About the second Droz-Farny circle

Profesor Ion Pătrașcu, Colegiul Național Frații Buzești, Craiova, România
Profesor Florentin Smarandache, Universitatea New Mexico, USA

In this article, we prove the theorem relative to the *second Droz-Farny circle*, and a sentence that generalizes it. The paper [1] informs that the following *Theorem* is attributed to J. Neuberg (*Mathesis*, 1911).

First Theorem. The circles with its centers in the middles of triangle ABC passing through its orthocenter H intersect the sides BC, CA and AB respectively in the points A_1, A_2, B_1, B_2 and C_1, C_2, situated on a concentric circle with the circle circumscribed to the triangle ABC (**the second Droz-Farny circle**).

Proof. We denote by M_1, M_2, M_3 the middles of ABC triangle's sides, see *Figure 1*. Because $AH \perp M_2M_3$ and H belongs to the circles with centers in M_2 and M_3, it follows that AH is the radical axis of these circles, therefore we have $AC_1 \cdot AC_2 = AB_2 \cdot AB_1$. This relation shows that B_1, B_2, C_1, C_2 are concyclic points, because the center of the circle on which they are situated is O, the center of the circle circumscribed to the triangle ABC, hence we have that:

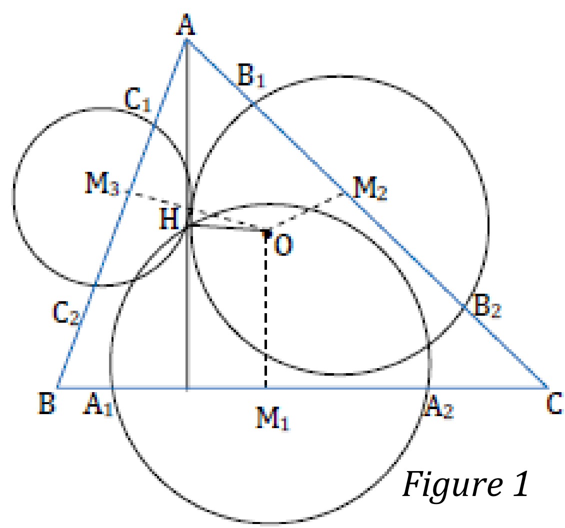

Figure 1

$OB_1 = OC_1 = OC_2 = OB_2$ (1).

Analogously, O is the center of the circle on which the points A_1, A_2, C_1, C_2 are situated, hence:

$$OA_1 = OC_1 = OC_2 = OA_2 \qquad (2).$$

Also, O is the center of the circle on which the points A_1, A_2, B_1, B_2 are situated, and therefore:

$$OA_1 = OB_1 = OB_2 = OA_2 \qquad (3).$$

The relations (1), (2), (3) show that the points $A_1, A_2, B_1, B_2, C_1, C_2$ are situated on a circle having the center in O, called *the second Droz-Farny circle*.

Proposition. The radius of the second Droz-Farny circle is given by:

$$R_2^2 = 5e^2 - \frac{1}{2}(a^2 + b^2 + c^2).$$

Proof. From the right triangle OM_1A_1, using Pitagora theorem, it follows that:

$$OA_1^2 = OM_1^2 + A_1M_1^2 = OM_1^2 + M_1M_2.$$

From the triangle BHC, using the median theorem, we have:

$$HM_1^2 = \frac{1}{4}[2(BH^2 + CH^2) - BC^2].$$

But in a triangle, $AH = 2OM_1, BH = 2OM_2, CH = 2OM_3$, hence:

$$HM_1^2 = 2OM_2^2 + 2OM_3^2 = \frac{a^2}{4}.$$

But $OM_1^2 = R^2 - \frac{a^2}{4}$; $OM_2^2 = R^2 - \frac{b^2}{4}$; $OM_3^2 = R^2 - \frac{c^2}{4}$, where R is the radius of the circle circumscribed to the triangle ABC.

We find that $OA_1^2 = R_2^2 = 5R^2 - \frac{1}{2}(a^2 + b^2 + c^2)$.

Remarks.

1. We can compute $OM_1^2 + M_1M_2$ using the median theorem in the triangle OM_1H for the median M_1O_9 (O_9 is the center of the nine points circle, i.e. the middle of (OH)). Because $O_9M_1 = \frac{1}{2}R$, we obtain: $R_2^2 = \frac{1}{2}(OM^2 + R^2)$. In this way, we can prove the *Theorem* computing OB_1^2 and OC_1^2.
2. The statement of the *First Theorem* was the subject no. 1 of the 49th International Olympiad in Mathematics, held at Madrid in 2008.
3. The *First Theorem* can be proved in the same way for an obtuse triangle, but it is obvious that for a right triangle, the second Droz-Farny circle coincides with the circle circumscribed to the triangle ABC.
4. The *First Theorem* appears as proposed problem in [2].

Second Theorem. The three pairs of points determined by the intersections of each circle with the center in the middle of triangle's side with the respective side are on a circle if and only these circles have as radical center the triangle's orthocenter.

Proof. Let M_1, M_2, M_3 the middles of the sides of triangle ABC and let $A_1, A_2, B_1, B_2, C_1, C_2$ the intersections with BC, CA, AB respectively of the circles with centers in M_1, M_2, M_3.

Let us suppose that $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclic points. The circle on which they are situated has evidently the center in O, the center of the circle circumscribed to the triangle ABC. The radical axis of the circles with centers M_2, M_3 will be perpendicular on the line of centers M_2M_3, and because A has equal powers in relation to these circles, since $AB_1 \cdot AB_2 = AC_1 \cdot AC_2$, it follows that the radical axis will be the perpendicular taken from A on M_2M_3, i.e. the height from A of triangle ABC.

Furthermore, it ensues that the radical axis of the circles with centers in M_1 and M_2 is the height from B of triangle ABC and consequently the intersection of the heights, hence the orthocenter H of the triangle ABC is the radical center of the three circles.

Reciprocal. If the circles having the centers in M_1, M_2, M_3 have the orthocenter with the radical center, it follows that the point A, being situated on the height from A which is the radical axis of the circles of centers M_2, M_3 will have equal powers in relation to these circles and, consequently, $AB_1 \cdot AB_2 = AC_1 \cdot AC_2$, a relation that implies that B_1, B_2, C_1, C_2 are concyclic points, and the circle on which these points are situated has O as its center. Similarly, $BA_1 \cdot BA_2 = BC_1 \cdot BC_2$, therefore A_1, A_2, C_1, C_2 are concyclic points on a circle of center O. Having $OB_1 = OB_2 = OC_1 = OC_2$ and $OA_1 \cdot OA_2 = OC_1 \cdot OC_2$, we get that the points $A_1, A_2, B_1, B_2, C_1, C_2$ are situated on a circle of center O.

Remarks.

1. The *First Theorem* is a particular case of the *Second Theorem*, because the three circles of centers M_1, M_2, M_3 pass through H, which means that H is their radical center.
2. The Problem 525 from [3] drives us to the following *Proposition* that provides the way to construct the circles of centers M_1, M_2, M_3 that intersect the sides in points that belong to a Droz-Farny circle of type 2.

Proposition.

The circles $C\left(M_1, \frac{1}{2}\sqrt{k+a^2}\right)$, $C\left(M_2, \frac{1}{2}\sqrt{k+b^2}\right)$, $C\left(M_3, \frac{1}{2}\sqrt{k+c^2}\right)$ intersect the sides BC, CA, AB respectively in six concyclic points; k is a conveniently chosen constant, and a, b, c are the lengths of the sides of triangle ABC.

Proof.

According to the *Second Theorem*, it is necessary to prove that the orthocenter H of triangle ABC is the radical center for the circles from hypothesis.

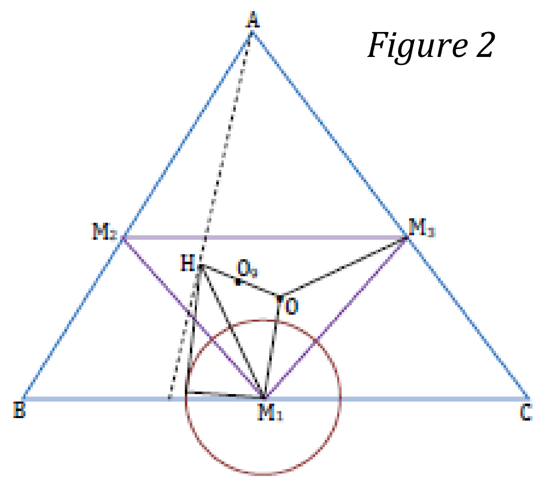

Figure 2

The power of H in relation with $C\left(M_1, \frac{1}{2}\sqrt{k+a^2}\right)$ is equal to $HM_1^2 - \frac{1}{4}(k + a^2)$. We observed that $M_1^2 = 4R^2 - \frac{b^2}{2} - \frac{c^2}{2} - \frac{a^2}{4}$, therefore $HM_1^2 - \frac{1}{4}(k+a^2) = 4R^2 - \frac{a^2+b^2+c^2}{4} - \frac{1}{4}k$. We use the same expression for the power of H in relation to the circles of centers M_2, M_3, hence H is the radical center of these three circles.

Bibliography.

[1] C. Mihalescu: *Geometria elementelor remarcabile [The Geometry of Outstanding Elements]*. Bucharest: Editura Tehnică, 1957.

[2] I. Pătraşcu: *Probleme de geometrie plană [Some Problems of Plane Geometry]*, Craiova: Editura Cardinal, 1996.

[3] C. Coşniţă: *Teoreme şi probleme alese de matematică [Theorems and Problems]*, Bucureşti: Editura de Stat Didactică şi Pedagogică, 1958.

[4] I. Pătraşcu, F. Smarandache: *Variance on Topics of Plane Geometry*, Educational Publishing, Columbus, Ohio, SUA, 2013.