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ON THE SMARANDACHE FUNCTION AND SQUARE COMPLEMENTS *

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Abstract The main purpose of this paper is using the elementary method to study the mean value properties of the Smarandache function, and give an interesting asymptotic formula.

Keywords: Smarandache function; Square complements; Asymtotic formula.

§1. Introduction

Let n be an positive integer, if $a(n)$ is the smallest integer such that $na(n)$ is a perfect square number, then we call $a(n)$ as the square complements of n. The famous Smarandache function $S(n)$ is defined as following:

$$S(n) = \min\{m : m \in N, n|m!\}.$$

In problem 27 of [1], Professor F. Smarandache let us to study the properties of the square complements. It seems no one know the relation between this sequence and the Smaradache function before. In this paper, we shall study the mean value properties of the Smarandache function acting on the square complements, and give an interesting asymptotic formula for it. That is, we shall prove the following conclusion:

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Theorem. For any real number $x \geq 3$, we have the asymptotic formula

$$\sum_{n\leq x} S(a(n)) = \frac{\pi^2 x^2}{12\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

§2. Proof of the theorem

To complete the proof of the theorem, we need some simple Lemmas. For convenience, we denote the greatest prime divisor of n by $p(n)$.

Lemma 1. If n is a square free number or $p(n) > \sqrt{n}$, then $S(n) = p(n)$.

Proof. (i) n is a square free number. Let $n = p_1 p_2 \cdots p_r p(n)$, then

$$p_i | p(n)!, \qquad i = 1, 2, \cdots, r.$$

So $n | p(n)!$, but $p(n) \dagger (p(n)-1)!$, so $n \dagger (p(n)-1)!$, that is, $S(n) = p(n)$;

(ii) $p(n) > \sqrt{n}$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p(n)$, so we have

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} < \sqrt{n}$$

then

$$p_i^{\alpha_i} | p(n)!, \qquad i = 1, 2, \cdots, r.$$

So $n | p(n)!$, but $p(n) \dagger (p(n)-1)!$, so $S(n) = p(n)$.

This proves Lemma 1.

Lemma 2. Let p be a prime, then we have the asymptotic formula

$$\sum_{\sqrt{x}\leq p\leq x} p = \frac{x^2}{2\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Proof. Let $\pi(x)$ denotes the number of the primes up to x. Noting that

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right),$$

from the Abel's identity [2], we have

$$\begin{aligned} \sum_{\sqrt{x}\leq p\leq x} p &= \pi(x)x - \pi(\sqrt{x})\sqrt{x} - \int_{\sqrt{x}}^{x} \pi(t)dt \\ &= \frac{x^2}{\ln x} - \frac{1}{2}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right) \\ &= \frac{x^2}{2\ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \end{aligned}$$

This proves Lemma 2.

Now we prove the theorem. First we have

$$\begin{aligned} \sum_{n\leq x} S(a(n)) &= \sum_{m^2 n\leq x} S(n)|\mu(n)| \\ &= \sum_{m\leq\sqrt{x}} \sum_{n\leq \frac{x}{m^2}} S(n)|\mu(n)|. \end{aligned} \quad (1)$$

To the inner sum, using the above lemmas we get

$$\begin{aligned}
&\sum_{n\leq \frac{x}{m^2}} S(n)|\mu(n)| \\
=\ & \sum_{\substack{np\leq \frac{x}{m^2}\\ p\geq \sqrt{np}}} p|\mu(n)| + O\left(x^{\frac{3}{2}}\right) \\
=\ & \sum_{\substack{np\leq \frac{x}{m^2}\\ p\geq \sqrt{\frac{x}{m^2}}}} p|\mu(n)| + O\left(x^{\frac{3}{2}}\right) \\
=\ & \sum_{n\leq \sqrt{\frac{x}{m^2}}} |\mu(n)| \sum_{\sqrt{\frac{x}{m^2}}\leq p\leq \frac{x}{nm^2}} p + O\left(x^{\frac{3}{2}}\right) \\
=\ & \sum_{n\leq \ln^2 x} \frac{|\mu(n)|x^2}{2n^2m^4\ln\frac{x}{nm^2}} + \sum_{\ln^2 x<n\leq \sqrt{\frac{x}{m^2}}} \frac{|\mu(n)|x^2}{2n^2m^4\ln\frac{x}{nm^2}} + O\left(\frac{x^2}{m^4\ln^2 x}\right) \\
=\ & \frac{\zeta(2)x^2}{2\zeta(4)m^4\ln x} + O\left(\frac{x^2}{m^4\ln^2 x}\right). \qquad (2)
\end{aligned}$$

Combining (1) and (2), we have

$$\begin{aligned}
\sum_{n\leq x} S(a(n)) &= \frac{\zeta(2)x^2}{2\zeta(4)\ln x}\sum_{m\leq\sqrt{x}}\frac{1}{m^4} + O\left(\frac{x^2}{\ln^2 x}\sum_{m\leq\sqrt{x}}\frac{1}{m^4}\right) \\
&= \frac{\zeta(2)x^2}{2\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).
\end{aligned}$$

Noting that $\zeta(2)=\frac{\pi^2}{6}$, so we have

$$\sum_{n\leq x} S(a(n)) = \frac{\pi^2x^2}{12\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

This completes the proof of Theorem.

Reference

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[3] Pan Chengdong and Pan Chengbiao, Element of the Analytic Number Theory, Science Press, Beijing, 1991.