SMARANDACHE NUMERICAL FUNCTIONS

by

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F. Smarandache defines [1] a numerical function $S : \mathbb{N}^* \longrightarrow \mathbb{N}$. $S(n)$ is the smallest integer m such that $m!$ is divisible by n. Using certain results on standardised structures, three kinds of Smarandache functions are defined and are etablished some compatibility relations between these functions.

1. Standardising functions. Let X be a nonvoid set, $r \subset X \times X$ an equivalence relation, $\hat{X}$ the corresponding quotient set and $(I,\le)$ a totally ordered set.

1.1 **Definition.** If $g : \hat{X} \longrightarrow I$ is an arbitrarely injective function, then $f : X \longrightarrow I$ defined by $f(x) = g(\hat{x})$ is a standardising function. In this case the set X is said to be $[r,(I,\le),f]$ standardised.

If r_1 and r_2 are two equivalence relations on X, then $r = r_1 \wedge r_2$ is defined as $x\, r\, y$ if and only if $x\, r_1\, y$ and $x\, r_2\, y$. Of course r is an equivalence relation.

In the following theorem we consider functions having the same monotonicity. The functions $f_i : X \longrightarrow I$, $i = \overline{1,s}$ are of the same monotonicity if for every x,y from X it results

$$f_k(x) \le f_k(y) \quad \text{if and only if} \quad f_j(x) \le f_j(y) \quad \text{for} \quad k,j = \overline{1,s}$$

1.2 Theorem. If the standardising functions $f_i : X \longrightarrow I$ corresponding to the equivalence relations r_i, $i = \overline{1,s}$, are of the some monotonicity then $f = \max_i \{ f_i \}$ is a standardising function corresponding to $r = \bigwedge_i r_i$, having the same monotonicity as f_i.

Proof. We give the proof of theorem in case $s = 2$. Let $\hat{x}_{r_1}$, $\hat{x}_{r_2}$, $\hat{x}_r$ be the equivalence clases of x corresponding to r_1, r_2 and to $r = r_1 \wedge r_2$ respectively and $\hat{X}_{r_1}$, $\hat{X}_{r_2}$, $\hat{X}_r$ the quotient sets on X.

We have $f_1(x) = g_1(\hat{x}_{r_1})$ and $f_2(x) = g_2(\hat{x}_{r_2})$, where

$g_i : \hat{X}_{r_i} \longrightarrow I$, $i=1,2$ are injective functions. The function $g : \hat{X}_r \longrightarrow I$ defined by $g(\hat{x}_r) = \max\{g_1(\hat{x}_{r_1}), g_2(\hat{x}_{r_2})\}$ is injective.

Indeed, if $\hat{x}^1_r \neq \hat{x}^2_r$ and $\max\{g_1(\hat{x}^1_{r_1}), g_2(\hat{x}^1_{r_2})\} =$ $= \max\{g_1(\hat{x}^2_{r_1}), g_2(\hat{x}^2_{r_2})\}$, then be cause of the injectivity of g_1 and g_2 we have for example $\max\{g_1(\hat{x}^1_{r_1}), g_2(\hat{x}^1_{r_2})\} =$ $= g_1(\hat{x}^1_{r_1}) = g_2(\hat{x}^2_{r_2}) = \max\{g_1(\hat{x}^2_{r_1}), g_2(\hat{x}^2_{r_2})\}$ and we obtain a contradiction because $f_1(x^2) = g_1(\hat{x}^2_{r_1}) < g_1(\hat{x}^1_{r_1}) = f_1(x^1)$

$f_2(x^1) = g_2(\hat{x}^1_{r_1}) < g_2(\hat{x}^2_{r_2}) = f_2(x^2)$, that is

f_1 and f_2 are not of the same monotonicity From the injectivity of g it results that $f : X \longrightarrow I$ defined by $f(x) = g(\hat{x}_r)$ is a standardising function. In addition we have $f(x^1) \leq f(x^2) \Longleftrightarrow$ $g(\hat{x}^1_r) \leq g(\hat{x}^2_r) \Longleftrightarrow \max\{g_1(\hat{x}^1_{r_1}), g_2(\hat{x}^1_{r_2})\} \leq \max\{g_1(\hat{x}^2_{r_1}), g_2(\hat{x}^2_{r_2})\} \Longleftrightarrow$ $\Longleftrightarrow \max\{f_1(x^1), f_2(x^1)\} \leq \max\{f_1(x^2), f_2(x^2)\} \Longleftrightarrow f_1(x^1) \leq f_1(x^2)$ and $f_2(x^1) \leq f_2(x^2)$ because f_1 and f_2 are of the same monotonicity.

Let us supose now that $\top$ and $\perp$ are two algebraic lows on X and I respectively.

1.3. Definition. The standardising function $f: X \longrightarrow I$ is said to be Σ -compatibile with $\top$ and $\perp$ if for every x,y in X the triplet $(f(x), f(y), f(x \top y))$ satisfies the condition Σ. In this case it is said that the function f Σ -standardise the struc-trure $(X, \top)$ in the structure $(I, \leq, \perp)$.

For example, if f is the Smarandache function $S: \mathbb{N}^* \longrightarrow \mathbb{N}$, ($S(n)$ is the smallest integer such that $(S(n))!$ is divisible by n) then we get the following Σ-stadardisations:

a) S Σ_1-standardise $(\mathbb{N}^*, .)$ in $(\mathbb{N}^*, \leq, +)$ because we have

$$\Sigma_1: S(a.b) \leq S(a) + S(b)$$

b) but S verifie also the relation

$$\Sigma_2; \ \max(S(a), S(b)) \leq S(a.b) \leq S(a).S(b)$$

so S Σ_2-standardise the structure $(\mathbb{N}^*, .)$ in $(\mathbb{N}^*, \leq, .)$

2. Smarandache functions of first kind. The Smarandache function S is defined by means of the following functions S_p ; for every prime number p let $S_p: \mathbb{N}^* \longrightarrow \mathbb{N}^*$ having the property that $(S_p(n))!$ is divisible by p^n and is the smallest positive integer with this property. Using the notion of standardising functions in this section we give some generalisa-sion of S_p.

2.1. Definition. For every $n \in \mathbb{N}^*$ the relation $r_n \subset \mathbb{N}^* \times \mathbb{N}^*$ is defi-ned as follows: i) if $n = u^i$ (u=1 or u=p number prime, $i \in \mathbb{N}^*$) and $a, b \in \mathbb{N}^*$ then $a \ r_n b$ if and only if it exists $k \in \mathbb{N}^*$ such that $k! = M u^{ia}$, $k! = M u^{ib}$ and k is the smallest positive integer with this property.

ii) if $n = p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_s^{i_s}$, then

$$r_n = r_{p_1^{i_1}} \wedge r_{p_2^{i_2}} \wedge \ldots \wedge r_{p_s^{i_s}}$$

2.2. Definition. For each $n \in \mathbb{N}^*$ the Smarandache function of first kind is the numerical function $S_n : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ defined as follows

i) if $n = u^i$ ($u=1$ or $u=p$ number prime) then $S_n(a) = k$, k being the smallest positive integer with the property that $k! = M u^{ia}$

ii) if $n = p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_s^{i_s}$, then $S_n(a) = \max_{1 \le j \le s} \{ S_{p_j^{i_j}}(a) \}$

Let us observe that :

a) the functions S_n are standardising functions corresponding to the equivalence relations r_n and for $n=1$ we get $\hat{x}_{r_1} = \mathbb{N}^*$ for every $x \in \mathbb{N}^*$ and $S_1(n) = 1$ for every n.

b) if $n=p$ then S_n is the function S_p defined by Smarandache.

c) the functions S_n are increasing and so, are of the same monotonicity in the sense given in the above section.

2.3. Theorem. The functions S_n, for $n \in \mathbb{N}^*$, Σ_1-standardise $(\mathbb{N}^*, +)$ in $(\mathbb{N}^*, \le, +)$ by Σ_1: $\max(S_n(a), S_n(b)) \le S_n(a+b) \le S_n(a) + S_n(b)$ for every $a, b \in \mathbb{N}^*$ and Σ_2-standardise $(\mathbb{N}^*, +)$ in $(\mathbb{N}^*, \le, \cdot)$ by Σ_2 : $\max(S_n(a), S_n(b)) \le S_n(a+b) \le S_n(a) \cdot S_n(b)$, for every $a, b \in \mathbb{N}^*$

Proof. Let, for instance, p be a prime number, $n=p^i$, $i \in \mathbb{N}^*$ and $a^* = S_{p^i}(a)$, $b^* = S_{p^i}(b)$, $k = S_{p^i}(a+b)$. Then by the definition of S_n (Definition 2.2.) the numbers a^*, b^*, k are the smallest positive integers such that $a^*! = Mp^{ia}$, $b^*! = Mp^{ib}$ and $k! = Mp^{i(a+b)}$.

Because $k! = Mp^{ia} = Mp^{ib}$ we get $a^* \le k$ and $b^* \le k$, so $\max(a^*, b^*) \le k$

That is the first inequalities in Σ_1 and Σ_2 holds.

Now, $(a^*+b^*)! = a^*!(a^*+1) \ldots (a^*+b^*) = Ma^*!\, b^*! = Mp^{i(a+b)}$ and so $k \le a^* + b^*$ which implies that Σ_1 is valide.

If $n = p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_s^{i_s}$, from the first case we have

Σ_1: $\max\{S_{p_j^{i_j}}(a), S_{p_j^{i_j}}(b)\} \le S_{p_j^{i_j}}(a+b) \le S_{p_j^{i_j}}(a) + S_{p_j^{i_j}}(b), j=\overline{1,s}$

in consequence

$\max\{\max_j S_{p_j^{i_j}}(a), \max_j S_{p_j^{i_j}}(b)\} \le \max_j\{S_{p_j^{i_j}}(a+b)\} \le \max_j\{S_{p_j^{i_j}}(a)\} +$

$\max_j\{S_{p_j^{i_j}}(b)\}$, $j = \overline{1,s}$. That is

$$\max\{S_n(a), S_n(b)\} \le S_n(a+b) \le S_n(a) + S_n(b)$$

For the proof of the second part in Σ_2 let us notice that

$(a+b)! \le (ab)! \iff a+b \le ab \iff a > 1$ and $b > 1$ and that

ours inequality is satisfied for $n=1$ because $S_1(a+b)=S_1(a)=$
$= S_1(b) = 1$.

Let now $n>1$. It results that for $a^* = S_n(a)$ we have $a^* > 1$. Indeed, if $n = p_1^{i_1} \; p_2^{i_2} \; \ldots \; p_s^{i_s}$ then $a^* = 1$ if and only if $S_n(a)=$
$= \max_j \{S_{p_j^{i_j}}(a)\} = 1$ which implies that $p_1 = p_2 = \ldots = p_s = 1$,

so $n=1$. It results that for every $n>1$ we have $S_n(a) = a^* > 1$ and

$S_n(b) = b^* > 1$. Then $(a^*+b^*)! \le (a^* \cdot b^*)!$ we obtain

$$S_n(a+b) \le S_n(a) + S_n(b) \le S_n(a) \cdot S_n(b) \quad \text{from } n > 1.$$

3. Smarandache functions of the second kind. For every $n \in \mathbb{N}^*$, let S_n by the Smarandache function of the first kind defined above.

3.1. Definition. The Smarandache functions of the second kind are the functions $S^k : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ defined by $S^k(n) = S_n(k)$, for $k \in \mathbb{N}^*$.

We observe that for $k=1$ the function S^k is the Smarandache function S defined in [1], with the modify $S(1) = 1$. Indeed for $n>1$

$$S^1(n) = S_n(1) = \max_j\{S_{p_j^{i_j}}(1)\} = \max_j\{S_{p_j}(i_j)\} = S(n).$$

3.2. Theorem. The Smarandache functions of the second kind Σ_3-standardise $(\mathbb{N}^*,\cdot)$ in $(\mathbb{N}^*,\le,+)$ by

Σ_3: $\max\{S^k(a),S^k(b)\} \le S^k(a.b) \le S^k(a) + S^k(b)$, for every $a,b \in \mathbb{N}^*$

and Σ_4-standardise $(\mathbb{N}^*,\cdot)$ in $(\mathbb{N}^*,\le,\cdot)$ by

Σ_4: $\max\{S^k(a),S^k(b)\} \le S^k(a.b) \le S^k(a).S^k(b)$, for every $a,b\in\mathbb{N}^*$

Proof. The equivalence relation corresponding to S^k is r^k, defined by $a\, r^k b$ if and only if there exists $a^* \in \mathbb{N}^*$ such that $a^*! = Ma^k$, $a^*! = Mb^k$ and a^* is the smallest integer with this property. That is, the functions S^k are standardising functions attached to the equivalence relations r^k.

This functions are not of the some monotonicity because, for example, $S^2(a) \le S^2(b) \leftrightarrow S(a^2) \le S(b^2)$ and from these inequalities $S^1(a) \le S^1(b)$ does not result.

Now for every $a,b \in \mathbb{N}^*$ let $S^k(a) = a^*$, $S^k(b) = b^*$, $S^k(a.b)= s$. Then a^*, b^*, s are respectively these smallest positive integers such that $a^*! = Ma^k$, $b^*! = Mb^*$, $s! = M(a^kb^k)$ and so $s! = Ma^k = Mb^k$, that is, $a^* \le s$ and $b^* \le s$, which implies that $\max\{a^*,b^*\}\le s$

or $$\max\{S^k(a),S^k(b)\} \le S^k(a.b) \qquad (3.1)$$

Because of the fact that $(a^*+b^*)! = M(a^*!\, b^*!) = M(a^kb^k)$, it results that $s \le a^* + b^*$, so

$$S^k(a.b) \le S^k(a) + S^k(b) \qquad (3.2)$$

From (3.1) and (3.2) it results that

$$\max\{S^k(a),S^k(b)\} \le S^k(a) + S^k(b) \qquad (3.3)$$

Which is the relation Σ_3.

From $(a^*b^*)! = M(a^*!.b^*!)$ it results that $S^k(a.b) \le S^k(a).S^k(b)$ and thus the relation Σ_4.

4. The Smarandache functions of the third kind.

We considere two arbitrary sequances (a) $1=a_1, a_2, \ldots, a_n, \ldots$

(b) $1=b_1, b_2, \ldots, b_n, \ldots$

with the properties that $a_{kn} = a_k . a_n$, $b_{kn} = b_k . b_n$. Obviously, there are infinitely many such sequences; because chosing an arbitrary value for a_2, the next terms in the net can be easily determined by the imposed condition.

Let now the function $f_a^b : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ defined by $f_a^b(n) = S_{a_n}(b_n)$, S_{a_n} is the Smarandache function of the first kind. Then it is easily to see that :

(i) for $a_n = 1$ and $b_n = n, n \in \mathbb{N}^*$ it results that $f_a^b = S_1$

(ii) for $a_n = n$ and $b_n = 1, n \in \mathbb{N}^*$ it results that $f_a^b = S^1$

4.1. Definition. The Smarandache functions of the third kind are the functions $S_a^b = f_a^b$ in the case that the sequances (a) and (b) are different from those concerned in the situation (i) and (ii) from above.

4.2. Theorem. The functions f_a^b Σ_3-standardise $(\mathbb{N}^*, \cdot)$ in $(\mathbb{N}^*, \leq, +, \cdot)$ by

$$\Sigma_3: \quad \max \{f_a^b(k), f_a^b(n)\} \leq f_a^b(k.n) \leq b_n . f_a^b(k) + b_k f_a^b(n) \quad .$$

Proof. Let $f_a^b(k) = S_{a_k}(b_k) = k^*$, $f_a^b(n) = S_{a_n}(b_n) = n^*$ and $f_a^b(kn) = S_{a_{kn}}(b_{kn}) = t$. Then k^*, n^* and t are the smallest positive integers such that $k^*! = M\, a_k^{b_k}$, $n^*! = M\, a_n^{b_n}$ and $t! = M\, a_{kn}^{b_{kn}} = M(a_k . a_n)^{b_k b_n}$. Of course,

$$\max\{k^*, n^*\} \leq t \qquad (4.1)$$

Now, because $(b_k . n^*)! = M(n^*!)^{b_k}$, $(b_n . k^*)! = M(k^*!)^{b_n}$ and $(b_k n^* + b_n k^*)! = M[(b_k n^*)!.(b_n k^*)!] = M[(n^*!)^{b_k}.(k^*!)^{b_n}] =$

$= M[(a_n^{b_n})^{b_k} . (a_k^{b_k})^{b_n}] = M[(a_k . a_n)^{b_k b_n}]$ it results that

$$t \le b_n k^* + b_k n^* \qquad (4.2)$$

From (4.1) and (4.2) we obtain

$$\max\{k^*, n^*\} \le t \le b_n k^* + b_k n^* \qquad (4.3)$$

From (4.3) we get Σ_3 , so the Smarandache functions of the third kind satisfy

Σ_σ: $\max\{S_a^b(k), S_a^b(n)\} \le S_a^b(kn) \le b_n S_a^b(k) + b_k S_a^b(n)$, for evry $k, n \in N^*$

4.3. Example. Let the sequances (a) and (b) defined by $a_n = b_n = n$, $n \in N^*$.

The corresponding Smarandache function of the third kind is

$S_\alpha^a : N^* \longrightarrow N^*$, $S_a^a(n) = S_n(n)$ and Σ_σ becomes

$$\max\{S_k(k), S_n(n)\} \le S_{kn}(kn) \le nS_k(k) + kS_n(n) \text{ ,for every } k, n \in N^*$$

This relation is equivalent with the following relation written by meens with the Smarandache function:

$$\max \{S(k^k), S(n^n)\} \le S[(kn)^{kn}] \le n.S(k^k) + k.S(n^n) .$$

References

[1] F.Smarandache,A Function in the Number Theory,An.Univ. Timisoara,seria st. mat Vol. XVIII,fasc.1,pp.79-88.1980.

[2] Smarandache Function-Journal-Vol.1 No.1,December 1990.