Ion Patrascu, Florentin Smarandache

An Extension of a Problem of Fixed Point

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In this article, we extend the requirement of the *Problem 9.2* proposed at *Varna 2015 Spring Competition*, both in terms of membership of the measure γ, and the case for the problem for the ex-inscribed circle $\mathcal{C}$. We also try to guide the student in the search and identification of the fixed point, for succeeding in solving any problem of this type.

The statement of the problem is as follows:

> "We fix an angle $\gamma \in (0, 90^0)$ and the line AB which divides the plane in two half-planes γ and $\bar{\gamma}$. The point C in the half-plane γ is situated such that $m(\widehat{ACB}) = \gamma$. The circle inscribed in the triangle ABC with the center I is tangent to the sides AC and BC in the points F and E, respectively. The point P is located on the segment line $(IE$, the point E between I and P such that $PE \perp BC$ and $PE = AF$. The point Q is situated on the segment line $(IF$, such that F is between I and Q; $QF \perp AC$ and $QF = BE$. Prove

> that the mediator of segment PQ passes through a fixed point." *(Stanislav Chobanov)*

Proof.

Firstly, it is useful to note that the point C varies in the half-plane ψ on the arc capable of angle γ; we know as well that $m(\widehat{AIB}) = 90^0 + \frac{\gamma}{2}$, so I varies on the arc capable of angle of measure $90^0 + \frac{\gamma}{2}$ situated in the half-plane ψ.

Another useful remark is about the segments AF and BE, which in a triangle have the lengths $p - a$, respectively $p - b$, where p is the half-perimeter of the triangle ABC with $AB = c$ – constant; therefore, we have $\Delta PEB \equiv \Delta AFQ$ with the consequence $PB = QA$. Considering the vertex C of the triangle ABC the middle of the arc capable of angle γ built on AB, we observe that PQ is parallel to AB; more than that, $ABPQ$ is an isosceles trapezoid, and segment PQ mediator will be a symmetry axis of the trapezoid, so it will coincide with the mediator of AB, which is a fixed line, so we're looking for the fixed point on mediator of AB.

Let D be the intersection of the mediators of segments PQ and AB, see *Figure 1*, where we considered $m(\hat{A}) < m(\hat{B})$. The point D is on the mediator of AB, so we have $DA = DB$; the point D is also on the mediator of PQ, so we have $DP = DQ$; it follows that: $\Delta PBD \equiv \Delta QAD$, a relation from where we get that $\sphericalangle QAD = \sphericalangle PBD$.

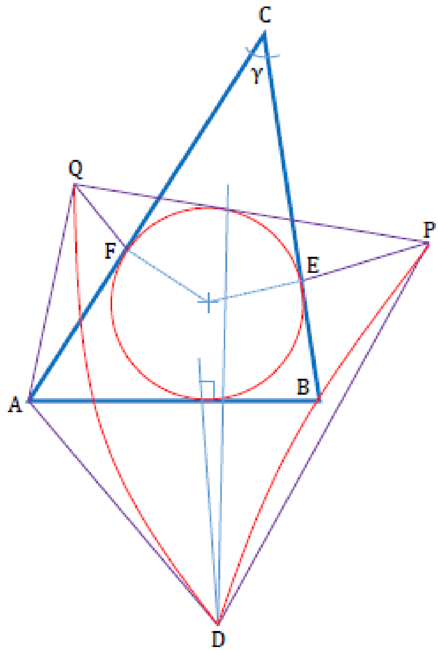

Figure 1

If we denote $m(\widehat{QAF}) = x$ and $m(\widehat{DAB}) = y$, we have $\widehat{QAD} = x + A + y$, $\widehat{PBD} = 360^0 - B - y - (90^0 - x)$.

From $x + A + y = 360^0 - B - y - 90^0 + x$, we find that $A + B + 2y = 270^0$, and since $A + B = 180^0 - \gamma$, we find that $2y = 90^0 - \gamma$, therefore the requested fixed point D is the vertex of triangle DAB, situated in $\bar{\psi}$ such that $m(\widehat{ADB}) = 90^0 - \gamma$.

1^{st} *Remark.*

If $\gamma = 90^0$, we propose to the reader to prove that the quadrilateral $ABPQ$ is a parallelogram; in this case, the requested fixed point does not exist (or is the point at infinity of the perpendicular lines to AB).

2^{nd} *Remark.*

If $\gamma \in (90^0, 180^0)$, the problem applies, and we find that the fixed point D is located in the half-plane ψ, such that the triangle DAB is isosceles, having $m(\widehat{AOB}) = \gamma - 90^0$.

We suggest to the reader to solve the following problem:

> We fix an angle $\gamma \in (0^0, 180^0)$ and the line AB which divides the plane in two half-planes, ψ and $\bar{\psi}$. The point C in the half-plane ψ is located such that $m(\widehat{ACB}) = \gamma$. The circle C – ex-inscribed to the triangle ABC with center I_c is tangent to the sides AC and BC in the points F and E, respectively. The point P is located on the line segment $(I_cE$, E is between I_c and P such that $PE \perp BC$ and $PE = AF$. The point Q is located on the line segment $(I_cF$ such that F is between I and Q, $QF \perp AC$ and $QF = BE$. Prove that the mediator of the segment PQ passes through a fixed point.

3rd Remark.

As seen, this problem is also true in the case $\gamma = 90^0$, more than that, in this case, the fixed point is the middle of AB. Prove!

References.

[1] Ion Patrascu: *Probleme de geometrie plană* [Planar Geometry Problems]. Craiova: Editura Cardinal, 1996.