Apollonius Circles of Rank -1

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In this article, we highlight some properties of the Apollonius circles of rank -1 associated with a triangle.

We recall some essential notions.

Definition 1.

It is called cevian of rank k in the triangle *ABC* a cevian *AD* with $D \in BC$ and $\frac{BD}{DC} = \left(\frac{AB}{AC}\right)^k$, $k \in \mathbb{R}$.

Remark.

The median is a cevian of rank 0. The bisector is a cevian of rank 1.

Definition 2.

The cevian of rank -1 is called *antibisector* and it is isotomic to the bisector.

The external cevian of rank -1 is called external *antibisector*.

Definition 3.

The circle built on the segment determined by the feet of the antibisector in *A* and of the external antibisector in *A* as diameter is called *A* – Apollonius circle of rank -1 associated to the triangle *ABC*.

Remark.

Three Apollonius circles of rank -1 correspond to a triangle.

Theorem 1.

The *A* – Apollonius circle of rank -1 associated to the triangle *ABC* is the geometric place of the points M from triangle's plane, with the property $\frac{MB}{MC} = \frac{AC}{AB}$.

For theorem proof, see [1].

Theorem 2.

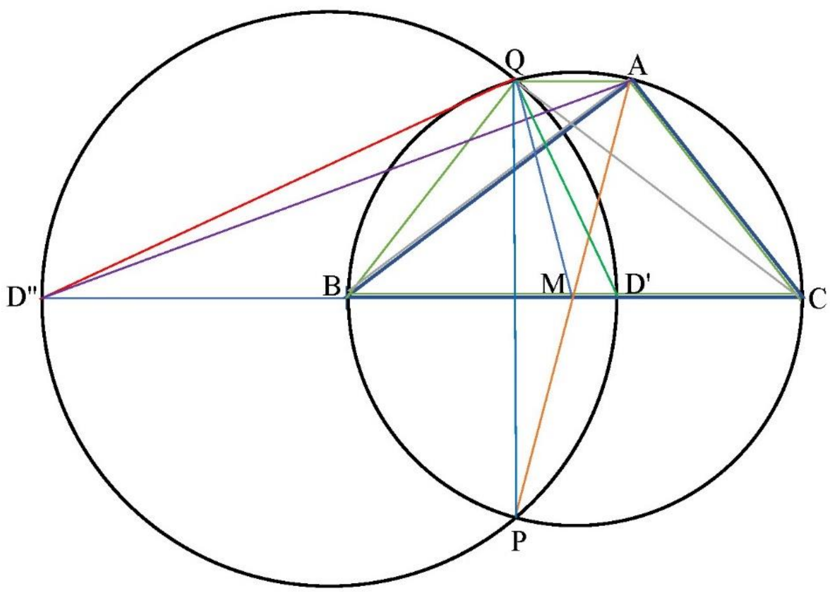
The Apollonius circles of rank -1 associated to the triangle *ABC* pass through two fixed points (they are part of a fascicle of the second type).

Theorem 3.

The *A* – Apollonius circle of rank -1 of the triangle *ABC* intersects its circumscribed circle in two points that belong respectively to the median in *A* of the triangle and the parallel taken through *A* to the side *BC*.

Proof.

Let Q be the intersection to a parallel taken through *A* to the *BC* with the circumscribed circle of the triangle *ABC*. Therefore, the quadrilateral *QACB* is an isosceles trapezoid, so $QC = AB$ and $QB = AC$.



Because $\frac{QB}{QC} = \frac{AC}{AB}$, it follows that the point Q belongs to the A – Apollonius circle of rank -1. We denote by *P* the intersection of median *AM* of the triangle *ABC* with its circumscribed circle. Because the median divides the triangle in two equivalent triangles, we have that the area of ΔABM is equal with the area of ΔACM and the area ΔPBM is equal with the area of ΔPCM. By addition, it follows that the area ΔABP is equal with ΔACP. But the area of $\Delta ABP = \frac{1}{2} \cdot AB \cdot PB \cdot sin\widehat{ABP}$, and the area of $\Delta ACP = \frac{1}{2} \cdot AC \cdot PC \cdot sin\widehat{ACP}$. As the angles *ACP* and *ABP* are supplementary, their sinuses are equal and consequently we obtain that $AB \cdot PB = AC \cdot PC$, i.e. $\frac{PB}{PC} = \frac{AC}{AB}$, and we such obtain that the point P belongs to the A – Apollonius circle of rank -1.

Proposition 1.

The *A* – Apollonius circle of rank -1 of the triangle *ABC* is an Apollonius circle for the triangle *QBC*, where *Q* is the intersection with the circumscribed circle of the triangle *ABC* with the parallel taken through *A* to *BC*.

Proof.

The quadrilateral *AQBC* is an isosceles trapezoid; therefore, $\sphericalangle BAC \equiv \sphericalangle QBC$, so QD' is bisector in *QBC* (D' is symmetric towards *M*, the middle of (*BC*), of the bisector feet taken from *A* of the triangle *ABC*. Since $D''Q \perp D'Q$, we have that $D''Q$ is an external bisector for $\sphericalangle BQC$ and therefore the *A* – Apollonius circle of rank -1 is the Apollonius circle of the *QBC* triangle.

Remarks.

1. From the previous proposition, it follows that *QP* is a simedian in the triangle *QBC*, therefore the quadrilateral *QBPC* is a harmonic quadrilateral.
2. The quadrilateral *QBPC* being harmonic, it follows that *PQ* is a simedian in the triangle *PBC*.
3. The Brocard circles of the triangles *ABC* and *QBC* are congruent. Indeed, if *O* is the center of the circumscribed circle of the triangle *ABC* and *M* the middle of the side *BC*, we have that the triangles *ABC* and *QBC* are symmetric to *OM*. Therefore, the simetric of *K* – the simedian center of *ABC* towards *OM*, will be *K'* the simedian center of *QBC*. The Brocard circles with diameters *OK* respectively *OK'*, from $OK = OK'$, it follows that they are congruent (they are symmetrical towards *OM*).

Bibliography

[1]. I. Patrașcu, F. Smarandache. **Apollonius Circles of rank *k***. In *Recreații matematice*, year XVIII, no. 1/2016, Iassi, Romania.

[2]. I. Patrascu, F. Smarandache. **Complements to Classic Topics of Circles Geometry**. Pons Editions, Brussels, 2016.