BENCZE MIHÁLY
FLORIN POPOVICI
FLORENTIN SMARANDACHE

About Very Perfect Numbers

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About Very Perfect Numbers[1]

A natural number n is named very perfect if $\sigma(\sigma(n))=2n$ (see [1]).

Theorem. The square of an odd prime number can't be very perfect number.

Proof. Let be $n=p^2$, where p is an odd prime number, t en $\sigma(n)=1+p+p^2$, $\sigma(\sigma(n)) = \sigma(1+p+p^2) = 2p^2$. We decompose $\sigma(n)$ in canonical form, from where $1 + p + p^2 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$. Because $p(p+1)+1$ is odd, in the canonical decompose must be only odd primes.

$$\sigma(\sigma(n)) = (1+p_1+\ldots+p_1^{\alpha_1})\ldots(1+p_k+\ldots+p_k^{\alpha_k}) = \frac{p_1^{\alpha_1+1}-1}{p_1-1}\cdots\frac{p_k^{\alpha_k+1}-1}{p_k-1} = 2p^2.$$

Because $\frac{p_1^{\alpha_1+1}-1}{p_1-1} > 2, \ldots, \frac{p_k^{\alpha_k+1}-1}{p_k-1} > 2$,

one gets that $2p^2$ can't be decomposed in more than two factors, so each one > 2, therefore $k \le 2$.

Case 1. For $k = 1$ we find $\sigma(n) = 1 + p + p^2 = p_1^{\alpha_1}$, from where one gets $p_1^{\alpha_1+1} = p_1(1+p+p^2)$ and

$$\sigma(\sigma(n)) = \frac{p_1^{\alpha_1+1}-1}{p_1-1} = 2p^2,$$

$p_1(1+p+p^2)-1=2p^2(p_1-1)$, from where $p_1-1 = p(pp_1-2p-p_1)$. The right side is divisible by p, thus $p_1 - 1$ is a p multiple. Because $p_1 > 2$ it results $p_1 \ge p-1$ and

$$p_1^2 \ge (p-1)^2 > p^2+p+1 = p_1^{\alpha_1},$$

thus $\alpha_1=1$ and $\sigma(n)=p^2+p+1=p_1$, $\sigma(\sigma(n)) = \sigma(p_1) = 1 + p_1$. If n is very perfect then $1 + p_1 = 2p^2$ or $p^2+p+2 = 2p^2$. The solutions of the equation are $p = -1$ and $p = 2$ which is a contradiction.

Case2. For k=2 we have $\sigma(n)=p^2+p+1 = p_1^{\alpha_1} p_2^{\alpha_2}$.

$$\sigma(\sigma(n)) = (1+p_1+\ldots+p_1^{\alpha_1})(1+p_2+\ldots+p_2^{\alpha_2}) = \frac{p_1^{\alpha_1+1}-1}{p_1-1}\cdot\frac{p_2^{\alpha_2+1}-1}{p_2-1} = 2p^2.$$

Because $\frac{p_1^{\alpha_1+1}-1}{p_1-1} > 2$ and $\frac{p_2^{\alpha_2+1}-1}{p_2-1} > 2$

it results

$$\frac{p_1^{\alpha_1+1}}{p_1-1} = p \text{ and } \frac{p_2^{\alpha_2+1}-1}{p_2-1} = 2p$$

(or inverse),

thus $p_1^{\alpha_1+1}-1 = p(p_1-1)$, $p_2^{\alpha_2+1}-1 = 2p(p_2-1)$,

then $p_1^{\alpha_1+1}p_2^{\alpha_2+1} - p_1^{\alpha_1+1} - p_2^{\alpha_2+1} + 1 = 2p^2(p_1-1)(p_2-1)$,

[1] Together with Mihály Bencze and Florin Popovici

thus $\sigma(n) = p^2+p+1 = p_1^{\alpha_1-1}p_2^{\alpha_2-1}$
and $p_1p_2(p^2+p+1) = 2p^2(p_1-1)(p_2-1) + p_1^{\alpha_1+1} + p_2^{\alpha_2+1} - 1$
or $p_1p_2p(p+1) + p_1p_2 - 1 = 2p^2(p_1-1)(p_2-1) + (p_1^{\alpha_1+1} - 1) + (p_2^{\alpha_2-1} - 1) =$
$2p^2(p_1-1)(p_2-1) + p(p_1-1) + 2p(p_2-1)$ accordingly p divides p_1p_2-1, thus $p_1p_2 > p+1$ and $p_1^2 p_2^2 \geq (p+1)^2 > p^2+p+1 = p_1^{\alpha_1} p_2^{\alpha_2}$ Hence:
Π_1) If $\alpha_1 = 1$ and $n = 2p^2$, then $\sigma(n) = p^2+p+1 = p_1p_2^{\alpha_2}$

$$\text{and } \frac{p_1^2-1}{p_1-1} = p \text{ and } \frac{p_2^{\alpha_2+1}-1}{p_2-1} = 2p,$$

thus $p_1+1=p$ wich is a contradiction.

Π_2) If $\alpha_2 = 1$ and $n = 2p^2$, then $\sigma(n) = p^2+p+1 = p_1^{\alpha_1} p_2$

$$\text{and } \frac{p_1^{\alpha_1+1}-1}{p_1-1} = p \text{ and } \frac{p_2^2-1}{p_2-1} = 2p,$$

thus $p_2+1 = 2p$, $p_2 = 2p-1$ and $\sigma(n) = p^2+p+1 = p_1^{\alpha_1}(2p+1)$,
from where $4\sigma(n)=(2p-1)(2p+3)+7= 4p_1^{\alpha}(2p-1)$, accordingly 7 is divisible by $2p-1$ and thus p is divisible by 4 which is a contradiction.

Reference :

1 Suryanarayama, Elemente der Mathematik, 1969.

["Octogon", Braşov, Vol. 5, No. 2, 53-4, October 1997.]