Smarandache Notions Journal, Vol. 7, No. 1-2-3, 1996, pp. 100-107.

SOME CONVERGENCE PROBLEMS INVOLVING THE SMARANDACHE FUNCTION

by

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In this paper we consider same series attashed to the Smarandache function (Dirichlet series and other (numerical) series).Asimptotic behaviour and convergence of these series is etablished.

1. INTRODUCTION. The Smarandache function $S : N^* \to N^*$ is defined [3] such that $S(n)$ is the smallest integer n with the property that n! is divisible by n.If

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_t^{\alpha_t} \tag{1.1}$$

is the decomposition into primes of the positiv integer n, then

$$S(n) = \max_i S(p_i^{\alpha_i}) \tag{1.2}$$

and more general if $n_1 \overset{d}{\vee} n_2$ is the smallest commun multiple of n_1 and n_2 then .

$$S(n_1 \overset{d}{\vee} n_2) = \max(S(n_1), S(n_2))$$

Let us observe that on the set N of non-negative integers, there are two latticeal structures generated respectively by $\vee = \max$, $\wedge = \min$ and $\overset{d}{\vee}$ = the last commun multiple, $\underset{d}{\wedge}$ = the greatest commun division. if we denote by $\leq$ and $\leq_d$ the induced orders in these lattices,It results

$$S(n_1 \overset{d}{\vee} n_2) = S(n_1) \vee S(n_2)$$

The calculus of $S(p^\alpha)$ depends closely of two numerical scale, namely the standard scale

$$(p) : 1, p, p^2, \ldots, p^n, \ldots$$

and the generalised numerical scale [p]

$$[p] : a_1(p), a_2(p), \ldots, a_n(p), \ldots$$

where $a_k(p) = (p^k-1)/(p-1)$. The dependence is in the sens that

$$S(p^\alpha) = p(\alpha_{[p]})_{(p)} \quad (1.3)$$

so, $S(p^\alpha)$ is obtained multiplyng p by the number obtained writing α in the scale [p] and "reading" it in the scale (p).

Let us observe that if $b_n(p) = p^n$ then the calculus in the scale [p] is essentialy different from the standard scale (p), because :

$$b_{n+1}(p) = pb_n(p) \quad \text{but } a_{n+1}(p) = pa_n(p) + 1$$

(for more details see [2]).

We have also [1] that

$$S(p^\alpha) = (p-1)\alpha + \sigma_{[p]}(\alpha) \quad (1.4)$$

where $\sigma_{[p]}(\alpha)$ is the sum of digits of the number α writen in the scale [p].

In [4] it is showed that if φ is Euler's totient function and we note $S_p(\alpha) = S(p^\alpha)$ then

$$S_p(p^{\alpha-1}) = \varphi(p^\alpha) + p \quad (1.5)$$

It results that $\varphi(p_i^{\alpha_i}) = S(p_i^{p_i^{\alpha_i-1}}) - p$ so

$$\varphi(n) = \prod_{i=1}^{r}\left(S(p_i^{p_i^{\alpha_i-1}}) - p_i\right).$$

In the same paper [4] the function S is extended to the set Q of rational numbers.

2. GENERATING FUNCTIONS. It is known that we may attashe to each numerical function f:N*-->C the Dirichlet serie :

$$D_f(z) = \sum_{n=1}^{\infty} \frac{f(n)}{n^z} \quad (2.1)$$

which for some $z = x + iy$ may be convergent or not.
The simplest Dirichlet series is:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad (2.2)$$

called Riemann's function or zeta function where is convergent for $Re(z) > 1$.

It is said for instance that if f is Möbius function ($\mu(1) = 1$, $\mu(p_1 p_2 \cdots p_i) = (-1)^i$ and $\mu(n) = 0$ if n is divisible by the squar of a prime number) then $D_\mu(z) = 1/\zeta(z)$ for $x>1$, and if f is Euler's totient function ($\varphi(n)$ = the number of positive integers not greater than and prime to the positive integer n) then $D_\varphi(z) = \zeta(z-1)/\zeta(z)$ for $x>2$).

We have also $D_d(z) = \zeta^2(z)$, for $x > 1$, where $d(n)$ is the number of divisors of n, including 1 and n, and $D_{\sigma_k}(n) = \zeta(z)\cdot\zeta(z-k)$ (for $x > 1$, $x > k+1$), where $\sigma_k(n)$ is the sum of the k-th powers of the divisors of n. We write $\sigma(n)$ for $\sigma_1(n)$.

In the sequel let we suppose that z is a real number, so $z = x$.

For the Smarandache function we have:

$$D_s(x) = \sum_{n=1}^{\infty} \frac{S(n)}{n^x}$$

If we note :

$$F_f^*(n) = \sum_{k \leq_d n} f(k)$$

it is said that Möbius function make a connection between f and F_f^* by the inversion formula:

$$f(n) = \sum_{k \leq_d n} F_f^{o}(k)\mu(\tfrac{n}{k}) \qquad (2.3)$$

The functions F_f^{o} are also called generating functions.

In [4] the Smarandache functions is regarded as a generating function and is constructed the function s_o such that:

$$s_o(n) = \sum_{k \leq_d n} S(k)\mu(\tfrac{n}{k})$$

2.1. PROPOSITION. For all $x > 2$ we have :

(i) $\zeta(x) \leq D_S(x) \leq \zeta(x-1)$

(ii) $1 \leq D_{so}(x) \leq D_S(x)$

(iii) $\zeta^2(x) \leq D_{FS}(x) \leq \zeta(x) \cdot \zeta(x-1)$

Proof. (i) The asertion results from the fact that $1 \leq S(n) \leq n$.

(ii) Using the multiplication of Dirichlet series we have:

$$\frac{1}{\zeta(x)} \cdot D_s(x) = \left(\sum_{k=1}^{\infty} \frac{\mu(k)}{k^x}\right)\left(\sum_{k=1}^{\infty} \frac{S(k)}{k^x}\right) = \mu(1)S(1) + \frac{\mu(1)S(2)+\mu(2)S(1)}{2^x} +$$

$$+ \frac{\mu(1)S(3)+\mu(3)S(1)}{3^x} + \frac{\mu(1)S(4)+\mu(2)S(2)+\mu(4)S(1)}{4^x} + \ldots = \sum_{k=1}^{\infty} \frac{s_o(k)}{k^x} = D_{so}(x)$$

and the asertion result using (i).

(iii) We have

$$\zeta(x) \cdot D_S(x) = \left(\sum_{k=1}^{\infty} \frac{1}{k^x}\right)\left(\sum_{k=1}^{\infty} \frac{S(k)}{k^x}\right) = S(1) + \frac{S(1)+S(2)}{2^x} + \frac{S(1)+S(3)}{3^x} + \ldots = D_{FS}(x)$$

so the inequalities holds using (i).

Let us observe that (iii) is equivalent to $D_d(x) \leq D_{FS} < D_\sigma(x)$. These inequalities can be deduced also observing that from $1 \leq S(n) \leq n$ it result:

$$\sum_{k \leq_d n} 1 \leq \sum_{k \leq_d n} S(k) \leq \sum_{k \leq_d n}$$

so,

$$d(n) \leq F_s(n) \leq \sigma(n) \qquad (2.4)$$

But from the fact that $F_S < n + 4$ (proved in [5]) we deduce

$$d(n) \leq F_s(n) \leq n + 4 \qquad (2.5)$$

Until now it is not known a closed formula for the calculus of the functions $D_s(x)$, $D_{s_0}(x)$ or $D_{rs}(x)$, but we can deduce asimptotic behaviour of these functions using the following well known results:

2.2. THEOREM. (i) $\zeta(z) = \frac{1}{z-1} + O(1)$

(ii) $\ln\zeta(z) = \ln\frac{1}{z-1} + O(z-1)$

(iii) $\zeta'(z) = -\frac{1}{(z-1)^2} + O(1)$

for all complex number.

Then from the proposition 2.1 we can get inequalities as the fallowings:

(i) $\frac{1}{x-1} + O(1) \leq D_s(x) \leq \frac{1}{x-2} + O(1)$

(ii) $1 \leq D_{s_0}(x) \leq \frac{x-1}{e^{A}(x-2)}$ for some positive constant A

(iii) $-\frac{1}{(x-1)^2} + O(1) \leq D'_s(x) \leq -\frac{1}{(x-2)^2} + O(1)$.

The Smarandache functions S may be extended to all the nonnegative integers defining S(-n) = S(n).

In [3] it is proved that the serie

$$\sum_{k=1}^{\infty} \frac{S(k)}{k!}$$

is convergent and has the sum q ∈ (e-1,2).

We can consider the function

$$f(z) = \sum_{k=1}^{\infty} \frac{S(k)}{(k+1)!} z^k$$

convergent for all z ∈ C because

$$\frac{a_{n+1}}{a_n} = \frac{S(n+1)}{(n+2)S(n)} \leq \frac{n+1}{(n+2)S(n)} \leq \frac{1}{S(n)}$$

and so $\frac{a_{n+1}}{a_n} \to 0$

2.3. PROPOSITION. The function f statisfies $|f(z)| \leq qz$ an the unit disc $U(0,1) = \{z \mid |z| < 1\}$.

Proof. A lema does to Schwartz asert that if the function f is olomorphe on the unit disc $U(0,1) = \{ z \mid |z| < 1\}$ and satisfies $f(0) = 0$, $|f(z)| \leq 1$ for $z \in U(0,1)$ then $|f(z)| \leq |z|$ on $U(0,1)$ and $|f'(0)| \leq 1$.

For $|z| < 1$ we fave $|f(z)| < q$ so $(1/q) f(z)$ satisfies the conditions of Schwartz lema.

3. SERIES INVOLVING THE SMARANDACHE FUNCTION. In this section we shall studie the convergence of some series concerning the function S.

Let b: N*—>N* be the function defined by: b(n) is the complement of n until the smallest factorial. From this definition it results that $b(n) = (S(n)!)/n$ for all $n \in N^*$.

3.1. PROPOSITION. The sequences $(b(n))_{n\geq 1}$ and also $(b(n)/n^k)_{n\geq 1}$ for $k\in R$, are divergent.

Proof. (i) The asertion results from the fact that $b(n!) = 1$ and if $(p_n)_{n\geq 1}$ is the sequence of prime members then

$$b(p_n) = \frac{S(p_n)!}{p_n} = \frac{p_n!}{p_n} = (p_n - 1)!$$

(ii) Let we note $x_n = b(n)/n^k$. Then

$$x_n = \frac{S(n)!}{n^{k+1}}$$

and for $k > 0$ it results

$$x_{n!} = \frac{S(n!)!}{(n!)^{k+1}} = \frac{n!}{(n!)^{k+1}} \to 0$$

$$x_{p_n} = \frac{p_n!}{(p_n)^{k+1}} = \frac{(p-1)!}{(p_n)^{k+1}} > \frac{p_1 \cdot p_2 \cdots p_{n-1}}{p_n^{k+1}} > p_n$$

because it in said [6] that $p_1\, p_2 \cdots p_{n-1} > p_n^{k+2}$ for n sufficiently large.

3.2. PROPOSITION. The sequence $T(n) = 1+ \sum_{i=2}^{n} \frac{1}{b(n)} - \ln b(n)$ is divergent.

Proof. If we suppose that $\lim_{n\to\infty} T(n) = l < \infty$, then because $\sum_{i=2}^{\infty} \frac{1}{b(n)} = \infty$ (see [3]) it results the contradiction $\lim_{n\to\infty} \ln b(n) = \infty$.

If we suppose $\lim_{n\to\infty} T(n) = -\infty$, from the equality $\ln b(n) = 1 + \sum_{i=2}^{\infty} \frac{1}{b(n)} - T(n)$ it results $\lim_{n\to\infty} \ln b(n) = \infty$.

We can't have $\lim_{n\to\infty} T(n) = +\infty$ because $T(n) < 0$. Indeed, from $i \leq S(i)!$ for $i \geq 2$ it results

$$i / S(i)! \leq 1 \text{ for all } i \geq 2$$

so

$$T(p_n) = 1 + \frac{2}{S(2)!} + \ldots + \frac{p_n}{S(p_n)!} - \ln((p_n - 1)!) < 1 + (p_n - 1) - \ln((p_n - 1)!) =$$

$$= p_n - \ln((p_n - 1)!).$$

But for k sufficiently large we have $e^k<(k-1)!$ that is there exists $m\in N$ so that $p_n< \ln((p_n - 1)!)$ for $n \geq m$. It results $p_n - \ln((p_n - 1)!) < 0$ for $n \geq m$, and so $T(n) < 0$.

Let now be the function

$$H_b(x) = \sum_{2\leq n\leq x} b(n).$$

3.3. PROPOSITION. The serie

$$\sum_{n\geq 2} H_b^{-1}(n) \tag{3.1}$$

is convergent.

Proof. the sequence $(b(2)+b(3)+ \ldots + b(n))_n$ is strictley increasing to ∞ and

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}>\frac{S(2)!}{2}$$

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}>\frac{S(5)!}{5}$$

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}+\frac{S(6)!}{6}>\frac{S(5)!}{5}$$

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}+\frac{S(6)!}{6}+\frac{S(7)}{7}>\frac{S(7)!}{7}$$

....

so we have

$$\sum_{n\geq 2} H_b^{-1}(n)=\frac{1}{\frac{S(2)!}{2}}+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}}+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}}+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}}+\dots$$

$$\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}+\dots+\frac{S(n)!}{n}}+\dots<$$

$$<\frac{2}{\frac{S(2)!}{2}}+\frac{1}{\frac{S(3)!}{3}}+\frac{2}{\frac{S(5)!}{5}}+\frac{4}{\frac{S(7)!}{7}}+\frac{2}{\frac{S(11)!}{11}}+\dots+\frac{p_{k+1}-p_k}{\frac{S(p_k)!}{p_k}}+\dots$$

$$<1+\sum_{k\geq 2}\frac{p_k(p_{k+1}-p_k)}{S(p_k)!}=1+\sum_{k\geq 2}\frac{p_k(p_{k+1}-p_k)}{p_k!}=1+\frac{1}{2}+\frac{1}{12}+\sum_{k\geq 2}\frac{p_k(p_{k+1}-p_k)}{p_k!}.$$

But $(p_n-1)!>p_1p_2\dots p_n$ for $n\geq 4$ and then

$$\sum_{n\geq 2} H_b^{-1}(n)<\frac{19}{12}+\sum_{k\geq 4}a_k$$

where $a_k=\frac{p_k(p_{k+1}-p_k)}{p_k!}=\frac{p_{k+1}-p_k}{1\cdot 2\cdot 3\cdot\ldots\cdot(p_k-1)}<\frac{p_{k+1}-p_k}{p_1p_2\dots p_k}<\frac{p_{k+1}}{p_1p_2\dots p_k}$

Because $p_1p_2\dots p_k>p^3_{k+1}$ for k sufficiently large, it results

$$a_k<\frac{p_{k+1}}{p_{k+1}^3}=\frac{1}{p_{k+1}^2}\text{ for }k\geq k_o$$

and the convergence of the serie (3.1) follows from the convergence of the serie $\sum_{k\geq k_0}\frac{1}{p_{k+1}^2}$.

In the followings we give an elementary proof of the convergence of the series $\sum_{k=2}^{\infty}\frac{1}{S(k)^{\alpha}\sqrt{S(k)!}}$, $\alpha\in R$, $\alpha>1$ provides information on the convergence behavior of the series $\sum_{k=2}^{\infty}\frac{1}{S(k)!}$.

3.4. PROPOSITION. The series $\sum_{k=2}^{\infty}\frac{1}{S(k)^{\alpha}\sqrt{S(k)!}}$, converges if $\alpha\in R$ and $\alpha>1$.

Proof.

$$\sum_{k=2}^{\infty}\frac{1}{S(k)^{\alpha}\sqrt{S(k)!}}=\frac{1}{2^{\alpha}\sqrt{2!}}+\frac{1}{3^{\alpha}\sqrt{3!}}+\frac{1}{4^{\alpha}\sqrt{4!}}+\frac{1}{5^{\alpha}\sqrt{5!}}+\frac{1}{3^{\alpha}\sqrt{3!}}+\frac{1}{7^{\alpha}\sqrt{7!}}+\frac{1}{4^{\alpha}\sqrt{4!}}+\dots=\sum_{t=2}^{\infty}\frac{m_t}{t^{\alpha}\sqrt{t!}}$$

where m_t denotes the number of elements of the set

M_t { $k\in N^*$, $S(k)=t$ } = { $k\in N^*$, $k\mid t$ and $k\mid(t-1)!$ }.

It follows that M_t { k∈N*, k | t } and there fore $m_t<d(t!)$.
Hence $m_t < 2\sqrt{t!}$ and consequently we have

$$\sum_{t=2}^{\infty} \frac{m_t}{t^{\alpha}\sqrt{t!}} < \sum_{t=2}^{\infty} \frac{2\sqrt{t!}}{t^{\alpha}\sqrt{t!}} = 2\sum_{t=2}^{\infty} \frac{1}{t^{\alpha}}$$

So, $\sum_{t=2}^{\infty} \frac{m_t}{t^{\alpha}\sqrt{t!}}$ converges.

3.5. PROPOSITION. $t^{\alpha}\sqrt{t!} < t!$ if $\alpha \in R$, $\alpha > 1$ and $t > t_* = [e^{2\alpha+1}]$, $t \in N^*$. (where [x] means the integer part of x).

Proof. $t^{\alpha}\sqrt{t!} < t! \Leftrightarrow t^{2\alpha}t! < (t!)^2 \Leftrightarrow t^{2\alpha} < t!$ (2)

On the other hand $t^{2\alpha} < (\frac{t}{e})^t \Leftrightarrow (e\cdot\frac{t}{e})^{2\alpha} < (\frac{t}{e})^t \Leftrightarrow e^{2\alpha}\cdot(\frac{t}{e})^{2\alpha} < (\frac{t}{e})^t \Leftrightarrow e^{2\alpha} < (\frac{t}{e})^{t-2\alpha}$ (3)

If $t>e^{2\alpha+1} \Rightarrow (\frac{t}{e})^{t-2\alpha} > (\frac{e^{2\alpha+1}}{e})^{t-2\alpha} = (e^{2\alpha})^{t-2\alpha} > (e^{2\alpha})^{e^{2\alpha+1}-2\alpha}$

Applyng the well-known result that $e^x > 1+x$ if $x > 0$ for $x = 2\alpha$ we have
$(e^{2\alpha})^{e^{2\alpha+1}-2\alpha} > (e^{2\alpha})^{2\alpha+1+1-2\alpha} = (e^{2\alpha})^2 = e^{4\alpha} > e^{2\alpha}$.
So, if $t > e^{2\alpha+1}$ we fave $e^{2\alpha} < (\frac{t}{e})^{t-2\alpha}$ (4)
It is well known that $(\frac{t}{e})^t < t!$ if $t\in N^*$. (5)

Now, the proof of the proposition is obtained as follows:

If $t > t_* = [e^{2\alpha+1}]$, $t \in N^*$ we have $e^{2\alpha} < (\frac{t}{e})^{t-2\alpha} \Leftrightarrow t^{2\alpha} < (\frac{t}{e})^t < t!$. Hence $t^{2\alpha} < t!$ if $t > t_*$ and this proves the proposition.

CONSEQUENCE. The series $\sum_{k=2}^{\infty} \frac{1}{S(k)!}$ converges.

Proof. $\sum_{k=2}^{\infty} \frac{1}{S(k)!} = \sum_{t=2}^{\infty} \frac{m_t}{t!}$ where m_t is defined as above.

If $t > t_*$ we have $t^{\alpha}\sqrt{t!} < t! \Leftrightarrow \frac{1}{t^{\alpha}\sqrt{t!}} > \frac{1}{t!} \Leftrightarrow \frac{m_t}{t^{\alpha}\sqrt{t!}} > \frac{m_t}{t!}$.

Since $\sum_{t=2}^{\infty} \frac{m_t}{t^{\alpha}\sqrt{t!}}$ converges it results that $\sum_{t=2}^{\infty} \frac{m_t}{t!}$ also converges.

REMARQUE. From the definition of the Smarandache function it results that

card { k∈N*: S(k)=t } = card { k∈N*: k | t and k | (t-1)!} = d(t!)-d((t-1)!)

so we get

$$\sum_{t=2}^{n} car(dS^{-1}(t)) = \sum_{k=2}^{n} (d(t!) - d((t-1)!)) = d(n!) - 1$$

ACKNOWLEDGEMENT:
The autors wish to express their gratitude to S.S Kim, vice-president and Ion Castravete, executive manager to Daewoo S.A. from Craiova, for having agreed cover ours traveling expenses for the conference as well as a great deal of the local expenses in Thessaloniki.

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