On some numerical functions

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In this paper we prove that the following numerical functions:

1. $F_S: N^* \to N$, $F_S(x) = \sum_{i=1}^{\pi(x)} S(p_i^x)$, where p_i are the prime natural numbers which are not greater than x and $\pi(x)$ is the number of them,

2. $\theta: N^* \to N$, $\theta(x) = \sum_{p_i \mid x} S(p_i^x)$, where p_i are the prime natural numbers which divide x,

3. $\tilde{\theta}: N^* \to N$, $\tilde{\theta}(x) = \sum_{p_i \nmid x} S(p_i^x)$, where p_i are the prime natural numbers which are smaller than x and do not divide x,

which involve the Smarandache function, does not verify the Lipschitz condition. These results are useful to study the behaviour of the numerical functions considered above.

Proposition 1 *The function* $F_S: N^* \to N$, $F_S(x) = \sum_{i=1}^{\pi(x)} S(p_i^x)$, *where* p_i *and* $\pi(x)$ *have the significence from above, does not verify the Lipschitz condition.*

Proof. Let $K > 0$ be a given real number, $x = p$ be a prime natural number, which verify $p > \left[\sqrt{K}+1\right]$ and $y = p-1$. It is easy to see that $\pi(p) = \pi(p-1)+1$, for every prime natural number p, since the prime natural numbers which are not greater than p are the same as those of $(p-1)$ in addition to p. We have:

$$|F_S(x) - F_S(y)| = F_S(p) - F_S(p-1) =$$
$$= \left[S(p_1^p) + S(p_2^p) + \cdots + S(p_{\pi(p-1)}^p) + S(p^p)\right] -$$
$$- \left[S(p_1^{p-1}) + S(p_2^{p-1}) + \cdots + S(p_{\pi(p-1)}^{p-1})\right] =$$

$$= \left[S(p_1^p) - S(p_1^{p-1})\right] + \cdots + \left[S(p_{\pi(p-1)}^p) - S(p_{\pi(p-1)}^{p-1})\right] + S(p^p).$$

But $S(p_i^p) \geq S(p_i^{p-1})$ for every $i \in \overline{1, \pi(p-1)}$, therefore we have

$$|F_S(x) - F_S(y)| \geq S(p^p).$$

Because $S(p^p) = p^2$, for every prime p, it follows:

$$|F_S(x) - F_S(y)| \geq S(p^p) = p^2 > K = K \cdot 1 = K\,(p - (p-1)) = K\,|x - y|.$$

We have proved that for every real $K > 0$ there exist the natural numbers $x = p$ and $y = p - 1$, chosen as above, so that $|F_S(x) - F_S(y)| > K\,|x - y|$, therefore F_S does not verify the Lipschitz condition.

Remark 1 *Another proof, longer and more technical, can be made using a rezult which asserts that the Smarandache function S also does not verify the Lipschitz condition. We have chosen this proof because it is more simple and free of another results.*

Proposition 2 *The function $\theta : N^* \to N$, $\theta(x) = \sum\limits_{p_i | x} S(p_i^x)$, where p_i are the prime natural numbers which divide x, does not verify the Lipschitz condition.*

Proof. Let $K > 0$ be a given real number, $x > 2$ be a natural number which has the prime factorization

$$x = p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_r}^{\alpha_r}$$

and $y = x \cdot p_k$ where $p_k > \max\{2, K\}$ is a prime natural number which does not divide x.

We have:

$$|\theta(x) - \theta(y)| = \left|\theta\left(p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_r}^{\alpha_r}\right) - \theta\left(p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_r}^{\alpha_r} \cdot p_k\right)\right| =$$

$$\left|S(p_{i_1}^x) + S(p_{i_2}^x) + \cdots + S(p_{i_r}^x) - S(p_{i_1}^y) - S(p_{i_2}^y) - \cdots - S(p_{i_r}^y) - S(p_k^y)\right|.$$

But $x < x \cdot p_k = y$ which implies that $S(p_{i_j}^x) \leq S(p_{i_j}^y)$, for $j = \overline{1, r}$ so that

$$|\theta(x) - \theta(y)| = \left[S(p_{i_1}^y) - S(p_{i_1}^x)\right] + \left[S(p_{i_2}^y) - S(p_{i_2}^x)\right] + \cdots$$

$$+ \left[S(p_{i_r}^y) - S(p_{i_r}^x)\right] + S(p_k^y) =$$

$$= \left[S(p_{i_1}^{x \cdot p_k}) - S(p_{i_1}^x)\right] + \left[S(p_{i_2}^{x \cdot p_k}) - S(p_{i_2}^x)\right] + \cdots$$

$$+ \left[S(p_{i_r}^{x \cdot p_k}) - S(p_{i_r}^x)\right] + S(p_k^y).$$

In [1] it is proved the following formula which gives a lower and an upper bound for $S(p^r)$, wher p is a prime natural number and r is a natural number:

$$(p-1)r+1 \leq S(p^r) \leq pr \tag{1}$$

Using this formula, we have:

$$\begin{aligned} S(p_{i_j}^{x \cdot p_k}) - S(p_{i_j}^{x}) &\geq (p_{i_j}-1) \cdot x \cdot p_k + 1 - p_{i_j} \cdot x = \\ &= x(p_k(p_{i_j}-1) - p_{i_j}) > 0, \ (\forall) j = \overline{1,r} \end{aligned}$$

because $p_k > 2 \geq \frac{p_{i_j}}{p_{i_j}-1}$, $(\forall) j = \overline{1,r}$.

Then, we have:

$$|\theta(x) - \theta(y)| \geq S(p_k^y) \geq (p_k-1) \cdot x \cdot p_k >> (p_k-1) \cdot x \cdot K = K(p_k \cdot x - x) = K|x-y|$$

Therefore we have proved that for every real number $K > 0$ there exist the natural numbers x, y such that: $|\theta(x) - \theta(y)| > K|x-y|$ which shows that the function θ does not verify the Lipschitz condition.

Proposition 3 *The function $\tilde{\theta} : N^* \to N$, $\tilde{\theta}(x) = \sum\limits_{p_i \nmid x} S(p_i^x)$, where p_i are the prime natural numbers which are smaller than x and do not divide x, does not verify the Lipschitz condition.*

Proof. Let $K > 0$ be a given real number. Then for $x > \frac{K}{2}$ and $y = 2 \cdot x$, using the Tchebycheff theorem we know that between x and y there exists a prime natural number p. It is clear that p does not dividex and $2x$, thus $\tilde{\theta}(y)$ contains, in the sum, besides all the terms of $\tilde{\theta}(x)$, also $S(p^y)$ as a term. We have:

$$\left|\tilde{\theta}(x) - \tilde{\theta}(y)\right| = \left|\tilde{\theta}(x) - \tilde{\theta}(2x)\right| = \tilde{\theta}(2x) - \tilde{\theta}(x) \geq \tilde{\theta}(x) + S(p^y) - \tilde{\theta}(x) = S(p^y) \geq$$

$$\geq (p-1)y + 1 = (p-1) \cdot 2x + 1 \geq x \cdot 2x + 1 = 2x^2 + 1 > x \cdot K = K|x-y|$$

therefore the function $\tilde{\theta}$ also does not verify the Lipschitz condition.

References

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