ABOUT THE SMARANDACHE SQUARE'S COMPLEMENTARY FUNCTION

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DEFINITION 1. *Let $a: \mathbf{N}^* \to \mathbf{N}^*$ be a numerical function defined by $a(n) = k$ where k is the smallest natural number such that nk is a perfect square: $nk = s^2$, $s \in \mathbf{N}^*$, which is called the Smarandache square's complementary function.*

PROPERTY 1.*For every $n \in \mathbf{N}^*$ $a(n^2) = 1$ and for every prime natural number $a(p)=p$.*

PROPERTY 2. *Let n be a composite natural number and $n = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdots p_{i_r}^{\alpha_{i_r}}$, $0 < p_{i_1} < p_{i_2} < \cdots < p_{i_r}$, $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r} \in \mathbf{N}$ it's prime factorization. Then*

$$a(n) = p_{i_1}^{\beta_{i_1}} \cdot p_{i_2}^{\beta_{i_2}} \cdots p_{i_r}^{\beta_{i_r}} \text{ where } \beta_{i_j} = \begin{cases} 1 \text{ if } \alpha_{i_j} \text{ is an odd natural number} \\ \\ 0 \text{ if } \alpha_{i_j} \text{ is an even natural number} \end{cases} \quad j = \overline{1,r}.$$

If we take into account of the above definition of the function a, it is easy to prove both the properties.

PROPERTY 3. $\frac{1}{n} \le \frac{a(n)}{n} \le 1$, for every $n \in \mathbf{N}^*$ *where a is the above defined function.*

Proof. It is easy to see that $1 \le a(n) \le n$ for every $n \in \mathbf{N}^*$, so the property holds.

CONSEQUENCE. $\sum_{n\ge 1} \frac{a(n)}{n}$ diverges.

PROPERTY 4. *The function $a: \mathbf{N}^* \to \mathbf{N}^*$ is multiplicative:*

$$a(x \cdot y) = a(x) \cdot a(y) \text{ for every } x, y \in \mathbf{N}^* \text{ whith } (x,y) = 1$$

Proof. For $x = 1 = y$ we have $(x,y) = 1$ and $a(1 \cdot 1) = a(1) \cdot a(1)$. Let $x = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdots p_{i_r}^{\alpha_{i_r}}$ and $y = q_{j_1}^{\gamma_{j_1}} \cdot q_{j_2}^{\gamma_{j_2}} \cdots q_{j_s}^{\gamma_{j_s}}$ be the prime factorization of x and y, respectively, and $x \cdot y \ne 1$. Because $(x,y) = 1$ we have $p_{i_h} \ne q_{j_k}$ for every $h = \overline{1,r}$ and $k = \overline{1,s}$. Then,

$$a(x)=p_{i_1}^{\beta_{i_1}}\cdot p_{i_2}^{\beta_{i_2}}\cdots p_{i_r}^{\beta_{i_r}} \text{ where } \beta_{i_j}=\begin{cases}1 \text{ if } \alpha_{i_j} \text{ is odd}\\ 0 \text{ if } \alpha_{i_j} \text{ is even}\end{cases},\ j=\overline{1,r},$$

$$a(y)=q_{j_1}^{\delta_{j_1}}\cdot q_{j_2}^{\delta_{j_2}}\cdots q_{j_s}^{\delta_{j_s}} \text{ where } \delta_{j_k}=\begin{cases}1 \text{ if } \gamma_{j_k} \text{ is odd}\\ 0 \text{ if } \gamma_{j_k} \text{ is even}\end{cases},\ k=\overline{1,s} \text{ and}$$

$$a(xy)=p_{i_1}^{\beta_{i_1}}\cdot p_{i_2}^{\beta_{i_2}}\cdots p_{i_r}^{\beta_{i_r}}\cdot q_{j_1}^{\delta_{j_1}}\cdot q_{j_2}^{\delta_{j_2}}\cdots q_{j_s}^{\delta_{j_s}}=a(x)\cdot a(y)$$

Property 5. *If $(x,y)=1$, x and y are not perfect squares and x,y>1 the equation a(x)=a(y) has not natural solutions.*

Proof. It is easy to see that $x\neq y$. Let $x=\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}$ and $y=\prod_{k=1}^{s}q_{j_k}^{\gamma_{j_k}}$, (where $p_{i_h}\neq q_{j_k}$, $\forall h=\overline{1,r}, k=\overline{1,s}$ be their prime factorization.

Then $a(x)=\prod_{h=1}^{r}p_{i_h}^{\beta_{i_h}}$ and $a(y)=\prod_{k=1}^{s}q^{\delta_{j_k}}$, where β_{i_h} for $h=\overline{1,r}$ and δ_{j_k} for $k=\overline{1,s}$ have the above signifiecance, but there exist at least $\beta_{i_h}\neq 0$ and $\delta_{j_k}\neq 0$. (**because *x* and *y* are** not perfect squares). Then $a(x)\neq a(y)$.

Remark. If $x=1$ from the above equation it results $a(y)=1$, so y must be a a perfect square (analogously for $y=1$).

Consequence. The equation $a(x)=a(x+1)$ has not natural solutions, because for $x>1$ x and $x+1$ are not both perfect squares and $(x, x+1)=1$.

Property 6. *We have $a(x\cdot y^2)=a(x)$, for every $x,y\in\mathbf{N}^*$.*

Proof. If $(x, y)=1$, then $(x,y^2)=1$ and using property 4 and property 1 we have $a(x\cdot y^2)=a(x)\cdot a(y^2)=a(x)$. If $(x,y)\neq 1$ we can write: $x=\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}}$ and $y=\prod_{k=1}^{s}q_{jk}^{\gamma_{jk}}\cdot\prod_{t=1}^{n}d_{l_t}^{\gamma_{l_t}}$ where $p_{i_h}\neq d_{l_t}$, $q_{jk}\neq d_{l_t}$, $p_{i_h}\neq q_{jk}$, $\forall h=\overline{1,r}$, $k=\overline{1,s}$, $t=\overline{1,n}$, but this

implies $\left(\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{k=1}^{s}q_{j_k}^{2\gamma_{j_k}},\ \prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}+2\gamma_{l_t}}\right)=1$ and

$$\left(\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}},\ \prod_{k=1}^{s}q_{j_k}^{2\gamma_{j_k}}\right)=1\Rightarrow a(xy^2)=a\left(\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{k=1}^{s}q_{j_k}^{2\gamma_{j_k}}\cdot\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}+2\gamma_{l_t}}\right)=$$

$$a\left(\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\cdot\prod_{k=1}^{s}q_{j_k}^{2\gamma_{j_k}}\right)\cdot a\left(\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}+2\gamma_{l_t}}\right)=a\left(\prod_{h=1}^{r}p_{i_h}^{\alpha_{i_h}}\right)\cdot a\left(\prod_{k=1}^{s}q_{j_k}^{2\gamma_{j_k}}\right)\cdot a\left(\prod_{t=1}^{n}d_{l_t}^{\alpha_{l_t}+2\gamma_{l_t}}\right)$$

$$a\left(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}\right)\cdot a\left(\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}-2\gamma_{l_t}}\right)=a\left(\prod_{h=1}^{r} p_{i_h}^{\alpha_{i_h}}\cdot\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}\right)=a(x)$$ because

$$a\left(\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}+2\gamma_{l_t}}\right)=\prod_{t=1}^{n} d_{l_t}^{\beta_{l_t}}=a\left(\prod_{t=1}^{n} d_{l_t}^{\alpha_{l_t}}\right),\ \text{where}\ \beta_{l_t}=\begin{cases}1 & \text{if } \alpha_{l_t}+2\gamma_{l_t} \text{ is odd}\\ 0 & \text{if } \alpha_{l_t}+2\gamma_{l_t} \text{ is even}\end{cases}$$

$$=\begin{cases}1 & \text{if } \alpha_{l_t} \text{ is odd}\\ 0 & \text{if } \alpha_{l_t} \text{ is even}\end{cases}$$

Consequence 1. For every $x\in\mathbf{N}^*$ and $n\in\mathbf{N}$, $a(x^n)=\begin{cases}1 & \text{if } n \text{ is even}\\ a(x) & \text{if } n \text{ is odd}\end{cases}$.

Consequence 2. If $\frac{x}{y}=\frac{m^2}{n^2}$ where $\frac{m}{n}$ is a simplified fraction, then $a(x)=a(y)$. It is easy to prove this, because $x=km^2$ and $y=kn^2$ and using the above property we have: $a(x)=a(km^2)=a(k)=a(kn^2)=a(y)$.

Property 7. *The sumatory numerical function of the function a is* $F(n)=\prod_{j=1}^{k}(H(\alpha_{i_j})(p_{i_j}+1)+\frac{1+(-1)^{\alpha_{i_j}}}{2})$ *where the prime factorization of n is* $n=p_{i_1}^{\alpha_{i_1}}\cdot p_{i_2}^{\alpha_{i_2}}\cdot\ldots\cdot p_{i_k}^{\alpha_{i_k}}$ *and* $H(\alpha)$ *is the number of the odd numbers which are smaller than* α.

Proof. The sumatory numerical function of a is defined as $F(n)=\sum_{d/n}a(d)$, because $(p_{i_1}^{\alpha_{i_1}},\prod_{t=2}^{k}p_{i_t}^{\alpha_{i_t}})=1$ we can use the property 4 and we obtain: $F(n)=\left(\sum_{d_1/p_{i_1}^{\alpha_{i_1}}}a(d_1)\right)\cdot\left(\sum_{d_2/p_{i_2}^{\alpha_{i_2}}\ldots p_{i_k}^{\alpha_{i_k}}}a(d_2)\right)$ and so on, making a finite number of steps we obtain $F(n)=\prod_{j=1}^{k}F(p_{i_j}^{\alpha_{i_j}})$. But we observe that

$$F(p^\alpha)=\begin{cases}\frac{\alpha}{2}(p+1)+1 & \text{if } \alpha \text{ is an even number}\\ \left(\left[\frac{\alpha}{2}\right]+1\right)(p+1) & \text{if } \alpha \text{ is an odd number}\end{cases}$$

where p is a prime number.

If we take into account of the definition of $H(\alpha)$ we find $H(\alpha)=\begin{cases}\frac{\alpha}{2} & \text{if } \alpha \text{ is even}\\ \left[\frac{\alpha}{2}\right]+1 & \text{if } \alpha \text{ is odd}\end{cases}$ so we can write $F(p^\alpha)=H(\alpha)\cdot(p+1)+\frac{1+(-1)^\alpha}{2}$,

therefore: $F(n)=\prod_{j=1}^{k}(H(\alpha_{i_j})(p_{i_j}+1)+\frac{1+(-1)^{\alpha_{i_j}}}{2}$.

In the sequel we study some equations which involve the function a.

1) Find the solutions of the equation: $xa(x)=m$, where $x,m \in \mathbf{N}^*$.

If m is not a perfect square then the above equation has not solutions.

If m is a perfect square, $m=z^2, z \in \mathbf{N}^*$, then we have to give the solutions of the equation $xa(x)=z^2$.

Let $z=p_{i_1}^{\alpha_1} \cdot p_{i_2}^{\alpha_2} \cdots p_{i_k}^{\alpha_k}$ be the prime factorization of z. Then $xa(x)=p_{i_1}^{2\alpha_1} \cdot p_{i_2}^{2\alpha_2} \cdots p_{i_k}^{2\alpha_k}$, so taking account of the definition of the function a, the equation has the following solutions: $x_1^{(0)}=p_{i_1}^{2\alpha_1} \cdot p_{i_2}^{2\alpha_2} \cdots p_{i_k}^{2\alpha_k}$ (because $a(x_1^{(0)})=1$), $x_1^{(1)}=p_{i_1}^{2\alpha_1-1} \cdot p_{i_2}^{2\alpha_2} \cdots p_{i_k}^{2\alpha_k}$ (because $a(x_1^{(1)})=p_{i_1}$), $x_2^{(1)}=p_{i_1}^{2\alpha_1} \cdot p_{i_2}^{2\alpha_2-1} \cdot p_{i_3}^{2\alpha_3} \cdots p_{i_k}^{2\alpha_k}$ (because $a(x_2^{(1)})=p_{i_2}$), ..., $x_k^{(1)}=p_{i_1}^{2\alpha_1} \cdot p_{i_2}^{2\alpha_2} \cdots p_{i_k}^{2\alpha_k-1}$ (because $a(x_k^{(1)})=p_{i_k}$), then $x_t^{(2)}=\dfrac{z^2}{p_{i_{j_1}} \cdot p_{i_{j_2}}}$, $j_1 \neq j_2$, $j_1, j_2 \in \{i_1,\dots,i_k\}$, $t=\overline{1,C_k^2}$ (because $a(x_t^{(2)})=p_{i_{j_1}} \cdot p_{i_{j_2}}$), and, in an analogue way, $x_t^{(3)}$, $t \in \overline{1,C_k^3}$ has as values $\dfrac{z^2}{p_{i_{j_1}} \cdot p_{i_{j_2}} \cdot p_{i_{j_3}}}$, where $j_1,j_2,j_3 \in \{i_1,\dots,i_k\}$ $j_1 \neq j_2, j_2 \neq j_3, j_3 \neq j_1$, and so on, $x_1^{(k)}=\dfrac{z^2}{p_{i_1} \cdot p_{i_2} \cdots p_{i_k}}=\dfrac{z^2}{z}=z$. So the above equation has $1+C_k^1+C_k^2+\cdots-C_k^k=2^k$ different solutions where k is the number of the prime divisors of m.

2) Find the solutions of the equation: $xa(x)+ya(y)=za(z)$, $x,y,z \in \mathbf{N}^*$.

Proof. We note $xa(x)=m^2$, $ya(y)=n^2$ and $za(z)=s^2$, $x,y,z \in \mathbf{N}^*$ and the equation

$$m^2+n^2=s^2, \quad m,n,s \in \mathbf{N}^* \qquad (*)$$

has the following solutions: $m=u^2-v^2$, $n=2uv$, $s=u^2+v^2$, $u>v>0$, $(u,v)=1$ and u and v have different evenes.

If (m,n,s) as above is a solution, then $(\alpha m, \alpha n, \alpha s)$, $\alpha \in \mathbf{N}^*$ is also a solution of the equation (*).

If (m,n,s) is a solution of the equation (*), then the problem is to find the solutions of the equation $xa(x)=m^2$ and we see from the above problem that there are 2^{k_1} solutions (where k_1 is the number of the prime divisors of m), then the solutions of the equations $ya(y)=n^2$ and respectively $za(z)=s^2$, so the number of the different solutions of the given equations, is $2^{k_1} \cdot 2^{k_2} \cdot 2^{k_3}=2^{k_1+k_2+k_3}$ (where k_2 and k_3 have the same signifience as k_1 , but concerning n and s , respectively).

For $\alpha>1$ we have $xa(x)=\alpha^2 m^2$, $ya(y)=\alpha^2 n^2$, $za(z)=\alpha^2 s^2$ and, using an analogue way as above, we find $2^{k_1+k_2+k_3}$ different solutions, where k_i, $i=\overline{1,3}$ is the number of the prime divisors of αm, αn and αs, respectively.

Remark . In the particular case $u=2$, $v=1$ we find the solution (3,4,5) for (*). So we must find the solutions of the equations $xa(x)=3^2\alpha^2$, $ya(y)=2^4\alpha^2$ and $za(z)=5^2\alpha^2$, for $\alpha \in \mathbf{N}^*$. Suppose that α has not 2,3 and 5 as prime factors in this prime factorization $\alpha=p_{i_1}^{\alpha_1} \cdot p_{i_2}^{\alpha_2} \cdots p_{i_k}^{\alpha_k}$. Then we have:

$$xa(x) = 3^2\alpha^2 \Rightarrow x \in \{3^2\alpha^2, \frac{3^2\alpha^2}{p_{i1}}, \ldots, \frac{3^2\alpha^2}{p_{ik}}, \frac{3^2\alpha^2}{p_{i1}\cdot p_{i2}}, \ldots, \frac{3^2\alpha^2}{p_{ik-1}\cdot p_{ik}}, \ldots, \frac{3^2\alpha^2}{p_{i1}\cdots p_{ik-1}}, \ldots, \frac{3^2\alpha^2}{p_{i2}\cdots p_{ik}},$$

$$\left. 3^2\alpha, 3\alpha^2, \frac{3\alpha^2}{p_{i1}}, \ldots, \frac{3\alpha^2}{p_{ik}}, \frac{3\alpha^2}{p_{i1}\cdot p_{i2}}, \ldots, \frac{3\alpha^2}{p_{ik-1}\cdot p_{ik}}, \ldots, \frac{3\alpha^2}{p_{i1}\cdots p_{ik-1}}, \ldots, \frac{3\alpha^2}{p_{i2}\cdots p_{ik}}, 3\alpha \right\}$$

$$ya(y) = 4^2\alpha^2 \Rightarrow y \in \{4^2\alpha^2, \frac{4^2\alpha^2}{p_{i1}}, \ldots, \frac{4^2\alpha^2}{p_{ik}}, \frac{4^2\alpha^2}{p_{i1}\cdot p_{i2}}, \ldots, \frac{4^2\alpha^2}{p_{ik-1}\cdot p_{ik}}, \ldots, \frac{4^2\alpha^2}{p_{i1}\cdots p_{ik-1}}, \ldots, \frac{4^2\alpha^2}{p_{i2}\cdots p_{ik}},$$

$$\left. 4^2\alpha, 8\alpha^2, \frac{8\alpha^2}{p_{i1}}, \ldots, \frac{8\alpha^2}{p_{ik}}, \frac{8\alpha^2}{p_{i1}\cdot p_{i2}}, \ldots, \frac{8\alpha^2}{p_{ik-1}\cdot p_{ik}}, \ldots, \frac{8\alpha^2}{p_{i1}\cdots p_{ik-1}}, \ldots, \frac{8\alpha^2}{p_{i2}\cdots p_{ik}}, 8\alpha \right\}$$

$$za(y) = 5^2\alpha^2 \Rightarrow z \in \{5^2\alpha^2, \frac{5^2\alpha^2}{p_{i1}}, \ldots, \frac{5^2\alpha^2}{p_{ik}}, \frac{5^2\alpha^2}{p_{i1}\cdot p_{i2}}, \ldots, \frac{5^2\alpha^2}{p_{ik-1}\cdot p_{ik}}, \ldots, \frac{5^2\alpha^2}{p_{i1}\cdots p_{ik-1}}, \ldots, \frac{5^2\alpha^2}{p_{i2}\cdots p_{ik}},$$

$$\left. 5^2\alpha, 5\alpha^2, \frac{5\alpha^2}{p_{i1}}, \ldots, \frac{5\alpha^2}{p_{ik}}, \frac{5\alpha^2}{p_{i1}\cdot p_{i2}}, \ldots, \frac{5\alpha^2}{p_{ik-1}\cdot p_{ik}}, \ldots, \frac{5\alpha^2}{p_{i1}\cdots p_{ik-1}}, \ldots, \frac{5\alpha^2}{p_{i2}\cdots p_{ik}}, 5\alpha \right\}$$

So any triplet (x_0, y_0, z_0) with x_0, y_0 and z_0 arbitrary of above corresponding values, is a solution for the equation (for example $(9,16,25)$, is a solution).

Definition. The triplets which are the solutions of the equation $xa(x)+ya(y)=za(z)$, $x,y,z \in \mathbf{Z}^*$ we call MIV numbers.

3) Find the natural numbers x such that $a(x)$ is a three - cornered, a squared and a pentagonal number.

Proof. Because 1 is the only number which is at the same time a three - cornered, a squared and a pentagonal number, then we must find the solutions of the equation $a(x)=1$, therefore x is any perfect square.

4) Find the solutions of the equation: $\frac{1}{xa(x)} + \frac{1}{ya(y)} = \frac{1}{za(z)}$, $x,y,z \in \mathbf{N}^*$.

Proof. We have $xa(x) = m^2, ya(y) = n^2, za(z) = s^2$, $m,n,s \in \mathbf{N}^*$.

The equation $\frac{1}{m^2} + \frac{1}{n^2} = \frac{1}{s^2}$ has the solutions:

$$m = t(u^2+v^2)2uv$$

$$n = t(u^2+v^2)(u^2-v^2)$$

$$s = t(u^2-v^2)2uv,$$

$u>v$, $(u, v)=1$, u and v have different eveness and $t \in \mathbf{N}^*$, so we have

$$xa(x) = t^2(u^2+v^2)^2 4u^2v^2$$
$$ya(y) = t^2(u^2+v^2)^2(u^2-v^2)^2$$
$$za(z) = t^2(u^2-v^2)^2 4u^2v^2$$

and we find x, y and z in the same way which is indicated in the first problem.

For example, if $u=2$, $v=1$, $t=1$ we have

$m=20$, $n=15$, $s=12$, so we must find the solutions of the following equations:

$$xa(x) = 20^2 = 2^4 \cdot 5^2 \Rightarrow x \in \{2^3 \cdot 5^2 = 200,\ 2^4 \cdot 5 = 80,\ 2^3 \cdot 5 = 40,\ 2^4 \cdot 5^2 = 400\}$$

$$ya(y) = 15^2 = 3^2 \cdot 5^2 \Rightarrow y \in \{15, 45, 75, 225\}$$

$$za(z) = 12^2 = 2^4 \cdot 3^2 \Rightarrow z \in \{24, 48, 72, 144\}$$

Therefore for this particular values of u, v and t we find $4 \cdot 4 \cdot 4 = 2^2 \cdot 2^2 \cdot 2^2 = 2^6 = 64$ solutions. (because $k_1 = k_2 = k_3 = 2$)

5) Find the solutions of the equation: $a(x)+a(y)+a(z) = a(x)a(y)a(z)$, $x, y, z \in \mathbf{N}^*$.

Proof. If $a(x) = m$, $a(y) = n$ and $a(z) = s$, the equation $m+n+s = m \cdot n \cdot s$, $m, n, s \in \mathbf{N}^*$ has a solutions the permutations of the set $\{1, 2, 3\}$ so we have:

$a(x) = 1 \Rightarrow x$ must be a perfect square, therefore $x = u^2$, $u \in \mathbf{N}^*$
$a(y) = 2 \Rightarrow y = 2v^2$, $v \in \mathbf{N}^*$
$a(z) = 3 \Rightarrow z = 3t^2$, $t \in \mathbf{N}^*$.

Therefore the solutions are the permutation of the sets $\{u^2, 2v^2, 3t^2\}$ where $u, v, t \in \mathbf{N}^*$.

6) Find the solutions of the equation $Aa(x)+Ba(y)+Ca(z) = 0$, $A, B, C \in \mathbf{Z}^*$.

Proof. If we note $a(x) = u, a(y) = v, a(z) = t$ we must find the solutions of the equation $Au + Bv + Ct = 0$.

Using the method of determinants we have:

$$\begin{vmatrix} A & B & C \\ A & B & C \\ m & n & s \end{vmatrix} = 0, \quad \forall m, n, s \in \mathbf{Z} \Rightarrow A(Bs - Cn) + B(Cm - As) + C(An - Bm) = 0,$$ and it

is known that the only solutions are
$$u = Bs - Cn$$
$$v = Cm - As$$
$$t = An - Bm, \quad \forall m, n, s \in \mathbf{Z}$$

so, we have
$$a(x) = Bs - Cn$$
$$a(y) = Cm - As$$
$$a(z) = An - Bm$$
and now we know to find x, y and z.

Example. If we have the following equation: $2a(x) - 3a(y) - a(z) = 0$, usind the above result we must find (with the above mentioned method) the solutions of the equations:

$a(x) = -3s + n$
$a(y) = -m - 2s$
$a(z) = 2n + 3m,\ \ m,n$ and $s \in \mathbf{Z}$.

For $m = -1,\ \ n = 2,\ s = 0\ :\ a(x) = 2,\ a(y) = 1,\ a(z) = 1$ so, the solution in this case is $(2\alpha^2, \beta^2, \gamma^2),\ \ \alpha, \beta, \gamma \in \mathbf{Z}^*$. For the another values of m,n,s we find the corresponding solutions.

7) The same problem for the equation $Aa(x) + Ba(y) = C,\ \ A,B,C \in \mathbf{Z}$.

Proof. $Aa(x) + Ba(y) - C = 0 \Leftrightarrow Aa(x) + Ba(y) + (-C)a(z) = 0$ with $a(z) = 1$ so we must have $An - Bm = 1$. If n_0 and m_0 are solutions of this equation $(An_0 - Bm_0 = 1)$ it remains us to find the solutions of the following equations:
$a(x) = Bs + Cn_0$
$a(y) = -Cm_0 - As,\ \ s \in \mathbf{Z}$, but we know how to find them.

Example. If we have the equation $2a(x) - 3a(y) = 5,\ x,y \in \mathbf{N}^*$ using the above results, we get: A=2, $B = -3,\ \ C = -5$ and $a(z) = 1 = 2n + 3m$. The solutions are $m = 2k + 1$ and $n = -1 - 3k,\ \ k \in \mathbf{Z}$. For the particular value $k = -1$ we have $m_0 = -1$ and $n_0 = 2$ so we find
$a(x) = -3 + 5 \cdot 2 = 10 - 3s$ and
$a(y) = -5(-1) - 2s = 5 - 2s$.

If $s_0 = 0$ we find $a(x) = 10 \Rightarrow x = 10u^2,\ \ u \in \mathbf{Z}^*$
$a(y) = 5 \Rightarrow y = 5v^2,\ \ v \in \mathbf{Z}^*$ and so on.

8) Find the solutions of the equation: $a(x) = ka(y)\ \ k \in \mathbf{N}^*\ \ k > 1$.

Proof. If k has in his prime factorization a factor which has an exponent ≥ 2, then the problem has not solutions.

If $k = p_{i_1} \cdot p_{i_2} \cdots p_{i_r}$ and the prime factorizarion of $a(y)$ is $a(y) = q_{j_1} \cdot q_{j_2} \cdots q_{j_u}$, then we have solutions only in the case $p_{i_1}, p_{i_2}, \dots p_{i_r} \notin \{q_{j_1}, q_{j_2}, \dots, q_{j_u}\}$.

This implies that $a(x) = p_{i_1} \cdot p_{i_2} \cdots p_{i_r} \cdot q_{j_1} \cdot q_{j_2} \cdots q_{j_u}$, so we have the solutions

$x = p_{i1} \cdot p_{i2} \cdots p_{ir} \cdot q_{j1} \cdot q_{j2} \cdots q_{ju} \cdot \alpha^2$
$y = q_{j_1} \cdot q_{j_2} \cdots q_{j_u} \cdot \beta^2,\ \ \alpha, \beta \in \mathbf{Z}^*$.

9) Find the solutions of the equation $a(x)$=x (the fixed points of the function a).

Proof. Obviously, $a(1)$=1. Let x >1 and let $x = p_{i_1}^{\alpha_{i1}} \cdot p_{i_2}^{\alpha_{i2}} \cdots p_{i_r}^{\alpha_{ir}},\ \ \alpha_{i_j} \geq 1$, for $j = \overline{1,r}$ be the prime factorization of x. Then $a(x) = p_{i_1}^{\beta_{i1}} \cdot p_{i2}^{\beta_{i2}} \cdots p_{i_r}^{\beta_{ir}}$ and $\beta_{i_j} \leq 1$ for $j = \overline{1,r}$. Because $a(x)$=x this implies that $\alpha_{i_j} = \beta_{i_j} = 1,\ \ \forall j \in \overline{1,r}$, therefore $x = p_{i_1} \cdot p_{i_2} \cdots p_{i_r}$, where $p_{i_j},\ j = \overline{1,r}$ are prime numbers.

REFERENCES

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