ABOUT THE CHARACTERISTIC FUNCTION OF A SET

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Abstract:

In this paper we give a method, based on the characteristic function of a set, to solve some difficult problems of set theory found in undergraduate studies.

Definition: Let's consider $A \subset E \neq \varnothing$ (a universal set), then $f_A : E \to \{0,\ 1\}$, where the function $f_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$ is called the characteristic function of the set A.

Theorem 1: Let's consider $A,\ B \subset E$. In this case $f_A = f_B$ if and only if $A = B$.

Proof.

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A = B \\ 0, & \text{if } x \notin A = B \end{cases} = f_B(x)$$

Reciprocally: For any $x \in A$, $f_A(x) = 1$, but $f_A = f_B$, therefore $f_B(x) = 1$, namely $x \in B$ from where $A \subset B$. The same way we prove that $B \subset A$, namely $A = B$.

Theorem 2: $f_{\tilde{A}} = 1 - f_A$, $\tilde{A} = C_E A$.

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$$f_{\tilde{A}}(x) = \begin{cases} 1, & \text{if } x \in \tilde{A} \\ 0, & \text{if } x \notin \tilde{A} \end{cases} = \begin{cases} 1, & \text{if } x \notin A \\ 0, & \text{if } x \in A \end{cases} = \begin{cases} 1-0, & \text{if } x \notin A \\ 1-1, & \text{if } x \in A \end{cases} = 1 - \begin{cases} 0, & \text{if } x \notin A \\ 1, & \text{if } x \in A \end{cases} = 1 - f_A(x)$$

Theorem 3: $f_{A \cap B} = f_A * f_B$.

Proof.

$$f_{A\cap B}(x)=\begin{cases}1, & \text{if } x\in A\cap B\\ 0, & \text{if } x\notin A\cap B\end{cases}=\begin{cases}1, & \text{if } x\in A \text{ and } x\in B\\ 0, & \text{if } x\notin A \text{ or } x\notin B\end{cases}=\begin{cases}1, & \text{if } x\in A,\ x\in B\\ 0, & \text{if } x\in A,\ x\notin B\\ 0, & \text{if } x\notin A,\ x\in B\\ 0, & \text{if } x\notin A,\ x\notin B\end{cases}=$$

$$=\left(\begin{cases}1 & \text{if } x\in A\\ 0 & \text{if } x\notin A\end{cases}\right)\cdot\left(\begin{cases}1 & \text{if } x\in B\\ 0 & \text{if } x\notin B\end{cases}\right)=f_A(x)f_B(x).$$

The theorem can be generalized by induction:

Theorem 4: $f_{\bigcap_{k=1}^{n} A_k}=\prod_{k=1}^{n} f_{A_k}$

Consequence. For any $n\in\mathbb{N}^*$, $f_M^n=f_M$.

Proof. In the previous theorem we chose $A_1=A_2=...=A_n=M$.

Theorem 5: $f_{A\cup B}=f_A+f_B-f_Af_B$.

Proof.

$$f_{A\cup B}=f_{\overline{\overline{A\cup B}}}=f_{\overline{\bar{A}\cap\bar{B}}}=1-f_{\bar{A}\cap\bar{B}}=1-f_{\bar{A}}f_{\bar{B}}=1-(1-f_A)(1-f_B)=f_A+f_B-f_Af_B$$

It can be generalized by induction:

Theorem 6: $f_{\bigcup_{k=1}^{n} A_k}=\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\le i_1<...<i_k\le n}^{n}(-1)^{k-1}f_{A_{i_1}}f_{A_{i_2}}\cdots f_{A_{i_k}}$

Theorem 7: $f_{A-B}=f_A(1-f_B)$

Proof. $f_{A-B}=f_{A\cap\bar{B}}=f_Af_{\bar{B}}=f_A(1-f_B)$.

It can be generalized by induction:

Theorem 8: $f_{A_1-A_2-...-A_n}=\sum_{k=1}^{n}(-1)^{k-1}f_{A_{i_1}}f_{A_{i_2}}...f_{A_{i_k}}$.

Theorem 9: $f_{A\Delta B}=f_A+f_B-2f_Af_B$

Proof.

$$f_{A\Delta B}=f_{A\cup B-A\cap B}=f_{A\cup B}(1-f_{A\cap B})=(f_A+f_B-f_Af_B)(1-f_Af_B)=f_A+f_B-2f_Af_B.$$

It can be generalized by induction:

Theorem 10: $F_{\Delta_{k=1}^{n}A_k}=\sum_{k=1}^{n}(-2)^{k-1}\sum_{1\le i_1<...<i_k\le n}f_{A_{i_1}A_{i_2}....A_{i_k}}$.

Theorem 11: $f_{A\times B}(x,y)=f_A(x)f_B(y)$.

Proof. If $(x,y)\in A\times B$, then $f_{A\times B}(x,y)=1$ and $x\in A$, namely $f_A(x)=1$ and $y\in B$, namely $f_B(y)=1$, therefore $f_A(x)f_B(y)=1$. If $(x,y)\notin A\times B$, then $f_{A\times B}(x,y)=0$ and $x\notin A$, namely $f_A(x)=0$ or $y\notin B$, namely $f_B(y)=0$, therefore $f_A(x)f_B(y)=0$.

This theorem can be generalized by induction.

Theorem 12: $f_{\times_{k=1}^{n} A_k}(x_1,\ x_2,\ldots,x_n)=\prod_{k=1}^{n} f_{A_k}(x_k)$.

Theorem 13: (De Morgan) $\overline{\bigcup_{k=1}^{n} A_k}=\bigcap_{k=1}^{n}\overline{A_k}$

Proof.

$$f_{\overline{\bigcup_{k=1}^{n} A_k}}=1-f_{\bigcup_{k=1}^{n} A_k}=1-\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\le i_1<\ldots<i_k\le n}^{n} f_{A_{i_1}}f_{A_{i_2}}\cdots f_{A_{i_k}}=\prod_{k=1}^{n}(1-f_{A_k})=\prod_{k=1}^{n} f_{\overline{A_k}}=f_{\bigcap_{k=1}^{n}\overline{A_k}}.$$

We prove in the same way the following theorem:

Theorem 14: (De Morgan) $\overline{\bigcap_{k=1}^{n} A_k}=\bigcup_{k=1}^{n}\overline{A_k}$.

Theorem 15: $\left(\bigcup_{k=1}^{n} A_k\right)\cap M=\bigcup_{k=1}^{n}\left(A_k\cap M\right)$.

Proof.

$$f_{\left(\bigcup_{k=1}^{n} A_k\right)\cap M}=f_{\bigcup_{k=1}^{n} A_k}f_M=\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\le i_1<\ldots<i_k\le n}^{n} f_{A_{i_1}}f_{A_{i_2}}\cdots f_{A_{i_k}}f_M=\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\le i_1<\ldots<i_k\le n}^{n} f_{A_{i_1}}f_{A_{i_2}}\cdots f_{A_{i_k}}f_M^{k}=$$

$$=\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\le i_1<\ldots<i_k\le n}^{n} f_{A_{i_1}\cap M}f_{A_{i_2}\cap M}\cdots f_{A_{i_k}\cap M}=f_{\bigcup_{k=1}^{n}(A_k\cap M)}$$

In the same way we prove that:

Theorem 16: $\left(\bigcap_{k=1}^{n} A_k\right)\cup M=\bigcap_{k=1}^{n}\left(A_k\cup M\right)$.

Theorem 17: $\left(\Delta_{k=1}^{n}A_k\right)\cap M=\Delta_{k=1}^{n}\left(A_k\cap M\right)$

Application.

$\left(\Delta_{k=1}^{n}A_k\right)\cup M=\Delta_{k=1}^{n}\left(A_k\cup M\right)$ if and only if $M=\Phi$.

Theorem 18: $M\times\left(\bigcup_{k=1}^{n} A_k\right)=\bigcup_{k=1}^{n}\left(M\times A_k\right)$

Proof.

$$f_{M\times\left(\bigcup_{k=1}^{n}A_k\right)}(x,y)=f_M(y)f_{\bigcup_{k=1}^{n}A_k}(x)=\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\le i_1<\ldots<i_k\le n}^{n}f_{A_{i_1}}(x)f_{A_{i_2}}(x)\ldots f_{A_{i_k}}(x)f_M(y)=$$

$$=\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\le i_1<\ldots<i_k\le n}^{n}f_{A_{i_1}}(x)f_{A_{i_2}}(x)\ldots f_{A_{i_k}}(x)f_M^k(y)=$$

$$=\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\le i_1<\ldots<i_k\le n}^{n}f_{A_{i_1}\times M}(x,y)\ldots f_{A_{i_k}\times M}(x,y)=f_{\bigcup_{k=1}^{n}(M\times A_k)}$$

In the same way we prove that:

Theorem 19: $M\times\left(\bigcap_{k=1}^{n}A_k\right)=\bigcap_{k=1}^{n}(M\times A_k)$.

Theorem 20: $M\times(A_1-A_2-\ldots-A_n)=(M\times A_1)-(M\times A_2)-\ldots-(M\times A_n)$.

Theorem 21: $(A_1-A_2)\cup(A_2-A_3)\cup\ldots\cup(A_{n-1}-A_n)\cup(A_n-A_1)=\bigcup_{k=1}^{n}A_k-\bigcap_{k=1}^{n}A_k$

Proof 1.

$$f_{(A_1-A_2)\cup\ldots\cup(A_n-A_1)}=\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\le i_1<\ldots<i_k\le n}^{n}f_{A_{i_1}-A_{i_2}}\cdots f_{A_{i_k}-A_{i_1}}=$$

$$=\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\le i_1<\ldots<i_k\le n}^{n}(f_{A_{i_1}}-f_{A_{i_2}}-f_{A_{i_1}}f_{A_{i_2}})\ldots(f_{A_{i_k}}-f_{A_{i_1}}-f_{A_{i_k}}f_{A_{i_1}})=$$

$$=\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\le i_1<\ldots<i_k\le n}^{n}f_{A_{i_1}}\cdots f_{A_{i_k}}\left(1-\prod_{p=1}^{n}f_{A_p}\right)=f_{\bigcup_{k=1}^{n}A_k}\left(1-f_{\bigcap_{k=1}^{n}A_k}\right)=f_{\bigcup_{k=1}^{n}A_k-\bigcap_{k=1}^{n}A_k}.$$

Proof 2. Let's consider $x\in\bigcup_{i=1}^{n}(A_i-A_{i+1})$, (where $A_{n+1}=A_1$), then there exists k such that $x\in(A_k-A_{k+1})$, namely $x\notin(A_k\cap A_{k+1})\subset A_1\cap A_2\cap\ldots\cap A_n$, namely $x\notin A_1\cap A_2\cap\ldots\cap A_n$, and $x\in\bigcup_{k=1}^{n}A_k-\bigcap_{k=1}^{n}A_k$.

Now we prove the inverse statement:

Let's consider $x\in\bigcup_{k=1}^{n}A_k-\bigcap_{k=1}^{n}A_k$, we show that there exists k such that $x\in A_k$ and $x\notin A_{k+1}$. On the contrary, it would result that for any $k\in\{1,2,\ldots,n\}$, $x\in A_k$ and $x\in A_{k+1}$ namely $x\in\bigcup_{k=1}^{n}A_k$, it results that there exists p such that $x\in A_p$, but from the previous reasoning it results that $x\in A_{p+1}$, and using this we consequently obtain that $x\in A_k$ for $k=\overline{p,n}$. But from $x\in A_n$ we obtain that $x\in A_1$, therefore, it results that $x\in A_k$, $k=\overline{1,p}$, from where $x\in A_k$, $k=\overline{1,n}$, namely $x\in A_1\cap\ldots\cap A_n$, that is a contradiction. Thus there exists r such that $x\in A_r$ and $x\notin A_{r+1}$, namely $x\in(A_r-A_{r+1})$ and therefore $x\in\bigcup_{k=1}^{n}(A_k-A_{k+1})$.

In the same way we prove the following theorem:

Theorem 22: $(A_1 \Delta A_2) \cup (A_2 \Delta A_3) \cup \ldots \cup (A_{n-1} \Delta A_n) = \bigcup_{k=1}^{n} A_k - \bigcap_{k=1}^{n} A_k$.

Theorem 23:

$$(A_1 \times A_2 \times \ldots \times A_k) \cap (A_{k+1} \times A_{k+2} \times \ldots \times A_{2k}) \cap (A_n \times A_1 \times \ldots \times A_{k-1}) = (A_1 \cap A_2 \cap \ldots \cap A_n)^k .$$

Proof. $f_{(A_1 \times \ldots \times A_k) \cap \ldots \cap (A_n \times A_1 \times \ldots \times A_{k-1})}(x_1, \ldots, x_n) =$

$= f_{A_1 \times \ldots \times A_k}(x_1, \ldots, x_n) \ldots f_{A_n \times \ldots \times A_{k-1}}(x_1, \ldots, x_n) =$

$= \left(f_{A_1}(x_1) \ldots f_{A_k}(x_k) \right) \ldots \left(f_{A_n}(x_n) \ldots f_{A_{k-1}}(x_{k-1}) \right) =$

$= f_{A_1}^{k}(x_1) \ldots f_{A_n}^{k}(x_n) = f_{A_1 \cap \ldots \cap A_n}^{k}(x_1, \ldots, x_n) =$

$= f_{(A_1 \cap \ldots \cap A_n)^k}(x_1, \ldots, x_n)$.

Theorem 24. $(P(E), \cup)$ is a commutative monoid.

Proof. For any $A, B \in P(E)$; $A \cup B \in P(E)$, namely the intern operation. Because $(A \cup B) \cup C = A \cup (B \cup C)$ is associative, $A \cup B = B \cup A$ commutative, and because $A \cup \varnothing = A$ then $\varnothing$ is the neutral element.

Theorem 25: $(P(E), \cap)$ is a commutative monoid.

Proof. For any $A, B \in P(E)$; $A \cap B \in P(E)$ namely intern operation. $(A \cap B) \cap C = A \cap (B \cap C)$ associative, $A \cap B = B \cap A$, commutative $A \cap E = A$, E is the neutral element.

Theorem 26: $(P(E), \Delta)$ is an abelian group.

Proof. For any $A, B \in P(E)$; $A \Delta B \in P(E)$, namely the intern operation. $A \Delta B = B \Delta A$ commutative. The proof of associativity is in the XII$^{\text{th}}$ grade manual as a problem. We'll prove it using the characteristic function of the set.

$f_{(A \Delta B) \Delta C} = 4 f_A f_B f_C - 2 f_A f_B + f_B f_C + f_C f_A + f_A + f_B + f_C = f_{A \Delta (B \Delta C)}$.

Because $A \Delta \varnothing = A$, $\varnothing$ is the neutral element and because $A \Delta A = \varnothing$; the symmetric element of A is A itself.

Theorem 27: $(P(E), \Delta, \cap)$ is a commutative Boole ring with a divisor of zero.

Proof. Because the previous theorem satisfies the commutative ring axioms, the first part of the theorem is proved. Now we prove that it has a divisor of zero. If $A \neq \varnothing$ and $B \neq \varnothing$ are two disjoint sets, then $A \cap B = \varnothing$, thus it has divisor of zero. From Theorem 17 we get that it is distributive for $n = 2$. Because for any $A \in P(E)$; $A \cap A = A$ and $A \Delta A = \varnothing$ it also satisfies the Boole-type axioms.

Theorem 28: Let's consider $H=\{f \mid f:E\to\{0,1\}\}$, then $(H,\oplus)$ is an abelian group, where $f_A\oplus f_B=f_A+f_B-2f_Af_B$ and $(P(E),\Delta)\cong(H,\oplus)$.

Proof. Let's consider $F:P(E)\to H$, where $f(A)=f_A$, then, from the previous theorem we get that it is bijective and because $F(A\Delta B)=f_{A\Delta B}=F(A)\oplus F(B)$ it is compatible.

Theorem 29: $card(A_1\Delta A_n)\le card(A_1\Delta A_2)+card(A_2\Delta A_3)+...+card(A_{n-1}\Delta A_n)$.

Proof. By induction. If $n=2$, then it is true, we show that for $n=3$ it is also true. Because $(A_1\cap A_2)\cup(A_2\cap A_3)\subseteq A_2\cup(A_1\cap A_3)$;

$card((A_1\cap A_2)\cup(A_2\cap A_3))\le card(A_2\cup(A_1\cap A_3))$ but

$card(M\cup N)=cardM+cardN-card(M\cap N)$, and thus

$cardA_2+card(A_1\cap A_3)-card(A_1\cap A_2)-card(A_2\cap A_3)\ge 0$, can be written as

$cardA_1+cardA_3-2card(A_1\cap A_3)\le$

$\le(cardA_1+cardA_2-2card(A_1\cap A_2))+(cardA_2+cardA_3-2card(A_2\cap A_3))$.

But because of

$(M\Delta N)=cardM+cardN-2card(M\cap N)$

then $card(A_1\Delta A_3)\le card(A_1\Delta A_2)+card(A_2\Delta A_3)$. The proof of this step of the induction relies on the above method.

Theorem 30: $(P^2(E),card(A\Delta B))$ is a metric space.

Proof. Let $d(A,B)=card(A\Delta B):P(E)\times P(E)\to\mathbb{R}$

1. $d(A,B)=0\Leftrightarrow card(A\Delta B)=0\Leftrightarrow card((A-B)\cup(B-A))=0$ but because $(A-B)\cap(B-A)=\varnothing$ we obtain $(A-B)+card(B-A)=0$ and because $(A-B)=0$ and $card(B-A)=0$, then $A-B=\varnothing$, $B-A=\varnothing$, and $A=B$.

2. $d(A,B)=d(B,A)$ results from $A\Delta B=B\Delta A$.

3. As a consequence of the previous theorem $d(A,C)\le d(A,B)+d(B,C)$.

As a result of the above three properties it is a metric space.

PROBLEMS

Problem 1.

Let's consider $A=B\cup C$ and $f:P(A)\to P(A)\times P(A)$, where $f(x)=(X\cup B,X\cup C)$. Prove that f is injective if and only if $B\cap C=\varnothing$.

Solution 1. If f is injective. Then

$f(\varnothing)=(\varnothing\cup B,\varnothing\cup C)=(B,C)=((B\cap C)\cup B,(B\cap C)\cup C)=f(B\cap C)$ from which we obtain $B\cap C=\varnothing$. Now reciprocally: Let's consider $B\cap C=\varnothing$, then $f(X)=f(Y)$; it results that $X\cup B=Y\cup B$ and $X\cup C=Y\cup C$ or

$X = X\cup\varnothing = X\cup\left(B\cap C\right) = (X\cup B)\cap(X\cup C) = (Y\cup B)\cap(Y\cup C) = Y\cup(B\cap C) = Y\cup\varnothing = Y$
namely it is injective.

Solution 2. Let's consider $B\cap C=\varnothing$ passing over the set function $f(X)=f(Y)$ if and only if $X\cup B=Y\cup B$ and $X\cup C=Y\cup C$, namely $f_{X\cup B}=f_{Y\cup B}$ and $f_{X\cup C}=f_{Y\cup C}$ or $f_X+f_B-f_Xf_B=f_Y+f_B-f_Yf_B$ and $f_X+f_C-f_Xf_C=f_Y+f_C-f_Yf_C$ from which we obtain $\left(f_X-f_Y\right)\left(f_B-f_C\right)=0$.

Because $A=B\cup C$ and $B\cap C=\varnothing$, we have

$$\left(f_B-f_C\right)(u)=\begin{cases}1, & \text{if } u\in B\\ -1, & \text{if } u\in C\end{cases}\neq 0$$

therefore $f_X-f_Y=0$, namely $X=Y$ and thus it is injective.

Generalization. Let $M=\bigcup_{k=1}^{n}A_k$ and $f:P(A)\to P^n(A)$, where

$f(X)=(X\cup A_1, X\cup A_2,\dots,X\cup A_n)$.

Prove that f is injective if and only if $A_1\cap A_2\cap\dots\cap A_n=\varnothing$.

Problem 2. Let $E\neq\varnothing$, $A\in P(E)$, and $f:P(E)\to P(E)\times P(E)$, where $f(X)=(X\cap A, X\cup A)$.

a. Prove that f is injective
b. Prove that $\{f(x), x\in P(E)\}=\{(M,N)\,|\,M\subset A\subset N\subset E\}=K$.
c. Let $g:P(E)\to K$, where $g(X)=f(X)$. Prove that g is bijective and compute its inverse.

Solution.

a. $f(X)=f(Y)$, namely $(X\cap A, X\cup A)=\left(Y\cap A, Y\cup A\right)$ and then $X\cap A=Y\cap A$, $X\cup A=Y\cup A$, from where $X\Delta A=Y\Delta A$ or $\left(X\Delta A\right)\Delta A=\left(Y\Delta A\right)\Delta A$, $X\Delta(A\Delta A)=Y\Delta(A\Delta A)$, $X\Delta\varnothing=Y\Delta\varnothing$ and thus $X=Y$, namely f is injective.

b. $\{f(X), X\in P(E)\}=f(P(E))$. We'll show that $f(P(E))\subset K$. For any $(M,N)\in f(P(E))$, $\exists\, X\in P(E): f(X)=(M,N)$; $(X\cap A, X\cup A)=(M,N)$.

From here $X\cap A=M$, $X\cup A=N$, namely $M\subset A$ and $A\subset N$ thus $M\subset A\subset N$, and, therefore $(M,N)\in X$.

Now, we'll show that $K\subset f(P(E))$, for any $(M,N)\in K$, $\exists\, X\in P(E)$ such that $f(X)=(M,N)$. $f(X)=(M,N)$, namely $(X\cap A, X\cup A)=(M,N)$ from where $X\cap A=M$ and $X\cup A=N$, namely $X\Delta A=N-M$, $\left(X\Delta A\right)\Delta A=(N-M)\Delta A$, $X\Delta\varnothing=(N-M)\Delta A$,

$X=(N-M)\Delta A$, $X=(N\cap\overline{M})\Delta A$,

$X=\left((N\cap\overline{M})-A\right)\cup\left(A-(N\cap\overline{M})\right)=\left((N\cap\overline{M})\cap A\right)\cup\left(A\cap(\overline{N\cap\overline{M}})\right)=$

$=\left(N\cap(\overline{M}\cap\overline{A})\right)\cup\left(A\cap(N\cap\overline{M})\right)=(N\cap\overline{A})\cup\left((A\cap\overline{N})\cup(A\cap M)\right)=$

$= (N \cap \overline{A}) \cup (\varnothing \cup M) = (N - A) \cup M$.

From here we get the unique solution: $X = (N - A) \cup M$.

We test $f\left((N - A) \cup M\right) = \left(((N - A) \cup M) \cap A, ((N - A) \cup M) \cup A\right)$

but

$$\left((N - A) \cup M\right) \cap A = \left((N \cap \overline{A}) \cup M\right) \cap A = \left((N \cap \overline{A}) \cap A\right) \cup (M \cap A) =$$

$$= \left((N \cap (\overline{A} \cap A)\right) \cup M = (N \cap \varnothing) \cup M = \varnothing \cup M = M$$

and

$$\left((N - A) \cup M\right) \cup A = (N - A) \cup (M \cup A) = (N - A) \cup A = (N \cap \overline{A}) \cup A =$$

$$= (N \cup A) \cap (\overline{A} \cup A) = N \cap E = N, \ f\left((N - A) \cup M\right) = (M, N).$$

Thus $f\left(P(E)\right) = K$.

c. From point a. we have that g is injective, from point b. we have that g surjective, thus g is bijective. The inverse function is:

$$g^{-1}(M, N) = (N - A) \cup M .$$

Problem 3. Let $E \neq \varnothing$, $A, B \in P(E)$ and $f : P(E) \to P(E) \times P(E)$, where $f(X) = (X \cap A, X \cap B)$.

a. Give the necessary and sufficient condition such that f is injective.

b. Give the necessary and sufficient condition such that f is surjective.

c. Supposing that f is bijective, compute its inverse.

Solution.

a. Suppose that f is injective. Then:

$$f(A \cup B) = \left((A \cup B) \cap A, \ (A \cup B) \cap B\right) = (A, B) = \left(E \cap A, \ E \cap B\right) = f(E),$$

from where $A \cup B = E$.

Now we suppose that $A \cup B = E$, it results that:

$$X = X \cap E = X \cap (A \cup B) = (X \cap A) \cup (X \cap B) = (Y \cap A) \cup (Y \cap B) = Y \cap (A \cup B) = Y \cap E = Y$$

namely from $f(X) = f(Y)$ we obtain that $X = Y$, namely f is injective.

b. Suppose that f is surjective, for any $M, N \in P(A) \times P(B)$, there exists

$X \in P(E), \ f(X) = (M, \ N), \ \left(X \cap A, \ X \cap B\right) = (M, \ N), \ X \cap A = M, \ X \cap B = N$.

In special cases $(M, \ N) = (A, \ \varnothing)$, there exists $X \in P(E)$, from

$X \supset A, \ \varnothing = X \cap B \supset A \cap B, \ A \cap B = \varnothing$.

Now we suppose that $A \cap B = \varnothing$ and show that it is surjective.

Let $\left(M, N\right) \in P(A) \times P(B)$, then $M \subset A, \ N \subset B$, $M \cap B \subset A \cap B = \varnothing$, and $N \cap A \subset B \cap A = \varnothing$, namely $M \cap B = \varnothing$, $N \cap A = \varnothing$ and

$f\left(M \cup N\right) = \left((M \cup N) \cap A, \ (M \cup N) \cap B\right) =$

$= \left((M \cap A) \cup (N \cap A), \ (M \cap B) \cup (N \cap B)\right) = \left(M \cup \varnothing, \ \varnothing \cup N\right) = \left(M, \ N\right),$

for any (M, N) there exists $X = M \cup N$ such that $f(X) = (M, N)$, namely f is surjective.

c. We'll show that $f^{-1}((M, N)) = M \cup N$.

Remark. In the previous two problems we can use the characteristic function of the set as in the first problem. We leave this method for the readers.

Application. Let $E \neq \varnothing$, $A_k \in P(E)$ $(k = 1, ..., n)$ and $f : P(E) \to P^n(E)$, where $f(X) = (X \cap A_1, X \cap A_2, ..., X \cap A_n)$.

Prove that f is injective if and only if $\bigcup_{k=1}^{n} A_k = E$.

Application. Let $E \neq \varnothing$, $A_k \in P(E)$, $(k = 1, ..., n)$ and $f : P(E) \to P^n(E)$, where $f(X) = (X \cap A_1, X \cap A_2, ..., X \cap A_n)$.

Prove that f is surjective if and only if $\bigcap_{k=1}^{n} \overline{A}_k = \varnothing$.

Problem 4. We name the set M convex if for any $x, y \in M$ $tx + (1-t)y \in M$, for any $t \in [0,1]$.

Prove that if A_k, $(k = 1, ..., n)$ are convex sets, then $\bigcap_{k=1}^{n} A_k$ is also convex.

Problem 5. If A_k, $(k = 1, ..., n)$ are convex sets, then $\bigcap_{k=1}^{n} A_k$ is also convex.

Problem 6. Give the necessary and sufficient condition such that if A, B are convex/concave sets, then $A \cup B$ is also convex/concave. Generalization for the $\mathbb{N}$ set.

Problem 7. Give the necessary and sufficient condition such that if A, B are convex/concave sets then $A \Delta B$ is also convex/concave. Generalization for the $\mathbb{N}$ set.

Problem 8. Let $f, g : P(E) \to P(E)$, where $f(x) = A - X$, and

$g(x) = A \Delta X$, $A \in P(E)$.

Prove that f, g are bijective and compute their inverse functions.

Problem 9. Let $A \circ B = \{(x, y) \in \square \times \square \mid \exists z \in \square : (x, z) \in A \text{ and } (z, y) \in B\}$. In a particular case let $A = \{(x, \{x\}) \mid x \in \square\}$ and $B = \{(\{y\}, y) \mid y \in \square\}$.

Represent the $A \circ A$, $B \circ A$, $B \circ B$ cases.

Problem 10.

i. If $A \cup B \cup C = D$, $A \cup B \cup D = C$, $A \cup C \cup D = B$, $B \cup C \cup D = A$, then $A = B = C = D$

ii. Are there different A, B, C, D sets such that $A \cup B \cup C = A \cup B \cup D = A \cup C \cup D = B \cup C \cup D$?

Problem 11. Prove that $A \Delta B = A \cup B$ if and only if $A \cap B = \varnothing$.

Problem 12. Prove the following identity.

$$\bigcap_{i,j=1,i<j}^{n} A_k \cup A_j = \bigcup_{i=1}^{n} \left(\bigcap_{j=1,j\neq i}^{n} A_j \right)$$

Problem 13. Prove the following identities.

$$(A \cup B) - (B \cap C) = (A - (B \cap C)) \cup (B - C) = (A - B) \cup (A - C) \cup (B - C)$$

and

$$A - [(A \cap C) - (A \cap B)] = (A - \overline{B}) \cup (A - C).$$

Problem 14. Prove that $A \cup (B \cap C) = (A \cup B) \cap C = (A \cup C) \cap B$ if and only if $A \subset B$ and $A \subset C$.

Problem 15. Prove the following identities:

$$(A - B) - C = (A - B) - (C - B),$$
$$(A \cup B) - (A \cup C) = B - (A \cap C),$$
$$(A \cap B) - (A \cap C) = (A \cap B) - C.$$

Problem 16. Solve the following system of equations:

$$\begin{cases} A \cup X \cup Y = (A \cup X) \cap (A \cup Y) \\ A \cap X \cap Y = (A \cap X) \cup (A \cap Y) \end{cases}.$$

Problem 17. Solve the following system of equations:

$$\begin{cases} A \Delta X \Delta B = A \\ A \Delta Y \Delta B = B \end{cases}.$$

Problem 18. Let $X, Y, Z \subseteq A$. Prove that:
$Z = (X \cap \overline{Z}) \cup (Y \cap \overline{Z}) \cup (\overline{X} \cap Z \cap \overline{Y})$ if and only if $X = Y = \varnothing$.

Problem 19. Prove the following identity:

$$\bigcup_{k=1}^{n} [A_k \cup (B_k - C)] = \left(\bigcup_{k=1}^{n} A_k \right) \cup \left[\left(\bigcup_{k=1}^{n} A_k \right) - C \right].$$

Problem 20. Prove that: $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$.

Problem 21. Prove that:

$$(A\Delta B)\Delta C=\left(A\cap\overline{B}\cap\overline{C}\right)\cup\left(\overline{A}\cap B\cap\overline{C}\right)\cup\left(\overline{A}\cap\overline{B}\cap C\right)\cup\left(A\cap B\cap C\right).$$

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