# A Neutrosophic Description Logic

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**Abstract.** Description Logics (DLs) are appropriate, widely used, logics for managing structured knowledge. They allow reasoning about individuals and concepts, *i.e.* set of individuals with common properties. Typically, DLs are limited to dealing with crisp, well defined concepts. That is, concepts for which the problem whether an individual is an instance of it is a yes/no question. More often than not, the concepts encountered in the real world do not have a precisely defined criteria of membership: we may say that an individual is an instance of a concept only to a certain degree, depending on the individual's properties. The DLs that deal with such fuzzy concepts are called fuzzy DLs. In order to deal with fuzzy, incomplete, indeterminate and inconsistent concepts, we need to extend the capabilities of fuzzy DLs further.

In this paper we will present an extension of fuzzy $\mathcal{ALC}$, combining Smarandache's neutrosophic logic with a classical DL. In particular, concepts become neutrosophic (here neutrosophic means fuzzy, incomplete, indeterminate and inconsistent), thus, reasoning about such neutrosophic concepts is supported. We will define its syntax, its semantics, describe its properties and present a constraint propagation calculus for reasoning in it.

**Keywords**: Description logic, fuzzy description logic, fuzzy $\mathcal{ALC}$, neutrosophic description logic.

## 1 Introduction

The modelling and reasoning with uncertainty and imprecision is an important research topic in the Artificial Intelligence community. Almost all the real world knowledge is imperfect. A lot of works have been carried out to extend existing knowledge-based systems to deal with such imperfect information, resulting in a number of concepts being investigated, a number of problems being identified and a number of solutions being developed [2, 6, 8, 9].

*Description Logics* (DLs) have been utilized in building a large amount of knowledge-based systems. DLs are a logical reconstruction of the so-called frame-based knowledge representation languages, with the aim of providing a simple well-established Tarski-style declarative semantics to capture the meaning of the most popular features of structured representation of knowledge. A main point is that DLs are considered as to be attractive logics in knowledge based applications as they are a good compromise between expressive power and computational complexity.

Nowadays, a whole family of knowledge representation systems has been build using DLs, which differ with respect to their expressiveness, their complexity and the completeness of their algorithms, and they have been used for building a variety of applications [10, 3, 1, 7].

The classical DLs can only deal with crisp, well defined concepts. That is, concepts for which the problem whether an individual is an instance of it is a yes/no question. More often than not, the concepts encountered in the real world do not have a precisely defined criteria of membership. There are many works attempted to extend the DLs using fuzzy set theory [12–14, 5, 15, 17]. These fuzzy DLs can only deal with fuzzy concepts but not incomplete, indeterminate, and inconsistent concepts (neutrosophic concepts). For example, "Good Person" is a neutrosophic concepts, in the sense that by different subjective opinions, the truth-membership degree of tom is good person is 0.6, and the falsity-membership degree of tom is good person is 0.6, which is inconsistent, or the truth- membership degree of tom is good person is 0.6, and the falsity-membership degree of tom is good person is 0.3, which is incomplete.

The set and logic that can model and reason with fuzzy, incomplete, indeterminate, and inconsistent information are called neutrosophic set and neutrosophic logic, respectively [11, 16]. In Smarandache's neutrosophic set theory,a neutrosophic set $A$ defined on universe of discourse $X$, associates each element $x$ in $X$ with three membership functions: truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$, and falsity-membership function $F_A(x)$, where $T_A(x), I_A(x), F_A(x)$ are real standard or non-standard subsets of $]^{-}0, 1^{+}[$, and $T_A(x), I_A(x), F_A(x)$ are independent. For simplicity, in this paper, we will extend Straccia's fuzzy DLs [12, 14] with neutrosophic logic, called neutrosophic DLs, where we only use two components $T_A(x)$ and $F_A(x)$, with $T_A(x) \in [0, 1], F_A(x) \in [0, 1], 0 \leq T_A(x) + F_A(x) \leq 2$. The neutrosophic DLs is based on the DL $\mathcal{ALC}$, a significant and expressive representative of the various DLs. This allows us to adapt it easily to the different DLs presented in the literature. Another important point is that we will show that the additional expressive power has no impact from a computational complexity point of view. The neutrosophic $\mathcal{ALC}$ is a strict generalization of fuzzy $\mathcal{ALC}$, in the sense that every fuzzy concept and fuzzy terminological axiom can be represented by a corresponding neutrosophic concept and neutrosophic terminological axiom, but not *vice versa*.

The rest of paper is organized as follows. In the following section we first introduce Straccia's $\mathcal{ALC}$. In section 3 we extend $\mathcal{ALC}$ to the neutrosophic case and discuss some properties in Section 4, while in Section 5 we will present a constraint propagation calculus for reasoning in it. Section 6 concludes and proposes future work.

## 2 A Quick Look to Fuzzy $\mathcal{ALC}$

We assume three alphabets of symbols, called *atomic concepts* (denoted by $A$), *atomic roles* (denoted by $R$) and *individuals* (denoted by $a$ and $b$). [1]

A *concept* (denoted by $C$ or $D$) of the language $\mathcal{ALC}$ is built out of atomic concepts according to the following syntax rules:

$$
\begin{aligned}
C, D \longrightarrow \quad & \top \mid \text{(top concept)} \\
& \bot \mid \text{(bottom concept)} \\
& A \mid \text{(atomic concept)} \\
& C \sqcap D \mid \text{(concept conjunction)} \\
& C \sqcup D \mid \text{(concept disjunction)} \\
& \neg C \mid \text{(concept negation)} \\
& \forall R.C \mid \text{(universal quantification)} \\
& \exists R.C \ \text{(existential quantification)}
\end{aligned}
$$

Fuzzy DL extends classical DL under the framework of Zadeh's fuzzy sets [18].A *fuzzy set* $S$ with respect to an universe $U$ is characterized by a *membership function* $\mu_S : U \rightarrow [0,1]$, assigning an $S$-membership degree, $\mu_S(u)$, to each element $u$ in $U$. In fuzzy DL, ($i$) a concept $C$, rather than being interpreted as a classical set, will be interpreted as a fuzzy set and, thus, concepts become *fuzzy*; and, consequently, ($ii$) the statement "$a$ is $C$", *i.e.* $C(a)$, will have a truth-value in $[0,1]$ given by the degree of membership of being the individual $a$ a member of the fuzzy set $C$.

### 2.1 Fuzzy Interpretation

A *fuzzy interpretation* is now a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is, as for the crisp case, the *domain*, whereas $\cdot^{\mathcal{I}}$ is an *interpretation function* mapping

1. individual as for the crisp case, *i.e.* $a^{\mathcal{I}} \neq b^{\mathcal{I}}$, if $a \neq b$;
2. a concept $C$ into a membership function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0,1]$;
3. a role $R$ into a membership function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0,1]$.

[1] Through this work we assume that every metavariable has an optional subscript or superscript.

If $C$ is a concept then $C^{\mathcal{I}}$ will naturally be interpreted as the *membership degree function* of the fuzzy concept (set) $C$ w.r.t. $\mathcal{I}$, *i.e.* if $d \in \Delta^{\mathcal{I}}$ is an object of the domain $\Delta^{\mathcal{I}}$ then $C^{\mathcal{I}}(d)$ gives us the degree of being the object $d$ an element of the fuzzy concept $C$ under the interpretation $\mathcal{I}$. Similarly for roles. Additionally, the interpretation function $.^{\mathcal{I}}$ has to satisfy the following equations: for all $d \in \Delta^{\mathcal{I}}$,

$$
\begin{aligned}
&\top^{\mathcal{I}}(d) &&= 1\\
&\bot^{\mathcal{I}}(d) &&= 0\\
&(C \sqcap D)^{\mathcal{I}}(d) &&= \min\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\}\\
&(C \sqcup D)^{\mathcal{I}}(d) &&= \max\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\}\\
&(\neg C)^{\mathcal{I}}(d) &&= 1 - C^{\mathcal{I}}(d)\\
&(\forall R.C)^{\mathcal{I}}(d) &&= \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{1 - R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\}\}\\
&(\exists R.C)^{\mathcal{I}}(d) &&= \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\}\}.
\end{aligned}
$$

We will say that two concepts $C$ and $D$ are said to be *equivalent* (denoted by $C \cong D$) when $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all interpretation $\mathcal{I}$. As for the crisp non fuzzy case, dual relationships between concepts hold: *e.g.* $\top \cong \neg\bot$, $(C \sqcap D) \cong \neg(\neg C \sqcup \neg D)$ and $(\forall R.C) \cong \neg(\exists R.\neg C)$.

### 2.2 Fuzzy Assertion

A *fuzzy assertion* (denoted by $\psi$) is an expression having one of the following forms $\langle \alpha \geq n \rangle$ or $\langle \alpha \leq m \rangle$, where $\alpha$ is an $\mathcal{ALC}$ assertion, $n \in (0, 1]$ and $m \in [0, 1)$. From a semantics point of view, a fuzzy assertion $\langle \alpha \leq n \rangle$ constrains the truth-value of $\alpha$ to be less or equal to $n$ (similarly for $\geq$). Consequently, *e.g.* $\langle$ $(\mathbf{Video} \sqcap \exists\mathbf{About.Basket})(\mathbf{v1}) \geq 0.8\rangle$ states that video **v1** is likely about basket. Formally, an interpretation $\mathcal{I}$ *satisfies* a fuzzy assertion $\langle C(a) \geq n \rangle$ (resp. $\langle R(a, b) \geq n \rangle$) iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n$ (resp. $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n$). Similarly, an interpretation $\mathcal{I}$ *satisfies* a fuzzy assertion $\langle C(a) \leq n \rangle$ (resp. $\langle R(a, b) \leq n \rangle$) iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \leq n$ (resp. $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \leq n$). Two fuzzy assertion $\psi_1$ and $\psi_2$ are said to be *equivalent* (denoted by $\psi_1 \cong \psi_2$) iff they are satisfied by the same set of interpretations. An *atomic fuzzy assertion* is a fuzzy assertion involving an atomic assertion (assertion of the form $A(a)$ or $R(a, b)$).

### 2.3 Fuzzy Terminological Axiom

From a syntax point of view, a *fuzzy terminological axiom* (denoted by $\tilde{\tau}$ is either a fuzzy concept specialization or a fuzzy concept definition. A *fuzzy concept specialization* is an expression of the form $A \prec C$, where $A$ is an atomic concept and $C$ is a concept. On the other hand, a *fuzzy concept definition* is an expression of the form $A :\approx C$, where $A$ is an atomic concept and $C$ is a concept. From a semantics point of view, a fuzzy interpretation $\mathcal{I}$ *satisfies* a fuzzy concept specialization $A \prec C$ iff

$$\forall d \in \Delta^{\mathcal{I}}, A^{\mathcal{I}}(d) \leq C^{\mathcal{I}}(d), \tag{1}$$

whereas $\mathcal{I}$ *satisfies* a fuzzy concept definition $A :\approx C$ iff

$$\forall d \in \Delta^{\mathcal{I}}, A^{\mathcal{I}}(d) = C^{\mathcal{I}}(d). \tag{2}$$

### 2.4 Fuzzy Knowlege Base, Fuzzy Entailment and Fuzzy Subsumption

A *fuzzy knowledge base* is a finite set of fuzzy assertions and fuzzy terminological axioms. $\Sigma_A$ denotes the set of fuzzy assertions in $\Sigma$, $\Sigma_T$ denotes the set of fuzzy terminological axioms in $\Sigma$ (the terminology), if $\Sigma_T = \emptyset$ then $\Sigma$ is *purely assertional*, and we will assume that a terminology $\Sigma_T$ is such that no concept $A$ appears more than once on the left hand side of a fuzzy terminological axiom $\tilde{\tau} \in \Sigma_T$ and that no cyclic definitions are present in $\Sigma_T$.

An interpretation $\mathcal{I}$ *satisfies* (is *a model of*) a set of fuzzy $\Sigma$ iff $\mathcal{I}$ satisfies each element of $\Sigma$. A fuzzy KB $\Sigma$ *fuzzy entails* a fuzzy assertion $\psi$ (denoted by $\Sigma \models_f \psi$) iff every model of $\Sigma$ also satisfies $\psi$.

Furthermore, let $\Sigma_T$ be a terminology and let $C, D$ be two concepts. We will say that $D$ *fuzzy subsumes* $C$ w.r.t. $\Sigma_T$ (denoted by $C \preceq_{\Sigma_T} D$) iff for every model $\mathcal{I}$ of $\Sigma_T$, $\forall d \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)$ holds.

## 3 A Neutrosophic DL

Our neutrosophic extension directly relates to Smarandache's work on neutrosophic sets [11, 16]. A neutrosophic set $S$ defined on universe of discourse $U$, associates each element $u$ in $U$ with three membership functions: truth-membership function $T_S(u)$, indeterminacy-membership function $I_S(u)$, and falsity-membership function $F_S(u)$, where $T_S(u), I_S(u), F_S(u)$ are real standard or non-standard subsets of $]^-0, 1^+[$, and $T_S(u), I_S(u), F_S(u)$ are independent. For simplicity, here we only use two components $T_S(u)$ and $F_S(u)$, with $T_S(u) \in [0,1], F_S(u) \in [0,1], 0 \leq T_S(u) + F_S(u) \leq 2$. It is easy to extend our method to include indeterminacy-membership function. $T_S(u)$ gives us an estimation of degree of $u$ belonging to $U$ and $F_S(u)$ gives us an estimation of degree of $u$ not belonging to $U$. $T_S(u) + F_S(u)$ can be 1 (just as in classical fuzzy sets theory). But it is not necessary. If $T_S(u) + F_S(u) < 1$, for all $u$ in $U$, we say the set $S$ is incomplete, if $T_S(u) + F_S(u) > 1$, for all $u$ in $U$, we say the set $S$ is inconsistent. According to Wang [16], the truth-membership function and falsity-membership function has to satisfy three restrictions: for all $u \in U$ and for all neutrosophic sets $S_1, S_2$ with respect to $U$

$$T_{S_1 \cap S_2}(u) = \min\{T_{S_1}(u), T_{S_2}(u)\}, F_{S_1 \cap S_2}(u) = \max\{F_{S_1}(u), F_{S_2}(u)\}$$

$$T_{S_1 \cup S_2}(u) = \max\{T_{S_1}(u), T_{S_2}(u)\}, F_{S_1 \cup S_2}(u) = \min\{F_{S_1}(u), F_{S_2}(u)\}$$

$$T_{\overline{S_1}}(u) = F_{S_1}(u), F_{\overline{S_1}}(u) = T_{S_1}(u),$$

where $\overline{S_1}$ is the complement of $S_1$ in $U$. Wang [16] gives the definition of $N$-norm and $N$-conorm of neutrosophic sets, min and max is only one of the choices. In general case, they may be the simplest and the best.

When we switch to neutrosophic logic, the notion of degree of truth-membership $T_S(u)$ of an element $u \in U$ w.r.t. the neutrosophic set $S$ over $U$ is regarded as the *truth-value* of the statement "$u$ is $S$", and the notion of degree of falsity-membership $F_S(u)$ of an element $u \in U$ w.r.t. the neutrosophic set $S$ over $U$ is regarded as the *falsity-value* of the statement "$u$ is $S$". Accordingly, in our neutrosophic DL, ($i$) a concept $C$, rather than being interpreted as a fuzzy set, will be interpreted as a neutrosophic set and, thus, concepts become *imprecise* (fuzzy, incomplete, and inconsistent); and, consequently, ($ii$) the statement "$a$ is $C$", *i.e.* $C(a)$ will have a truth-value in $[0, 1]$ given by the degree of truth-membership of being the individual $a$ a member of the neutrosophic set $C$ and a falsity-value in $[0, 1]$ given by the degree of falsity-membership of being the individual $a$ not a member of the neutrosophic set $C$.

### 3.1 Neutrosophic Interpretation

A em neutrosophic interpretation is now a tuple $\mathcal{I} = (\Delta^{\mathcal{I}}, (\cdot)^{\mathcal{I}}, |\cdot|^t, |\cdot|^f)$, where $\Delta^{\mathcal{I}}$ is, as for the fuzzy case, the *domain*, and

1. $(\cdot)^{\mathcal{I}}$ is an *interpretation function* mapping
   (a) individuals as for the fuzzy case, *i.e.* $a^{\mathcal{I}} \neq b^{\mathcal{I}}$, if $a \neq b$;
   (b) a concept $C$ into a membership function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0, 1] \times [0, 1]$;
   (c) a role $R$ into a membership function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1] \times [0, 1]$.
2. $|\cdot|^t$ and $|\cdot|^f$ are *neutrosophic valuation*, *i.e.* $|\cdot|^t$ and $|\cdot|^f$ map
   (a) every atomic concept into a function from $\Delta^{\mathcal{I}}$ to $[0, 1]$;
   (b) every atomic role into a function from $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to $[0, 1]$.

If $C$ is a concept then $C^{\mathcal{I}}$ will naturally be interpreted as a pair of membership functions $\langle |C|^t, |C|^f \rangle$ of the neutrosophic concept (set) $C$ w.r.t. $\mathcal{I}$, *i.e.* if $d \in \Delta^{\mathcal{I}}$ is an object of the domain $\Delta^{\mathcal{I}}$ then $C^{\mathcal{I}}(d)$ gives us the degree of being the object $d$ an element of the neutrosophic concept $C$ and the degree of being the object $d$ not an element of the neutrosophic concept $C$ under the interpretation $\mathcal{I}$. Similarly for roles. Additionally, the interpretation function $(\cdot)^{\mathcal{I}}$ has to satisfy the following equations: for all $d \in \Delta^{\mathcal{I}}$,

$$
\begin{aligned}
\top^{\mathcal{I}}(d) &= \langle 1, 0 \rangle \\
\bot^{\mathcal{I}}(d) &= \langle 0, 1 \rangle \\
(C \sqcap D)^{\mathcal{I}}(d) &= \langle \min\{|C|^t(d), |D|^t(d)\}, \max\{|C|^f(d), |D|^f(d)\} \rangle \\
(C \sqcup D)^{\mathcal{I}}(d) &= \langle \max\{|C|^t(d), |D|^t(d)\}, \min\{|C|^f(d), |D|^f(d)\} \rangle \\
(\neg C)^{\mathcal{I}}(d) &= \langle |C|^f(d), |C|^t \rangle \\
(\forall R.C)^{\mathcal{I}}(d) &= \langle \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{|R|^f(d, d'), |C|^t(d')\}\}, \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{|R|^t(d, d'), |C|^f(d')\}\} \rangle \\
(\exists R.C)^{\mathcal{I}}(d) &= \langle \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{|R|^t(d, d'), |C|^t(d')\}\}, \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{|R|^f(d, d'), |C|^f(d')\}\} \rangle
\end{aligned}
$$

Note that the semantics of $\forall R.C$

$$
(\forall R.C)^{\mathcal{I}}(d) = \langle \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{|R|^f(d, d'), |C|^t(d')\}\}, \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{|R|^t(d, d'), |C|^f(d')\}\} \rangle \tag{3}
$$

is the result of viewing $\forall R.C$ as the open first order formula $\forall y.\neg F_R(x,y) \vee F_C(y)$, where the universal quantifier $\forall$ is viewed as a conjunction over the elements of the domain. Similarly, the semantics of $\exists R.C$

$$(\exists R.C)^{\mathcal{I}}(d) = \langle \sup_{d' \in \Delta^{\mathcal{I}}} \{\min\{|R|^t(d,d'), |C|^t(d')\}\}, \inf_{d' \in \Delta^{\mathcal{I}}} \{\max\{|R|^f(d,d'), |C|^f(d')\}\}\rangle \tag{4}$$

is the result of viewing $\exists R.C$ as the open first order formula $\exists y.F_R(x,y) \wedge F_C(y)$ and the existential quantifier $\exists$ is viewed as a disjunction over the elements of the domain. Moreover, $|\cdot|^t$ and $|\cdot|^f$ are extended to complex concepts as follows: for all $d \in \Delta^{\mathcal{I}}$

$$\begin{aligned}
|C \sqcap D|^t(d) &= \min\{|C|^t(d), |D|^t(d)\} \\
|C \sqcap D|^f(d) &= \max\{|C|^f(d), |D|^f(d)\} \\[1ex]
|C \sqcup D|^t(d) &= \max\{|C|^t(d), |D|^t(d)\} \\
|C \sqcup D|^f(d) &= \min\{|C|^f(d), |D|^f(d)\} \\[1ex]
|\neg C|^t(d) &= |C|^f(d) \\
|\neg C|^f(d) &= |C|^t(d) \\[1ex]
|\forall R.C|^t(d) &= \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{|R(d,d')|^f, |C|^t(d)\}\} \\
|\forall R.C|^f(d) &= \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{|R(d,d')|^t, |C|^f(d)\}\} \\[1ex]
|\exists R.C|^t(d) &= \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{|R(d,d')|^t, |C|^t(d)\}\} \\
|\exists R.C|^f(d) &= \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{|R(d,d')|^f, |C|^f(d)\}\}
\end{aligned}$$

We will say that two concepts $C$ and $D$ are said to be *equivalent* (denoted by $C \cong^n D$) when $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all interpretation $\mathcal{I}$. As for the fuzzy case, dual relationships between concepts hold: *e.g.* $\top \cong^n \neg\bot, (C \sqcap D) \cong^n \neg(\neg C \sqcup \neg D)$ and $(\forall R.C) \cong^n \neg(\exists R.\neg C)$.

### 3.2 Neutrosophic Assertion

A *neutrosophic assertion* (denoted by $\varphi$) is an expression having one of the following form $\langle \alpha :\geq n, \leq m\rangle$ or $\langle \alpha :\leq n, \geq m\rangle$, where $\alpha$ is an $\mathcal{ALC}$ assertion, $n \in [0,1]$ and $m \in [0,1]$. From a semantics point of view, a neutrosophic assertion $\langle \alpha :\geq n, \leq m\rangle$ constrains the truth-value of $\alpha$ to be greater or equal to $n$ and falsity-value of $\alpha$ to be less or equal to $m$ (similarly for $\langle \alpha :\leq n, \geq m\rangle$). Consequently, *e.g.* $\langle(\mathbf{Poll} \sqcap \exists \mathbf{Support.War\_x})(\mathbf{p1}) :\geq 0.8, \leq 0.1\rangle$ states that poll **p1** is close to support War_x. Formally, an interpretation $\mathcal{I}$ *satisfies* a neutrosophic assertion $\langle \alpha :\geq n, \leq m\rangle$ (resp. $\langle R(a,b) :\geq n, \leq m\rangle$) iff $|C|^t(a^{\mathcal{I}}) \geq n$ and $|C|^f(a^{\mathcal{I}}) \leq m$ (resp. $|R|^t(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n$ and $|R|^f(a^{\mathcal{I}}, b^{\mathcal{I}}) \leq m$). Similarly, an interpretation $\mathcal{I}$ *satisfies* a neutrosophic assertion $\langle \alpha :\leq n, \geq m\rangle$ (resp. $\langle R(a,b) :\leq n, \geq m\rangle$) iff $|C|^t(a^{\mathcal{I}}) \leq n$ and $|C|^f(a^{\mathcal{I}}) \geq m$ (resp. $|R|^t(a^{\mathcal{I}}, b^{\mathcal{I}}) \leq n$

and $|R|^f(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq m$). Two fuzzy assertion $\varphi_1$ and $\varphi_2$ are said to be *equivalent* (denoted by $\varphi_1 \cong^n \varphi_2$) iff they are satisfied by the same set of interpretations. Notice that $\langle \neg C(a) :\geq n, \leq m\rangle \cong^n \langle C(a) :\leq m, \geq n\rangle$ and $\langle \neg C(a) :\leq n, \geq m\rangle \cong^n \langle C(a) :\geq m, \leq n\rangle$. An *atomic neutrosophic assertion* is a neutrosophic assertion involving an atomic assertion.

### 3.3 Neutrosophic Terminological Axiom

Neutrosophic terminological axioms we will consider are a natural extension of fuzzy terminological axioms to the neutrosophic case. From a syntax point of view, a *neutrosophic terminological axiom* (denoted by $\hat{\tau}$) is either a neutrosophic concept specialization or a neutrosophic concept definition. A *neutrosophic concept specialization* is an expression of the form $A \prec^n C$, where $A$ is an atomic concept and $C$ is a concept. On the other hand, a *neutrosophic concept definition* is an expression of the form $A :\approx^n C$, where $A$ is an atomic concept and $C$ is a concept. From a semantics point of view, we consider the natural extension of fuzzy set to the neutrosophic case [11, 16]. A neutrosophic interpretation $\mathcal{I}$ *satisfies* a neutrosophic concept specialization $A \prec^n C$ iff

$$\forall d \in \Delta^{\mathcal{I}}, |A|^t(d) \leq |C|^t(d), |A|^f(d) \geq |C|^f(d), \tag{5}$$

whereas $\mathcal{I}$ *satisfies* a neutrosophic concept definition $A :\approx^n C$ iff

$$\forall d \in \Delta^{\mathcal{I}}, |A|^t(d) = |C|^t(d), |A|^f(d) = |C|^f(d). \tag{6}$$

### 3.4 Neutrosophic Knowledge Base, Neutrosophic Entailment and Neutrosophic Subsumption

A *neutrosophic knowledge base* is a finite set of neutrosophic assertions and neutrosophic terminological axioms. As for the fuzzy case, with $\Sigma_A$ we will denote the set of neutrosophic assertions in $\Sigma$, with $\Sigma_T$ we will denote the set of neutrosophic terminological axioms in $\Sigma$ (the terminology), if $\Sigma_T = \emptyset$ then $\Sigma$ is *purely assertional*, and we will assume that a terminology $\Sigma_T$ is such that no concept $A$ appears more than once on the left hand side of a neutrosophic terminological axiom $\hat{\tau} \in \Sigma_T$ and that no cyclic definitions are present in $\Sigma_T$.

An interpretation $\mathcal{I}$ *satisfies* (is *a model of*) a neutrosophic $\Sigma$ iff $\mathcal{I}$ satisfies each element of $\Sigma$. A neutrosophic KB $\Sigma$ *neutrosophically entails* a neutrosophic assertion $\varphi$ (denoted by $\Sigma \models^n \varphi$) iff every model of $\Sigma$ also satisfies $\varphi$.

Furthermore, let $\Sigma_T$ be a terminology and let $C, D$ be two concepts. We will say that $D$ *neutrosophically subsumes* $C$ w.r.t. $\Sigma_T$ (denoted by $C \preceq^n_{\Sigma_T} D$) iff for every model $\mathcal{I}$ of $\Sigma_T$, $\forall d \in \Delta^{\mathcal{I}}, |C|^t(d) \leq |D|^t(d)$ and $|C|^f(d) \geq |D|^f(d)$ holds.

Finally, given a neutrosophic KB $\Sigma$ and an assertion $\alpha$, we define the *greatest lower bound* of $\alpha$ w.r.t. $\Sigma$ (denoted by $glb(\Sigma, \alpha)$) to be $\langle \sup\{n : \Sigma \models^n \langle\alpha :\geq n, \leq m\rangle\}, \inf\{m : \Sigma \models^n \langle\alpha :\geq n, \leq m\rangle\}\rangle$. Similarly, we define the *least upper bound* of $\alpha$ with respect to $\Sigma$ (denoted by $lub(\Sigma, \alpha)$) to be $\langle \inf\{n : \Sigma \models^n \langle\alpha :\leq n, \geq m\rangle\}, \sup\{m : \Sigma \models^n \langle\alpha :\leq n, \geq m\rangle\}\rangle$ ($\sup \emptyset = 0, \inf \emptyset = 1$). Determing the *lub* and the *glb* is called the *Best Truth-Value Bound* (BTVB) problem.

# 4 Some Properties

In this section, we discuss some properties of our neutrosophic $\mathcal{ALC}$.

## 4.1 Concept Equivalence

The first ones are straightforward: $\neg\top \approx^n \bot, C \sqcap \top \approx^n C, C \sqcup \top \approx^n \top, C \sqcap \bot \approx^n \bot, C \sqcup \bot \approx^n C, \neg\neg C \approx^n C, \neg(C \sqcap D) \approx^n \neg C \sqcup \neg D, \neg(C \sqcup D) \approx^n \neg C \sqcap \neg D, C_1 \sqcap (C_2 \sqcup C_3) \approx^n (C_1 \sqcap C_2) \sqcup (C_1 \sqcap C_3)$ and $C_1 \sqcup (C_2 \sqcap C_3) \approx^n (C_1 \sqcup C_2) \sqcap (C_1 \sqcup C_3)$. For concepts involving roles, we have $\forall R.C \approx^n \neg\exists R.\neg C, \forall R.\top \approx^n \top, \exists R.\bot \approx^n \bot$ and $(\forall R.C) \sqcap (\forall R.D) \approx^n \forall R.(C \sqcap D)$. Please note that we do not have $C \sqcap \neq C \approx^n \bot$, nor we have $C \sqcup \neg C \approx^n \top$ and, thus, $(\exists R.C) \sqcap (\forall R.\neg C) \approx^n \bot$ and $(\exists R.C) \sqcup (\forall R.\neg C) \approx^n \top$ do not hold.

## 4.2 Entailment Relation

Of course, $\Sigma \models^n \langle \alpha :\geq n, \leq m\rangle$ iff $glb(\Sigma, \alpha) = \langle f, g\rangle$ with $f \geq n$ and $g \leq m$, and similarly $\Sigma \models^n \langle \alpha :\leq n, \geq m\rangle$ iff $lub(\Sigma, \alpha) = \langle f, g\rangle$ with $f \leq n$ and $g \geq m$. Concerning roles, note that $\Sigma \models^n \langle R(a, b) :\geq n, \leq m\rangle$ iff $\langle R(a, b) :\geq f, \leq g\rangle \in \Sigma$ with $f \geq n$ and $g \leq m$. Therefore,

$$glb(\Sigma, R(a, b)) = \langle \max\{n : \langle R(a, b) :\geq n, \leq m\rangle \ \in \Sigma\}, \\ \min\{m : \langle R(a, b) :\geq n, \leq m\rangle \ \in \Sigma\}\rangle \quad (7)$$

while the same is not true for the $\langle R(a, b) :\leq n, \geq m$ case. While $\langle R(a, b) :\leq f, \geq g\rangle \in \Sigma$ and $f \leq n, g \geq m$ imply $\Sigma \models^n \langle R(a, b) :\leq n, \geq m\rangle$, the converse is false (*e.g.* $\{\langle \forall R.A(a) :\geq 1, \leq 0\rangle, \langle A(b) :\leq 0, \geq 1\rangle\} \models^n \langle R(a, b) :\leq 0, \geq 1\rangle$).

Furthermore, from $\Sigma \models^n \langle C(a) :\leq n, \geq m\rangle$ iff $\Sigma \models^n \langle \neg C(a) :\geq m, \leq n\rangle$, it follows $lub(\Sigma, C(a)) = \langle f, g\rangle$ iff $glb(\Sigma, \neg C(a)) = \langle g, f\rangle$. Therefore, $lub$ can be determined through $glb$ (and vice versa). The same reduction to $glb$ does not hold for $lub(\Sigma, R(a, b))$ as $\neg R(a, b)$ is not an expression of our language.

*Modus ponens on concepts* is supported: if $n > g$ and $m < f$ then $\{\langle C(a) :\geq n, \leq m\rangle, \langle(\neg C \sqcup D)(a) :\geq f, \leq g\rangle\} \models^n \rangle D(a) :\geq f, \leq g\rangle$ holds.

*Modus ponens on roles* is supported: if $n > g$ and $m < f$ then $\{\langle R(a, b) :\geq n, \leq m\rangle, \langle \forall R.D(a) :\geq f, \leq g\rangle\} \models^n \langle D(b) :\geq f, \leq g\rangle$ and $\{\langle \exists R.C(a) :\geq n, \leq m\rangle, \langle \forall R.D(a) :\geq f, \leq g\rangle\} \models^n \langle \exists R.(C \sqcap D)(a) :\geq \min\{n, f\}, \leq \max\{m, g\}\rangle$ hold. Moreover, $\{\langle \forall R.C(a) :\geq n, \leq m\rangle, \langle \forall R.D(a) :\geq f, \leq g\rangle\} \models^n \langle \forall (R.(C \sqcap D))(a) :\geq \min\{n, f\}, \leq \max\{m, g\}\rangle$ holds.

*Modus ponens on specialization* is supported. The following degree bounds propagation through a taxonomy is supported. If $C \preceq^n_\Sigma D$ then (*i*) $\Sigma \cup \{\langle C(a) :\geq n, \leq m\rangle\} \models^n \langle D(a) :\geq n, \leq m\rangle\}$; and (*ii*) $\Sigma \cup \{\langle D(a) :\leq n, \geq m\rangle\} \models^n \langle C(a) :\leq n, \geq m\rangle$ hold.

### 4.3 Soundness and Completeness of the Semantics

Our neutrosophic semantics is *sound* and *complete* w.r.t. fuzzy semantics. First we must note that the neutrosophic $\mathcal{ALC}$ is a strict generalization of fuzzy $\mathcal{ALC}$, in the sense that every fuzzy concept and fuzzy terminological axiom can be represented by a corresponding neutrosophic concept and neutrosophic terminological axiom, but not *vice versa*. It is easy to verify that,

**Proposition 1.** *A classical fuzzy $\mathcal{ALC}$ can be simulated by a neutrosophic $\mathcal{ALC}$, in the way that a fuzzy assertion $\langle \alpha \geq n \rangle$ represented by a neutrosophic assertion $\langle \alpha :\geq n, \leq 1-n \rangle$, a fuzzy assertion $\langle \alpha \leq n \rangle$ represented by a neutrosophic assertion $\langle \alpha :\leq n, \geq 1-n \rangle$ and a fuzzy terminological axiom $\tilde{\tau}$ represented by a neutrosophic terminological axiom $\hat{\tau}$ in the sense that if $\mathcal{I}$ is a fuzzy interpretation then $|C|^t(a) = C^{\mathcal{I}}(a)$ and $|C|^f(a) = 1 - C^{\mathcal{I}}(a)$.* ⊣

Let us consider the following transformations $\sharp(\cdot)$ and $\star(\cdot)$ of neutrosophic assertions into fuzzy assertions,

$$\begin{aligned}
\sharp\langle \alpha :\geq n, \leq m \rangle &\mapsto \langle \alpha \geq n \rangle, \\
\star\langle \alpha :\geq n, \leq m \rangle &\mapsto \langle \alpha \leq m \rangle, \\
\sharp\langle \alpha :\leq n, \geq m \rangle &\mapsto \langle \alpha \leq n \rangle, \\
\star\langle \alpha :\leq n, \geq m \rangle &\mapsto \langle \alpha \geq m \rangle,
\end{aligned}$$

We extend $\sharp(\cdot)$ and $\star(\cdot)$ to neutrosophic terminological axioms as follows: $\sharp\hat{\tau} = \tilde{\tau}$ and $\star\hat{\tau} = \tilde{\tau}$. Finally, $\sharp\Sigma = \{\sharp\varphi : \varphi \in \Sigma_A\} \cup \{\sharp\hat{\tau} : \hat{\tau} \in \Sigma_T\}$ and $\star\Sigma = \{\star\varphi : \varphi \in \Sigma_A\} \cup \{\star\hat{\tau} : \hat{\tau} \in \Sigma_T\}$.

**Proposition 2.** *Let $\Sigma$ be a neutrosophic KB and let $\varphi$ be a neutrosophic assertion ($\langle \alpha :\geq n, \leq m \rangle$ or $\langle \alpha :\leq n, \geq m \rangle$). Then $\Sigma \models^n \varphi$ iff $\sharp\Sigma \models \sharp\varphi$ and $\star\Sigma \models \star\varphi$.* ⊣

*Proof.* ($\Rightarrow$): Let $\varphi$ be $\langle \alpha :\geq n, \leq m \rangle$. Consider a fuzzy interpretation $\mathcal{I}$ satisfying $\sharp\Sigma$ and $\mathcal{I}'$ satisfying $\star\Sigma$. $\langle \mathcal{I}, \mathcal{I}' \rangle$ is also a neutrosophic interpretation such that $a^{\mathcal{I}} = a^{\mathcal{I}'}$, $C^{\mathcal{I}}(a) = |C|^t(a)$ and $C^{\mathcal{I}'}(a) = |C|^f(a)$, $R^{\mathcal{I}}(d,d') = |R|^t(d,d')$ and $R^{\mathcal{I}'}(d,d') = |R|^f(d,d')$ hold. By induction on the structure of a concept $C$ it can be shown that $\mathcal{I}$ ($\mathcal{I}'$) satisfies $C(a)$ iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n$ ($C^{\mathcal{I}'}(a^{\mathcal{I}'} \geq n$) for fuzzy assertion $\langle C(a) \geq n \rangle$ and $C^{\mathcal{I}}(a^{\mathcal{I}}) \leq n$ ($C^{\mathcal{I}'}(a^{\mathcal{I}'})$ for fuzzy assertion $\langle C(a) \leq n \rangle$. Similarly for roles. By the definition of $\sharp(\cdot)$ and $\star(\cdot)$, therefore $\langle \mathcal{I}, \mathcal{I}' \rangle$ is a neutrosophic interpretation satisfying $\Sigma$. By hypothesis, $\langle \mathcal{I}, \mathcal{I}' \rangle$ satisfies $\langle \alpha :\geq n, \leq m \rangle$. Therefore, $\mathcal{I}$ satisfies $\sharp\varphi$ and $\mathcal{I}'$ satisfies $\sharp\varphi$. The proof is similar for $\varphi = \langle \alpha :\leq n, \geq m \rangle$.

($\Leftarrow$): Let $\varphi$ be $\langle \alpha :\geq n, \leq m \rangle$. Consider a neutrosophic $\mathcal{I}$ satisfying $\Sigma$. $\mathcal{I}$ can be regarded as two fuzzy interpretations $\mathcal{I}'$ and $\mathcal{I}''$ such that $a^{\mathcal{I}} = a^{\mathcal{I}'} = a^{\mathcal{I}''}$, $C^{\mathcal{I}'}(d) = |C|^t(d)$ and $C^{\mathcal{I}''}(d) = |C|^f(d)$, $R^{\mathcal{I}'}(d,d') = |R|^t(d,d')$ and $R^{\mathcal{I}''}(d,d') = |R|^f(d,d')$hold. By induction on the structure of a concept $C$ it can be shown that $\mathcal{I}$ satisfies $C(a)$ iff $|C|^t(a^{\mathcal{I}}) \geq n, |C|^f(a^{\mathcal{I}}) \leq m$ for neutrosophic assertion

$\langle C(a) :\geq n, \leq m\rangle$ and $|C|^t(a^{\mathcal{I}}) \leq n, |C|^f(a^{\mathcal{I}}) \geq m$ for neutrosophic assertion $\langle C(a) :\leq n, \geq m\rangle$. Similarly for roles. By the definition of $\sharp(\cdot)$ and $\star(\cdot)$, therefore, $\mathcal{I}^{'}$ is a fuzzy interpretation satisfying $\sharp\Sigma$ and $\mathcal{I}^{"}$ satisfying $\star\Sigma$. By hypothesis, $\mathcal{I}^{'}$ satisfies $\sharp\varphi$ and $\mathcal{I}^{"}$ satisfies $\star\varphi$. And according to the definition of $\sharp(\cdot)$ and $\star(\cdot)$, $\mathcal{I}$ satisfies $\langle \alpha :\geq n, \leq m\rangle$. The proof is similar for $\varphi = \langle \alpha :\leq n, \geq m\rangle$. □

### 4.4 Subsumption

As for the fuzzy case, subsumption between two concepts $C$ and $D$ w.r.t. a terminology $\Sigma_T$, *i.e.* $C \preceq^n_{\Sigma_T} D$, can be reduced to the case of an empty terminology, *i.e.* $C' \preceq^n_{\emptyset} D'$.

*Example 1.* Suppose we have two polls *p1* and *p2* about two wars war_x and war_y, separately. By the result of *p1*, it establishes that, to some degree n people in the country support the war_x and to some degree m people in the country do not support the war_x, whereas by the result of *p2*, it establishes that, to some degree f people in the country support the war_y and to some degree g people in the country do not support the war_y. Please note that, truth-degree and falsity-degree give a quantitative description of the *supportness* of a poll w.r.t. a war, *i.e.* the supportness is handled as a neutrosophic concept. So, let us consider

$$\Sigma = \{\langle p1 : \exists Support.war\_x :\geq 0.6, \leq 0.5\rangle, \langle p2 : \exists Support.war\_y :\geq 0.8, \leq 0.1\rangle,$$
$$war\_x \prec^n War, war\_y \prec^n War\}$$

where the axioms specify that both war_x and war_y are a War. According to the expansion process, $\Sigma$ will be replaced by

$$\Sigma^{'} = \{\langle p1 : \exists Support.war\_x :\geq 0.6, \leq 0.5\rangle, \langle p2 : \exists Support.war\_y :\geq 0.8, \leq 0.1\rangle,$$
$$war\_x :\approx^n War \sqcap war\_x^*, war\_y :\approx^n War \sqcap war\_y^*\},$$

which will be simplified to

$$\Sigma^{"} = \{\langle p1 : \exists Support.(War \sqcap war\_x^*) :\geq 0.6, \leq 0.5\rangle,$$
$$\langle p2 : \exists Support.(War \sqcap war\_y^*) :\geq 0.8, \leq 0.1\rangle\}.$$

Now, if we are looking for supportness of polls of War, then from $\Sigma$ we may infer that $\Sigma \models^n \langle p1 : \exists Support.War :\geq 0.6, \leq 0.5\rangle$ and $\Sigma \models^n \langle p2 : \exists Support.War :\geq 0.8, \leq 0.1\rangle$. Furthermore, it is easily verified that $\Sigma^{"} \models^n \langle p1 : \exists Support.War :\geq 0.6, \leq 0.5\rangle$ and $\Sigma^{"} \models^n \langle p2 : \exists Support.War :\geq 0.8, \leq 0.1\rangle$ hold as well. Indeed, for any neutrosophic assertion $\varphi$, $\Sigma \models^n \varphi$ iff $\Sigma^{"} \models^n \varphi$ holds. □

## 5 Decision Algorithms in Neutrosophic $\mathcal{ALC}$

Deciding whether $\Sigma \models^n \langle \alpha :\geq n, \leq m\rangle$ or $\Sigma \models^n \langle \alpha :\leq n, \geq m\rangle$ requires a calculus. Without loss of generality we will consider purely assertional neutrosophic KBs only.

We will develop a calculus in the style of the constraint propagation method, as this method is usually proposed in the context of DLs[4] and fuzzy DLs[12, 14]. We first address the entailment problem, then the subsumption problem and finally the BTVB problem. Both the subsumption problem and the BTVB problem will be reduced to the entailment problem.

### 5.1 A Decision Procedure for the Entailment Problem

Consider a new alphabet of $\mathcal{ALC}$ *variables*. An interpretation is extended to variables by mapping these into elements of the interpretation domain. An $\mathcal{ALC}$ *object* (denoted by $\omega$) is either an individual or a variable.[2]

A *constraint* (denoted by $\alpha$ is an expression of the form $C(\omega)$ or $R(\omega, \omega^{'})$, where $\omega, \omega^{'}$ are objects, $C$ is an $\mathcal{ALC}$ concept and $R$ is a role. A *neutrosophic constraint* (denoted by $\varphi$) is an expression having one of the following four forms: $\langle \alpha :\geq n, \leq m\rangle, \langle \alpha :\leq n, \geq m\rangle, \langle \alpha :> n, < m\rangle, \langle \alpha :< n, > m\rangle$. Note that neutrosophic assertions are neutrosophic constraints.

The definitions of satisfiability of a constraint, a neutrosophic constraint, a set of constraints, a set of neutrosophic constraints, atomic constraint and atomic neutrosophic constraint are obvious.

It is quite easily verified that the neutrosophic entailment problem can be reduced to the unsatisfiability problem of a set of neutrosophic constraints:

$$\Sigma \models^n \langle \alpha :\geq n, \leq m\rangle \text{ iff } \Sigma \cup \{\langle \alpha :< n, > m\rangle\} \text{ not satisfiable} \tag{8}$$

$$\Sigma \models^n \langle \alpha :\leq n, \geq m\rangle \text{ iff } \Sigma \cup \{\langle \alpha :> n, < m\rangle\} \text{ not satisfiable} \tag{9}$$

Our calculus, determining whether a finite set $S$ of neutrosophic constraints is satisfiable or not, is based on a set of constraint propagation rules transforming a set $S$ of neutrosophic constraints into "simpler" satisfiability preserving sets $S_i$ until either all $S_i$ contain a clash (indicating that from all the $S_i$ no model of $S$ can be build) or some $S_i$ is completed and clash-free, that is, no rule can be further applied to $S_i$ and $S_i$ contains no clash (indicating that from $S_i$ a model of $S$ can be build).

A set of neutrosophic constraints $S$ contains a *clash* iff it contains either one of the constraints in Table 1 or $S$ contains a conjugated pair of neutrosophic constraints. Each entry in Table 2 says us under which condition the row-column pair of neutrosophic constraints is a *conjugated pair*. Given a neutrosophic constraint $\varphi$, with $\varphi^c$ we indicate a conjugate of $\varphi$ (if there exists one). Notice that a conjugate of a neutrosophic constraint may be not unique, as there could be infinitely many. For instance, both $\langle C(a) :< 0.6, > 0.3\rangle$ and $\langle C(a) :\leq 0.7, \geq 0.4\rangle$ are conjugates of $\langle C(a) :\geq 0.8, \leq 0.1\rangle$.

Concerning the rules, for each connective $\sqcap, \sqcup, \neg, \forall, \exists$ there is a rule for each relation $\langle \geq, \leq\rangle, \langle >, <\rangle, \langle \leq, \geq\rangle, \langle <, >\rangle$, *i.e.* there are 20 rules. The rules have the form:

$$\Phi \rightarrow \Psi \text{ if } \Gamma \tag{10}$$

[2] In the following, if there is no ambiguity, $\mathcal{ALC}$ variables and $\mathcal{ALC}$ objects are called variables and objects, respectively.

$\langle \bot(\omega) :\geq n, \leq m\rangle$, where $n > 0$ or $m < 1$
$\langle \top(\omega) :\leq n, \geq m\rangle$, where $n < 1$ or $m > 0$
$\langle \bot(\omega) :> n, < m\rangle$, $\langle \top(\omega) :< n, > m\rangle$
$\langle C(\omega) :< 0, > m\rangle$, $\langle C(\omega) :> 1, < m\rangle, \langle C(\omega) :< n, > 1\rangle, \langle C(\omega) :> n, < 0\rangle$

**Table 1.** Clashes

| | $\langle \alpha :< f, > g\rangle$ | $\langle \alpha :\leq f, \geq g$ |
|---|---|---|
| $\langle \alpha :\geq n, \leq m$ | $n \geq f$ or $m \leq g$ | $n > f$ or $m < g$ |
| $\langle \alpha :> n, < m$ | $n \geq f$ or $m \leq g$ | $n \geq f$ or $m \leq g$ |

**Table 2.** Conjugated Pairs

where $\Phi$ and $\Psi$ are sequences of neutrosophic constraints and $\Gamma$ is a condition. A rule fires only if *the condition $\Gamma$ holds, if the current set $S$ of neutrosophic constraints contains neutrosophic constraints matching the precondition $\Phi$ and the consequence $\Psi$ is not already in $S$*. After firing, the constraints from $\Psi$ are added to $S$. The rules are the following:

$$\begin{array}{ll} (\neg_{\langle \geq, \leq \rangle}) & \langle \neg C(\omega) :\geq n, \leq m\rangle \rightarrow \langle C(\omega) :\leq m, \geq n\rangle \\ (\neg_{\langle >, < \rangle}) & \langle \neg C(\omega) :> n, < m\rangle \rightarrow \langle C(\omega) :< m, > n\rangle \\ (\neg_{\langle \leq, \geq \rangle}) & \langle \neg C(\omega) :\leq n, \geq m\rangle \rightarrow \langle C(\omega) :\geq m, \leq n\rangle \\ (\neg_{\langle <, > \rangle}) & \langle \neg C(\omega) :< n, > m\rangle \rightarrow \langle C(\omega) :> m, < n\rangle \end{array} \tag{11}$$

$$\begin{array}{ll} (\sqcap_{\langle \geq, \leq \rangle}) & \langle (C \sqcap D)(\omega) :\geq n, \leq m\rangle \rightarrow \langle C(\omega) :\geq n, \leq m\rangle, \langle D(\omega) :\geq n, \leq m\rangle \\ (\sqcap_{\langle >, < \rangle}) & \langle (C \sqcap D)(\omega) :> n, < m\rangle \rightarrow \langle C(\omega) :> n, < m\rangle, \langle D(\omega) :> n, < m\rangle \\ (\sqcap_{\langle \leq, \geq \rangle}) & \langle (C \sqcap D)(\omega) :\leq n, \geq m\rangle \rightarrow \langle C(\omega) :\leq n, \geq m\rangle, \langle D(\omega) :\geq n, \leq m\rangle | \\ & \langle C(\omega) :\geq n, \leq m\rangle, \langle D(\omega) :\leq n, \geq m\rangle | \\ & \langle C(\omega) :\leq n, \geq 0\rangle, \langle C(\omega) :\geq 0, \leq m\rangle, \langle D(\omega) :\geq n, \leq 1\rangle, \langle D(\omega) :\leq 1, \geq m\rangle | \\ & \langle C(\omega) :\geq n, \leq 1\rangle, \langle C(\omega) :\leq 1, \geq m\rangle, \langle D(\omega) :\geq 0, \leq m\rangle, \langle D(\omega) :\leq n, \geq 0\rangle \\ (\sqcap_{\langle <, > \rangle}) & \langle (C \sqcap D)(\omega) :< n, > m\rangle \rightarrow \langle C(\omega) :< n, > m\rangle, \langle D(\omega) :\geq n, \leq m\rangle | \\ & \langle C(\omega) :\geq n, \leq m\rangle, \langle D(\omega) :< n, > m\rangle | \\ & \langle C(\omega) :< n, > 0\rangle, \langle C(\omega) \geq 0, \leq m\rangle, \langle D(\omega) :\geq n, \leq 1\rangle, \langle D(\omega) :< 1, > m\rangle | \\ & \langle C(\omega) :\geq n, \leq 1\rangle, \langle C(\omega) :< 1, > m\rangle, \langle D(\omega) :< n, > 0\rangle, \langle D(\omega) :\geq 0, \leq m\rangle \end{array}$$

$$\begin{array}{ll} (\sqcup_{\langle \geq, \leq \rangle}) & \langle (C \sqcup D)(\omega) :\geq n, \leq m\rangle \rightarrow \langle C(\omega) :\geq n, \leq m\rangle, \langle D(\omega) :\leq n, \geq m\rangle | \\ & \langle C(\omega) :\leq n, \geq m\rangle, \langle D(\omega) :\geq n, \leq m\rangle | \\ & \langle C(\omega) :\geq n, \leq 1\rangle, \rangle C(\omega) :\leq 1, \geq m\rangle, \langle D(\omega) :\leq n, \geq 0\rangle, \langle D(\omega) :\geq 0, \leq m\rangle | \\ & \langle C(\omega) :\geq 0, \leq m\rangle, \langle C(\omega) :\leq n, \geq 0\rangle, \langle D(\omega) :\geq n, \leq 1\rangle, \langle D(\omega) :\leq 1, \geq m\rangle \\ (\sqcup_{\langle >, < \rangle}) & \langle (C \sqcup D)(\omega) :> n, < m\rangle \rightarrow \langle C(\omega) :> n, < m\rangle, \langle D(\omega) :\leq n, \geq m\rangle | \end{array}$$

$$\langle C(\omega) :\leq n, \geq m\rangle, \langle D(\omega) :> n, < m\rangle |$$
$$\langle C(\omega) :> n, < 1\rangle, \langle C(\omega) :\leq 1, \geq m\rangle, \langle D(\omega) :\leq n, \geq 0\rangle, \langle D(\omega) :> 0, < m\rangle |$$
$$\langle C(\omega) :\leq n, \geq 0\rangle, \langle C(\omega) :> 0, < m\rangle, \langle D(\omega) :> n, < 1\rangle, \langle D(\omega) :\leq 1, \geq m\rangle$$

$(\sqcup_{\langle\leq,\geq\rangle})$ $\langle (C \sqcup D)(\omega) :\leq n, \geq m\rangle \rightarrow \langle C(\omega) :\leq n, \geq m\rangle, \langle D(\omega) :\leq n, \geq m\rangle$

$(\sqcup_{\langle<,>\rangle})$ $\langle (C \sqcup D)(\omega) :< n, > m\rangle \rightarrow \langle C(\omega) :< n, > m\rangle, \langle D(\omega) :< n, > m\rangle$

$(\forall_{\langle\geq,\leq\rangle})$ $\langle (\forall R.C)(\omega_1) :\geq n, \leq m\rangle, \langle R(\omega_1, \omega_2) :\geq f, \leq g\rangle \rightarrow \langle C(\omega_2) :\geq n, \leq m\rangle$
if $f > m$ and $g < n$

$(\forall_{\langle>,<\rangle})$ $\langle (\forall R.C)(\omega_1) :> n, < m\rangle, \langle R(\omega_1, \omega_2) :\geq f, \leq g\rangle \rightarrow \langle C(\omega_2) :> n, < m\rangle$
if $f \geq m$ and $g \leq n$

$(\exists_{\langle\leq,\geq\rangle})$ $\langle (\exists R.C)(\omega_1) :\leq n, \geq m\rangle, \langle R(\omega_1, \omega_2) :\geq f, \leq g\rangle \rightarrow \langle C(\omega_2) :\leq n, \geq m\rangle$
if $f > n$ and $g < m$

$(\exists_{\langle<,>\rangle})$ $\langle (\exists R.C)(\omega_1) :< n, > m\rangle, \langle R(\omega_1, \omega_2) :\geq f, \leq g\rangle \rightarrow \langle C(\omega_2) :< n, > m\rangle$
if $f \geq n$ and $g \leq m$

$(\exists_{\geq,\leq})$ $\langle (\exists R.C)(\omega) :\geq n, \leq m\rangle \rightarrow \langle R(\omega, x) :\geq n, \leq m\rangle, \langle C(x) :\geq n, \leq m\rangle$
if $x$ is new variable and there is no $\omega^{'}$ such that both
$\langle R(\omega, \omega^{'}) :\geq n, \leq m\rangle$ and $\langle C(\omega^{'}) :\geq n, \leq m\rangle$ are already in the constraint set

$(\exists_{>,<})$ $\langle (\exists R.C)(\omega) :> n, < m\rangle \rightarrow \langle R(\omega, x) :> n, < m\rangle, \langle C(x) :> n, < m\rangle$
if $x$ is new variable and there is no $\omega^{'}$ such that both
$\langle R(\omega, \omega^{'}) :> n, < m\rangle$ and $\langle C(\omega^{'}) :> n, < m\rangle$ are already in the constraint set

$(\forall_{\leq,\geq})$ $\langle (\forall R.C)(\omega) :\leq n, \geq m\rangle \rightarrow \langle R(\omega, x) :\geq m, \leq n\rangle, \langle C(x) :\leq n, \geq m\rangle$
if $x$ is new variable and there is no $\omega^{'}$ such that both
$\langle R(\omega, \omega^{'}) :\geq m, \leq n\rangle$ and $\langle C(\omega^{'}) :\leq n, \geq m\rangle$ are already in the constraint set

$(\forall_{<,>})$ $\langle (\forall R.C)(\omega) :< n, > m\rangle \rightarrow \langle R(\omega, x) :> m, < n\rangle, \langle C(x) :< n, > m\rangle$
if $x$ is new variable and there is no $\omega^{'}$ such that both
$\langle R(\omega, \omega^{'}) :> m, < n\rangle$ and $\langle C(\omega^{'}) :< n, > m\rangle$ are already in the constraint set

A set of neutrosophic constraints $S$ is said to be *complete* if no rule is applicable to it. Any complete set of neutrosophic constraints $S_2$ obtained from a set of neutrosophic constraints $S_1$ by applying the above rules (11) is called a *completion* of $S_1$. Due to the rules $(\sqcup_{\geq,\leq}), (\sqcup_{>,<}), (\sqcap_{\leq,\geq})$ and $(\sqcap_{<,>})$, more than one completion can be obtained. These rules are called *nondeterministic rules*. All other rules are called *deterministic rules*.

It is easily verified that the above calculus has the *termination property, i.e.* any completion of a finite set of neutrosophic constraints $S$ can be obtained after a finite number of rule applications.

*Example 2.* Consider Example 1 and let us prove that $\Sigma^{"} \models^n \langle (\exists Support.War)(p1) \geq 0.6, \leq 0.5\rangle$. We prove the above relation by verifying that all completions of

$S = \Sigma'' \cup \{\langle(\exists Support.War)(p1) :< 0.6, > 0.5\rangle\}$ contain a clash. In fact, we have the following sequence.

| | |
|---|---|
| (1) $\langle(\exists Support.(War \sqcap war\_x^*))(p1) :\geq 0.6, \leq 0.5\rangle$ | Hypothesis:$S$ |
| (2) $\langle(\exists Support.(War \sqcap war\_y^*))(p2) :\geq 0.8, \leq 0.1\rangle$ | |
| (3) $\langle(\exists Support.War)(p1) :< 0.6, > 0.5\rangle$ | |
| (4) $\langle Support(p1, x) :\geq 0.6, \leq 0.5\rangle, \langle(War \sqcap war\_x^*)(x) :\geq 0.6, \leq 0.5\rangle$ | $(\exists_{\geq,\leq}) : (1)$ |
| (5) $\langle War(x) :< 0.6, > 0.5\rangle$ | $(\exists_{<,>}) : (3), (4)$ |
| (6) $\langle War(x) :\geq 0.6, \leq 0.5\rangle, \langle war\_x^*(x) :\geq 0.6, \leq 0.5\rangle$ | $(\sqcap_{\geq,\leq}) : (4)$ |
| (7) clash | $(5), (6)$ |

□

**Proposition 3.** *A finite set of neutrosophic constraints $S$ is satisfiable iff there exists a clash free completion of $S$.* ⊣

From a computational complexity point of view, the neutrosophic entailment problem can be proven to be a PSPACE-complete problem, as is the classical entailment problem and fuzzy entailment problem.

**Proposition 4.** *Let $\Sigma$ be a neutrosophic KB and let $\varphi$ be a neutrosophic assertion. Determining whether $\Sigma \models^n \varphi$ is a PSPACE-complete problem.* ⊣

*Proof.* By the Proposition 1, $\Sigma \models^n \varphi$ iff $\sharp\Sigma \models \sharp\varphi$ and $\star\Sigma \models \star\varphi$. From the PSPACE-completeness of the entailment problem in fuzzy $\mathcal{ALC}$[14], PSPACE-completeness of the neutrosophic entailment problems follows. □

This result establishes an important property about our neutrosophic DLs. In effect, it says that no additional computational cost has to be paid for the major expressive power.

### 5.2 A Decision Procedure for the Subsumption Problem

In this section we address the subsumption problem, *i.e.* deciding whether $C \preceq^n_{\Sigma_T} D$, where $C$ and $D$ are two concepts and $\Sigma_T$ is a neutrosophic terminology. As we have seen (see Example 1), $C \preceq^n_{\Sigma_T} D$ can be reduced to the case of an empty terminology by applying the KB expansion process. So, without loss of generality, we can limit our attention to the case $C \preceq^n_\emptyset D$.

It can easily be shown that

**Proposition 5.** *Let $C$ and $D$ be two concepts. It follows that $C \preceq^n_\emptyset D$ iff for all $n, m$, $\langle C(a) :\geq n, \leq m\rangle \models^n \langle D(a) :\geq n, \leq m\rangle$, where $a$ is a new individual.* ⊣

*Proof.* ($\Rightarrow$) Assume that $C \preceq^n_\emptyset D$ holds. Suppose to the contrary that $\exists n, m$ such that $\langle C(a) :\geq n, \leq m\rangle \models^n \langle D(a) :\geq n, \leq m\rangle$ does not hold. Therefore, there is an interpretation $\mathcal{I}$ and an $n, m$ such that $|C|^t(a^\mathcal{I}) \geq n$ and $|D|^t(a^\mathcal{I}) < n$ or $|C|^f(a^\mathcal{I}) \leq m$ and $|D|^f(a^\mathcal{I}) > m$. But, from the hypothesis $n \leq |C|^t(a^\mathcal{I}) \leq |D|^t(a^\mathcal{I}) < n$ or $m \geq |C|^f(a^\mathcal{I}) \geq |D|^f(a^\mathcal{I}) > m$ follow. Absurd.
($\Leftarrow$) Assume that for all $n, m, \langle C(a) :\geq n, \leq m\rangle \models^n \langle D(a) :\geq n, \leq m\rangle$ holds. Suppose to the contrary that $C \preceq^n_\emptyset D$ does not hold. Therefore, there is an interpretation $\mathcal{I}$ and $d \in \Delta^\mathcal{I}$ such that $|C|^t(d) > |D|^t(d) \geq 0$ or $|C|^f(d) < |D|^f(d) \leq 1$. Let us extent $\mathcal{I}$ to $a$ such that $a^\mathcal{I} = d$ and consider $\overline{n} = |C|^t(d)$ and $\overline{m} = |C|^f(d)$. Of course, $\mathcal{I}$ satisfies $\langle C(a) :\geq \overline{n}, \leq \overline{m}\rangle$. Therefore, from the hypothesis it follows that $\mathcal{I}$ satisfies $\langle D(a) :\geq \overline{n}, \leq \overline{m}\rangle$, *i.e.* $|D|^t(d) \geq \overline{n} = |C|^t(d) > |D|^t(d)$ or $|D|^f(d) \leq \overline{m} = |C|^f(d) < |D|^f(d)$. Absurd. $\square$

How can we check whether for all $n, m, \langle C(a) :\geq n, \leq m\rangle \models^n \langle D(a) :\geq n, \leq m\rangle$ holds? The following proposition shows that

**Proposition 6.** *Let $C$ and $D$ be two concepts, $n_1, m_1 \in \{0, 0.25, 0.5, 0.75, 1\}$ and let $a$ be an individual. It follows that for all $n, m\langle C(a) :\geq n, \leq m\rangle \models^n \langle D(a) :\geq n, \leq m\rangle$ iff $\langle C(a) :\geq n_1, \leq m_1\rangle \models^n \langle D(a) :\geq n_1, \leq m_1\rangle$ holds.* $\dashv$

As a consequence, the subsumption problem can be reduced to the entailment problem for which we have a decision algorithm.

### 5.3 A Decision Procedure for the BTVB Problem

We address now the problem of determining $glb(\Sigma, \alpha)$ and $lub(\Sigma, \alpha)$. This is important, as computing , *e.g.* $glb(\Sigma, \alpha)$, is in fact the way to answer a query of type "to which degree is $\alpha$ (at least) true and (at most) false, given the (imprecise) facts in $\Sigma$?".

Without loss of generality, we will assume that all concepts are in NNF (*Negation Normal Form*).

**Proposition 7.** *Let $\Sigma$ be a set of neutrosophic assertions in NNF and let $\alpha$ be an assertion. Then $glb(\Sigma, \alpha) \in N^\Sigma$ and $lub(\Sigma, \alpha) \in M^\Sigma$, where*

$$N^\Sigma = \{\langle n, m\rangle : \langle \alpha :\geq n, \leq m'\rangle \in \Sigma, \langle \alpha :\geq n', \leq m\rangle \in \Sigma\}$$

$$M^\Sigma = \{\langle n, m\rangle : \langle \alpha :\leq n, \geq m'\rangle \in \Sigma, \langle \alpha :\leq n', \geq m\rangle \in \Sigma\}$$

$\dashv$

The algorithm computing $glb(\Sigma, \alpha)$ and $lub(\Sigma, \alpha)$ are described in Table 3.

## 6 Conclusions and Future Work

In this paper, we have presented a quite general neutrosophic extension of the fuzzy DL $\mathcal{ALC}$, a significant and expressive representative of the various DLs.

| |
|---|
| **Algorithm** $glb(\Sigma, \alpha)$<br>Set $Min := \langle 0, 1 \rangle$ and $Max := \langle 1, 0 \rangle$.<br>1. Pick $\langle n, m \rangle \in M^{\Sigma}$ such that *first element of Min* $< n <$ *first element of Max* and *second element of Max* $< m <$ *second element of Min*. If there is no such $\langle n, m \rangle$, then set $glb(\Sigma, \alpha) := Min$ and exit.<br>2. If $\Sigma \models^{n} \langle \alpha :\geq n, \leq m \rangle$ then set $Min = \langle n, m \rangle$, else set $Max = \langle n, m \rangle$. Go to Step 1.<br><br>**Algorithm** $lub(\Sigma, \alpha)$<br>Set $Min := \langle 1, 0 \rangle$ and $Max := \langle 0, 1 \rangle$.<br>1. Pick $\langle n, m \rangle \in N^{\Sigma}$ such that *first element of Max* $< n <$ *first element of Min* and *second element of Min* $< m <$ *second element of Max*. If there is no such $\langle n, m \rangle$, then set $lub(\Sigma, \alpha) := Min$ and exit.<br>2. If $\Sigma \models^{n} \langle \alpha :\leq n, \geq m \rangle$ then set $Min = \langle n, m \rangle$, else set $Max = \langle n, m \rangle$. Go to Step 1. |

**Table 3.** Algorithms $glb(\Sigma, \alpha)$ and $lub(\Sigma, \alpha)$

Our neutrosophic DL enables us to reason in presence of imprecise (fuzzy, incomplete, and inconsistent) $\mathcal{ALC}$ concepts, *i.e.* neutrosophic $\mathcal{ALC}$ concepts. From a semantics point of view, neutrosophic concepts are interpreted as neutrosophic sets, *i.e.* given a concept $C$ and an individual $a$, $C(a)$ is interpreted as the truth-value and falsity-value of the sentence "$a$ is $C$". From a syntax point of view, we allow to specify lower and upper bounds of the truth-value and falsity-value of $C(a)$. Complete algorithms for reasoning in it have been presented, that is, we have devised algorithms for solving the entailment problem, the subsumption problem as well as the best truth-value bound problem.

An important point concerns computational complexity. The complexity result shows that the additional expressive power has no impact from a computational complexity point of view.

This work can be used as a basis both for extending existing DL and fuzzy DL based systems and for further research. In this latter case, there are several open points. For instance, it is not clear yet how to reason both in case of neutrosophic specialization of the general form $C \prec^{n} D$ and in the case cycles are allowed in a neutrosophic KB. Another interesting topic for further research concerns the semantics of neutrosophic connectives. Of course several other choices for the semantics of the connectives $\sqcap, \sqcup, \neg, \exists, \forall$ can be considered.

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