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Convergence of A Family of Series

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CONVERGENCE OF A FAMILY OF SERIES

In this article we will construct a family of expressions $\mathcal{E}(n)$. For each element $E(n)$ from $\mathcal{E}(n)$, the convergence of the series $\sum_{n \geq n_E} E(n)$ could be determined in accordance to the theorems from this article.

This article gives also applications.

(1) Preliminary

To render easier the expression, we will use the recursive functions. We will introduce some notations and notions to simplify and reduce the size of this article.

(2) Definitions: lemmas.

We will construct recursively a family of expressions $\mathcal{E}(n)$.
For each expression $E(n) \in \mathcal{E}(n)$, the degree of the expression is defined recursively and is denoted $d^0 E(n)$, and its dominant (leading) coefficient is denoted $c(E(n))$.

1. If a is a real constant, then $a \in \mathcal{E}(n)$.
   $d^0 a = 0$ and $c(a) = a$.
2. The positive integer $n \in \mathcal{E}(n)$.
   $d^0 n = 1$ and $c(n) = 1$.
3. If $E_1(n)$ and $E_2(n)$ belong to $\mathcal{E}(n)$ with $d^0 E_1(n) = r_1$ and $d^0 E_2(n) = r_2$, $c(E_1(n)) = a_1$ and $c(E_2(n)) = a_2$, then:
   a) $E_1(n)E_2(n) \in \mathcal{E}(n)$; $d^0(E_1(n)E_2(n)) = r_1 + r_2$; $c(E_1(n)E_2(n))$ which is $a_1 a_2$.
   b) If $E_2(n) \neq 0 \;\; \forall n \in \mathbb{N}(n \geq n_{E_2})$, then $\frac{E_1(n)}{E_2(n)} \in \mathcal{E}(n)$ and
   $$d^0\left(\frac{E_1(n)}{E_2(n)}\right) = r_1 - r_2, \quad c\left(\frac{E_1(n)}{E_2(n)}\right) = \frac{a_1}{a_2}.$$
   c) If α is a real constant and if the operation used is well defined, $\left(E_1(n)\right)^\alpha$ (for all $n \in N,\ n \geq n_{E_1}$), then:
   $\left(E_1(n)\right)^\alpha \in \mathcal{E}(n)$, $d^0\left(\left(E_1(n)\right)^\alpha\right) = r_1 \alpha$, $c\left(\left(E_1(n)\right)^\alpha\right) = a_1^\alpha$
   d) If $r_1 \neq r_2$, then $E_1(n) \pm E_2(n) \in \mathcal{E}(n)$, $d^0\left(E_1(n) \pm E_2(n)\right)$ is the max of r_1 and r_2, and $c\left(E_1(n) \pm E_2(n)\right) = a_1$, respectively a_2 resulting that the grade is r_1 and r_2.
   e) If $r_1 = r_2$ and $a_1 + a_2 \neq 0$, then $E_1(n) + E_2(n) \in \mathcal{E}(n)$, $d^0\left(E_1(n) + E_2(n)\right) = r_1$ and $c\left(E_1(n) + E_2(n)\right) = a_1 + a_2$.

f) If $r_1 = r_2$ and $a_1 - a_2 \neq 0$, then $E_1(n) - E_2(n) \in \mathcal{E}(n)$, $d^0\left(E_1(n) - E_2(n)\right) = r_1$ and $c\left(E_1(n) - E_2(n)\right) = a_1 - a_2$.

4. All expressions obtained by applying a finite number of step 3 belong to $\mathcal{E}(n)$.

Note 1. From the definition of $\mathcal{E}(n)$ it results that, if $E(n) \in \mathcal{E}(n)$ then $c\left(E(n)\right) \neq 0$, and that $c\left(E(n)\right) = 0$ if and only if $E(n) = 0$.

Lemma 1. If $E(n) \in \mathcal{E}(n)$ and $c\left(E(n)\right) > 0$, then there exists $n' \in \mathbb{N}$, such that for all $n > n'$, $E(n) > 0$.

Proof: Let's consider $c\left(E(n)\right) = a_1 > 0$ and $d^0\left(E(n)\right) = r$.

If $r > 0$, then $\lim\limits_{n\to\infty} E(n) = \lim\limits_{n\to\infty} n^r \dfrac{E(n)}{n^r} = \lim\limits_{n\to\infty} a_1 n^r = +\infty$, thus there exists $n' \in \mathbb{N}$ such that, for any $n > n'$ we have $E(n) > 0$.

If $r < 0$, then $\lim\limits_{n\to\infty} \dfrac{1}{E(n)} = \lim\limits_{n\to\infty} \dfrac{n^{-r}}{\dfrac{E(n)}{n^r}} = \dfrac{1}{a_1} \lim\limits_{n\to\infty} n^{-r} = +\infty$ thus there exists $n' \in \mathbb{N}$, such that for all $n > n'$, $\dfrac{1}{E(n)} > 0$ we have $E(n) > 0$.

If $r = 0$, then $E(n)$ is a positive real constant, or $\dfrac{E_1(n)}{E_2(n)} = E(n)$, with $d^0 E_1(n) = d^0 E_2(n) = r_1 \neq 0$, according to what we have just seen, $c\left(\dfrac{E_1(n)}{E_2(n)}\right) = \dfrac{c\left(E_1(n)\right)}{c\left(E_2(n)\right)} = c\left(E(n)\right) > 0$.

Then: $c\left(E_1(n)\right) > 0$ and $c\left(E_2(n)\right) < 0$: it results

$$\left.\begin{array}{l}\text{there exists } n_{E_1} \in \mathbb{N},\ \forall n \in \mathbb{N} \text{ and } n \geq n_{E_1},\ E_1(n) > 0 \\ \text{there exists } n_{E_2} \in \mathbb{N},\ \forall n \in \mathbb{N} \text{ and } n \geq n_{E_2},\ E_2(n) > 0\end{array}\right\} \Rightarrow$$

there exists $n_E = \max(n_{E_1}, n_{E_2}) \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_E,\ E(n) \dfrac{E_1(n)}{E_2(n)} > 0$

then $c\left(E_1(n)\right) < 0$ and $c\left(E_2(n)\right) < 0$ and it results:

$E(n) = \dfrac{E_1(n)}{E_2(n)} = \dfrac{-E_1(n)}{-E_2(n)}$ which brings us back to the precedent case.

Lemma 2: If $E(n) \in \mathcal{E}(n)$ and if $c\left(E(n)\right) < 0$, then it exists $n' \in \mathbb{N}$, such that for any $n > n'$, $E(n) < 0$.

Proof:

The expression $-E(n)$ has the propriety that $c\left(-E(n)\right)>0$, according to the recursive definition. According to lemma 1: there exists $n^{'}\in\mathbb{N}$, $n\geq n^{'}$, $-E(n)>0$, i.e. $+E(n)<0$, q. e. d.

Note 2. To prove the following theorem, we suppose known the criterion of convergence of the series and certain of its properties

(3) Theorem of convergence and applications.

Theorem: Let's consider $E(n)\in\mathcal{E}(n)$ with $d^{0}\left(E(n)\right)=r$ having the series

$$\sum_{n\geq n_{\varepsilon}} E(n)\,,\quad E(n)\not\equiv 0\,.$$

Then:

A) If $r<-1$ the series is absolutely convergent.
B) If $r\geq -1$ it is divergent where $E(n)$ is well defined $\forall n\geq n_E, n\in\mathbb{N}$.

Proof: According to lemmas 1 and 2, and because:

the series $\sum_{n\geq n_E} E(n)$ converge $\Leftrightarrow$ the series $-\sum_{n\geq n_E} E(n)$ converge,

we can consider the series $\sum_{n\geq n_E} E(n)$ like a series with positive terms.

We will prove that the series $\sum_{n\geq n_E} E(n)$ has the same nature as the series $\sum_{n\geq 1}\frac{1}{n^{-r}}$.

Let us apply the second criterion of comparison:

$$\lim_{n\to\infty}\frac{E(n)}{\frac{1}{n^{-r}}}=\lim_{n\to\infty}\frac{E(n)}{n^{r}}=c\left(E(n)\right)\neq\pm\infty\,.$$

According to the note 1 if $E(n)\not\equiv 0$ then $c\left(E(n)\right)\neq 0$ and then the series $\sum_{n\geq n_E} E(n)$ has the same nature as the series $\sum_{n\geq 1}\frac{1}{n^{-r}}$, i.e.:

A) If $r<-1$ then the series is convergent;
B) If $r>-1$ then the series is divergent;

For $r<-1$ the series is absolute convergent because it is a series with positive terms.

Applications:

We can find many applications of these. Here is an interesting one:

If $P_q(n)$, $R_s(n)$ are polynomials of n of degree q,s, and that $P_q(n)$ and $R_s(n)$ belong to $\mathcal{E}(n)$:

1) $$\sum_{n \ge n_{PR}} \frac{\sqrt[k]{P_q(n)}}{\sqrt[h]{R_s(n)}} \quad \text{is} \quad \begin{cases} \text{convergent, if } s/h - q/k > 1 \\ \text{divergent, } \quad \text{if } s/h - q/k \le 1 \end{cases}$$

2) $$\sum_{n \ge n_R} \frac{1}{R_s(n)} \quad \text{is} \quad \begin{cases} \text{convergent, if } s > 1 \\ \text{divergent, } \quad \text{if } s \le 1 \end{cases}$$

Example: The series $\sum_{n \ge 2} \frac{\sqrt[2]{n+1} \cdot \sqrt[3]{n-7} + 2}{\sqrt[5]{n^2} - 17}$ is divergent because $\frac{2}{5} - \left(\frac{1}{2} + \frac{1}{3}\right) < 1$ and if we call $E(n)$ the quotient of this series, $E(n)$ belongs to $\mathcal{E}(n)$ and it is well defined for $n \ge 2$.

References:

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