A FUNCTION IN THE NUMBER THEORY

Summary

In this paper I shall construct a function η having the following properties:

(1) $\forall\, n \in Z \quad n \neq 0 \quad (\eta(n))! = M\, n.$

(2) $\eta(n)$ is the smallest natural number with the property (1).

We consider: $N = \{0, 1, 2, 3, \dots\}$ and $N^* = \{1, 2, 3, \dots\}$.

Lemma 1. $\forall\, k, p \in N^*$, $p \neq 1$, k is uniquely written under the shape: $k = t_1 a_{n_1}^{(p)} + \dots + t_\ell a_{n_\ell}^{(p)}$ where $a_{n_i}^{(p)} = \dfrac{p^{n_i}-1}{p-1}$, $i = \overline{1,\ell}$, $n_1 > n_2 > \dots > n_\ell > 0$ and $1 \le t_j \le p - 1$, $j = \overline{1,\ell-1}$, $1 \le t_\ell \le p$, $n_i, t_i \in N$, $i = \overline{1,\ell,}$ $\ell \in N^*$.

Proof. The string $(a_n^{(p)})_{n \in N^*}$ consists of strictly increasing infinite natural numbers and $a_{n+1}^{(p)} - 1 = p \cdot a_n^{(p)}$, $\forall\, n \in N^*$, p is fixed,

$$a_1^{(p)} = 1,\ a_2^{(p)} = 1 + p,\ a_3^{(p)} = 1 + p + p^2,\ \dots$$

$\Rightarrow N^* = \bigcup_{n \in N^*} ([a_n^{(p)}, a_{n+1}^{(p)}) \cap N^*)$ where $[a_n^{(p)}, a_{n+1}^{(p)}) \cap$

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$$\cap\ [a_{n+1}^{(p)},\ a_{n+2}^{(p)}) = \emptyset$$

because $a_n^{(p)} < a_{n+1}^{(p)} < a_{n+2}^{(p)}$.

Let $k \in N^*$, $N^* = \bigcup_{n \in N^*} ([a_n^{(p)}, a_{n+1}^{(p)}) \cap N^*) \Rightarrow \exists! n_1 \in N^*: k \in$

$\in \left[a_{n_1}^{(p)}, a_{n_1+1}^{(p)} \right) \Rightarrow$ k is uniquely written under

the shape $k = \left[\dfrac{k}{a_{n_1}^{(p)}} \right] a_{n_1}^{(p)} + r_1$ (integer division theorem).

We note $\left[\dfrac{k}{a_{n_1}^{(p)}} \right] = t_1 \Rightarrow k = t_1 a_{n_1}^{(p)} + r_1,\ r_1 < a_{n_1}^{(p)}$.

If $r_1 = 0$, as $a_{n_1}^{(p)} \le k \le a_{n_1+1}^{(p)} - 1 \Rightarrow 1 \le t_1 \le p$ and Lemma

1 is proved.

If $r_1 \neq 0 \Rightarrow \exists\ !\ n_2 \in N^*:\ r_1 \in \left[a_{n_2}^{(p)}, a_{n_2+1}^{(p)} \right)$;

$a_{n_1}^{(p)} > r_1 \Rightarrow n_1 > n_2$, $r_1 \neq 0$ and $a_{n_1}^{(p)} \le k \le a_{n_1+1}^{(p)} - 1 \Rightarrow 1 \le t_1 \le$

$\le p - 1$ because we have $t_1 \le (a_{n_1+1}^{(p)} - 1 - r_1) : a_n^{(p)} < p$.

The procedure continues similarly. After a finite number of steps ℓ, we achieve $r_\ell = 0$, as k = finite, $k \in N^*$

and $k > r_1 > r_2 > \ldots > r_\ell = 0$ and between 0 and k there is only a finite number of distinct natural numbers.

Thus:

k is uniquely written: $k = t_1 a_{n_1}^{(p)} + r_1$, $1 \le t_1 \le p - 1$,

r is uniquely written: $r_1 = t_2 a_{n_2}^{(p)} + r_2$, $n_2 < n_1$,

$$1 \le t_2 \le p - 1,$$

⋮

$r_{\ell-1}$ is uniquely written: $r_{\ell-1} = t_\ell a_{n_\ell}^{(p)} + r_\ell$ and $r_\ell = 0$,

$$n_\ell < n_{\ell-1}, \ 1 \le t_\ell \le p,$$

⇒k is uniquely written under the shape $k = t_1 a_{n_1}^{(p)} + \ldots +$

$$+ \ldots + t_\ell a_{n_\ell}^{(p)}$$

with $n_1 > n_2 > \ldots > n_\ell > 0$; $n_\ell > 0$ because $n_\ell \in N^*$, $1 \le t_j \le$ $\le p - 1$, $j = \overline{1, \ell-1}$, $1 \le t_\ell \le p$, $\ell \ge 1$.

Let $k \in N^*$, $k = t_1 a_{n_1}^{(p)} + \ldots + \bar{t}_\ell a_{n_\ell}^{(p)}$ with $a_{n_i}^{(p)} = \dfrac{p^{n_i} - 1}{p-1}$,

$i = \overline{1,\ell}$, $\ell \geq 1$, n_i, $t_i \in N^*$, $i = \overline{1,\ell}$, $n_1 > n_2 > \ldots > n_\ell >$

$1 \leq t_j \leq p - 1$, $j = \overline{1,\ell-1}$, $1 \leq t_\ell \leq p$.

I construct the function η_p, p = prime > 0, $\eta_p: N^* \to N$ thus:

$$\forall n \in N^* \quad \eta_p(a_n^{(p)}) = p^n ,$$

$$\eta_p(t_1 a_{n_1}^{(p)} + \ldots + t_\ell a_{n_\ell}^{(p)}) = t_1 \eta_p(a_{n_1}^{(p)}) + \ldots$$

$$+ t_\ell \eta_p (a_{n_\ell}^{(p)}) .$$

NOTE 1. The function η_p is well defined for each natural number.

Proof

LEMMA 2. $\forall$ k $\in N^*$ ⇒ k is uniquely written as $k = t_1 a_{n_1}^{(p)} + \ldots + t_\ell a_{n_\ell}^{(p)}$ with the conditions from Lemma 1 ⇒ $\exists!$ $t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} = \eta_p (t_1 a_{n_1}^{(p)} + \ldots + t_\ell a_{n_\ell}^{(p)})$ and $t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} \in N^*$.

LEMMA 3. $\forall\, k \in N^*$, $\forall\, p \in N$, p = prime $\Rightarrow k = t_1 a_{n_1}^{(p)} + \dots + t_\ell a_{n_\ell}^{(p)}$ with the conditions from Lemma 2 $\Rightarrow \eta_p(k) = t_1 p^{n_1} + \dots + t_\ell p^{n_\ell}$.

It is known that $\left[\dfrac{a_1 + \dots + a_n}{b}\right] \geq \left[\dfrac{a_1}{b}\right] + \dots + \left[\dfrac{a_n}{b}\right]$ $\forall\, a_i, b \in N^*$ where through $[\alpha]$ we have written the integer side of the number α. I shall prove that p's powers sum from the natural numbers which make up the result factors $(t_1 p^{n_1} + \dots + t_\ell p^{n_\ell})!$ is $\geq k$;

$$\left[\frac{t_1 p^{n_1} + \dots + t_\ell p^{n_\ell}}{p}\right] \geq \left[\frac{t_1 p^{n_1}}{p}\right] + \dots + \left[\frac{t_\ell p^{n_\ell}}{p}\right] = t_1 p^{n_1 - 1} + \dots + t_\ell p^{n_\ell - 1}$$

$\vdots$

$$\left[\frac{t_1 p^{n_1} + \dots + t_\ell p^{n_\ell}}{p^{n}}\right] \geq \left[\frac{t_1 p^{n_1}}{p^{n_\ell}}\right] + \dots + \left[\frac{t_\ell p^{n_\ell}}{p^{n_\ell}}\right] = t_1 p^{n_1 - n_\ell} + \dots + t_\ell p^{0}$$

$$\left[\frac{t_1 p^{n_1} + \dots + t_\ell p^{n_\ell}}{p^{n_1}}\right] \geq \left[\frac{t_1 p^{n_1}}{p^{n_1}}\right] + \dots + \left[\frac{t_\ell p^{n_\ell}}{p^{n_1}}\right] = t_1 p^0 + \dots$$

$$+ \left[\frac{t_\ell p^{n_\ell}}{p^{n_1}}\right] .$$

Adding → p's powers sum is $\geq t_1(p^{n_1-1} + \dots + p^0) + \dots +$

$+ t_\ell\,(p^{n_\ell-1} + \dots + p^0) = t_1 a_{n_1}^{(p)} + \dots + t_\ell a_{n_\ell}^{(p)} = k$.

THEOREM 1. the function η_p, p = prime, defined previously, has the following properties:

(1) $\forall\, k \in N^*$, $(\eta_p(k))! = \mathcal{M}p^k$.

(2) $\eta_p(k)$ is the smallest number with the property (1).

Proof

(1) results from Lemma 3.

(2) $\forall\, k \in N^*$, $p \geq 2 \Rightarrow k = t_1 a_{n_1}^{(p)} + \dots + t_\ell a_{n_\ell}^{(p)}$ (by Lemma 2) is uniquely written, where:

$$n_i,\ t_i \in N^*,\ n_1 > n_2 > \dots > n_\ell > 0,\ a_{n_i}^{(p)} = \frac{p^{n_i}-1}{p-1} \in N^*,$$

$$i = \overline{1,\ell}\ ,\ 1 \le t_j \le p-1,\ j = \overline{1,\ell-1}\ ,\ 1 < t_\ell < p.$$

$$\Rightarrow \eta_p(k) = t_1 p^{n_1} + \dots + t_\ell p^{n_\ell}.\quad \text{I note:}\quad z = t_1 p^{n_1} +$$

$$+ \dots + t_\ell p^{n_\ell}.$$

Let us prove that z is the smallest natural number with the property (1). I suppose by the method of reductio ad absurdum that $\exists\ \gamma \in N,\ \gamma < z$:

$\gamma! = Mp^k;$

$\gamma < z \Rightarrow \gamma \le z - 1 \Rightarrow (z-1)! = Mp^k.$

$z - 1 = t_1 p^{n_1} + \dots + t_\ell\, p_\ell^{n_\ell} - 1;\ n_1 > n_2 > \dots > n_\ell \ge 1$ and

$n_j \in N,\ j = \overline{1,\ell}$;

$$\left[\frac{z-1}{p}\right] = t_1 p^{n_1-1} + \dots + t_{\ell-1} p^{n_{\ell-1}-1} + t_\ell p^{n_\ell-1} - 1 \text{ as } \left[\frac{-1}{p}\right] = -1$$

because $p \ge 2$,

$$\left[\frac{z-1}{p^{n_\ell}}\right] = t_1p^{n_1-n_\ell} + \dots + t_{\ell-1}p^{n_{\ell-1}-n_\ell} + t_\ell p^0 - 1 \text{ as } \left[\frac{-1}{p^{n_\ell}}\right] = -1$$

as $p \geq 2,\ n_\ell \geq 1$,

$$\left[\frac{z-1}{p^{n_\ell+1}}\right] = t_1p^{n_1-n_\ell-1} + \dots + t_{\ell-1}p^{n_{\ell-1}-n_\ell-1} + \left[\frac{t_\ell p^{n_\ell}-1}{p^{n_\ell+1}}\right] =$$

$$= t_1p^{n_1-n_\ell-1} + \dots + t_{\ell-1}p^{n_{\ell-1}-n_\ell-1} \quad \text{because}$$

$$0 < t_\ell p^{n_\ell} - 1 \leq p \cdot p^{n_\ell} - 1 < p^{n_\ell+1} \quad \text{as } t_\ell < p\ ;$$

$\vdots$

$$\left[\frac{z-1}{p^{n_{\ell-1}}}\right] = t_1p^{n_1n_{\ell-1}} + \dots + t_{\ell-1}p^0 + \left[\frac{t_\ell p^{n_\ell}-1}{p^{n_{\ell-1}}}\right] = t_1p^{n_1-n_{\ell-1}} +$$

$$+ \dots + t_{\ell-1}p^0 \text{ as } n_{\ell-1} > n_\ell\ ,$$

$\vdots$

$$\left[\frac{z-1}{p^{n_1}}\right] = t_1p^0 + \left[\frac{t_2p^{n_2} + \dots + t_\ell p^{n_\ell}-1}{p^{n_1}}\right] = t_1p^0\ .$$

Because $0 < t_2p^{n_2} + \dots + t_\ell p^{n_\ell}-1 \leq (p-1)\,p^{n_2} + \dots +$

$$+ (p-1)p^{n_{\ell-1}} + p\cdot p^{n_\ell} - 1 \leq (p-1)\cdot \sum_{i=n_{\ell-1}}^{n_2} p^i + p^{n_\ell+1} - 1 \leq$$

$$\leq (p-1)\frac{p^{n_2+1}}{p-1} = p^{n_2+1} - 1 < p^{n_1} - 1 < p^{n_1} \Rightarrow$$

$$\Rightarrow \left[\frac{t_2 p^{n_2} + \ldots + t_\ell p^{n_\ell} - 1}{p^{n_1}}\right] = 0$$

$$\left[\frac{z-1}{p^{n_1+1}}\right] = \left[\frac{t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} - 1}{p^{n_1+1}}\right] = 0 \text{ because:}$$

$0 < t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} - 1 < p^{n_1+1} - 1 < p^{n_1+1}$ according to a reasoning similar to the previous one.

Adding $\Rightarrow$ p's powers sum in the natural numbers which make up the product factors $(z-1)!$ is:

$$t_1 (p^{n_1-1} + \ldots + p^0) + \ldots + t_{\ell-1}' (p^{n_{\ell-1}-1} + \ldots + p^0) +$$

$$+ t_\ell (p^{n_\ell-1} + \ldots + p^0) - 1\cdot n_\ell = k - n_\ell < k - 1 < k \text{ because}$$

$n_\ell > 1 \Rightarrow (z-1)! \neq \mathcal{M}p^k$, this contradicts the supposition made.

- $\eta_p(k)$ is the smallest natural number with the property $(\eta_p(k))! = \mathcal{M}p^k$.

I construct a new function $\eta: Z\setminus\{0\} \to N$ defined as follows:

$$\begin{cases} \eta(\pm 1) = 0, \\ \forall\, n = \epsilon\, p_1^{\alpha_1} \ldots p_s^{\alpha_s} \text{ with } \epsilon = \pm 1,\ p_i = \text{prime}, \\ p_i \neq p_j \text{ for } i \neq j,\ \alpha_i \geq 1,\ \ i = \overline{1,s}\ ,\ \eta(n) = \\ = \max\limits_{i=\overline{1,s}} \{\eta_{p_i}(\alpha_i)\}. \end{cases}$$

NOTE 2. η is well defined and defined overall.

Proof

(a) $\forall\, n \in Z$, $n \neq 0$, $n \neq \pm 1$, n is uniquely written, independent of the order of the factors, under the shape of $n = \epsilon\, p_1^{\alpha_1} \ldots p_s^{\alpha_s}$ with $\epsilon = \pm 1$ where p_i = prime, $p_i \neq p_j$, $\alpha_i \geq$ ≥ 1 (decompose into prime factors in Z = factorial ring)).

- $\exists!\ \eta(n) = \max\limits_{\overline{1,s}} \{\eta_{p_i}(\alpha_i)\}$ as s = finite and $\eta_{p_i}(\alpha_i) \in N^*$

and $\exists \underset{i=\overline{1,s}}{\max} \{\eta_{p_i}(\alpha_i)\}$

(b) $n = \pm 1 \Rightarrow \exists ! \ \eta(n) = 0.$

THEOREM 2. The function η previously defined has the following properties:

(1) $(\eta(n))! = \mathcal{M} n, \ \forall n \in Z\setminus\{0\}$;

(2) $\eta(n)$ is the smallest natural number with this property.

<u>Proof</u>

(a) $\eta(n) = \underset{i=\overline{1,s}}{\max} \{\eta_{p_i}(\alpha_i)\}, \ n = \epsilon \cdot p_1^{\alpha_1} \dots p_s^{\alpha_s}$,

$(n \neq \pm 1)$,

$(\eta_{p_1}(\alpha_1))! = \mathcal{M} p_1^{\alpha_1}$,

$\vdots$

$(\eta_{p_s}(\alpha_s))! = \mathcal{M} p_s^{\alpha_s}$.

Supposing $\underset{i=\overline{1,s}}{\max} \{\eta_{p_i}(\alpha_1)\} = \eta_{p_{i_0}}(\alpha_{i_0}) \Rightarrow (\eta_{p_{i_0}}(\alpha_{i_0}))! =$

$= \mathcal{M} p_{i_0}^{\alpha_{i_0}}), \eta_{p_{i_0}}(\alpha_{i_0}) \in N^*$ and because $(p_i, p_j) = 1, \ i \neq j$,

$$\Rightarrow (\eta_{p_{i_0}}(\alpha_{i_0}))! = \mathcal{M} p_j^{\alpha_j}, \; j = \overline{1,s}.$$

$$\Rightarrow (\eta_{p_{i_0}}(\alpha_{i_0}))! = \mathcal{M} p_1^{\alpha_1} \dots p_s^{\alpha_s}.$$

(b) $n = \pm 1 \Rightarrow \eta(n) = 0;\ 0! = 1,\ 1 = \mathcal{M}\epsilon \cdot 1 = \mathcal{M}n.$

(2) (a) $n \neq \pm 1 \Rightarrow n = \epsilon\, p_1^{\alpha_1} \dots p_s^{\alpha_s} \Rightarrow \eta(n) = \max\limits_{i=\overline{1,s}} \eta_{p_i}.$

Let $\max\limits_{i=\overline{1,s}} \{\eta_{p_i}(\alpha_i)\} = \eta_{p_{i_0}}(\alpha_{i_0}),\ 1 \le i \le s;$

$\eta_{p_{i_0}}(\alpha_{i_0})$ is the smallest natural number with the property:

$$(\eta_{p_{i_0}}(\alpha_{i_0}))! = \mathcal{M}\, p_{i_0}^{\alpha_{i_0}} \Rightarrow \forall \gamma \in N,\ \gamma < \eta_{p_{i_0}}(\alpha_{i_0}) \Rightarrow$$

$$\gamma! \neq \mathcal{M} p_{i_0}^{\alpha_{i_0}} \Rightarrow \gamma! \neq \mathcal{M}\, \epsilon \cdot p_1^{\alpha_1} \dots p_{i_0}^{\alpha_{i_0}} \dots p_s^{\alpha_s} = \mathcal{M}\, n$$

$\Rightarrow \eta_{p_{i_0}}(\alpha_{i_0})$ is the smallest natural number with the property.

(b) $n = \pm 1 \Rightarrow \eta(n) = 0$ and it is the smallest natural number $\Rightarrow$ 0 is the smallest natural number with the property $0! = \mathcal{M}(\pm 1)$.

NOTE $\underline{3}$. The functions η_p are increasing, not injective, on $N^* \to \{p^k \mid k = 1, 2, \ldots\}$ they are surjective.

The function η is increasing, it is not injective, it is surjective on $Z \setminus \{0\} \to N \setminus \{1\}$.

CONSEQUENCE. Let $n \in N^*$, $n > 4$. Then $n = \text{prime} \Leftrightarrow \eta(n) = n$.

<u>Proof</u>

"$\Rightarrow$"

$n = \text{prime}$ and $n \geq 5 \Rightarrow \eta(n) = \eta_n(1) = n$.

"$\Leftarrow$"

Let $\eta(n) = n$ and suppose by absurd that $n \neq \text{prime} \Rightarrow$

(a) or $n = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$ with $s \geq 2$, $\alpha_i \in N^*$, $i = \overline{1,s}$,

$$\eta(n) = \max_{i=\overline{1,s}} \{\eta_{p_i}(\alpha_i)\} = \eta_{p_{i_0}}(\alpha_{i_0}) < \alpha_{i_0} p_{i_0} < n$$

contradicts the assumption; or

(b) $n = p_1^{\alpha_1}$ with $\alpha_1 \geq 2 \Rightarrow \eta(n) = \eta_{p_1}(\alpha_1) \leq p_1 \cdot \alpha_1 < p_1^{\alpha_1} = n$

because $\alpha_1 \geq 2$ and $n > 4$ and it contradicts the hypothesis.

<u>Application</u>

$\underline{1}$. Find the smallest natural number with the property:

$$n! = \mathcal{M}(\pm 2^{31} \cdot 3^{27} \cdot 7^{13}) .$$

Solution

$$\eta(\pm\ 2^{31} \cdot 3^{27} \cdot 7^{13}) = \max\ \{\eta_2(31),\ \eta_3(27),\ \eta_7(13)\}.$$

Let us calculate $\eta_2(31)$; we make the string $(a_n^{(2)})_{n \in N^*} =$
$= 1, 3, 7, 15, 31, 63, \ldots$
$31 = 1 \cdot 31 \Rightarrow \eta_2(31) = \eta_2(1 \cdot 31) = 1 \cdot 2^5 = 32.$
Let's calculate $\eta_3(27)$ making the string $(a_n^{(3)})_{n \in N^*} =$
$= 1, 4, 13, 40, \ldots; 27 = 2 \cdot 13 + 1 \Rightarrow \eta_3^{(27)} = \eta_3(2 \cdot 13 + 1 \cdot 1) =$
$= 2 \cdot \eta_3(13) + 1 \cdot \eta_3(1) = 2 \cdot 3^3 + 1 \cdot 3^1 = 54 + 3 = 57.$
Let's calculate $\eta_7(13)$; making the string $(a_n^{(7)})_{n \in N^*} =$
$= 1, 8, 57, \ldots; 13 = 1 \cdot 8 + 5 \cdot 1 \Rightarrow \eta_7(13) = 1 \cdot \eta_7(8) + 5 \cdot \eta_7(1)$
$= 1 \cdot 7^2 + 5 \cdot 7^1 = 49 + 35 = 84 \Rightarrow \eta(\pm\ 2^{31} \cdot 3^{27} \cdot 7^{13}) = \max\ \{32, 57,$
$84\} = 84 \Rightarrow 84! = M(\pm\ 2^{31} \cdot 3^{27} \cdot 7^{13})$ and 84 is the smallest number with this property.

2. Which are the numbers with the factorial ending in 1000 zeros?

Solution

$n = 10^{1000}$, $(\eta(n))! = M10^{1000}$ and it is the smallest number with this property.

$\eta(10^{1000}) = \eta(2^{1000} \cdot 5^{1000}) = \max\ \{\eta_2\ (1000),\ \eta_5\ (1000)\} =$

$= \eta_5(1000) = \eta_5(1 \cdot 781 + 1 \cdot 156 + 2 \cdot 31 + 1) = 1 \cdot 5^5 + 1 \cdot 5^4 +$

$+ 2 \cdot 5^3 + 1 \cdot 5^7 = 4005$, 4005 is the smallest number with this property. 4006, 4007, 4008, 4009 verify the property but 4010 does not because 4010! = 4009! 4010 has 1001 zeros.

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