texttt{ or } \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1, p_2, \cdots, p_r, p$ are distinct primes, and $\alpha_1, \alpha_2, \cdots, \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

Dr. Xu Zhefeng [5] studied the value distribution problem of $S(n)$, and proved the following conclusion:

Let $P(n)$ denotes the largest prime factor of n, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}(S(n)-P(n))^2=\frac{2\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}}{3\ln x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

On the other hand, Lu Yaming [6] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \geq 2$, the equation

$$S(m_1+m_2+\cdots+m_k)=S(m_1)+S(m_2)+\cdots+S(m_k)$$

has infinite groups positive integer solutions $(m_1, m_2, \cdots, m_k)$.

Jozsef Sandor [7] proved for any positive integer $k \geq 2$, there exist infinite groups positive integer solutions $(m_1,\ m_2,\ \cdots,\ m_k)$ satisfied the following inequality:

$$S(m_1+m_2+\cdots+m_k)>S(m_1)+S(m_2)+\cdots+S(m_k).$$

Also, there exist infinite groups of positive integer solutions $(m_1, m_2, \cdots, m_k)$ such that

$$S(m_1+m_2+\cdots+m_k)<S(m_1)+S(m_2)+\cdots+S(m_k).$$

The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of $[S(n)-S(S(n))]^2$, and give an interesting mean value formula for it. That is, we shall prove the following conclusion:

Theorem. Let k be any fixed positive integer. Then for any real number $x > 2$, we have the asymptotic formula

$$\sum_{n\leq x}[S(n)-S(S(n))]^2=\frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot x^{\frac{3}{2}}\cdot\sum_{i=1}^{k}\frac{c_i}{\ln^i x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1}x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, c_i $(i = 1,\ 2, \cdots,\ k)$ are computable constants and $c_1 = 1$.

§2. Proof of the Theorem

In this section, we shall prove our theorem directly. In fact for any positive integer $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then from [3] we know that

$$S(n)=\max\{S(p_1^{\alpha_1}),\ S(p_2^{\alpha_2}),\ \cdots,\ S(p_s^{\alpha_s})\}\equiv S(p^{\alpha}). \tag{2}$$

Now we consider the summation

$$\sum_{n\leq x}[S(n)-S(S(n))]^2=\sum_{n\in A}[S(n)-S(S(n))]^2+\sum_{n\in B}[S(n)-S(S(n))]^2, \tag{3}$$

where A and B denote the subsets of all positive integer in the interval $[1,\ x]$. A denotes the set involving all integers $n \in [1,\ x]$ such that $S(n) = S(p^2)$ for some prime p; B denotes the set involving all integers $n \in [1,\ x]$ such that $S(n) = S(p^\alpha)$ with $\alpha = 1$ or $\alpha \geq 3$. If $n \in A$, then $n = p^2 m$ with $P(m) < 2p$, where $P(m)$ denotes the largest prime factor of m. So from the definition of $S(n)$ we have $S(n) = S(mp^2) = S(p^2) = 2p$ and $S(S(n)) = S(2p) = p$ if $p > 2$.

From (2) and the definition of A we have

$$\begin{aligned}
&\sum_{n\in A} [S(n) - S(S(n))]^2 \\
=& \sum_{\substack{n\leq x \\ p^2\|n,\ \sqrt{n}<p^2}} \left[S(p^2) - S(S(p^2))\right]^2 + \sum_{\substack{n\leq x \\ p^2\|n,\ p^2\leq\sqrt{n}}} \left[S(p^2) - S(S(p^2))\right]^2 \\
=& \sum_{\substack{p^2 n\leq x \\ n<p^2,\ (p,\ n)=1}} \left[S(p^2) - S(S(p^2))\right]^2 + \sum_{\substack{p^2 n\leq x \\ p^2\leq n,\ (p,\ n)=1}} \left[S(p^2) - S(S(p^2))\right]^2 \\
=& \sum_{\substack{p^2 n\leq x \\ n<p^2,\ (p,\ n)=1}} p^2 + \sum_{\substack{p^2 n\leq x \\ n\geq p^2,\ (p,\ n)=1}} p^2 + O(1) \\
=& \sum_{n\leq\sqrt{x}} \sum_{n<p^2\leq \frac{x}{n}} p^2 + O\left(\sum_{m\leq x^{\frac{1}{4}}} \sum_{p\leq\left(\frac{x}{m}\right)^{\frac{1}{3}}} p^2 \right) + O\left(\sum_{p\leq x^{\frac{1}{4}}} \sum_{p^2\leq n\leq \frac{x}{p^2}} p^2 \right) \\
=& \sum_{n\leq\sqrt{x}} \sum_{p\leq\sqrt{\frac{x}{n}}} p^2 + O\left(\frac{x^{\frac{5}{4}}}{\ln x}\right), \qquad (4)
\end{aligned}$$

where $p^2\|n$ denotes $p^2|n$ and $p^3 \dagger n$.

By the Abel's summation formula (See Theorem 4.2 of [8]) and the Prime Theorem (See Theorem 3.2 of [9]):

$$\pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1,\ 2, \cdots, k)$ are computable constants and $a_1 = 1$.

We have

$$\begin{aligned}
\sum_{p\leq\sqrt{\frac{x}{n}}} p^2 &= \frac{x}{n} \cdot \pi\left(\sqrt{\frac{x}{n}}\right) - \int_{\frac{3}{2}}^{\sqrt{\frac{x}{n}}} 2y \cdot \pi(y) dy \\
&= \frac{1}{3} \cdot \frac{x^{\frac{3}{2}}}{n^{\frac{3}{2}}} \cdot \sum_{i=1}^{k} \frac{b_i}{\ln^i \sqrt{\frac{x}{n}}} + O\left(\frac{x^{\frac{3}{2}}}{n^{\frac{3}{2}} \cdot \ln^{k+1} x}\right), \qquad (5)
\end{aligned}$$

where we have used the estimate $n \leq \sqrt{x}$, and all b_i are computable constants and $b_1 = 1$.

Note that $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} = \zeta\left(\frac{3}{2}\right)$, and $\sum_{n=1}^{\infty} \frac{\ln^i n}{n^{\frac{3}{2}}}$ is convergent for all $i = 1,\ 2,\ 3, \cdots, k$. So from

(4) and (5) we have

$$\begin{aligned}
&\sum_{n\in A}[S(n)-S(S(n))]^2\\
=&\sum_{n\le\sqrt{x}}\left[\frac{1}{3}\cdot\frac{x^{\frac{3}{2}}}{n^{\frac{3}{2}}}\cdot\sum_{i=1}^{k}\frac{b_i}{\ln^i\sqrt{\frac{x}{n}}}+O\left(\frac{x^{\frac{3}{2}}}{n^{\frac{3}{2}}\cdot\ln^{k+1}x}\right)\right]+O\left(\frac{x^{\frac{5}{4}}}{\ln x}\right)\\
=&\frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot x^{\frac{3}{2}}\cdot\sum_{i=1}^{k}\frac{c_i}{\ln^i x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1}x}\right),
\end{aligned}\tag{6}$$

where c_i $(i=1,2,\ 3,\cdots,k)$ are computable constants and $c_1=1$.

Now we estimate the summation in set B. For any positive integer $n\in B$, if $S(n)=S(p)=p$, then $[S(n)-S(S(n))]^2=[S(p)-S(S(p))]^2=0$; If $S(n)=S(p^\alpha)$ with $\alpha\ge 3$, then

$$[S(n)-S(S(n))]^2=[S(p^\alpha)-S(S(p^\alpha))]^2\le\alpha^2p^2$$

and $\alpha\le\ln x$. So that we have

$$\sum_{n\in B}[S(n)-S(S(n))]^2\ll\sum_{\substack{np^\alpha\le x\\ \alpha\ge 3}}\alpha^2\cdot p^2\ll x\cdot\ln^2 x.\tag{7}$$

Combining (3), (6) and (7) we may immediately deduce the asymptotic formula

$$\sum_{n\le x}[S(n)-S(S(n))]^2=\frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot x^{\frac{3}{2}}\cdot\sum_{i=1}^{k}\frac{c_i}{\ln^i x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^{k+1}x}\right),$$

where c_i $(i=1,2,\ 3,\cdots,k)$ are computable constants and $c_1=1$.

This completes the proof of Theorem.

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