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THE NORMAL BEHAVIOR OF THE SMARANDACHE FUNCTION

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Let $S(n)$ be the smallest integer k so that $n|k!$. This is known as the Smarandache function and has been studied by many authors. If $P(n)$ denotes the largest prime factor of n, it is clear that $S(n) \geqslant P(n)$. In fact, $S(n) = P(n)$ for most n, as noted by Erdös [E]. This means that the number, $N(x)$, of $n \leqslant x$ for which $S(n) \neq P(n)$ is $o(x)$. In this note we prove an asymptotic formula for $N(x)$.

First, denote by $\rho(u)$ the Dickman function, defined by

$$\rho(u) = 1 \quad (0 \leqslant u \leqslant 1), \qquad \rho(u) = 1 - \int_1^u \frac{\rho(v-1)}{v}\, dv \quad (u > 1).$$

For $u > 1$ let $\xi = \xi(u)$ be defined by

$$u = \frac{e^\xi - 1}{\xi}.$$

It can be easily shown that

$$\xi(u) = \log u + \log_2 u + O\left(\frac{\log_2 u}{\log u}\right),$$

where $\log_k x$ denotes the kth iterate of the logarithm function. Finally, let $u_0 = u_0(x)$ be defined by the equation

$$\log x = u_0^2 \xi(u_0).$$

The function $u_0(x)$ may also be defined directly by

$$\log x = u_0 \left(x^{1/u_0^2} - 1\right).$$

It is straightforward to show that

$$(1) \qquad u_0 = \left(\frac{2\log x}{\log_2 x}\right)^{\frac{1}{2}} \left(1 - \frac{\log_3 x}{2\log_2 x} + \frac{\log 2}{2\log_2 x} + O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^2\right)\right).$$

We can now state our main result.

Theorem 1. *We have*

$$N(x) \sim \frac{\sqrt{\pi}(1+\log 2)}{2^{3/4}}(\log x \log_2 x)^{3/4} x^{1 \;\; 1/u_0} \rho(u_0).$$

There is no way to write the asymptotic formula in terms of "simple" functions, but we can get a rough approximation.

Corollary 2. *We have*

$$N(x) = x \exp\left\{-(\sqrt{2}+o(1))\sqrt{\log x \log_2 x}\right\}.$$

The asymptotic formula can be made a bit simpler, without reference to the function ρ as follows.

Corollary 3. *We have*

$$N(x) \sim \frac{e^{\gamma}(1+\log 2)}{2\sqrt{2}}(\log x)^{\frac{1}{2}}(\log_2 x) x^{1 \;\; 2/u_0} \exp\left\{\int_0^{\frac{\log x}{u_0^2}} \frac{e^v - 1}{v}\, dv\right\},$$

where $\gamma = 0.5772\ldots$ *is the Euler-Mascheroni constant.*

This will follow from Theorem 1 using the formula in Lemma 2 which relates $\rho(u)$ and $\xi(u)$.

The distribution of $S(n)$ is very closely related to the distribution of the function $P(n)$. We begin with some standard estimates of the function $\Psi(x,y)$, which denotes the number of integers $n \leqslant x$ with $P(n) \leqslant y$.

Lemma 1 [HT, Theorem 1.1]. *For every* $\epsilon > 0$,

$$\Psi(x,y) = x\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right), \quad u = \frac{\log x}{\log y},$$

uniformly in $1 \leqslant u \leqslant \exp\{(\log y)^{3/5 \;\; \epsilon}\}$.

Lemma 2 [HT, Theorem 2.1]. *For* $u \geqslant 1$,

$$\begin{aligned}\rho(u) &= \left(1 + O\left(\frac{1}{u}\right)\right)\sqrt{\frac{\xi'(u)}{2\pi}} \exp\left\{\gamma - \int_1^u \xi(t)\, dt\right\} \\ &= \exp\left\{-u\left(\log u + \log_2 u - 1 + O\left(\frac{\log_2 u}{\log u}\right)\right)\right\}.\end{aligned}$$

Lemma 3 [HT, Corollary 2.4]. *If* $u > 2$, $|v| \leqslant u/2$, *then*

$$\rho(u-v) = \rho(u)\exp\{v\xi(u) + O((1+v^2)/u)\}.$$

Further, if $u > 1$ *and* $0 \leqslant v \leqslant u$ *then*

$$\rho(u-v) \ll \rho(u)e^{v\xi(u)}.$$

We will show that most of the numbers counted in $N(x)$ have

$$P(n) \approx \exp\left\{\sqrt{\tfrac{1}{2}\log x \log_2 x}\right\}.$$

Let

$$Y_1 = \exp\left\{\tfrac{1}{3}\sqrt{\log x \log_2 x}\right\}, \quad Y_2 = Y_1^6 = \exp\left\{2\sqrt{\log x \log_2}\right\}.$$

Let N_1 be the number of n counted by $N(x)$ with $P(n) \leqslant Y_1$, let N_2 be the number of n with $P(n) \geqslant Y_2$, and let $N_3 = N(x) - N_1 - N_2$. By Lemmas 1 and 2,

$$N_1 \leqslant \Psi(x, Y_1) = x\exp\{-(1.5 + o(1))\sqrt{\log x \log_2 x}\}.$$

For the remaining $n \leqslant x$ counted by $N(x)$, let $p = P(n)$. Then either $p^2|n$ or for some prime $q < p$ and $b \geqslant 2$ we have $q^b \parallel n$, $q^b \nmid p!$. Since $p!$ is divisible by $q^{[p/q]}$ and $b \leqslant 2\log x$, it follows that $q > p/(3\log x) > p^{1/2}$. In all cases n is divisible by the square of a prime $\geqslant Y_2/(3\log x)$ and therefore

$$N_2 \leqslant \sum_{p \geqslant \frac{Y_2}{3\log x}} \frac{x}{p^2} \leqslant \frac{6x\log x}{Y_2} \ll x\exp\left\{-1.9\sqrt{\log x \log_2 x}\right\}.$$

Since $q > p^{1/2}$ it follows that $q^{[p/q]} \parallel p!$. If n is counted by N_3, there is a number $b \geqslant 2$ and prime $q \in [p/b, p]$ so that $q^b|n$. For each $b \geqslant 2$, let $N_{3,b}(x)$ be the number of n counted in N_3 such that $q^b \parallel n$ for some prime $q \geqslant p/b$. We have

$$\sum_{b \geqslant 6} N_{3,b} \ll x\left(\frac{3\log x}{Y_1}\right)^5 \ll x\exp\left\{-(5/3 + o(1))\sqrt{\log x \log_2 x}\right\}.$$

Next, using Lemma 1 and the fact that ρ is decreasing, for $3 \leqslant b \leqslant 5$ we have

$$\begin{aligned} N_{3,b} &= \sum_{Y_1 < p < Y_2}\left(\Psi\left(\frac{x}{p^b}, p\right) + \sum_{p/b \leqslant q < p}\Psi\left(\frac{x}{pq^b}, q\right)\right) \\ &\ll x\sum_{Y_1 < p < Y_2}\left(\frac{1}{p^b}\rho\left(\frac{\log x}{\log p} - b\right) + \sum_{p/2 < q < p}\frac{1}{pq^b}\rho\left(\frac{\log x - \log p - b\log q}{\log p}\right)\right) \\ &\ll x\sum_{Y_1 < p < Y_2} p^{\ \ b}\rho\left(\frac{\log x}{\log p} - (b+1)\right). \end{aligned}$$

By partial summation, the Prime Number Theorem, Lemma 2 and some algebra,

$$N_{3,b} \ll \exp\left\{-(1.5 + o(1))\sqrt{\log x \log_2 x}\right\}.$$

The bulk of the contribution to $N(x)$ will come from $N_{3,2}$. Using Lemma 1 we obtain

(2)

$$N_{3,2} = \sum_{Y_1<p<Y_2} \left(\Psi\left(\frac{x}{p^2}, p\right) + \sum_{\frac{p}{2}<q<p} \Psi\left(\frac{x}{pq^2}, q\right) \right)$$
$$= \left(1+O\left(\sqrt{\tfrac{\log_2 x}{\log x}}\right)\right) x \sum_{Y_1<p<Y_2} \left(\frac{\rho\left(\frac{\log x}{\log p}-2\right)}{p^2} + \sum_{p/2<q<p} \frac{\rho\left(\frac{\log x \ \log p}{\log q}-2\right)}{pq^2} \right).$$

By Lemma 3, we can write

$$\rho\left(\frac{\log x - \log p}{\log q} - 2\right) = \rho\left(\frac{\log x}{\log q} - 3\right)\left(1+O\left(\sqrt{\frac{\log_2 x}{\log x}}\right)\right).$$

The contribution in (2) from p near Y_1 or Y_2 is negligible by previous analysis, and for fixed $q \in [Y_1, Y_2/2]$ the Prime Number Theorem implies

$$\sum_{q<p<2q} \frac{1}{p} = \frac{\log 2}{\log q} + O((\log q)^{\ 2}) = \frac{\log 2}{\log p} + O\left(\frac{1}{\log^2 Y_1}\right).$$

Reversing the roles of p, q in the second sum in (2), we obtain

$$N_{3,2} = \left(1+O\left(\sqrt{\tfrac{\log_2 x}{\log x}}\right)\right) x \sum_{Y_1<p<Y_2} \frac{1}{p^2}\left(\rho\left(\frac{\log x}{\log p}-2\right) + \frac{\log 2}{\log p}\rho\left(\frac{\log x}{\log p}-3\right)\right).$$

By partial summation, the Prime Number Theorem with error term, and the change of variable $u = \log x/\log p$,

$$(3) \qquad N_{3,2} = \left(1+O\left(\sqrt{\tfrac{\log_2 x}{\log x}}\right)\right) x \int_{u_1}^{u_2} \left(\frac{\rho(u-2)}{u} + \frac{\log 2}{\log x}\rho(u-3)\right) x^{\ 1/u} du,$$

where

$$u_1 = \frac{1}{2}\sqrt{\frac{\log x}{\log_2 x}}, \qquad u_2 = 6u_1.$$

The integrand attains its maximum value near $u = u_0$ and we next show that the most of the contribution of the integral comes from u close to u_0. Let

$$w_0 = \frac{u_0}{100}, \quad w_1 = K\sqrt{u_0}, \quad w_2 = w_1\left(\frac{\log_3 x}{\log_2 x}\right)^{1/2},$$

where K is a large absolute constant. Let I_1 be the contribution to the integral in (3) with $|u-u_0| > w_0$, let I_2 be the contribution from $w_1 < |u-u_0| \leqslant w_0$, let I_3 be the contribution from $w_2 < |u-u_0| \leqslant w_1$, and let I_4 be the contribution from $|u-u_0| \leqslant w_2$. First, by Lemma 2, the integrand in (3) is

$$\exp\left\{-\left(\frac{1}{c} - \frac{c}{2} + o(1)\right)\sqrt{\log x \log_2 x}\right\}, \quad c = \left(\frac{\log_2 x}{\log x}\right) u.$$

The function $1/c + c/2$ has a minimum of $\sqrt{2}$ at $c = \sqrt{2}$, so it follows that

$$I_1 \ll \exp\left\{-\left(\sqrt{2} + 10 \ ^{5}\right)\sqrt{\log x \log_2 x}\right\}.$$

Let $u = u_0 - v$. For $w_1 \leqslant |v| \leqslant w_0$, Lemma 2 and the definition (1) of u_0 imply that the integrand in (3) is

$$\begin{aligned}
&\leqslant \rho(u_0)\exp\left\{v\xi(u_0) - \frac{\log x}{u_0}\left(1 + \frac{v}{u_0} + \frac{v^2}{u_0^2} + \frac{v^3}{u_0^3}\right) + O\left(\frac{v^2}{u_0} + \log u_0\right)\right\} \\
&\ll \rho(u_0)x^{\ \ 1/u_0}\exp\left\{-\frac{v^2}{u_0^3}\log x + O\left(\frac{v^2}{u_0} + \log u_0\right)\right\} \\
&\ll \rho(u_0)x^{\ \ 1/u_0}\exp\left\{-0.9\frac{v^2}{u_0^3}\log x\right\}
\end{aligned}$$

for K large enough. It follows that

$$I_2 \ll u_0\rho(u_0)x^{\ \ 1/u_0}\exp\{-20\log_2 x\} \ll (\log x)^{\ \ 10}\rho(u_0)x^{\ \ 1/u_0}.$$

For the remaining u, we first apply Lemma 3 with $v = 2$ and $v = 3$ to obtain

$$I_3 + I_4 = \left(1 + O\left(\sqrt{\tfrac{\log_2 x}{\log x}}\right)\right)\int_{u_0 \ \ w_1}^{u_0+w_1} \rho(u)x^{\ \ 1/u}\left(\frac{e^{2\xi(u)}}{u} + \frac{\log 2}{\log x}e^{3\xi(u)}\right) du$$

We will show that $I_3 + I_4 \gg \rho(u_0)x^{\ \ 1/u_0}(\log x)^{3/2}$, which implies

$$\text{(4)} \quad N(x) = \left(1 + O\left(\sqrt{\tfrac{\log_2 x}{\log x}}\right)\right)\int_{u_0 \ \ w_1}^{u_0+w_1} \rho(u)x^{\ \ 1/u}\left(\frac{e^{2\xi(u)}}{u} + \frac{\log 2}{\log x}e^{3\xi(u)}\right) du.$$

This provides an asymptotic formula for $N(x)$, but we can simplify the expression somewhat at the expense of weakening the error term. First, we use the formula

$$\xi(u) = \log u + \log_2 u + O\left(\frac{\log_2 u}{\log u}\right),$$

and then use $u = u_0 + O(u_0^{1/2})$ and (1) to obtain

$$I_3 + I_4 = \left(1 + O\left(\tfrac{\log_3 x}{\log_2 x}\right)\right)\frac{\sqrt{2}}{4}(1 + \log 2)x(\log x)^{\frac{1}{2}}(\log_2 x)^{\frac{3}{2}}\int_{u_0 \ \ w_1}^{u_0+w_1} \rho(u)x^{\ \ 1/u}\, du.$$

By Lemma 3, when $w_2 \leqslant |v| \leqslant w_1$, where $u = u_0 - v$, we have

$$\begin{aligned}
\rho(u_0 - v)x^{\ \ \frac{1}{u_0 \ \ v}} &\ll \rho(u_0)x^{\ \ \frac{1}{u_0}}\exp\left\{v\xi(u_0) - \frac{\log x}{u_0}\left(\frac{v}{u_0} + \frac{v^2}{u_0^2} + \frac{v^3}{u_0^3}\right)\right\} \\
&\ll \rho(u_0)x^{\ \ \frac{1}{u_0}}\exp\left\{-\frac{v^2}{u_0^3}\log x\right\} \\
&\ll \rho(u_0)x^{\ \ \frac{1}{u_0}}\exp\left\{-\frac{w_2^2}{u_0^3}\log x\right\} \\
&\ll \rho(u_0)x^{\ \ \frac{1}{u_0}}(\log_2 x)^{\ \ 3}
\end{aligned}$$

provided K is large enough. This gives

$$\int_{w_2 \leqslant |u \;\; u_0| \leqslant w_1} \rho(u) x^{\;\,1/u}\, du \ll \rho(u_0) x^{\;\,1/u_0} (\log x)^{1/4} (\log_2 x)^{\;\,3.5}.$$

For the remaining v, Lemma 3 gives

$$\rho(u_0 - v) x^{\;\,1/(u_0 \;\; v)} = \left(1 + O\left(\tfrac{\log_3 x}{\log_2 x}\right)\right) \rho(u_0) x^{\;\,1/u_0} \exp\left\{-\frac{v^2}{u_0^3} \log x\right\}.$$

Therefore,

$$\rho(u_0)^{\;\,1} x^{\frac{1}{u_0}} \int_{u_0 \;\; w_2}^{u_0 + w_2} \rho(u) x^{\;\,1/u}\, du = \left(1 + O\left(\tfrac{\log_3 x}{\log_2 x}\right)\right) \int_{\;\, w_2}^{w_2} \exp\left\{-v^2 \frac{\log x}{u_0^3}\right\} dv.$$

The extension of the limits of integration to $(-\infty, \infty)$ introduces another factor $1 + O((\log_2 x)^{\;\,1})$, so we obtain

$$I_3 + I_4 = \left(1 + O\left(\tfrac{\log_3 x}{\log_2 x}\right)\right) \frac{\sqrt{\pi}(1 + \log 2)}{2^{3/4}} (\log x \log_2 x)^{3/4} \rho(u_0) x^{\;\,\frac{1}{u_0}}$$

and Theorem 1 follows. Corollary 2 follows immediately from Theorem 1 and (1). To obtain Corollary 3, we first observe that $\xi'(u) \sim u^{\;\,1}$ and next use Lemma 2 to write

$$\rho(u_0) \sim \frac{e^\gamma}{\sqrt{2\pi u_0}} \exp\left\{-\int_1^{u_0} \xi(t)\, dt\right\}.$$

By the definitions of ξ and u_0 we then obtain

$$\begin{aligned} \int_1^{u_0} \xi(t)\, dt &= \int_0^{\xi(u_0)} e^v - \frac{e^v - 1}{v}\, dv \\ &= e^{\xi(u_0)} - 1 - \int_0^{\xi(u_0)} \frac{e^v - 1}{v}\, dv \\ &= \frac{\log x}{u_0} - \int_0^{\frac{\log x}{u_0^2}} \frac{e^v - 1}{v}\, dv. \end{aligned}$$

Corollary 3 now follows from (1).

References

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