Pluckings from the Tree
of
Smarandache Sequences and Functions

By Charles Ashbacher

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Preface

In writing a book, one encounters and overcomes many obstacles. Not the least of which is the occasional case of writer's block. This is especially true in mathematics where sometimes the answer is currently and may for all time be unknown. There is nothing worse than writing yourself into a corner where your only exit is to build a door by solving unsolved problems. In any case, it is my hope that you will read this volume and come away thinking that I have overcome enough of those obstacles to make the book worthwhile. As always, your comments and criticisms are welcome. Feel free to contact me using any of the addresses listed below, although e-mail is the preferred method.

Being the third in a series on the Smarandache Notions, it is a tribute to the mind of Florentin Smarandache that there seems to be no end to the chain of problems. He is to be commended for contributing so many problems in so many areas. It will be at least decades before most of the problems that he has posed will be resolved. If you found this book interesting, I strongly encourage you to examine the references listed at the end of this book. There is much more there that remains unexplored. To keep up to date with the latest results in the area of Smarandache notions, consult the **Smarandache Notions Journal**, published by American Research Press.

Once again, my thanks go to the staff of American Research Press for their help and patience as this book began to take shape. It is essentially impossible for one person to create a book, and without their support this one would not exist. The occasional letter containing a new problem or approach was always an inspiration, even when the problem appeared complex. There is no doubt that I will never be at a loss for problems to study. Of course, any errors that remain are the total responsibility of the author.

Special thanks to Rose Slaymaker at nemec.com, inc. for converting this book to an e-book format.

Additional thanks go to all those people who had a hand in my education. There is no greater role in the world than training someone so that they may go out and exceed your accomplishments. Sometimes, education is nothing more than telling someone that they can do something. For that, I owe Chemistry professor Leonardo Lim of Mount Mercy College in Cedar Rapids, Iowa, USA an eternal debt. He richly deserves one of the simplest of all accolades, "he is a good man."

As I completed the last section of this book, one of the problems that is listed as unsolved was resolved in the problem section of the May, 1998 issue of **School Science and Mathematics**. The problem was

Does the equation $Z(n) = Z(n + 1)$ have any solutions?

where $Z(n)$ is the Pseudo-Smarandache function. It was given as part (b) of problem number 4625 and the published solution was listed as a composite by John Koker, N. J. Kuenzi, Heinz-Jürgen Seiffert and David R. Stone. The proof confirms that there are no numbers satisfying the equation.

Finally, I would like to again dedicate this book to my lovely daughter Katrina. She is without question "the best little girl in the whole wide world." No mathematical proof can match her beauty and elegance. The joy she gives me is uncountable by any system of measure.

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Chapter I

Introduction and Old Business

Once again, we embark together on a journey of exploration. This book, the third in a series of works exploring the set of problems called Smarandache Notions, continues the job begun in the first two. However, in this case, there is a concerted effort to delve more deeply into the fundamental mathematics of the problems and resolve the issues. Therefore, the level of difficulty here will be somewhat higher than that of the previous books. But that should not deter you from reading on. Sometimes harder tasks are more fun. They certainly are more interesting.

For review, we start with the definition of the Smarandache function, which is the basis for many of the problems in this book.

Definition: For any integer n $\leq$ 1, the value of the Smarandache function S(n) = m is the smallest integer m $\neq$ 0 such that n divides m factorial.

Examples:

S(10) = 5 since 10 divides 120 but not any of the previous factorials, 1, 2, 6, or 24.
S(5) = 5 for the same reasons.

There will be places where reference will be made to previous work. When appropriate, sufficient background material will be given. However, in the interests of moving forward at a reasonable pace, there may be times where the reader will have to consult one of the previous books in this series [1] [2] or one of the other references [3] [4] [5].

We will start with some problems that were tentatively explored in the second book of this series [2]. Some of these problems also appear in [4] and [5] and deal with how many prime members there are in sequences of numbers.

I.1 The Smarandache Pierced Chain Sequence

In a previous work in this series[2], the Smarandache Pierced Chain(SPC) sequence was discussed. The numbers in this sequence are:

101, 1010101, 10101010101, 101010101010101, 1010101010101010101, ...

As a function the sequence can be defined as

SPC(n) = 101 with n instances of the string 0101 concatenated on the right and n starting at zero.

Clearly, every element of SPC(n) is divisible by 101, so the question posed by Smarandache was:

How many elements of { m | m = SPC(n)/101, n $\leq$ 0 } are prime?

This topic was discussed, but not resolved in the previous volume.

Kenichiro Kashihara, a mathematician and medical student in Tokyo, Japan has written a book on Smarandache notions called, **Comments and Topics on Smarandache Notions and Problems**[6]. That book is highly recommended and on page 7, he proves that there are no primes in this set.

I.2 The Number Of Primes in Some Smarandache Sequences

Each of the following problems is from Dumitrescu and Seleacu[4]. The numbering used here matches that in their book.

(1) The Smarandache Circular Sequence is given by

> 1, 12, 123, 1234, 12345, 123456, 1234567, 12345678, 123456789, 12345678910, 1234567891011, . . .

How many primes are there in this sequence?

(3) Smarandache Symmetric Sequence:

> 1, 11, 121, 12321, 1234321, 123454321, 12345654321, . . .

How many primes are there in this sequence?

(5) Smarandache Mirror Sequence:

> 1, 212, 32123, 4321234, 543212345, 65432123456, 7654321234567, . . .

How many primes are there in this sequence?

As Pal Erdös explains in an unpublished letter to T. Yau[7], it is very difficult to prove significant results concerning the number of primes in such sequences. Other than even/odd parity, there is no pattern of divisibility used in the creation of the sequence, and the numbers grow exponentially. He said that barring the appearance of some unusual property, there is not now and may not ever be a hope of proof.

And it is easy to see how quickly the numbers exceed modern computing capability. For the Smarandache Mirror Sequence, the concatenation of a single digit number adds two digits, but once the two digit numbers are encountered the concatenation adds four digits.

Additional sets of problems just as difficult can be created if one uses a base other than ten. For example, in base eight, the analogous problems are:

(1) Smarandache Circular Sequence

> 1, 12, 123, 1234, 12345, 123456, 1234567, 123456710, 12345671011, . . .

How many primes are there in this sequence?

(3) Smarandache Symmetric Sequence

1, 11, 121, 12321, 1234321, 123454321, 12345654321, 1234567654321, 1234567107654321, 12345671011107654321, . . .

How many primes are there in this sequence?

(5) Smarandache Mirror Sequence

1, 212, 32123, 4321234, 543212345, 65432123456, 7654321234567, 10765432123456710, 1110987654321234567891011, . . .

How many primes are there in this sequence?

Such problems appear to be just as hard as the base ten versions.

I.3 The Fibonacci and Lucas Sequences and Some Smarandache Sequences.

Definition: The Fibonacci sequence is defined in the following way:

$F(0) = 0, F(1) = 1, F(n+2) = F(n+1) + F(n)$ for $n \leq 2$.

And one can also ask if any element, (other than the trivial first element), of any of the three sequences above is a Fibonacci number. To explore this issue, a computer program was written to search for elements of the Smarandache Circular Sequence that are also Fibonacci numbers. In examining all Smarandache Circular numbers from 12 up through 1234….29982999 no number that is simultaneously a Fibonacci was found.

Conjecture: The only number that is simultaneously a Fibonacci number and a member of the Smarandache Circular Sequence is the trivial case of m = 1.

Similar questions can be asked for the other two sequences.

Definition: The Lucas numbers are defined in a similar way:

$$L(0) = 2, L(1) = 1, L(n+2) = L(n+1) + L(n)$$

The program was modified and rerun to search for all numbers in the same range that are simultaneously a member of the Smarandache Circular Sequence and a Lucas number. In this case, L(11) = 123 is also a member of the Smarandache Circular Sequence.

Conjecture: The only numbers that are simultaneously Lucas and Smarandache Circular are m = 1 and m = 123.

And again, similar questions can be asked for the other two sequences.

A problem originally proposed by T. Yau[8] was examined in the first volume of this series.

For what triplets (n, n+1, n+2) does the Smarandache function satisfy the Fibonacci relationship

S(n) + S(n+1) = S(n+2)?

A computer search by C. Ashbacher up through 1,000,000 yielded 8 solutions and based on the form of the solutions he conjectured that the number of solutions is in fact infinite. Henry Ibstedt from Sweden has conducted a more extensive search, finding many other solutions. His conclusion was, "This study strongly indicates that the set of solutions is infinite." A paper summarizing these results appeared in **Smarandache Notions Journal**[9] and this was also a primary topic in his book, **Surfing on the Ocean of Numbers**[10].

I.4 The Radu Problem, "Is There Always A Prime Number Between S(n) and S(n+1)?"

Another problem that was also explored in the first volume was originally proposed by I. M. Radu[11].

Show that, except for a finite set of numbers, there exists at least one prime number between S(n) and S(n+1)?

Based on the evidence from a small computer search, Ashbacher conjectured the converse, namely that there is an infinite number of integers n such that there are no primes between S(n) and S(n+1). Ibstedt performed a more extensive computer search and found many cases where there was no prime between S(n) and S(n+1). In this case his conclusion was, "A very large set of solutions was obtained. There is no indication that the set would be finite." This conclusion has also been published in **Smarandache Notions Journal**[12] and in his book[10].

Chapter II

New Problems and Results

II.1 Primes and Powers in the Smarandache Quotient Sequence

The following definition appeared in the collection by Dumitrescu and Seleacu[4].

Definition: Given any integer n $\geq$ 1, the Smarandache Quotient SQ(n) = k is the smallest integer such that nk is a factorial.

And the following problems were posed by Kashihara[6].

1) How many elements of SQ are powers of a prime?
2) How many elements of SQ are square, cubic, etc.?

It is easy to prove that the answer is infinite in both cases. Both results are based on the following theorem.

Theorem: Let k be any integer such that k > 1. Then there is some number n such that SQ(n) = k.

Proof: Let k > 1 and p a prime such that

$$k < \frac{p!}{k} = n < p!.$$

Then k will be the smallest integer such that nk is a factorial and by definition SQ(n) = k.

Since the range of SQ is all integers greater than one, both questions are answered. In fact, the result holds for any set of integers.

II.2 Decimal Digits and The Smarandache Counter

In his collection of unsolved problems[5], Smarandache makes the following definition:

Definition: For a any decimal digit and b an integer, the Smarandache counter C(a,b) = n is the number of times a appears as a digit in b.

He then asks about the particular instances of C(1,n!) and C($1,n^n$).

A computer program was written to determine the smallest value of n such that C(d,n!) $\geq$ 10 for d every non-zero decimal digit. The solutions are

$C(1, 47!) = 10$
$C(2, 49!) = 10$
$C(3, 69!) = 10$
$C(4, 56!) = 13$
$C(5, 55!) = 11$
$C(6, 63!) = 11$
$C(7, 71!) = 10$
$C(8, 56!) = 11$
$C(9, 70!) = 12$

Another program was written to determine the smallest value of n such that
$C(d, n^n) \geq 10$ for all non-zero decimal digits d. The solutions are

$C(1, 29^{29}) = 11$
$C(2, 48^{48}) = 11$
$C(3, 41^{41}) = 13$
$C(4, 45^{45}) = 15$
$C(5, 34^{34}) = 10$
$C(6, 44^{44}) = 13$
$C(7, 42^{42}) = 12$
$C(8, 47^{47}) = 10$
$C(9, 39^{39}) = 15$

There are many other functions that could be used with the Smarandache counter.

II.3 Primes In the Smarandache Deconstructive Sequence

One sequence that was not examined in the previous work is the Smarandache Deconstructive Sequence[4].

1, 23, 456, 7891, 23456, 789123, 4567891, 23456789, 123456789, 1234567891, ...

where the kth term has k digits formed by taking the digits 1-9 in sequential circular order. If we write out the first thirty terms, we see a pattern develop regarding the first and last digits of the terms.

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23456789123456789123456
789123456789123456789123
4567891234567891234567891
23456789123456789123456789
123456789123456789123456789
1234567891234567891234567891
23456789123456789123456789123
456789123456789123456789123456

Notice that the terminal digits follow the pattern

1, 3, 6, 1, 6, 3, 1, 9, 9, 1, 3, 6, 1, 6, 3, 1, 9, 9, 1, 3, 6, 1, . . .

and the initial digits the pattern

1, 2, 4, 7, 2, 7, 4, 2, 1, 1, 2, 4, 7, 2, 7, 4, 2, 1, 1, 2, 4, 7, . . .

which is of course a consequence of the first pattern.

It is easy to prove that these patterns repeat indefinitely and that

$3 \mid SDS(n)$ if and only if $3 \mid n$.

The main question concerning this sequence is

How many primes does the Smarandache Deconstructive Sequence contain?

Of the first 23 elements, 6 are prime if 1 is not considered a prime. And of those prime numbers, 2 ended in 3, 2 in 9 and 2 in 1. Given that so many of the initial numbers are prime, the following conjecture seems reasonable, although it is admitted that the evidence is slim.

Conjecture: The Smarandache Deconstructive Sequence contains infinitely many primes.

Two out of every 9 numbers end in 6. In examining the factorizations, we see that 456 is divisible by 2^3, 23456 by 2^5, and all others by 2^7. This prompts the question:

Question: Does every element of the Smarandache Deconstructive Sequence ending with a 6 contain at least 3 instances of the prime 2 as a factor?

Or even more specifically,

Question: If we form a sequence from the elements of SDS(n) that end in 6, do the powers of 2 that divide them form a monotonically increasing sequence?

Examining the divisors of the elements of the sequence leads to the additional question:

Question: Let k be the largest integer such that $3^k \mid n$ and j the largest integer such that $3^j \mid SDS(n)$. Is it true that k is always equal to j?

II.4 The Smarandache Odd, Prime, Even and Reverse Sequences

Another sequence that we can ask similar questions about is formed by concatenating the odd positive integers in increasing order.

Smarandache Odd Sequence: The sequence of numbers formed by repeatedly concatenating the odd positive integers.

1, 13, 135, 1357, 13579, 1357911, 135791113, 13579111315, . . .

Clearly, a functional description of the elements of this set is given by

OS(n) = 1...(2n-1).

There is a fairly obvious question to ask.

Unsolved Question: How many primes are there in the Odd Sequence?

In running a simple UBASIC program to test for primes, the following were found

13, 135791113151719, 1357911131517192123252729 31

in a search up through

13579111315171921232527293133353739 41.

The following theorem is easy to prove.

Theorem: 3 | OS(n) if and only if 3 | n.

Proof: If we examine the positive odd integers modulo three, we see the recurring pattern

1, 0, 2, 1, 0, 2, 1, 0, 2, 1, 0, 2, 1, 0, 2, 1, 0, 2, 1, 0, 2, 1, . . .

Which gives the modulo three pattern for OS(n)

1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, . . . ?

And of course if 5 | n then 5 | OS(n).

The computer program mentioned previously was modified to search for numbers in the Smarandache Odd Sequence that are also Fibonacci numbers. The program was run for all numbers 135 up through 135... 29972999 and no solutions were found. This leads to the conjecture.

Conjecture: Except for the trivial case of n = 1, there are no numbers in the Smarandache Odd Sequence that are also Fibonacci numbers.

Repeating the same search for Lucas numbers, no solutions are found.

Conjecture: Except for the trivial case of n = 1, there are no numbers in the Smarandache Odd Sequence that are also Lucas numbers.

Concatenating the prime numbers will create a related sequence.

Smarandache Prime Sequence:

2, 23, 235, 2357, 235711, 23571113, 2357111317, 235711131719,

Unsolved Question: How many primes are there in the Prime Sequence?

Once again a UBASIC program was written to examine the first few members of this sequence looking for primes. Up through 23571113171923293137414347 the only primes found were 2, 23, 2357.

Given that there is no ordering similar to that of the previous problem, finding a pattern to exploit in filtering out additional numbers is very difficult.

The next sequence that we will deal with concerns only positive even integers.

Smarandache Even Sequence:

The sequence of numbers formed by repeatedly concatenating the positive even integers.

2, 24, 246, 2468, 246810, 24681012, 2468101214, 246810121416, . . .

Where the functional form is clearly

ES(n) = 2 . . . 2n.

Obviously, only the first term is prime. The main question to ask here concerns perfect powers.

Unsolved Question: Are there any perfect powers in ES(n)?

It seems a safe bet that the answer to this question is no. In general, perfect powers tend to either follow a repeating pattern when there are patterns of repeated digits in the base or have near-random distributions. The digits of the elements of this sequence follow a rigid, non-repeating pattern.

However, another related question that may be of more interest concerns the numbers that are twice a prime.

Unsolved Question: How many elements of ES(n) are twice a prime?

A search for solutions using a UBASIC program was performed. Up through

24681012141618202224262830 32

only

2468101214 = 2 * 1234050607

was found to be twice a prime.

Conjecture: There are other values of n such that ES(n) = 2p for p a prime.

The previously mentioned computer program was modified to look for members of the Smarandache even sequence that are also Fibonacci numbers. The search was conducted up through

2468. . . . 29962998

and the only number found to be in both sets is n = 2.

Conjecture: The only number in the Smarandache even sequence and a Fibonacci number is the trivial case of n = 2.

After being modified to look for numbers in the same range that were also Lucas numbers, the program was rerun. Again, the only number found to be in the Smarandache even sequence and a Lucas number was the trivial case of n = 2.

Conjecture: The only number in the Smarandache even sequence and a Lucas number is the trivial case of n = 2.

The final sequence that we will deal with here is simply the reverse of one of the previous ones.

Smarandache Reverse Sequence:

The sequence of numbers formed by concatenating the increasing integers on the left side.

1, 21, 321, 4321, 54321, 654321, 7654321, 87654321, 987654321, 10987654321, . . .

These first 35 elements of this sequence were checked and no primes were found. The obvious question to consider is then

Question: Does this sequence contain a prime?

Chapter III

Other Problems and Results

III.1 Sequences Involving the Smarandache Function

The first new problem we will deal with is one that always seemed to present another interesting twist every time another avenue was explored.

This problem is number (2067) of the collection edited by Muller[3].

Let {an} be the sequence defined in the following way

$a_0 = 1$, $a_1 = 2$ and
$a_{n+1} = a_{S(n)} + S(a_n)$ for $n > 1$.

where S(n) is the Smarandache function.

Are there infinitely many pairs of integers (m,n) $m \neq n$, such that $a_m = a_n$?

For example, $a_6 = a_9 = 20$

Note: There is an error in the problem as stated in the reference. The example given there is $a_9 = a_{13} = 16$. However, as can easily be verified, $a_9 = 20$.

To investigate this problem a C++ program to compute the terms of this sequence was written. Since this program is going to be used in several different ways, a documented listing follows.

```
#include<stdio.h>

/* To allow for the computation of large numbers of elements of the sequence, a dynamically
allocated doubly-linked list is used. The data section of each node contains the value of that
element of the sequence in value as well as the subscript of that element stored in number. */

class seqnode
{
  public:
  unsigned long value;
  unsigned long number;
 seqnode *pprev,*pnext;
};

// This function computes the value of the Smarandache function.

unsigned long smarvalue(unsigned long );

void main()
{
  FILE *fp1;
```

```
// This is the subscript of the sequence element currently being computed.
  unsigned long num;
// These two values are used for temporary storage in computing the next element of the
// sequence.
  unsigned long temp1,temp2;
// This identifier stores the number of pairs found where am  an .
  unsigned long totalcount;
// Pointers to instances of seqnode that are used to traverse the linked list.
  seqnode *pnode,*tnode;
// Pointers to the head and tail of the linked list of nodes.
  seqnode *head_node_list,*tail_node_list;
// Open the file to store the data for this sequence.
  fp1=fopen("seqdat.dat","w");
  head_node_list=tail_node_list=NULL;

// Create the first two nodes of the list
  pnode=new seqnode;
  pnode->value=1;
  pnode->number=0;
  pnode->pprev=pnode->pnext=NULL;
  head_node_list=tail_node_list=pnode;
  pnode=new seqnode;
  pnode->value=2;
  pnode->number=1;
  pnode->pprev=tail_node_list;
  pnode->pnext=NULL;
  tail_node_list->pnext=pnode;
  tail_node_list=pnode;
// Initialize the counter to the first value and create a loop to compute the additional
// elements of the sequence up to but not including the element with subscript n.
 num=2;
 while(num<100)
   {
// Compute S(n) for the last element of the list.This will be the second term of the
// relation that generates the remaining elements.
    temp1=smarvalue(tail_node_list->value);
// Compute S(n-1) to obtain the subscript of the first term of the generating relation.
    temp2=smarvalue(n-1);
// Use the tnode pointer to scan the list to find the node whose subscript matches that
// already computed.
   tnode=head_node_list;
  while(tnode->number!=temp2)
   {
     tnode=tnode->pnext;
    }
// Create a new node, assign the proper values and place it in the list.
   pnode=new seqnode;
   pnode->value=tnode->value+temp1;
   pnode->number=num;
   pnode->pprev=tail_node_list;
```

```
   pnode->pnext=NULL;
   tail_node_list->pnext=pnode;
   tail_node_list=pnode;
   num++;
  }

// Now that the first n elements of the sequence are computed, we go back and look for
// pairs that are equal.

// Initialize the counter of matches
   totalcount=0;
   pnode=head_node_list;
   while(pnode!=tail_node_list)
   {
    tnode=pnode->pnext;
    while(tnode!=NULL)
    {
     if(pnode->value==tnode->value)
    {
// We have a pair of nodes whose value match. The counter is incremented and the data
// sections of both nodes is dumped to the open file.
     totalcount++;
     fprintf(fp1,"%ld %ld %ld %ld\n",pnode->number,pnode->value,pnode->number,
                                              pnode->value);
    }
     tnode=tnode->pnext;
    }
    pnode=pnode->pnext;
   }
  fprintf(fp1,"The value of totalcount is %ld\n",totalcount);
// The following are dumped to the monitor for checking.
  printf("The value of n is %ld\n",n);
  printf("The value of totalcount is %ld\n",totalcount);
// Now we need to free up the memory dynamically allocated for the nodes.
  pnode=head_node_list;
 while(pnode!=NULL)
 {
   tnode=pnode;
   pnode=pnode->pnext;
  delete tnode;
 }
 fclose(fp1);
}
/* The following function computes the value of S(num) for details on the algorithm,
    consult[1].  */

unsigned long smarvalue(unsigned long num)
{
// Initially set to the value of num. Stores the value that is left as the factors are rmoved.
  unsigned long temp1;
```

```
// Used as a loop counter.
  short i;
 // This stores the number of factors in the input number num
  short factcount;
// Used to store the value of the current potential divisor of num.
  unsigned long divisor;
// Used to determine how many instances of a prime have been encountered.
  unsigned long startnum;
// Stores the factors of num.
  unsigned long factors[30];
// Stores the exponents of each of the factors of num.
  short exps[30];
// Stores the values of S(factors[i]exps[i] )
  unsigned long smars[30];
// Stores the maximum value found in smars[30].
  unsigned long max;
// Used as temporary storage.
  unsigned long tempsum;
// Stores the number of instances of  a factor.
  unsigned long sum_of_factor;
// Used as a flag to terminate the operation of a loop.
  unsigned char found;
  factcount=0;
// Clear all arrays
  for(i=0;i<30;i++)
  {
    factors[i]=0;
    exps[i]=0;
    smars[i]=0;
  }
 // Remove all instances of the even prime 2 from num.
  temp1=num;
  if((temp1%2)==0)
  {
    temp1=temp1/2;
    factors[0]=2;
    exps[0]=1;
    while(((temp1%2)==0)&&(temp1>1))
    {
     temp1=temp1/2;
     exps[0]++;
    }
  factcount=1;
  } // end of the temp1%2==0 if

// Remove all odd divisors.
 divisor=3;
  while(temp1>1)
  {
   if((temp1%divisor)==0)
```

```
  {
   factors[factcount]=divisor;
   exps[factcount]=1;
   temp1=temp1/divisor;
   while(((temp1%divisor)==0)&&(temp1>1))
   {
     temp1=temp1/divisor;
     exps[factcount]++;
    }
   factcount++;
  } // end of the if on temp1%divisor==0
  divisor=divisor+2;
 } // end of the while loop to remove all odd divisors

// Num has now been factored into the associated prime factors. The next step is to
// compute the values of S(factors[i]exps[i]) and place them in smars[i].

 for(i=0;i<factcount;i++)
  {
// If the value of the exponent is less than the prime, the computation is simple.
  if(exps[i]<factors[i])
  {
   smars[i]=exps[i]*factors[i];
  }
  else
  {
// Otherwise we have to count the instances of the prime in the number of instances.
    startnum=exps[i]/2;
    if(startnum<1)
    {
     startnum=1;
    }
   found=0;
/ / Now perform the iteration that counts the number of instances of the prime.
    while(found==0)
    {
     sum_of_factor=startnum;
     tempsum=startnum/factors[i];
     while(tempsum>0)
     {
       sum_of_factor=sum_of_factor+tempsum;
       tempsum=tempsum/factors[i];
     } // end of the while loop on tempsum
    if(sum_of_factor>=exps[i])
    {
    found=1;
    }
    else
    {
     startnum++;
```

```
    }
   } // end of the loop on found==0
 // We can now compute the value of the function for this prime factor
   smars[i]=startnum*factors[i];
   } // end of the else
  } // end of the while loop on i

// Now that the values of S have been computed for all prime factors and placed in the
// array smars[i], we simply need take the largest number in that array.
  max=0;
  for(i=0;i<factcount;i++)
  {
  if(smars[i]>max)
  {
   max=smars[i];
   }
  }
 return(max);
  }
```

Running the program for all values of n < 100, we get the following table of 63 pairs of equal elements

Table 1

m	n	common value
6	9	20
8	14	22
8	15	22
8	27	22
10	11	25
13	36	16
14	15	22
14	27	22
15	27	22
16	37	26
17	75	33
18	39	44
19	28	31
19	50	31
20	78	62
22	56	34
23	29	42
23	34	42
23	96	42
24	30	49
24	46	49
24	73	49
26	63	48

28	50	31
29	34	42
29	96	42
30	46	49
30	73	49
31	45	29
31	66	29
31	92	29
31	97	29
32	71	58
32	98	58
33	99	51
34	96	42
35	52	40
35	69	40
38	65	39
38	91	39
40	49	27
40	55	27
41	44	24
42	54	28
42	57	28
45	66	29
45	92	29
45	97	29
46	73	49
47	51	56
48	68	63
48	74	63
49	55	27
52	69	40
54	57	28
65	91	39
66	92	29
66	97	29
68	74	63
70	79	47
71	98	58
72	82	87
92	97	29

In examining this table, note three things.

a) The sequence is not monotonic. For n an integer, all three of the relations $a_n < a_{n+1}$, $a_n > a_{n+1}$ and $a_n = a_{n+1}$ are true for some value of n.

b) For n an integer, it is possible for there to be more than one m such that $a_n = a_m$.

c) Pairs of elements (a_m,a_n) such that $a_m = a_n$ are very frequent in this region.

The program was rerun for num in increments of 100 from 100 through 2000 and the results are summarized in the following table

Table 2

maximum value of num	number of equal (a_m,a_n) pairs	ratio of (number pairs/num)
100	63	0.63
200	276	1.38
300	625	2.08
400	1000	2.50
500	1505	3.01
600	2271	3.785
700	3056	4.366
800	3874	4.8425
900	4725	5.25
1000	5757	5.757
1100	6782	6.17
1200	7897	6.58
1300	9082	6.986
1400	10250	7.32
1500	11625	7.75
1600	12942	8.089
1700	14414	8.48
1800	15907	8.84
1900	17494	9.21
2000	19150	9.58

When looking at the table, keep in mind that it is possible for the number of pairs to exceed the number of elements. This is a consequence of a basic principle of counting.

For example, if we have a set of 5 items, { a,b,c,d,e }, then the set of possible pairs where the order of the pair is irrelevant is { (a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e) }. The formula for computing the number of such pairs is well-known. If we are given n items, the number of possible pairs is given by

$$\frac{n(n-1)}{2}$$

And now, going back to the table, we see a very interesting pattern. The ratio

$$\frac{\text{number of pairs}}{\text{num}}$$

is monotonically increasing at an approximate rate of 0.37 per 100.

To determine if this is consistent across a greater range, the program was rerun for additional values of num that were multiples of 1000.

Table 3

maximum value of num	number of equal (a_m,a_n) pairs	ratio of (number pairs/num)
3000	32861	10.95
4000	56615	14.15
5000	84071	16.81
6000	119138	19.86
7000	156785	22.40
8000	201370	25.17
9000	252312	28.03
10000	308237	30.82

which does show a drop in the rate of increase even though it still exists.

During the run for num = 10000, three counters were used to determine the relationship between a_n and a_{n+1}. The counts for each of the three possible relationships were

$a_n > a_{n+1}$ 4411
$a_n < a_{n+1}$ 5519
$a_n = a_{n+1}$ 69

Furthermore, the largest number in the sequence up to term numbered 10,000 is 551.

Given the nature of S(n) and the definition of the sequence, it should come as no surprise that the relationships between successive terms exhibits this relationship. The Smarandache function is one where n - S(n) can be anywhere from zero to an arbitrarily large number. For a proof of this see[1].

Based on this evidence, the following conjecture seems to be a very safe one.

Conjecture 1: There are an infinite number of pairs (m,n), $m \neq n$ such that

$$a_m = a_n.$$

A natural question to ask at this point concerns the growth in the size of the elements of the sequence.

Question 1: Is there a number M such that $a_n < M$ for all $n > 0$?

Of course an affirmative answer to question 1 will also verify the conjecture.

To investigate this question the program was rerun for additional values of num all evenly divisible by 10,000. The results hint at a resolution of conjecture 1 and question 1.

Table 4

maximum value of num	value of largest element in the sequence

20,000	634
30,000	758
40,000	758
50,000	758
60,000	866
70,000	866
80,000	866
90,000	866
100,000	866
110,000	866
120,000	898
130,000	898
140,000	898
150,000	1354
160,000	1354
200,000	1354

To explain this apparently bizarre behavior, it is necessary to reconsider the Smarandache function as well as the definition of the sequence. As is well-known, as the primes p get larger, the number of integers m such that $S(m) = p$ gets large very quickly[1]. Therefore, if a_q is a member of the sequence where q is a "large" prime, then q is a result of $S(n) = q$ only when n is a multiple of q. If a_q is then a "large" number in the sequence, $S(n) = q$ and a_n a prime, it is easily possible for the result of

$$a_{S(n)} + S(a_n)$$

to suddenly exceed the previous bound on the terms of the sequence after spending a great deal of time within that bound.

For example, suppose that $a_{863} = 500$, where of course 863 is prime. Then, only numbers of the form 863k are solutions to $S(m) = 863$. As the program cycles upward, it is then only necessary to get a number $a_n = a_{863k}$ where $S(a_n) > 366$ to jump over what appears to be an upper limit of 866 in the middle of the tested range. While it may take some time, it is quite likely that this will happen eventually.

It turns out that it is in fact quite easy to prove that the answer to question 1 is negative. We start with two simple lemmas.

Lemma 1: $a_n > 0$ for all elements in the sequence

$a_0 = 1$, $a_1 = 2$ and
$a_{n+1} = a_{S(n)} + S(a_n)$ for $n > 1$

and $a_n > 1$ for all $n > 0$.

Proof: Clear

Lemma 2: For any integer $n > 2$, there is another integer $m > n$ such that $S(m) = n$.

Proof: We start with the three well-known facts:

1) If $m = p_1^{a1} p_2^{a2} \ldots p_k^{ak}$ then

$$S(m) = \max\{ S(p_1^{a1}), \ldots, S(p_k^{ak}) \}.$$

2) $S(m) \leq m$ for all $m \geq 0$ where $S(m) = m$ for all primes and the number 4.

3) The range of $S(m)$ is the set of all nonnegative integers with the exception of 1.

If $n = S(m) < m$, we are done. So assume $n = S(m) = m$. Then by the second fact, m is either 4 or prime.

Case 1: If $m = 4$ then $S(6) = 4$ and we are done.

Case 2: $n = m = p$, where p is prime. By the hypothesis, p must be odd. Therefore,

$$S(2n) = S(2p) = S(m) = p = n.$$

And we can now easily prove that there is no such maximum number.

Theorem 1: There is in fact no number M such that $a_n < M$ for all a_n in the sequence

$$a_0 = 1,\ a_1 = 2 \text{ and}$$
$$a_{n+1} = a_{S(n)} + S(a_n) \text{ for } n > 1.$$

Proof: Suppose that such an M exists and let a_k represent the largest element in the sequence. By the previous lemma, there is another number $m > k$ such that $S(m) = k$. Therefore, there must be another number in the sequence

$$a_{m+1} = a_{s(m)} + S(a_m)$$

where $a_{s(m)} = a_k$. Since $S(a_m)$ 0, $a_{m+1} > a_n$ and we are done.

Many sequences defined in such a manner exhibit other interesting behaviors when the initial values are changed. Given the program, it is a simple matter to change the initial two values and rerun the program to investigate.

If $a_0 = 1$ and $a_1 = 1$, the sequence is very dull, as every element has the value 1.

The sequence where $a_0 = 2$ and $a_1 = 2$ is only slightly different from that where $a_0 = 1$ and $a_1 = 3$. Only the values of the first two elements are different as can be seen from tables of the first 10 values.

$a_0 = 2 \quad a_1 = 2$

Table 5

n $\quad$ a_n

0	2
1	2
2	4
3	8
4	12
5	16
6	22
7	19
8	38
9	31

$a_0 = 1 \quad a_1 = 3$

Table 6

n	a_n
0	1
1	3
2	4
3	8
4	12
5	16
6	22
7	19
8	38
9	31

And it is a simple matter to verify that the remaining elements must be identical.

The program was then rerun for the new initial values of $a_0 = 2$ and $a_1 = 2$ for several values of num. All results are summarized in the table below.

Table 7

maximum value of num	value of largest element in the sequence
10,000	778
20,000	1402
30,000	1402
40,000	1402
50,000	1402
60,000	1402
70,000	1402

However, theorem 1 can also be applied to this sequence to conclude that despite the initial appearance, there is no maximum element in the sequence.

III.2 Expressions Involving the Smarandache Function

The next problem that we deal with is (17) of the collection edited by Muller[3].

Are there integers m, n, p, q, with $m \neq n$ or $p \neq q$, for which

$$S(m) + S(m+1) + ... + S(m+p) = S(n) + S(n+1) + ...+ S(n+q)?$$

Clearly, if m = n, the only possible way to obtain equality is when p = q.

There is an immediate infinite family of solutions to this problem.

If m = 1 and n = 2, then every ordered pair (p,q) = (q+1,q) satisfies the conditions of the problem. That this must be true should be clear from the fact that they differ only in the first term and S(1) = 0.

Since the previous family is in some sense trivial, the next obvious question concerns the possible existence of other solutions, where m and n are both not 1. Also relevant is whether the number of such solutions is infinite.

A computer program was written to search for additional solutions and many were found within the small range searched. Of particular interest was the collection where m = 23, $m < n < 100$, $0 \leq p < 30$ and $0 \leq q < 30$. Some solutions were

m = 23, n = 24, p = 1 and q =2 with common sum 27
m = 23, n = 24, p = 22 and q = 22 with common sum 362
m = 23, n = 25, p = 0 and q = 1 with common sum 23
m = 23, n = 26, p = 13 and q = 14 with common sum 180
m = 23, n = 26, p = 14 and q=14 with common sum 217

Notice that there are solutions with p = q and where n = m + 1.

Even with these restrictions on the values of p and q, the only values of n in this range where a solution was not found were

30, 36, 40, 45, 48, 53, 57, 58, 63, 67, 68, 70, 76, 78, 79, 84, 86, 91, 93, 96, 97, 98

Which leads to the obvious question that is much more restrictive than the original one.

Question 2: Given m = 23, what is the set of numbers S = { n | there are integers p and q such that

$$S(m) + S(m+1) + ... + S(m+p) = S(n) + S(n+1) + ...+ S(n+q)?$$

Or to be more specific

a) Is S an infinite set?
b) If the answer to (a) is yes, is that set all integers greater than 23?

The previously mentioned program was rerun for a larger range of values for p and q and the solutions

m = 23, n = 30, p = 42 and q = 38 with common sum 804
m = 23, n = 36, p = 106 and q = 97 with common sum 3069
m = 23, n = 40, p = 53 and q = 44 with common sum 1145
m = 23, n = 45, p = 206 and q = 190 with common sum 9187
m = 23, n = 48, p = 37 and q = 26 with common sum 684
m = 23, n = 53, p = 87 and q = 72 with common sum 2392
m = 23, n = 57, p = 31 and q = 18 with common sum 554
m = 23, n = 58, p = 31 and q = 18 with common sum 554
m = 23, n = 63, p = 283 and q = 249 with common sum 15117
m = 23, n = 67, p = 33 and q = 16 with common sum 572
m = 23, n = 68, p = 33 and q = 18 with common sum 572
m = 23, n = 70, p = 33 and q = 18 with common sum 572
m = 23, n = 76, p = 65 and q = 40 with common sum 1494
m = 23, n = 78, p = 48 and q = 28 with common sum 1000
m = 23, n = 79, p = 72 and q = 47 with common sum 1722
m = 23, n = 84, p = 87 and q = 60 with common sum 2392
m = 23, n = 86, p = 38 and q = 20 with common sum 745
m = 23, n = 91, p = 73 and q = 44 with common sum 1730
m = 23, n = 93, p = 42 and q = 18 with common sum 804
m = 23, n = 96, p = 51 and q = 27 with common sum 1116
m = 23, n = 97, p = 49 and q = 24 with common sum 1006
m = 23, n = 98, p = 31 and q = 11 with common sum 554

were all found.

Running the program for m = 23 and n > 100 and restrictions on the value of p and q yields solutions for almost all values of n. In most of the cases where no solution was found for a specific value of n, increasing the allowable values for p and q led to a solution. This is a natural lead in to the following conjecture and question.

Conjecture: For m = 23, the set of all positive integers n such that there exist integers p and q satisfying

$$S(m = 23) + S(m+1) + ... + S(m+p) = S(n) + S(n+1) + ...+ S(n+q)$$

is infinite.

Question: Is there a value of n such that there is no ordered pair of integers (p,q) where

$$S(m = 23) + S(m+1) + ... + S(m+p) = S(n) + S(n+1) + ...+ S(n+q)?$$

In running the program for a few other fixed values of m, the result is always that within a small range of possible values for p and q, and for nearly every n > m, there is an ordered pair (p,q) such that

$$S(m) + S(m+1) + ... + S(m+p) = S(n) + S(n+1) + ...+ S(n+q)$$

Which leads to the following theorem and unsolved problems.

Theorem: Given any integer m > 2, it always possible to find numbers n,p and q such that

$$S(m) + S(m+1) + ... + S(m+p) = S(n) + S(n+1) + ...+ S(n+q).$$

In fact, for any m and arbitrary $p \leq 0$, such a solution is always possible.

Proof: Forming the sequence

$$S = \{ S(m), S(m) + S(m+1), S(m) + ... + S(m+p), . . . \}$$

we have a strictly increasing infinite set of numbers starting at S(m) > 2. Since the range of S consists of all nonnegative integers except 1, for every s ϵ S, there is some number k such that S(k) = s. Simply set n = k and q = 0.

Unsolved Problem: Given any integer m, how many ordered triples (p,n,q) such that

$$S(m) + S(m+1) + ... + S(m+p) = S(n) + S(n+1) + ...+ S(n+q)$$

are there?

Unsolved Problem: Given any pair of integers (m,n) where both are greater than 1 and $m \neq n$, is it always possible to find another pair of integers (p,q) such that

$$S(m) + S(m+1) + ... + S(m+p) = S(n) + S(n+1) + ...+ S(n+q)?$$

Of course, an affirmative result for the second resolves the first.

It is the opinion of the author that the answer to the second question is yes. If we form the two infinite sets of numbers

$$S_m = \{ S(m), S(m) + S(m+1), . . . ,S(m) + . . . + S(m+p), . . . \}$$

$$S_n = \{ S(n), S(n) + S(n+1), . . . ,S(n) + . . . + S(n+q), . . . \}$$

both are strictly increasing in size. However, given the nature of the Smarandache function the differences S(n+1) - S(n) can be arbitrarily large and arbitrarily small. Furthermore, the absolute difference is large only if one of n+1 or n is prime. In addition, since the sets of numbers

$$\{ m, m+1, m+2, . . . \} \text{ and } (n, n+1, n+2, . . . \}$$

involve all prime factors, there is no clear divisibility restriction. Putting all of this together, we have two infinite families of integers where each has a rate of increase that varies substantially up and down. Lacking a divisibility or other restriction, the probability that there will be a common element in S_m and S_n is 1.

Moving on, the next problem is (18) of the collection edited by Muller[3].

Are there integers m,n,p,k with $m \neq n$ and p > 0 such that:

$$\frac{S(m)^2 + S(m+1)^2 + \ldots + S(m+p)^2}{S(n)^2 + S(n+1)^2 + \ldots + S(n+p)^2} = k?$$

A computer program was written to search for solutions within a small range and many were found. Some examples are:

m = 6, n = 5, p = 4 and k = 1
m = 6, n = 5, p = 9 and k = 1
m = 8, n = 5, p = 3 and k = 2
m = 73, n = 5, p = 1 and k = 197

The program was run for several fixed values of n, keeping m in the range $n+1 \leq m < 100$ and $p < 20$. The number of solutions for each value of n is summarized in the following table.

Table 8

n	number of solutions in the given ranges
1	26
2	20
3	17
4	20
5	17
6	10
7	7
8	6
9	11
10	8
11	3

This is another problem where it would appear that there are an infinite number of ordered quadruples that satisfy the conditions of the problem. We now make two conjectures along those lines.

Conjecture: There are an infinite number of ordered quadruples (m,n,p,k), $m \neq n$ and $p > 0$ such that

$$\frac{S(m)^2 + S(m+1)^2 + \ldots + S(m+p)^2}{S(n)^2 + S(n+1)^2 + \ldots + S(n+p)^2} = k.$$

A companion conjecture has much tighter parameters and implies the previous one.

Conjecture: For every integer $n > 1$, there are corresponding integers $m \neq n$, p and k such that

$$\frac{S(m)^2 + S(m+1)^2 + \ldots + S(m+p)^2}{S(n)^2 + S(n+1)^2 + \ldots + S(n+p)^2} = k?$$

The arguments in favor here are similar to those used before. If we fix an integer n and choose a value for p, then letting m move through all numbers larger than n creates an infinite series of numbers where the elements have no specific divisibility restrictions. Similarly, modifying the value of p creates an infinite series of numbers with no divisibility restrictions. Minus such restrictions the two sequences almost certainly will have points in common.

A companion problem can be raised concerning the possible values for k.

Unsolved Problem: What is the set of all integers k such that there exists an ordered triple of integers (m,n,p) such that

$$\frac{S(m)^2 + S(m+1)^2 + \ldots + S(m+p)^2}{S(n)^2 + S(n+1)^2 + \ldots + S(n+p)^2} = k?$$

It is well-known that the Smarandache function tends to increase in the sense that for any number M there is only a finite number of integers m such that $S(m) \leq M$. Therefore, it would appear that this set is infinite. However, while it may prove possible to resolve the previous problem using such arguments, the following seems particularly hard.

Unsolved Problem: Find a value of k where it is not possible to find an ordered triple satisfying the equation of

$$\frac{S(m)^2 + S(m+1)^2 + \ldots + S(m+p)^2}{S(n)^2 + S(n+1)^2 + \ldots + S(n+p)^2} = k?$$

As a brief extension of this problem, we can modify the expression so that we consider cubes rather than squares. Modifying the program so that cubes are used rather than squares did not seem to modify the number of solutions for the few sample runs executed.

Question: Does the equation

$$\frac{S(m)^j + S(m+1)^j + \ldots + S(m+p)^j}{S(n)^j + S(n+1)^j + \ldots + S(n+p)^j} = k$$

have solutions (m,n,p,k) for every integer $j > 1$?

Question: As can be seen from the table of solutions, m = 8, n = 5, p = 3 and k =2 is a solution where the value of k matches the exponents. Is it true that for every integer $j > 1$, there is a solution (m,n,p,k) where k = j?

Problem (19) of the collection by Muller[3] deals with concatenating the values of the Smarandache function to create primes.

How many primes have the form

$$S(n) \circ S(n+1) \circ S(n+2) \circ \ldots \circ S(n+k)$$

for a fixed integer k?

Note: The operator $\circ$ in this context denotes concatenation. For example, S(2) = 2, S(3) = 3 and 23 is prime.

A computer program was written to explore this question for k = 1 and $2 \le n < 10{,}000$. There were 1612 instances where

$$S(n) \circ S(n+1)$$

is prime. Such results are reasonable. A large percentage of the values of the Smarandache function are prime for this range and the concatenation of two numbers where the second has a high likelihood of being prime would often be prime.

Question: Is the set of integers CN = { n | where S(n) $\circ$ S(n+1) is prime } an infinite set?

Rerunning the program in the ranges $2 \le n < 20{,}000$, $2 \le n < 30{,}000$ and $2 \le n < 40{,}000$ yielded 3038, 4386 and 5728 solutions respectively. Therefore, it seems safe to conjecture that the answer to the question is yes.

III.3 The Percentage of Values of the Smarandache Function That Are Prime

It is well-known that the number of primes less than or equal to x, (x), is given by the expression

$$\mathrm{II}(x) \sim \frac{x}{\log(x)}$$

An interesting analog to this is the question concerning the percentage of S(m) that are prime.

Definition: Let SII(x) be the number of integers y, where $2 \le y < x$ such that S(y) is prime.

Theorem: S(x) > II(x) for x > 6.

Proof: Clear from the knowledge that S(p) = p for p a prime and S(6) = 3.

Many problems involving the Smarandache function involve the value of SII(x). To get some idea of the behavior of SII(x), a program was written to count the values for various ranges.

To explore this problem a computer program was written that cycles through a range of numbers and determines the number of times S(m) was prime. The results are summarized in the following table. The first column is the upper limit on the tested range, the second the number of values m in the range where S(m) was prime and the third is the ratio of the two values.

Table 9

10000	9406	0.9406

20000	19052	0.9526
30000	28760	0.9587
40000	38490	0.9623
50000	48245	0.9649
60000	58014	0.9669
70000	67797	0.9685
100000	97193	0.9719
200000	195501	0.9775
500000	491578	0.9832
750000	738877	0.9852

The numbers in this table hint at a result that is the point of the following unsolved problem.

Unsolved Problem: Does the limit

$$\lim_{n\to\infty} \frac{\mathrm{SII}(x)}{x}$$

exist? If it does, what is it?

From the table, it appears that the limit does exist and is in fact 1. This is made more explicit in the following conjecture.

Conjecture:

$$\lim_{n\to\infty} \frac{\mathrm{SII}(x)}{x} = 1.$$

III.4 Creating Sequences Using the Smarandache Function

Problem (21) in the collection compiled by Muller is our next challenge.

Are there m,n,k non-null positive integers, $m \neq 1 \neq n$ for which

$$S(m*n) = m^k * S(n)?$$

A computer program was run for all values $2 \leq m < 2000$ and $2 \leq n < 2000$ and $k = 1$. The only solution found was $m = n = 2$ where $k = 1$. This leads to the next unsolved question.

Unsolved Problem: Is there another solution to problem(21)?

Definition: An A-sequence is an integer sequence

$$1 \leq a_1 < a_2 < \ldots$$

such that no element a_i is the sum of a set of distinct elements of the sequence that does not contain a_i.

Problem (34) of the collection by Muller deals with A-sequences

Is it possible to construct an A-sequence $a_1, a_2, \ldots$ such that $S(a_1), S(a_2), \ldots$ is also an A-sequence?

Theorem 4: There are an infinite number of A-sequences $a_1, a_2, \ldots$ such that $S(a_1), S(a_2), \ldots$ is also an A-sequence.

Proof: Perform the following algorithm

1) Start with any set S of k-1 numbers where both the numbers and their Smarandache values form finite A-sequences. It is trivial that such sequences exist.

2) Form the product P, of all of the elements of S.

3) By the infinitude of the primes, we can find a prime p such that $p > P$. Let a_k be an arbitrary multiple of p. Clearly, a_k cannot be the sum of any subset of S. a_k is also larger than all elements of S and $S(a_k)$ is some multiple of p. Furthermore, $S(a_k)$ is larger than the product of all $S(a_i)$ for all elements in S. Therefore, the new set $S' = S \cup \{a_k\}$ and the set of corresponding Smarandache function values are also finite A-sequences.

4) Go to step 2 with S= S'. ?

Problems (35), (36) and (37) from the collection by Muller all have a similar theme.

Find the greatest n such that if $a_1, a_2, \ldots, a_n$ is a p-sequence then $S(a_1), S(a_2), \ldots, S(a_n)$ is also a p-sequence. In this case a p-sequence is one of the following:

a) Arithmetic progression.
b) Geometric progression.
c) A complete system of modulo n residues.

As a specific example of part (c), a computer program was written that searched for a series of numbers $n, n+1, \ldots, n+k$ such that $S(n), S(n+1), \ldots, S(n+k)$ is a complete residue system modulo k+1. The program was run and many solutions were found for k=1,2,3 and 4. For example, there are 27 solutions in the range $1 \leq n < 1000$, with the last being 800, 801, 802, 803 and 804. However, a run with the upper limit of 100,000 for k = 5 failed to find a single solution. Which is the lead in to the next unsolved question.

Unsolved Question: Is there a sequence of numbers n, n+1, n+2, n+3, n+4, n+5 such that S(n), S(n+1), S(n+2), S(n+3), S(n+4), S(n+5) is a complete system of residues modulo 6?

To further complicate the problem and therefore make it more interesting, a run of the program for k = 6 yielded 5 solutions for n < 1000 with the smallest being 24, 25, 26, 27, 28, 29, 30. A run for n < 10000 yielded 15 solutions. However, runs with k = 7, 8, 9 and 10 for n < 10000 all failed to yield a solution.

Unsolved Question: For how many values of k is there a set of numbers $n, n+1, n+2, n+3, \ldots, n+k$ such that $S(n), S(n+1), \ldots, S(n+k)$ is a complete system of residues modulo k+1?

Conjecture: The number of integers k such that there is some number n such that S(n), S(n+1), . . . , S(n+k) is a complete system of residues modulo k+1 is finite.

If we remove the restriction that the original numbers not be sequential, then problem (c) becomes much simpler. To motivate the proof, we first repeat the well-known theorem of Dirichlet.

Dirichlet's Theorem: Let $d > 2$ and $a \neq 0$ be two numbers relatively prime to each other. Then the sequence

$$a, a + d, a + 2d, a + 3d, \ldots$$

contains infinitely many primes.

Theorem: There is no limit to the size of n where $a_1, a_2, \ldots, a_n$ is a complete system of residue systems modulo n and $S(a_1), S(a_2), \ldots, S(a_n)$ is also a complete system of residues modulo n.

Proof: Choose an odd prime p of arbitrary size. Then, the set of numbers 1, 2, 3, . . . , p is a complete system of residues modulo p. Clearly, each of the numbers { 1, 2, 3, . . . , p-1 } is relatively prime to p. Therefore, by Dirichlet's Theorem, each of the sequences

$$1 + kp, 2 + kp, \ldots, (p-1) + kp \quad \text{for } k = 0, 1, 2, 3, \ldots$$

contains an infinite number of primes. Furthermore, for every element of every sequence

$$j + kp \equiv j(\text{modulo } p)$$

Let p_1 be a prime in the sequence $1 + kp$, p_2 a prime in the sequence $2 + kp$, and so on. Then, $p_1, p_2, \ldots, p_{p-1}, p$ is a set of numbers that is a complete set of residues modulo p such that $S(p_1), S(p_2), \ldots S(p_{p-1}), S(p)$ is as well. Since p can be of arbitrary size, there is no limit to the length.

For part(a), the following theorem is clear.

Theorem: If there is a sequence of primes $p_1, p_2, \ldots, p_k$ such that the primes are all in arithmetic progression, then $S(p_1), S(p_2), \ldots, S(p_k)$ is also in arithmetic progression.

Unfortunately, the following has never been proven, although most experts consider it to be true.

Unsolved Question: For every $k > 3$, there exists at least one arithmetic progression consisting of k prime numbers.

According to Ribenboim[6], the largest value of k known to date is k = 19.

III.5 Combining the Smarandache Function and Other Sequences

Problem (5) in the collection edited by Muller deals with multiple sets of monotonic integers.

Let A be a set of consecutive positive integers. Find the largest set of numbers { n, n+1, n+2, . . . } such that { S(n), S(n+1), S(n+2), . . . } is monotonic.

In order to obtain some data for study, a computer program was written that searches for sequences that are monotonically decreasing. The first sequences of lengths 8, 9 and 10 numbers in length are given below.

n	12721	12722	12723	12724	12725	12726	12727	12728
S(n)	12721	6361	4241	3181	509	101	89	43

n	109453	109454	109455	109456	109457	109458	109459	109460	109461
S(n)	109453	54727	7297	6841	4759	2027	823	421	107

n	586951	586952	586953	586954	586955	586956	586957	586958
S(n)	586951	73369	21739	9467	1319	1193	1181	1091

n	586959	586960
S(n)	677	29

And the search was continued for all n < 2,044,000 and no longer sequence was discovered.

There are several clues here to a resolution of the problem. As is well-known, S(n) tends to grow as n grows, albeit much more slowly. Furthermore, since S(p) = p for p a prime and S(p+1) < p for p a prime, there are an enormous number of two element sequences that can start a longer one. Each sequence above has a common characteristic, in that the first term is a prime and the second twice a prime. This makes sense, as it is also well-known that if p is a prime S(p+1) $\leq$ (p+1)/2 with equality if and only if p + 1 is twice a prime. Having the second twice a prime leaves as much room as possible for the next sequence of numbers to be monotonically decreasing.

Another point against finding large sequences among the smaller numbers is due to the higher frequency of primes. Clearly, if p and p+k are both prime, then the largest such sequence starting at p can have length at most k-1.

Putting all of this together, the following conjecture is made, although the author readily admits that there is very little basis for it.

Conjecture: There is no largest value of k such that the set of numbers

{ S(n), S(n+1), S(n+2), . . . , S(n+k) }

is monotonically decreasing.

The aforementioned computer program was modified so that the search was for the largest sequence of numbers { n, n+1, n+2, . . . , n+k } so that S(n) < S(n+1) < S(n+2) .< . . . < S(n+k). When run through n = 2,646,000, the largest sequence found was for k=9.

n	721970	721971	721972	721973	721974	721975	721976	721977	721978	721979
S(n)	73	827	907	6067	10939	28879	90247	240659	360989	721979

It should come as no surprise that the largest sequence found for the numbers increasing is the same size as that for decreasing. All of the points raised for the decreasing sequence also hold for the increasing sequence. Therefore, the following analogous conjecture is also made for the increasing sequence.

Conjecture: There is no largest value of k such that the set of numbers

$$\{ S(n), S(n+1), S(n+2), \ldots, S(n+k) \}$$

is monotonically increasing.

III.6 Combining the Smarandache Function and Other Sequences

Let NS(k), k $\geq$ 1 be a generic expression for sequences of integers expressed in functional form. For example, the Fibonacci numbers

$$F_0 = 0, F_1 = 1 \quad F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 2$$

can be placed in functional form. Other examples include the Lucas numbers, triangular numbers and Cullen numbers.

It is possible to form the composition of any of these functions with the Smarandache function S(NS(k)). It is then natural to ask the question

Are there any numbers k such that S(NS(k)) = k?

For one such sequence, the result is immediate.

Theorem: Let $NS(k) = \frac{k(k+1)}{2}$, the kth triangular number. Then, there are an infinite number of integers k such that

$$S(NS(k)) = k.$$

Proof: If k is prime, then (k+1) is even and $\frac{(k+1)}{2}$ contains primes less than k. Therefore,

$S\left(\frac{(k+1)}{2}\right) < S(k) = k$. Putting it all together,

$$S\left(\frac{k(k+1)}{2}\right) = S(k) = k .$$

The next obvious question to ask is if there are solutions for k composite. A computer program was written to search for triangular numbers

$$\frac{k(k+1)}{2} = n$$

such that S(n) = k and k is composite. No solutions were found for n within the upper limits of a 32 bit unsigned long integer, n $\leq$ 4,000,000,000+. This is not simply coincidence and it is easy to prove that no such k is exists.

Theorem: There is no composite number k such that

$$S\left(\frac{k(k+1)}{2}\right) = k .$$

Proof: If k+1 is prime, then

$$S\left(\frac{k(k+1)}{2}\right) = k+1 .$$

Therefore, we can assume that both k and k+1 are composite. Clearly, these two numbers have no common factors. Therefore, we know that

$$S\left(\frac{k(k+1)}{2}\right) \leq \max\{S(k), S(k+1)\}.$$

It is also known that for n $\geq$ 10 and composite,

$$S(n) \leq \frac{n}{2} .$$

From this we know that

$$S\left(\frac{k(k+1)}{2}\right) \leq \frac{k+1}{2}$$, which is clearly less than k.

The set of polygonal numbers are defined by the general formula

$$z_n = \frac{n}{2}[2 + (n-1)d]$$

where d = 1, 2, 3, . . .

If d is fixed at 1, the sequence for n = 1, 2, 3, . . . forms the triangular numbers, d = 2 the square numbers and so on. For each of these sets of numbers, the following theorem should be obvious.

Theorem: Choose a "small" value of d in the general formula for polygonal numbers

$$z_n = \frac{n}{2}[2 + (n - 1)d]$$

Then, there are an infinite number of integers k such that $S(z_k) = k$.

Proof: If n is prime, then n divides $\frac{n}{2}[2 + (n - 1)d]$. If d is small enough, then n is the largest prime factor of the product z_n, there are no instances of n in $[2 + (n - 1)d]$ and $S([2 + (n - 1)d]) \leq S(n)$. Therefore, $S(z_n) = n$.

Sequences of numbers can also be defined using recurrence relations and two of the most widely known are the

Fibonacci $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

and

Lucas $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

A computer search for Fibonacci numbers $F_k \leq 4{,}000{,}000{,}000$ such that $S(F_k) = k$ yielded only the solution $F_5 = 5$. A similar search of Lucas numbers yielded only the solution $L_6 = 18$. The problem concerning the possible existence of additional solutions to either expression appears to be very hard.

The rate of growth of the Fibonacci and Lucas sequences are both well-known and the ratios

$$\frac{F_n}{n} \text{ and } \frac{L_n}{n}$$

both increase in a monotonic manner. Not so with the Smarandache function. The ratio

$$\frac{n}{S(n)}$$

can get arbitrarily large and is not monotonic, with a minimum value of 1. Clearly, if $S(F_n)=n$, then the Smarandache function must be the inverse of the Fibonacci function at that point. If it could be proved that after some upper limit M,

$$\frac{F_m}{m} > \frac{F_m}{S(F_m)} \quad \text{for } m > M,$$

then a search through all values up to M would complete the proof. Unfortunately, it does not seem likely that such a proof is possible. It is certainly beyond the ability of the author at this time.

Consider a number m that is the index of a Fibonacci number. What we are interested in here is the set of all numbers k such that $S(k) = m$. As was shown in Ashbacher[1], this set grows very large as m grows. Furthermore, the size of the largest number K such that $S(K) = m$ grows on the order of a factorial. Since that growth is faster than the growth of the Fibonacci function, it seems likely the order operator than can be placed between the ratios

$$\frac{F_m}{m} \text{ and } \frac{F_m}{S(F_m)}$$

will be both < and > infinitely many times. By chance, it is very possible that the ratios are identical.

Unsolved Question: Are there additional indices m such that $S(F_m) = m$?

Unsolved Question: Are there additional indices m such that $S(L_m) = m$?

Another number sequence that can be used is the set of numbers known as the Lucky numbers. To construct this set, perform the following sieving procedure.

1) Start with the set of numbers

1, 2, 3, 4, 5, . . .

2) Cross out every second(even) number to obtain

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, . . .

3) The next remaining number is 3, so cross out every third number in the remaining list

1, 3, 7, 9, 13, 15, 19, 21, 25, . . .

4) The next remaining number is 7, so strike out every seventh number in the remaining list

1, 3, 7, 9, 13, 15, 21, . . .

5) Repeat indefinitely.

After this process is completed, all numbers that remain are said to be Lucky. The first few elements of this set are

1, 3, 7, 9, 13, 15, 21, 25, 31, 33, 37, 43, . . .

If we index the Lucky numbers, we can express it in the form of a function with domain the natural numbers and range the Lucky numbers

L1 = 1, L2 = 3, L3 = 7 , etc.

We can then ask the question, are there any indices k such that

$S(L_k) = k$?

A computer program was written to examine this question for all Lucky numbers less than 50,000. Only two solutions were found

$Lucky_7 = 21$ and $Lucky_{41} = 205$.

Unsolved Question: Are there additional Lucky numbers such that $S(Lucky_k) = k$?

III.7 The Smarandache Prime Additive Complement

Each of the problems in the next section are from the collection by Kashihara[6].

The following definition appears in the collection by Dumetrescu and Seleacu[3].

Definition: For $n \geq 1$, the Smarandache Prime Additive Complement(SPAC(n)) is the smallest integer k such that n + k is prime.

The first few elements of this sequence are

1, 0, 0, 1, 0, 1, 3, 2, 1, 0, 1, 0, 3, 2, 1, ...

Since it is well-known that there are arbitrarily large gaps between primes, is is clear that there is no upper bound on the elements of SPAC.

In his book[6], Kashihara defines an additional function based on SPAC.

Definition: For $n \geq 1$, $A_n = \{ SPAC(1) + . . . + SPAC(n) \}/ n$.

And makes the following conjecture.

Conjecture: The sequence A_n is unbounded.

To test this conjecture, a computer program was written that will compute the value of n for selected values. The following table summarizes the computations. The natural logarithm is also included for comparison.

Table 10

n	A_n	ln(n)
100,000	7.80006	11.5129
200,000	8.49281	12.2060
300,000	8.90214	12.6115
400,000	9.16731	12.8992
500,000	9.35069	13.1223
600,000	9.53770	13.3046
700,000	9.69327	13.4588
800,000	9.80510	13.5923
900,000	9.92477	13.7101
1,000,000	10.01927	13.8155
1,500,000	10.42764	14.2209
2,000,000	10.74229	14.5086

3,000,000	11.11621	14.9141

Given the numeric evidence of the table, the conjecture seems a safe one. Notice how the last two columns maintain a fairly constant difference of approximately 3.78. Since the distribution of primes is described using the natural logarithm function, this is not surprising.

III.8 Series That Are Inverses

The following problem is also in the book by Kashihara.

Let $1 \leq a_1 \leq a_2 \leq \ldots \leq a_k \leq \ldots$ be an infinite sequence of integers so that no three members form an arithmetic progression. Florentin Smarandache asked the question:

Is it always true that

$$\sum_{k=1}^{\infty} \frac{1}{a_k} \leq 2 \, ?$$

The answer to this question is no. Consider the sequence

1, 2, 4, 5, 10, 20, 40, 80, . . .

Clearly, the subsequence

5, 10, 20, 40, . . .

is a geometric sequence with a = 5 and r = 2. Therefore, it is not possible to find three numbers that form a three term arithmetic progression.

Adding in the other terms, it is still clear that there are no three that form a three term arithmetic progression.

Forming the inverses of the geometric sequence and summing

$$\frac{1}{5} + \frac{1}{10} + \frac{1}{20} + \ldots$$

this is a geometric series with a = $\frac{1}{5}$ and r = $\frac{1}{2}$. The sum of this series is $\frac{2}{5}$.

Summing all terms,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{2}{5}$$

which is clearly greater than 2.

This is not the smallest such sequence. If 17 is added to the previous initial sequence, then the no three terms in arithmetic progression property still holds and the sum is even larger. This brings up the next unsolved question.

Unsolved Problem: Given a sequence of integers

$$a_1 \le a_2 \le \ldots \le a_k \le \ldots$$

where no three form an arithmetic progression, is there any bound on the sum

$$\sum_{k=1}^{\infty} \frac{1}{a_k} \ ?$$

III.9 Additional Equations Involving the Smarandache Function

The following two theorems involve properties of the Smarandache function.

Theorem: It is not possible to find a number n such that

$$S(n) * S(n + 1) = n.$$

Proof: Given the following well-known results.

1) If $n = p^k$, then $S(n) = mp$, where $m \le k$. If $k = 1$, then $S(n) = n$.

2) If $n = p_1^{a1} \ldots p_k^{ak}$, *then* $S(n) = \max\{S(p_i^{ak})\}$. If p_i is the prime factor using in the computation of S(n), it is said to be the prime of concern of n.

Using these results,

$$S(n + 1) = rq$$

where q is a prime factor of n + 1. From basic principles of divisibility, we know that q cannot divide n. Therefore, there is no solution to the equation

$$S(n) * S(n + 1) = n.$$

Problem: Find a solution to

$$\frac{S(n) * S(n+1)}{2} = n.$$

From the previous theorem, if there is a solution, 2 must be the prime of concern of n + 1. Using $(n + 1) = 2^k$ in a search for solutions, we quickly find

$$n = 15,\ S(15) = 5,\ S(16) = 6.$$

Smarandache also defined a series of problems based on concatentating together numbers in a sequence.

Definition: The Fibonacci numbers are the sequence of integers defined by the following recurrence relation.

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

Definition: The Smarandache Concatenated Fibonacci Sequence SCFS(n) is the sequence formed by the following recurrence. The symbol is used to represent concatenation.

$$SCFS(1) = F_1\ ,\ SCFS(n) = SCFS(n-1) \circ F_n \text{ where } F_n \text{ is the nth Fibonacci number and } n > 1.$$

Question: Other than the trivial case of the first element, is there any integer n such that SCFS(n) is a Fibonacci number?

A program was written to examine this problem for small values of n. It was run for all $n \leq 400$ and no additional solutions were found. At that point SCFS(401) consisted of 16906 digits. Given the size of SCFS(n) at this point, the following conjecture seems very safe.

Conjecture: Aside from the trivial SCFS(1) there is no integer n such that SCFS(n) is a $\frac{1}{6n}$.

Argument: Given the definition of the Fibonacci numbers, it is easy to show that for k any number of digits, there are in fact several Fibonacci numbers having that number of digits. Furthermore, it is also easy to show that there are at most two k-digit Fibonacci numbers where the leading digit is 1. Since all of the remaining 16905+ digits would then have to match those of a Fibonacci number exactly, the probability of such an event happening is extremely small.

III.10 The Smarandache Inferior And Superior Square Roots

Definition: For any integer $n \geq 0$, the Smarandache Inferior Square Root (SISR(n)), is the largest perfect square less than or equal to n.

Definition: For any integer $n \geq 0$, the Smarandache Superior Square Root(SSSR(n)) is the smallest perfect square greater than or equal to n.

The following definitions appeared in the collection by Kashihara.

Definition:

$$S_n = \frac{\sum_{k=1}^{n} \mathrm{SSSR}(k)}{n}$$

Definition:

$$I_n = \frac{\sum_{k=1}^{n} \mathrm{SISR}(k)}{n}$$

Along with the problems

a) Determine $\lim_{n\to\infty} (S_n - I_n)$

b) Determine $\lim_{n\to\infty} \frac{S_n}{I_n}$

Theorem:

a) $\lim_{n\to\infty} (S_n - I_n) = \infty$

b) $\lim_{n\to\infty} \frac{S_n}{I_n} = 1$

Proof: The following results will be used in both proofs.

(i) $$I_{n^2} = \frac{\left[n^2 + \sum_{k=1}^{n-1} (2k + 1) \; k^2 \right]}{n^2}$$ for n $\geq$ 2.

(ii) $$S_{n^2} = \frac{\left[\sum_{k=1}^{n} (2k - 1) \; k^2 \right]}{n^2}$$ for n $\geq$ 2.

Proof of (i):

By induction on n.

Basis n = 2

$$I_{2^2} = \frac{7}{4} = \frac{2^2 + 3}{4}$$

and with

$$\sum_{k=1}^{1} (2k + 1)k = 3 * 1 = 3$$

it is clear that the result holds for n = 2.

Inductive Step: Assume that the formula holds for j, i.e.

$$I_{j^2} = \frac{\left[j^2 + \sum_{k=1}^{j-1} (2k + 1) k^2 \right]}{j^2}.$$

Split the j^2 term from the numerator and consider the elements being added as the summing process moves from j^2 to $(j + 1)^2$.

Number	Inferior Square Part
j^2	j^2
$j^2 + 1$	j^2
$j^2 + 2$	j^2
.	.
.	.
.	.
$(j + 1)^2 - 1$	j^2
$(j + 1)^2$	$(j + 1)^2$

The sum of the inferior square parts when moving from j^2 to $((j + 1)^2 - 1)$ is $j^2 * (2j + 1)$. Adding this to the summation yields

$$\sum_{k=1}^{j} (2k + 1) k^2 .$$

Since $SSIR((j + 1)^2) = (j + 1)^2$, we add that in and have the desired formula for j + 1 in the numerator.

Therefore, by the principle of mathematical induction, the formula is true for all n.

Proof of (ii):

Again, by induction on n.

Basis Step: For n = 2, $S_4 = \frac{13}{4}$, and

$$\sum_{k=1}^{2} (2k - 1)k^2 = 13 .$$

Inductive Step: Assume that for j

$$S_{j^2} = \frac{\left[\sum_{k=1}^{j} (2k - 1)\, k^2 \right]}{j^2} .$$

Now, consider the elements when moving from $j^2 + 1$ to $(j + 1)^2$.

Number	Superior Square
$j^2 + 1$	$(j + 1)^2$
$j^2 + 2$	$(j + 1)^2$
.	.
.	.
$(j + 1)^2$	$(j + 1)^2$

since there are 2j + 1 of these, the amount added is $(2j + 1)(j + 1)^2$, which can be rewritten in the form $(2(j+1) - 1)(j + 1)^2$.

Adding this to the formula for j, we have verified the formula for j + 1. Therefore, by the principle of mathematical induction, the formula is true for all n.

It is well-known that

$$\sum_{k=1}^{n} k^3 = \frac{1}{4} n^2 (n + 1)^2$$

and

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n + 1)\left(2n + 1\right).$$

Armed with these formulas, we can now obtain closed-form expressions for $S_n{}^2$ and $I_n{}^2$.

$$I_{n^2} = \frac{n^2 + \sum_{k=1}^{n-1} (2k + 1)k^2}{n^2} =$$

$$\frac{n^2 + 2\sum_{k=1}^{n-1} k^3 + \sum_{k=1}^{n-1} k^2}{n^2} =$$

$$\frac{n^2 + 2\left[\frac{1}{4}(n-1)^2 n^2\right] + \left[\frac{1}{6}(n-1)n(2n-1)\right]}{n^2} =$$

$$\frac{n^2 + \frac{1}{2}n^4 - \frac{2}{3}n^3 + \frac{1}{6n}}{n^2} =$$

$$\frac{1}{2}n^2 - \frac{2}{3}n + 1 + \frac{1}{6n}.$$

$$S_{n^2} = \frac{\sum_{k=1}^{n} (2k - 1)k^2}{n^2} =$$

$$\frac{2\sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k^2}{n^2} =$$

$$\frac{\frac{1}{2}n^4 - \frac{2}{3}n^3 - \frac{1}{6n}}{n^2}$$

$$\frac{1}{2}n^2 - \frac{2}{3}n - \frac{1}{6n}$$

Now that we have these results, the solutions are easy.

$$S_{n^2} - I_{n^2} = \frac{1}{2}n^2 + \frac{2}{3}n - \frac{1}{6n} - \frac{1}{2}n^2 + \frac{2}{3}n - 1 = \frac{4}{3}n + \frac{1}{6n} - 1$$

which clearly goes to infinity as n goes to infinity. Using the obvious fact that

SSSR(k) $\geq$ SISR(k) for all k,

$$\lim_{n\to\infty}(S_n - I_n) = +\infty.$$

b) $$\frac{S_{n^2}}{I_{n^2}} = \frac{\frac{1}{2}n^2 + \frac{2}{3}n - 1 + \frac{1}{6n}}{\frac{1}{2}n^2 - \frac{2}{3}n + 1 - \frac{1}{6n}}$$

which clearly goes to 1 as n goes to infinity. The final step is to determine what happens between the squares. Choose an interval from n^2 to $(n + 1)^2$. We need consider only the summations in the numerators

$$\sum_{k=1}^{n^2} SSSR(k) = \frac{1}{4}n^4 + \frac{1}{2}n^3 - \frac{1}{6n}$$

$$\sum_{k=1}^{n^2} SISR(k) = n^2 + \frac{1}{4}n^4 - \frac{2}{3}n^3 + \frac{1}{6n}$$

Moving from n^2 to $(n + 1)^2$, each step adds n^2 to the sum of SISR(k) and $(n + 1)^2$ to the sum of SSSR(k). Therefore, the largest difference of these two sums over that interval occurs at $(n + 1)^2$ - 1. However, since $(n + 1)^2$ adds the same to both sums, the difference remains the same as that after adding the values for the $(n + 1)^2$ - 1 step. From this, it is clear that

$$\lim_{n\to\infty}\frac{S_n}{I_n} = 1.$$

Given the following definitions:

Definition: $s_n = \sqrt[n]{SSSR(0) + \ldots + SSSR(n)}$

Definition: $i_n = \sqrt[n]{SISR(0) + \ldots + SISR(n)}$

the following problems can be posed:

a) Determine $\lim\limits_{n \to \infty} (\mathrm{s_n} - \mathrm{i_n})$.

b) Determine $\lim\limits_{n \to \infty} \frac{\mathrm{s_n}}{\mathrm{i_n}}$.

Solution to (a):

The following inequalities are obvious

SISR(k) $\leq$ k for all k $\geq$ 0.

$$\sum_{k=1}^{n} \mathrm{SISR}(\mathrm{k}) \leq \sum_{k=1}^{n} \mathrm{k} = \frac{n(n+1)}{2}$$

$$i_n = \sqrt[n]{\mathrm{SISR}(1) + \ldots + \mathrm{SISR}(\mathrm{n})} \leq \sqrt[n]{\frac{n(n+1)}{2}} \leq \sqrt[n]{n^3} = n^{\frac{3}{n}}$$

It is well-known that

$$\lim_{n \to \infty} \mathrm{n}^{\frac{3}{n}} = 1.$$

The following inequalities are also obvious

SSSR(k) $\leq$ k^2

$$\sum_{k=1}^{n} \mathrm{SISR}(\mathrm{k}) \leq \sum_{k=1}^{n} \mathrm{k}^2 = \frac{1}{6}\mathrm{n}(\mathrm{n}+1)(2\mathrm{n}+1)$$

$$s_n = \sqrt[n]{\mathrm{SSSR}(0) + \ldots + \mathrm{SSSR}(n)} \le \sqrt[n]{\frac{1}{6} n(n+1)(2n+1)} \le$$

$$\sqrt[n]{\frac{2n^3 + 3n^2 + n}{6}} \le$$

$$\sqrt[n]{\frac{6n^3}{6}} = \sqrt[n]{n^3} = n^{\frac{3}{n}}$$

And once again,

$$\lim_{n \to \infty} n^{\frac{3}{n}} = 1$$

as well as

$s_n \ge 1$ for $n \ge 1$.

Using basic addition on limits gives the result

$$\lim_{n \to \infty} (s_n - i_n) = 0.$$

Solution to (b):

Easily, $i_n \ge 1$ for all $n \ge 1$ and $s_n \ge 1$ for all $n \ge 1$. Also, $s_n \ge i_n$ for $n \ge 0$. Since the numerator of the fraction

$$\frac{s_n}{i_n}$$

goes to 1 in the limit, it follows that

$$\lim_{n \to \infty} \frac{s_n}{i_n} = 1.$$

Chapter IV

The Pseudo-Smarandache Function

IV.1 Definition, Computation and Basic Theorems

In the previously mentioned book by Kashihara[6], he defines a new function similar to the Smarandache function.

Definition: Given any integer n $\geq$ 1, the value of the Pseudo-Smarandache function Z(n) is the smallest number m such that

$$n \mid \sum_{k=1}^{m} k.$$

It is well-known that this is equivalent to

Z(n) = m is the smallest number m such that $n \mid \frac{m(m+1)}{2}$.

This function has many properties similar to the Smarandache function and Kashihara does a good job in stating and proving many of them. Our purpose here is to explore some of the questions left open in that book.

The first point to address is a simple program to compute the values of Z(n). Since the limitation of an unsigned long integer in C is slightly over 4 billion, there is a computational limit of around 63,245. Therefore, the program given here will be in extended precision UBASIC that allows for much larger numbers.

```
100 n=2
110 m=1
120 t1=int((m*(m+1))/2)
130 t1=t1\n
140 if res=0 then goto 170
150 m=m+1
160 goto 110
170 print"For n equal to ";n," the value of Z(n) is ";m
180 input z%
190 n=n+1
200 goto 110
```

We start with a few simple theorems to provide some background.

Theorem:

$$Z(2^k) = 2^{k+1} - 1.$$

Proof: Since only one of m and (m+1) can be even, it follows that Z(2k) is the smallest number m such that the even number in the numerator of m(m+1)2 contains k+1 instances of 2. The smallest such number is clearly 2k+1 and the value of m is smallest when m+1 = 2k+1.

Theorem:

$Z(n) \leq n - 1$ for n an odd number.

Proof: Let n be an odd number. It is clear that in the fraction $\frac{(n-1)n}{2}$, (n - 1) is even and divisible by 2. Therefore

$$n \mid \frac{(n-1)n}{2}.$$

Theorem: If $n = p^k$, where p is an odd prime, then $Z(n) = p^k - 1$.

Proof:
Let $n = p^k$, where p is an odd prime. Then of course the smallest number m such that $p^k \mid m$ is $m = p^k$. Furthermore, p cannot divide (m - 1) and since (m - 1) is even, $2 \mid (m - 1)$. Therefore, $Z(p^k) = p^k - 1$.

Corollary:
If p is an odd prime, then Z(p) = p - 1.

IV.2 The Pseudo-Smarandache Function and Numbers that Contain Powers of 2

Theorem: If k > 0

$$Z(2^k * 3) = \begin{cases} 2^{k+1} - 1 \text{ if k is odd} \\ 2^{k+1} \text{ if k is even} \end{cases}$$

Proof: It is easy to verify that if k is odd, $3 \mid 2^{k+1} - 1$ and if k is even, $3 \mid 2^{k+1} + 1$. Therefore, $2^k * 3 \mid \frac{(2^{k+1}-1) * 2^{k+1}}{2}$ if k is odd and $2^k * 3 \mid \frac{(2^{k+1}-1) * \left(2^{k+1}+1\right)}{2}$ if k is even. Since 2^{k+1} is the smallest power of two that can be divided by 2 and still be divisible by 2^k, both are minimal.

Note that in both cases, we have that Z(n) < n.

Theorem: If k > 0, $Z(2^k * 5) =$

a) 2^{k+2} if k is congruent to 0 modulo 4
b) 2^{k+1} if k is conguent to 1 modulo 4
c) $2^{k+2} - 1$ if k is congruent to 2 modulo 4
d) $2^{k+1} - 1$ if k is congruent to 3 modulo 4

Note once again that in all cases, Z(n) < n.

Proof: We first state an easily verifiable fact and then proceed on a case-by-case basis.

It is easy to verify the following facts concerning powers of 2 greater than zero

2^k is congruent to 1 modulo 5 if k is congruent to 0 modulo 4
2^k is congruent to 2 modulo 5 if k is congruent to 1 modulo 4
2^k is congruent to 3 modulo 5 if k is congruent to 3 modulo 4
2^k is congruent to 4 modulo 5 is k is congruent to 2 modulo 4

Note that in all cases, the smallest power of two that can be divided by 2 and then by 2^k is 2^{k+1}.

a) Since k is congruent to 0 modulo 4, 2^{k+1} is congruent to 2 modulo 5, so neither 2^{k+1} - 1 or 2^{k+1} + 1 is divisible by 5. However, 2^{k+2} is congruent to 4 modulo 5, so 2^{k+2} + 1 is evenly divisible by 5. Therefore,

$$2^k * 5 \mid \frac{2^{k+2} * (2^{k+2} + 1)}{2}$$

and 2^{k+2} is the smallest such number.

b) Since k is congruent to 1 modulo 4, 2^{k+1} is congruent to 4 modulo 5, so 2^{k+1} + 1 is divisible by 5. Therefore,

$$2^k * 5 \mid \frac{2^{k+1} * (2^{k+1} + 1)}{2}$$

and 2^{k+1} is the smallest such number.

c) Since k is congruent to 2 modulo 4, 2^{k+1} is congruent to 3 modulo 5 and neither 2^{k+1} -1 or 2^{k+1} + 1 is divisible by 5. However, 2^{k+2} - 1 is divisible by 5, so

$$2^k * 5 \mid \frac{(2^{k+2} - 1) * 2^{k+2}}{2}$$

and 2^{k+2} - 1 is the smallest such number.

d) Since k is congruent to 3 modulo 4, 2^{k+1} is congruent to 4 modulo 5 so

$$2^k * 5 \mid \frac{(2^{k+1} - 1) * 2^{k+1}}{2}$$

and 2^{k+1} - 1 is the smallest such number.

IV.3 The Bounds on Abs(Z(n+1) - Z(n)) and Z(n+1) Divided By Z(n)

The following two unsolved questions appeared in the book by Kashihara.

Unsolved Question: Is the following bounded or unbounded

$$| Z(n+1) - Z(n) |?$$

Theorem:

$$| Z(n+1) - Z(n) | \text{ is unbounded.}$$

Proof: Let $n + 1 = 2^k$. Then, as a consequence of two of the above theorems, $Z(n+1) = 2^{k+1} - 1$ and $Z(n) \leq n$. Therefore, $|Z(n+1) - Z(n)| \geq 2^{k+1} - 1 - 2^k + 1 = 2^k$. Clearly, this number is unbounded as k goes to ∞.

Unsolved Question: Is the following bounded or unbounded

$$\left|\frac{Z(n+1)}{Z(n)}\right|?$$

In this case, a computer program was written that computes the largest ratio of

$$\frac{Z(n+1)}{Z(n)}$$

for all n in the range [1,m].

For example, in the range [1,1000] the largest value for the ratio was for the pair

$Z(990) = 44$, $Z(991) = 990$, ratio = 22.5

In the range [1,2000], the largest value was for the pair

$Z(1830) = 60$, $Z(1831) = 1830$, ratio = 30.5

The program was run up to 41,000 and the increasing values of n that formed the largest ratios are summarized in the table below.

Table 11

n	Z(n)	n+1	Z(n+1)	ratio
990	44	991	990	22.5
1830	60	1831	1830	30.5
2926	76	2927	2926	38.5

3916	88	3917	3916	44.5
4095	90	4096	8191	91.0
17578	187	17579	17578	94.0
20706	203	20707	20706	102.0
21736	208	21737	21736	104.5
22366	211	22367	22366	106.0
24976	223	24977	24976	112.0
26106	228	26107	26106	114.5
27966	236	27967	27966	118.5
31626	251	31627	31626	126.0
33930	260	33931	33930	130.5
34980	264	34981	34980	132.5
36856	271	36857	36856	136.0
37950	275	37951	37950	138.0
39340	280	39341	39340	140.5
40470	284	40471	40470	142.5

From this table, the following conjecture seems reasonable.

Conjecture:

$$\left|\frac{Z(n+1)}{Z(n)}\right|$$

is unbounded.

IV.4 Series Involving The Pseudo-Smarandache Function

The following theorem is easy to prove.

Theorem: $\sum_{k=1}^{\infty} \frac{1}{Z(k)}$ is divergent.

Proof: If p is an odd prime, then Z(p) = p - 1. Which means that

$$\frac{1}{Z(p)} \geq \frac{1}{p}.$$

Since the sum of the inverses of the primes is divergent the sum of the inverses of Z(n) must be as well.

The following question appears to be very hard.

Unsolved Question: Forming the set of partial sums

$$\sum_{k=1}^{n} \frac{1}{Z(k)}$$

the sum is an integer for n = 1. Is it an integer for any other value of n?

However, this one may be tractable.

Unsolved Question: Is the following convergent or divergent?

$$\sum_{k=1}^{\infty} \frac{1}{(Z(n))^2}$$

It is easy to prove that $Z(n)^2 > n$, for $n > 1$, so this series is component-wise less than the harmonic series. For every n that is not a power of two, $n^2 > Z(n)^2$ so for most terms it is greater than the sum of the inverse squares. Running a computer program to examine some partial sums, we see that the sums for the first 1,000 and 10,000 terms are 2.158853 and 2.210704 respectively. This strongly indicates that the series is in fact convergent and the sum is fairly small.

It is most likely that the answers to the following two questions will never be known.

Question: What is the smallest value of r such that

$$\sum_{k=1}^{\infty} \frac{1}{(Z(k))^r}$$

is convergent?

Question: What is the smallest value of r such that

$$\sum_{k=1}^{\infty} \frac{1}{(S(k))^r}$$

is convergent?

IV.5 Palindromes and the Pseudo-Smarandache Function

Definition: A number is said to be palindromic if it reads the same forwards and backwards. Examples of palindromes are

121, 34566543, 1111111111

One very interesting problem involves palindromes and iterations of the Pseudo-Smarandache function.

There are some palindromic numbers n such that Z(n) is also palindromic. For example,

$Z(909) = 404$ $\quad$ $Z(2222) = 1111$

A simple computer program was written to search for palindromic values of n such that Z(n) is also a palindrome over the range $10 \leq n \leq 10000$. Of the 189 palindromic values of n, for 37 or slightly over 19%, Z(n) was also palindromic.

Furthermore it is sometimes possible to repeat the function again and get another palindrome.

$Z(909) = 404, Z(404) = 303, Z(303) = 101$

Once again, a computer program was run looking for palindromic values of n in the same range such that Z(n) and Z(Z(n)) are also palindromic. In this case, 9 of the 37 values previously found, or slightly over 24%, satisfied the extended criteria.

Doing this one more time looking for palindromic values of n such that Z(n), Z(Z(n)), and Z(Z(Z(n))) are all palindromic, 2 of the 9 solutions to the previous question or roughly 22% exhibited the desired properties.

Definition: Let $Z^k(n) = Z(Z(Z(Z(\ldots n))))$ where the Z function is executed k times. For notational purposes, let $Z^0(n) = n$.

The computer program was modified to look for solutions in the range $10 \leq n \leq 10{,}000$ such that $Z^i(n)$ is palindromic for i = 0,1,2,3, and 4. No solutions were found in this range. If the percentages cited earlier, have any general applicability, this result is reasonable. Expanding the search to an upper limit of 100,000 one solution was found

$Z(86868) = 17271, Z(17271) = 2222, Z(2222) = 1111, Z(1111) = 505$

Since Z(505) = 100, this is the largest such sequence in this region.

Unsolved Question: Given a palindromic number n, what is the largest number of times one can perform the iteration

$Z(n), Z(Z(n)), Z(Z(Z(n))), \ldots$

and obtain a palindrome each time?

Unsolved Question: Do the percentages determined previously have any general applicability?

Conjecture: There is no largest value of m such that $Z^m(n)$ is palindromic for all

$k = 0,1,2,3, \ldots k.$

There are reasonable arguments in favor of this conjecture. Palindromes tend to be divisible by other palindromes. In conducting the search, numbers like 11. . .11, 22 ... 22 and 10 ... 01 appeared in the range of Z quite often.

IV.6 Concatenating The Values of the Pseudo-Smarandache Function

The following theorem concerning the Smarandache function is well-known.

Theorem: Given the values of the Smarandache function

$$S(1) = 0, S(2) = 2, S(3) = 3, S(4) = 4, S(5) = 5, S(6) = 3, ...$$

construct the number r by concatenating the values in the following way.

$$r = 0.023453...$$

The number r is irrational.

And the similar question can be posed for the Pseudo-Smarandache function.

Theorem: Given the values of the Pseudo-Smarandache function

$$Z(1) = 1, Z(2) = 3, Z(3) = 2, Z(4) = 7, Z(5) = 4, ...$$

construct the number r by concatenating the values in the following way

$$r = 0.13274 ...$$

The number r is irrational.

Proof: Once again, there is the appeal to the well-known theorem of Dirichlet.

If $a \neq 0$ and $d > 1$ are relatively prime integers, then the arithmetic sequence

$$a, a + d, a + 2d, a + 3d, ...$$

contains an infinite number of prime numbers.

Suppose that r is rational. Then at some point, the digits must repeat. Let m represent the number of digits in the repeated sequence of digits. Choose a to be a number with m + 2 digits whose last digit is odd. Let $d = 10^{m+2}$. Clearly, a and d are relatively prime and the last m + 2 digits of all numbers in the arithmetic sequence

$$a, a + d, a + 2d, a + 3d, a + 4d, ...$$

are all the same. Applying the theorem of Dirichlet, this sequence contains an infinite number of prime numbers. Since it is known that $Z(p) = p - 1$ and the last digit of all of these primes is greater than zero, using this in the construction of r will cause an infinitely repeated sequence of digits greater than m. Since there is more than one choice for a, this contradicts the supposition of a repeated sequence of digits.

IV.7 Equations Involving the Pseudo-Smarandache Function

A family of equations that can be created using the Pseudo-Smarandache function involves an integer constant.

Question: For what values of k $\geq$ 0 is there an integer n such that

$$k * Z(n) = n?$$

The first three possible values of k are easily dealt with.

Theorem: There are an infinite number of solutions to the equation

$$2 * Z(n) = n.$$

Proof: Let p be a prime of the form p = 4k + 3. Clearly,

$$2 * (4k+3) \mid \frac{(4k+3)(4k+4)}{2}$$

and 4k + 3 is the smallest such number. Therefore, Z(2p) = p, or equivalently, 2 * Z(p) = n. It is well-known that there are an infinite number of primes of this form, so we are done.

Theorem: There are an infinite number of solutions to the equation

$$3 * Z(n) = n.$$

Proof: Let p be an odd prime of the form p = 3k+2. It is well-known that there are an infinite number of such primes. Consider n = 3p. Clearly,

$$n = 3(3k+2) \mid \frac{(3k+2)(3k+3)}{2}$$

and this is the smallest possible choice for m.

Theorem: There are an infinite number of solutions to the equation

$$4 * Z(n) = n.$$

Proof: Let m be of the form 8k - 1 and consider the number n = 4m. Clearly,

$$n = 4(8k-1) \mid \frac{(8k-1)8k}{2}$$

and (8k - 1) is the smallest such number. Therefore, Z(n) = Z(4m) = m.

A computer program was run testing for solutions for all values of k in the set
k = { 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15} and many solutions were found for each value of k. With this numeric evidence, perhaps the following would be a more appropriate question.

Unsolved Question: Is there a value for k where there are only a finite number of solutions to the equation

$$k * Z(n) = n?$$

The following question is somewhat related to the previous.

Question: What the largest value of the ratio

$$\frac{Z(2n)}{Z(n)}$$

as n ranges over all positive integers?

A computer search was performed for all values of n up through 20,000 and the largest value of the ratio was

n = 8128, Z(n) = 127, Z(2n) = 16128

$$\frac{Z(2n)}{Z(n)} = 126.992126$$

and the growth of the ratio exhibits no pattern that can be used to predict the maximum.

Definition: Let ALTERZ(n) be the function that is the sum of alternating values of Z(i) from n down to 1.

$$\text{ALTERZ(n)} = \text{Z(n)} - \text{Z(n - 1)} + \text{Z(n - 2)} - \ldots \pm \text{Z(1)}.$$

For example,

ALTERZ(4) = Z(4) - Z(3) + Z(2) - Z(1).

Now

ALTERZ(3) = 2 - 3 + 1 = 0
ALTERZ(7) = 6 - 3 + 4 - 7 + 2 - 3 + 1 = 0

A computer search up through 20,000 failed to reveal any additional solutions.

Question: Are there any more values of n such that

ALTERZ(n) = 0?

It is also possible to create polynomial expressions using the Pseudo-Smarandache function.

Question: What are the solutions to the expression

$(Z(n))^2 + Z(n) = 2n$?

It turns out that there is an infinite number of solutions and it is a well-known family of integers.

Theorem: The triangular numbers,

$$T(r) = \frac{r(r+1)}{2}$$

are solutions to the expression

$$(Z(n))^2 + Z(n) = 2n.$$

Proof: Let n = $T(r) = \frac{r(r+1)}{2}$ be a triangular number. It is clear that 2n = r(r+1) and Z(n) = r. Simple substitution will then give

$$r^2 + r = r(r+1).$$

Question: Is there a solution to the expression

$$(Z(n))^2 + Z(n) = 2n$$

that is not a triangular number?

A brief computer search yielded no solutions to this question.

IV.8 Equations That Combine the Smarandache and Pseudo-Smarandache Functions

The following theorem that combines the Pseudo-Smarandache and Smarandache functions is easy to prove.

Theorem: There are an infinite number of solutions to the equation S(n) = Z(n).

Proof: Given the following well-known facts.

1) There are an infinite number of primes of the form p = 4k + 3.
2) S(2p) = p for p an odd prime.

Let p be a prime of the form p = 4k + 3. Then 4 | p + 1 and 2p divides $\frac{p(p+1)}{2}$. Since p - 1 is of the form 4k + 2, 2p cannot divide $\frac{(p-1)p}{2}$.

Clearly, it is not possible for 2p to divide $\frac{m(m+1)}{2}$ for m any number less than (p - 1).
Therefore, Z(2p) = p = S(2p).

Note that since Z(p) = p - 1 and S(p) = p for p an odd prime, Z(n) < S(n) infinitely often. Furthermore, since $Z(2^k) = 2^{k+1} - 1$ and $S(2^k) \leq 2^k$, Z(n) > S(n) infinitely often.

At this time, the following problem is still unsolved, although the answer has been conjectured to be false.

Unsolved Problem: Are there any integers n such that S(n) = S(n+1)?

Kashihara asks the similar question about the Pseudo-Smarandache function.

Unsolved Problem: Are there any integers n such that Z(n) = Z(n+1)?

A computer program was written to search for solutions and it was run up through n = 450,000. No solutions were found.

Combining these two functions and reversing the order of the composition, we have the problem.

Problem: How many solutions are there to the equation Z(S(n)) = S(Z(n))?

A computer program was created to search for solutions. Twenty eight were found in the range $2 \leq n \leq 10{,}000$. A complete list follows.

Table 12

n	Z(S(n))
2	3
3	2
4	7
5	4
10	4
25	4
50	4
56	6
75	4
84	6
100	4
150	4
168	6
300	4
440	10
616	10

1100	10
1925	10
2200	10
3080	10
3328	12
3850	10
4352	16
7700	10
8320	12
8704	16
8800	10
9856	10

I can discern no pattern in these solutions. Therefore, I leave it to the reader to determine if there are an infinite number of solutions to this problem.

IV.9 Infinite Series That Combine the Smarandache and Pseudo-Smarandache Functions

Infinite series can also be constructed that involve both Z(n) and S(n).

Theorem: $\sum_{k=1}^{\infty} \frac{1}{Z(n)+S(n)}$

is divergent.

Proof: If p is an odd prime, then Z(p) = p - 1 and S(p) = p. From this,

$$\frac{1}{Z(p)+S(p)} = \frac{1}{p-1+p} = \frac{1}{2p-1} > \frac{1}{2p}.$$

Therefore, restricting the sum only to the primes

$$\sum \frac{1}{2p} = \frac{1}{2} \sum \frac{1}{p}$$

and the sum of the inverses of the primes is known to be divergent. Since this is a partial sum of the series in question, it must be divergent.

The following question is probably unresolvable.

Question: What is the smallest value of r such that

$$\sum_{k=1}^{\infty} \frac{1}{(Z(k)+S(k))^r}$$

is convergent?

Question: Is the following sum convergent or divergent?

$$\sum_{n=1}^{\infty} \frac{1}{Z(n) * S(n)}$$

Given that the difference n - S(n) can be made arbitrarily large and Z(n) < n for almost all n, the inverse of the product is most likely divergent.

It has already been proven that there are an infinite number of solutions to the expression, Z(n) = S(n), so there are an infinite number of values of n where Z(n) * S(n) is a perfect square. However, the problem can still be made interesting if the problem is restated.

Question: Are there any values of n where S(n) $\neq$ Z(n) such that S(n) * Z(n) is a perfect square?

Running a computer program up through n = 300, several solutions were found.

Table 13

n	S(n)	Z(n)
99	11	44
110	11	44
112	7	63
165	11	44
192	8	128
209	19	76
252	7	63
261	29	116
275	11	99

Strongly supporting the conjecture

Conjecture: There are an infinite number of integers n such that Z(n) * S(n) is a perfect square and Z(n) $\neq$ S(n).

Chapter V

The Psuedo-Smarandache Function and the Classic Functions Of Number Theory

V.1 Equations Involving the Pseudo-Smarandache and Euler Phi Functions

Definition: Given any integer $n \geq 2$, the Euler phi function $\phi(n)$ is the number of integers k, $1 \leq k < n$, such that k and n are relatively prime.

Theorem: There are an infinite number of integers n such that $Z(n) = \phi(n)$.

Proof: It is well-known that if p is an odd prime, $\phi(p) = p-1$. Combining this with the corollary to a previous theorem, we have that if p is an odd prime

$$Z(p) = p - 1 = \phi(p).$$

In the previous case, we used odd primes only. That is not the only infinite family of solutions, as can be seen from the following theorem.

Theorem: There are an infinite number of composite integers n such that $Z(n) = \phi(n)$.

Proof: Let $n = 2p$, where p is an odd prime of the form $p = 4k + 1$. Consider the fraction

$$\frac{(p-1)p}{2}.$$

Replacing the values by the form of p, we have

$$\frac{(4k+1-1)(4k+1)}{2} = \frac{4k(4k+1)}{2}\ 2k(4k+1).$$

Clearly,

$$2(4k + 1) \mid 2k(4k+1)$$

and $p = 4k + 1$ is the smallest such number. Therefore, $Z(2p) = p - 1$. It is well-known that $\phi(2p) = p - 1$ for p an odd prime.

However, there is another question that can be asked.

Question: Are there any other solutions to the equation $Z(n) = \phi(n)$?

A related, but not identical question can also be posed.

Question: Is there another infinite family of solutions to the equation $Z(\phi(n)) = \phi(Z(n))$?

Forming the sum of the Pseudo-Smarandache and Euler phi functions, we can pose and prove a simple theorem.

Theorem: There are an infinite number of solutions to the expression

$$Z(n) + \phi(n) = n.$$

Proof: Let $n = 2^{2j} + 2^{2j+1}$ where $j \geq 1$. Factoring it, we have

$$n = 2^{2j} * 3.$$

Using the well-known formula for the computation of the function,

$$\phi(n) = (2 - 1)2^{2j-1} (3 - 1)3^{0} = 2^{2j}.$$

It is easy to verify that if k is odd, then

$$3 \mid 2^{k} + 1.$$

From this, it follows that

$$2^{2j} * 3 \mid \frac{2^{2j+1}\left(2^{2j} + 1\right)}{2}$$

and it is easy to see that 2^{2j+1} is the smallest such m. Therefore,

$$Z(2^{2j} * 3) = 2^{2j+1}$$

and

$$Z(n) + \phi(n) = n.$$

V.2 Equations Involving the Pseudo-Smarandache and Sum of Divisors Functions

Another classic number theoretic function is the sigma or sum of divisors function.

Definition: Given any integer $n \geq 1$, $\sigma(n)$ is the sum of all the divisors of n.

Theorem: If $n = 2^{k}$ where $k \geq 0$ then $\sigma(n) = Z(n)$.

Proof: It has already been proven that $Z(n) = 2^{k+1} - 1$ if $n = 2^{k}$ and it is well-known that $\sigma(p^{k}) = p^{k+1} - 1$.

In conducting a computer search for additional solutions, none were found.

Question: Are there any solutions to the equation $\sigma(n) = Z(n)$ where n is not a power of 2?

A computer program was written to search for solutions to the equation $\sigma(Z(n)) = Z(\sigma(n))$. It was run for all $n \leq 10{,}000$ and the following solutions were found.

Table 14

n	σ (Z(n))
132	63
171	39
730	296
848	216
1387	480
1679	480
1975	960
2250	1520
2436	384
2615	1440
2911	1728
3077	728
3084	2064
3267	399
3330	1520
3601	1112
4625	1520
5053	1024
5307	960
5327	3936
5753	1440
6130	2456
6154	728
6266	2904
7202	1112
7992	1824
8279	1952
8938	440
9023	2880
9250	1520
9937	1440

Again, I can discern no pattern to these solutions.

Question: Is there an infinite number of solutions to the equation $Z(\sigma(n)) = \sigma(Z(n))$?

V.3 Equations Involving the Pseudo-Smarandache and Number of Divisors Functions

Another classic function of number theory is the number of integral divisors function.

Definition: Let $n \geq 1$. The divisors function d(n) is the number of integers m, where $1 \leq m \leq n$ such that m divides n.

There are many problems that can be created involving d(n) and the Pseudo-Smarandache function.

Question: For how many integers n, is it true that

$$d(n) = Z(n)?$$

A computer program was written to search for solutions and the only ones found up through 10,000 were

$$n = 1, 3, \text{ and } 10.$$

Question: Is there another solution to

$$d(n) = Z(n)?$$

Question: How many solutions are there to the expression

$$d(n) + Z(n) = n?$$

A computer search up through 10,000 yielded only the number

$$n = 56,\ d(56) = 8,\ Z(56) = 48$$

Question: Is there another solution to

$$d(n) + Z(n) = n?$$

Combining d(n) and Z(n) in another way, we obtain an infinite family of expressions

$$(Z(n) + d(n))^2 = kn.$$

Question: For what values of k is there a value of n such that

$$(Z(n) + d(n))^2 = kn?$$

A computer search up through 20,000 for k = 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 yielded only the solutions

$$k = 4,\ n = 1,\ Z(n) = 1,\ d(n) = 1$$
$$k = 5,\ n = 45,\ Z(n) = 9,\ d(n) = 6$$

While it is certain that there are other solutions, the following problem is more open.

Question: Is there a value of k such that there are an infinite number of integers n that satisfy the expression

$$(Z(n) + d(n))^2 = kn?$$

Combining the Smarandache function S(n) with d(n) in the expression

$$S(n) + d(n) = n$$

we have an expression where the complete set of solutions is rather easy to determine.

V.4 All Solutions to the Equation S(n) + d(n) = n

Theorem: The only solutions to the equation

$$S(n) + d(n) = n, \; n > 0$$

are 1, 8 and 9.

Proof: Since S(1) = 0 and d(1) = 1 we have verified the special case of n = 1.

Furthermore, with S(p) = p for p a prime, it follows that any solution must be composite.

The following results are well-known.

a) $d(p_1^{a1} \ldots p_k^{ak}) = (a1 + 1) \ldots (ak + 1)$
b) $S(p^k) \leq kp$
c) $S(p_1^{a1} \ldots p_k^{ak}) = \max\{ S(p_1^{a1}) \ldots S(p_k^{ak}) \}$

Examining the first few powers of 2.

$S(2^2) = 4, \; d(2^2) = 3$
$S(2^3) = 4$ and $d(2^3) = 4$ which is a solution.
$S(2^4) = 6, \; d(2^4) = 5$

and in general

$$S(2^k) \leq 2k \text{ and } d(2^k) = k + 1.$$

It is an easy matter to verify that $2k + k + 1 = 3k + 1 < 2^k$ for $k > 4$.

Examining the first few powers of 3

$S(3^2) = 6$ and $d(3^2) = 3$, which is a solution.
$S(3^3) = 9, \; d(3^3) = 4$

and in general, $S(3^k) \leq 3k$ and $d(3^k) = k + 1$.

It is again an easy matter to verify that

$$3k + k + 1 < 3^k$$

for $k > 3$.

Consider $n = p^k$ where $p > 3$ is prime and $k > 1$. The expression becomes

$$S(p^k) + d(p^k) \leq kp + k + 1 = k(p+1) + 1.$$

Once again, it is easy to verify that this is less than p^k for $p \geq 5$.

Now, assume that $n = p_1^{a1} \ldots p_k^{ak}$, $k > 1$ is the unique prime factorization of n.

Case 1: $n = p_1p_2$, where $p_2 > p_1$. Then $S(n) = p_2$ and $d(n) = 2 * 2 = 4$. Forming the sum,

$$p_2 + 4$$

we then examine the subcases.

Subcase 1: $p_1 = 2$. The first few cases are

$n = 2 * 3$, $S(n) + d(n) = 7$
$n = 2 * 5$, $S(n) + d(n) = 9$
$n = 2 * 7$, $S(n) + d(n) = 11$
$n = 2 * 11$, $S(n) + d(n) = 15$

and it is easy to verify that $S(n) + d(n) < n$, for p_2 a prime greater than 11.

Subcase 2: $p_1 = 3$. The first few cases are

$n = 3 * 5$, $S(n) + d(n) = 5 + 4$
$n = 3 * 7$, $S(n) + d(n) = 7 + 4$
$n = 3 * 11$, $S(n) + d(n) = 11 + 4$

and it is easy to verify that $S(n) + d(n) < n$ for p_2 a prime greater than 11.

Subcase 3: It is easy to verify that

$$p_2 + 4 < p_1p_2$$

for $p_1 \geq 5$, $p_2 > p_1$.

Therefore, there are no solutions for $n = p_1p_2$, $p_1 < p_2$.

Case 2: $n = p_1p_2^{a2}$, where $a_2 > 1$ and $p_1 < p_2$ Then $S(n) \leq a_2p_2$ and $d(n) = 2(a_2 + 1)$.

$$S(n) + d(n) \leq a_2p_2 + 2(a_2 + 1) = a_2p_2 + 2a_2 + 2$$

We now induct on a_2 to prove the general inequality

$$a_2p_2 + 2a_2 + 2 < p_1p_2^{a2}$$

Basis Step: $a_2 = 2$. The formula becomes

$2p_2 + 4 + 2 = 2p_2 + 6$ on the left and

$p_1p_2p_2$ on the right. Since $p_2 \geq 3$, $2 + \frac{6}{p_2} \leq 4$ and $p_1p_2 \geq 6$. Therefore,

$$2 + \frac{6}{p_2} < p_1p_2p_2$$

and if we multiply everything by p_2, we have

$$2p_2 + 6 < p_1p_2p_2 .$$

Inductive Step: Assume that the inequality is true for $k \geq 2$

$$kp_2 + 2k + 2 < p_1p_2^k .$$

and examine the case where the exponent is $k + 1$.

$$(k + 1)\, p_2 + 2(k + 1) + 2 = kp_2 + p_2 + 2k + 2 + 2 = (kp_2 + 2k + 2) + p_2 + 2$$

$< p_1p_2^k + p_2 + 2$ by the inductive hypothesis.

Since $p_1p_2^k$ when $k \geq 2$ is greater than $p_2 + 2$ is follows that

$$p_1p_2^k + p_2 + 2 < p_1p_2^{k+1}.$$

Therefore, $S(n) + d(n) < n$, where $n = p_1p_2^k$, $k \geq 2$.

Case 3: $n = p_1^{a1}p_2$ where $a_1 > 1$.

We have two subcases for the value of $S(n)$, depending on the circumstances

Subcase 1: $S(n) \leq a_1p_1$.

Subcase 2: $S(n) = p_2$.

In all cases, $d(n) = 2(a_1 + 1)$.

Subcase 1: $S(n) + d(n) \leq a_1p_1 + 2(a_1 + 1) = a_1p_1 + 2a_1 + 2$.

Using an induction argument very similar to that applied in case 2, it is easy to prove that the inequality

$$a_1p_1 + 2a_1 + 2 < p_1^{a1}p_2$$

is true for all $a_1 \geq 2$.

Subcase 2: $S(n) + d(n) = p_2 + 2(a_1 + 1) = p_2 + 2a_1 + 2$

It is again a simple matter to verify that the inequality

$$p_2 + 2a_1 + 2 < p_1^{a1}p_2$$

is true for all $a_1 \geq 2$.

Case 4: $n = p_1^{a1}p_2^{a2}$ where $p_1 < p_2$ and $a_1, a_2 \geq 2$.

$$d(n) = (a_1 + 1)(a_2 + 1)$$

Subcase 1: $S(n) \leq a_1p_1$

$$S(n) + d(n) \leq a_1p_1 + (a_1 + 1)(a_2 + 1) < p_1^{a1} + p_1^{a1}(a_2 + 1) = p_1^{a1}(a_2 + 2) < p_1^{a1}p_2^{a2}$$

Subcase 2: $S(n) = a_2p_2$

$$S(n) + d(n) = a_2p_2 + (a_1 + 1)(a_2 + 1) < p_2^{a2} + p_2^{a2}(a_1 + 1) = p_2^{a2}(a_1 + 2) < p_1^{a1}p_2^{a2}$$

Case 5: $n = p_1^{a1} \ldots p_k^{ak}$, where $k \geq 2$.

The proof is by induction on k.

Basis Step: Completed in the first four cases.

Inductive Step: Assume that for $n = p_1^{ak} \ldots p_k^{ak}$, $k \geq 2$

$$a_ip_i + (a_1 + 1) \ldots (a_k + 1) < n_1$$

where $S(n_1) \leq a_ip_i$. Which means that

$$S(n_1) + d(n_1) < n_1.$$

Consider $n_2 = p_1^{a1} \ldots p_k^{ak}p_{k+1}^{ak+1}$.

Subcase 1: $S(n_2) = S(n_2)$. Since $p_{k+1} \geq 5$, it follows that $(a_{k+1} + 1) < p_{k+1}^{ak+1}$ and we can use this in combination with the inductive hypothesis to conclude

$$a_ip_i + (a_1 + 1) \ldots (a_k + 1)(a_{k+1} + 1) < n_1(p_{k+1})^{ak+1},$$

which implies that $S(n_2) + d(n_2) < n_2$.

Subcase 2: $S(n_2) > S(n_1)$, which implies that $S(n_2) \leq a_{k+1}p_{k+1}$. Starting with the inductive hypotheses

$$a_ip_i + (a_1 + 1) \ldots (a_k + 1) < p_1^{a1} \ldots p_k^{ak}$$

and multiply both sides by $a_{k+1}p_{k+1}$ to obtain the inequality

$$a_ip_ia_{k+1}p_{k+1} + a_{k+1}p_{k+1}(a_1 + 1) \ldots (a_k + 1) < p_1^{a1} \ldots p_k^{ak}a_{k+1}p_{k+1}$$

Since $p_{k+1} \geq 5$, it follows that

$$p_1^{a1} \ldots p_k^{ak}a_{k+1}p_{k+1} \leq p_1^{a1} \ldots p_k^{ak}p_{k+1}^{ak+1}$$

and with $a_{k+1}p_{k+1} > (a_{k+1} + 1)$, we have

$$a_{k+1}p_{k+1} + (a_1 + 1) \ldots (a_k + 1)\,(a_{k+1} + 1) < a_ip_ia_{k+1}p_{k+1} + a_{k+1}p_{k+1}(a_1 + 1) \ldots (a_k + 1).$$

Combining the inequalities, we have

$$a_{k+1}p_{k+1} + (a_1 + 1) \ldots (a_k + 1)\,(a_{k+1} + 1) < p_1^{a1} \ldots p_k^{ak}p_{k+1}^{ak+1}$$

which implies

$$S(n_2) + d(n_2) < n.$$

Therefore, the only solutions to the equation

$$S(n) + d(n) = n$$

are 1, 8 and 9.

V.5 Equations Involving Several Functions of Number Theory

Question: How many solutions are there to the expression

$$Z(n) + \phi(n) = d(n)?$$

A computer search up through 10,000 yielded no solutions so we frame it in the form of an unsolved question.

Unsolved Question: Are there any solutions to

$$Z(n) + \phi(n) = d(n)?$$

Question: How many solutions are there to the expression

$$Z(n) + d(n) = \phi(n)?$$

A computer search up through 10,000 yielded the solutions

$n = 88$, $Z(n) = 32$, $d(n) = 8$, $\phi(n) = 40$
$n = 494$, $Z(n) = 208$, $d(n) = 8$, $\phi(n) = 216$
$n = 728$, $Z(n) = 272$, $d(n) = 16$, $\phi(n) = 288$
$n = 1240$, $Z(n) = 464$, $d(n) = 16$, $\phi(n) = 480$

n = 4230, Z(n) = 1080, d(n) = 24, ϕ (n) = 1104

Question: Are there an infinite number of solutions to

$$Z(n) + d(n) = \phi(n)?$$

Question: How many solutions are there to the expression

$$\phi(n) + d(n) = Z(n)?$$

A computer search up through 10,000 yielded the following list of solutions.

Table 15

n	ϕ (n)	d(n)	Z(n)
2	1	2	3
42	12	8	20
98	42	6	48
144	48	15	63
840	192	32	224
1932	528	24	552
2360	928	16	944
3224	1440	16	1456
5660	2208	16	2224
5960	2368	16	2384
7960	3184	16	3184
9160	3648	16	3664

And the most obvious characteristic of the solutions is that in almost all cases, the value of d(n) is a power of 2. I can see no pattern that can be used to prove that the number of solutions to the expression is infinite. Perhaps some reader will be able to prove it.

Another computer program was written to search for solutions to the equation

$$Z(d(n)) = d(Z(n))$$

and used to search for solutions for all n, $1 \le n \le 500$. A list of solutions in that range appears in the following table.

Table 16

n	Z(d(n))
1	1
4	2
5	3
45	3
100	8

150	8
175	3
228	8
232	15
245	3
300	8
304	4
306	8
325	3
342	8
364	8
464	4
490	8
500	8

Other than the obvious consequence that the application of both functions often substantially reduces the value, I am unable to discern a pattern in this list.

However, from this evidence, the following conjecture seems safe.

Conjecture: There are an infinite number of integers n such that

Z(d(n)) = d(Z(n)).

V.6 Sum and Products of the Pseudo-Smarandache Values of the Digits of a Number

If we define Z(0) = 0, then we can define the following functions

Definition: For n $\geq$ 0, we define a function

SUMZ(n) = $\sum Z(d_i)$ where dare all the digits of n.

Definition: For n $\geq$ 0, we define a function

PRODZ(n) = $\prod Z(d_i)$ where d are all the digits of n.

Question: Are there any numbers n where

SUMZ(n) = Z(n)?

A computer search up through 10,000 yielded only the solution

n = 15 where Z(n) = 15, Z(1) = 1 and Z(5) = 4.

Question: Are there any other numbers n that satisfy this expression?

Given the rate of growth of the Z(n) function and the relative slow growth of any sum of digits operation, it would appear that the answer to this question is negative.

Question: Are there any numbers n such that

PRODZ(n) = Z(n)?

A computer search up through 10,000 yielded the solutions

n = 14, Z(n) = 7
n = 75, Z(n) = 24
n = 224, Z(n) = 63
n = 292, Z(n) = 72
n = 392, Z(n) = 48
n = 579, Z(n) = 192
n = 657, Z(n) = 72
n = 9436, Z(n) = 336

V.7 Sequences Where The Values Of the Pseudo-Smarandache Function Are Monotonically Increasing Or Decreasing

In searching for sequences of integers n, n + 1, n + 2, ... where Z(n), Z(n + 1), ... are either monotonically increasing or decreasing, Kashihara could find sequences of length at most three. He then asked the question as to whether longer sequences exist.

Another program was written to search for such solutions and in going up through n = 124380, the largest monotonically increasing sequence found was of length 7.

n	Z(n)
18886	1064
18887	1716
18888	3935
18889	5811
18890	7555
18891	8396
18892	14168

Going the other way, a search for a sequence where the numbers Z(k) are monotonically decreasing up through n = 60,860 was conducted. The largest sequence found was also of length 7.

n	Z(n)
7561	7560
7562	3780
7563	2520
7564	1952
7565	1869
7566	1260

7567	1127

From two of the previous theorems, we know that both Z(n) > n and Z(n) < n occur infinitely often.

V.8 Iterations Of the Pseudo-Smarandache Function

Definition: Let $Z^k(n)$ represent the repeated application of the Pseudo-Smarandache function k times Z(Z(Z(.....))).

Looking at the smallest values, we see a loop Z(2) = 3 and Z(3) = 2. Furthermore, Z(6) = 3 is an entry point into this loop. Examining some other small numbers

Z(4) = 7, Z(7) = 6, Z(6) = 3
Z(5) = 4, Z(4) = 7, Z(7) = 6, Z(6) = 3
Z(8) = 15, Z(15) = 5, etc.
Z(9) = 8, etc.

which leads to the question.

Question: Are there any integers n such that there is not some k for which

$Z^k(n) = 3$?

A program was written to examine this question and run for all numbers up through n = 126535. All cycle down to 3 and the largest number of iterations that it took was 28 for n = 123884. While hardly conclusive, this does indicate that there are no such numbers.

Conjecture: There is no value of n for which the repeated application of the Pseudo-Smarandache does not lead to 3.

We now come to the end of yet another journey through many additional Smarandache notions. I sincerely hope that you enjoyed reading it as much as I enjoyed writing it!

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