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DSm field and linear algebra of refined labels

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Abstract: *This chapter presents the DSm Field and Linear Algebra of Refined Labels (FLARL) in DSmT framework in order to work precisely with qualitative labels for information fusion. We present and justify the basic operators on qualitative labels (addition, subtraction, multiplication, division, root, power, etc).*

2.1 Introduction

Definitions of group, field, algebra, vector space, and linear algebra used in this paper can be found in [1, 2, 4]. Let L_1, L_2, ..., L_m be labels, where $m \geq 1$ is an integer. We consider a relation of order defined on these labels which can be "smaller", "less in quality", "lower", etc., $L_1 < L_2 <, \ldots, < L_m$. Let's extend this set of labels with a minimum label L_0, and a maximum label L_{m+1}. In the case when the labels are equidistant, i.e. the qualitative distance between any two consecutive labels is the same, we get an exact qualitative result, and a qualitative basic belief assignment (bba) is considered normalized if the sum of all its qualitative masses is equal to $L_{\max} = L_{m+1}$. If the labels are not equidistant, we still can use all qualitative operators defined in the FLARL, but the qualitative result is approximate, and a qualitative bba is considered quasi-normalized if the sum of all its masses is equal to $L_{\max}$. Connecting them to the classical interval $[0, 1]$, we have:

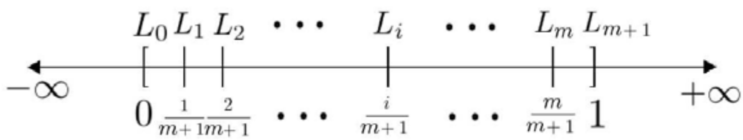

Figure 2.1: Ordered set of labels in $[0, 1]$.

So, $0 \equiv L_0 < L_1 < L_2 < \ldots < L_i < \ldots < L_m < L_{m+1} \equiv 1$, and $L_i = \frac{i}{m+1}$ for $i \in \{0, 1, 2, \ldots, m, m+1\}$.

1. **Ordinary labels**: The set of labels $\tilde{L} \triangleq \{L_0, L_1, L_2, \ldots, L_i, \ldots, L_m, L_{m+1}\}$ whose indexes are positive integers between 0 and $m+1$, is called the set of *1-Tuple labels*. We call a set of labels to be *equidistant labels*, if the geometric distance between any two consecutive labels is the same, i.e. $L_{i+1} - L_i =$ Constant for any i.

   And, the opposite definition: a set of labels is of *non-equidistant labels* if the distances between consecutive labels is not the same, i.e. there exists $i \neq j$ such that $L_{i+1} - L_i \neq L_{j+1} - L_j$.

   For simplicity and symmetry of the calculations, we further consider the case of equidistant labels. But the same procedures can *approximately* work for non-equidistant labels.

   This set of 1-Tuple labels is isomorphic with the numerical set $\{\frac{i}{m+1}, i = 0, 1, \ldots, m+1\}$ through the isomorphism $f_{\tilde{L}}(L_i) = \frac{i}{m+1}$.

2. **Refined labels**: We theoretically extend the set of labels $\tilde{L}$ to the left and right sides of the interval $[0, 1]$ towards $-\infty$ and respectively $+\infty$. So, we define:

$$L_{\mathbb{Z}} \triangleq \{\frac{j}{m+1}, j \in \mathbb{Z}\}$$

where $\mathbb{Z}$ is the set of all positive and negative integers (zero included).

Thus:

$$L_{\mathbb{Z}} = \{\ldots, L_{-j}, \ldots, L_{-2}, L_{-1}, L_0, L_1, L_2, \ldots, L_j, \ldots\} = \{L_j, j \in \mathbb{Z}\},$$

i.e. the set of extended labels with positive and negative indexes.

Similarly, one defines $L_{\mathbb{Q}} \triangleq \{L_q, q \in \mathbb{Q}\}$ as the set of labels whose indexes are fractions. $L_{\mathbb{Q}}$ is isomorphic with $\mathbb{Q}$ through the isomorphism $f_{\mathbb{Q}}(L_q) = \frac{q}{m+1}$ for any $q \in \mathbb{Q}$.

Even more general, we can define:

$$L_{\mathbb{R}} \triangleq \{\frac{r}{m+1}, r \in \mathbb{R}\}$$

where $\mathbb{R}$ is the set of all real numbers. $L_{\mathbb{R}}$ is isomorphic with $\mathbb{R}$ through the isomorphism $f_{\mathbb{R}}(L_r) = \frac{r}{m+1}$ for any $r \in \mathbb{R}$.

2.2 DSm field and linear algebra of refined labels

We will prove that $(L_{\mathbb{R}}, +, \times)$ is a field, where $+$ is the vector addition of labels, and $\times$ is the vector multiplication of labels which is called the *DSm field of refined labels*. Therefore, for the first time we introduce decimal or refined labels, i.e. labels whose index is decimal. For example: $L_{\frac{3}{2}}$ which is $L_{1.5}$ means a label in the middle of the label interval $[L_1, L_2]$. We also theoretically introduce *negative* labels, L_{-i} which is equal to $-L_i$, that occur in qualitative calculations.

Even more, $(L_{\mathbb{R}}, +, \times, \cdot)$, where $\cdot$ means scalar product, is a commutative linear algebra over the field of real numbers $\mathbb{R}$, with unit element, and whose each non-null element is invertible with respect to the multiplication of labels. This is called the *DSm Linear Algebra of Refined labels* (DSm-LARL for short).

2.2.1 Qualitative operators on DSm-LARL

Let's define the *qualitative operators* on this linear algebra. Let a, b, c in $\mathbb{R}$, and the labels $L_a = \frac{a}{m+1}$, $L_b = \frac{b}{m+1}$ and $L_c = \frac{c}{m+1}$. Let the scalars α, β in $\mathbb{R}$.

- **Vector Addition (addition of labels)**:

$$L_a + L_b = L_{a+b} \tag{2.1}$$

since $\frac{a}{m+1} + \frac{b}{m+1} = \frac{a+b}{m+1}$.

- **Vector Subtraction (subtraction of labels)**:

$$L_a - L_b = L_{a-b} \tag{2.2}$$

since $\frac{a}{m+1} - \frac{b}{m+1} = \frac{a-b}{m+1}$.

- **Vector Multiplication (multiplication of labels)**:

$$L_a \times L_b = L_{(ab)/(m+1)} \tag{2.3}$$

since $\frac{a}{m+1} \cdot \frac{b}{m+1} = \frac{(ab)/(m+1)}{m+1}$.

- **Scalar Multiplication (number times label)**:

$$\alpha \cdot L_a = L_a \cdot \alpha = L_{\alpha a} \tag{2.4}$$

since $\alpha \cdot \frac{a}{m+1} = \frac{\alpha a}{m+1}$.
As a particular case, for $\alpha = -1$, we get: $-L_a = L_{-a}$.
Also, $\frac{L_a}{\beta} = L_a \div \beta = \frac{1}{\beta} \cdot L_a = L_{\frac{a}{\beta}}$.

- **Vector Division (division of labels)**:

$$L_a \div L_b = L_{(a/b)(m+1)} \tag{2.5}$$

since $\frac{a}{m+1} \div \frac{b}{m+1} = \frac{a}{b} = \frac{(a/b)(m+1)}{m+1}$.

- **Scalar Power**:

$$(L_a)^p = L_{a^p/(m+1)^{p-1}} \tag{2.6}$$

since $(\frac{a}{m+1})^p = \frac{a^p/(m+1)^{p-1}}{m+1}$, $\forall p \in \mathbb{R}$.

- **Scalar Root**:

$$\sqrt[k]{L_a} = (L_a)^{\frac{1}{k}} = L_{a^{\frac{1}{k}}/(m+1)^{\frac{1}{k}-1}} \tag{2.7}$$

which results from replacing $p = \frac{1}{k}$ in the power formula (2.6), $\forall k$ integer ≥ 2.

2.2.2 The DSm field of refined labels

Since $(L_{\mathbb{R}}, +, \times)$ is isomorphic with the set of real numbers $(\mathbb{R}, +, \times)$, it results that $(L_{\mathbb{R}}, +, \times)$ is a field, called *DSm field of refined labels*. The field isomorphism: $f_{\mathbb{R}} : L_{\mathbb{R}} \to \mathbb{R}$, $f_{\mathbb{R}}(L_r) = \frac{r}{m+1}$ satisfies the axioms:

Axiom A1:

$$f_{\mathbb{R}}(L_a + L_b) = f_{\mathbb{R}}(L_a) + f_{\mathbb{R}}(L_b) \tag{2.8}$$

since $f_{\mathbb{R}}(L_a+L_b) = f_{\mathbb{R}}(L_{a+b}) = \frac{a+b}{m+1}$ and $f_{\mathbb{R}}(L_a)+f_{\mathbb{R}}(L_b) = \frac{a}{m+1}+\frac{b}{m+1} = \frac{a+b}{m+1}$.

Axiom A2:

$$f_{\mathbb{R}}(L_a \times L_b) = f_{\mathbb{R}}(L_a) \cdot f_{\mathbb{R}}(L_b) \tag{2.9}$$

since $f_{\mathbb{R}}(L_a \times L_b) = f_{\mathbb{R}}(L_{(ab)/(m+1)}) = \frac{ab}{m+1}$ and $f_{\mathbb{R}}(L_a) \cdot f_{\mathbb{R}}(L_b) = \frac{a}{m+1} \cdot \frac{b}{m+1} = \frac{ab}{(m+1)^2}$.

$(L_{\mathbb{R}}, +, \cdot)$ is a *vector space of refined labels* over the field of real numbers $\mathbb{R}$, since $(L_{\mathbb{R}}, +)$ is a commutative group, and the scalar multiplication (which is an external operation) verifies the axioms:

Axiom B1:

$$1 \cdot L_a = L_{1 \cdot a} = L_a \tag{2.10}$$

Axiom B2:

$$(\alpha \cdot \beta) \cdot L_a = \alpha \cdot (\beta \cdot L_a) \tag{2.11}$$

since both, left and right sides, are equal to $L_{\alpha\beta a}$

Axiom B3:

$$\alpha \cdot\ (L_a + L_b) = \alpha \cdot\ L_a + \alpha \cdot\ L_b \tag{2.12}$$

since $\alpha \cdot\ (L_a + L_b) = \alpha \cdot L_{a+b} = L_{\alpha(a+b)} = L_{\alpha a+\alpha b} = L_{\alpha a} + L_{\alpha b} = \alpha \cdot L_a + \alpha \cdot L_b$.

Axiom B4:

$$(\alpha + \beta) \cdot\ L_a = \alpha \cdot\ L_a + \beta \cdot\ L_a \tag{2.13}$$

since $(\alpha + \beta) \cdot\ L_a = L_{(\alpha+\beta)a} = L_{\alpha a+\beta a} = L_{\alpha a} + L_{\beta a} = \alpha \cdot\ L_a + \beta \cdot\ L_a$.

$(L_{\mathbb{R}}, +, \times, \cdot)$ is a *a Linear Algebra of Refined Labels* over the field $\mathbb{R}$ of real numbers, called *DSm Linear Algebra of Refined Labels* (DSm-LARL for short), which is commutative, with identity element (which is L_{m+1}) for vector multiplication, and whose non-null elements (labels) are invertible with respect to the vector multiplication. This occurs since $(L_{\mathbb{R}}, +, \cdot)$ is a vector space, $(L_{\mathbb{R}}, \times)$ is a commutative group, the set of scalars $\mathbb{R}$ is well-known as a field, and also one has:

- The vector multiplication is associative:

  Axiom C1 (Associativity of vector multiplication):

$$L_a \times (L_b \times L_c) = (L_a \times L_b) \times L_c \tag{2.14}$$

  since $L_a \times (L_b \times L_c) = L_a \times L_{(b\cdot c)/(m+1)} = L_{(a\cdot b\cdot c)/(m+1)^2}$ while $(L_a \times L_b) \times L_c = L_{(ab)/(m+1)} \times L_c = L_{(a\cdot b\cdot c)/(m+1)^2}$ as well.

- The vector multiplication is distributive with respect to addition:

  Axiom C2:

$$L_a \times (L_b + L_c) = L_a \times L_b + L_a \times L_c \tag{2.15}$$

  since $L_a \times (L_b + L_c) = L_a \times L_{b+c} = L_{(a\cdot(b+c))/(m+1)}$ and $L_a \times L_b + L_a \times L_c = L_{(ab)/(m+1)} + L_{(ac)/(m+1)} = L_{(ab+ac)/(m+1)} = L_{(a(b+c))/(m+1)}$.

  Axiom C3:

$$(L_a + L_b) \times L_c = L_a \times L_c + L_b \times L_c \tag{2.16}$$

  since $(L_a + L_b) \times L_c = L_{a+b} \times L_c = L_{((a+b)c))/(m+1)} = L_{(ac+bc)/(m+1)} = L_{(ac)/(m+1)+(bc)/(m+1)} = L_{(ac)/(m+1)} + L_{(bc)/(m+1)} = L_a \times L_c + L_b \times L_c$.

  Axiom C4:

$$\alpha \cdot (L_a \times L_b) = (\alpha \cdot L_a) \times L_b = L_a \times (\alpha \cdot L_b) \tag{2.17}$$

  since $\alpha \cdot (L_a \times L_b) = \alpha \cdot L_{(ab)/(m+1)} = L_{(\alpha ab)/(m+1)} = L_{((\alpha a)b)/(m+1)} = L_{\alpha a} \times L_b = (\alpha \cdot L_a) \times L_b$; but also $L_{(\alpha ab)/(m+1)} = L_{(a(\alpha b))/(m+1)} = L_a \times L_{\alpha b} = L_a \times (\alpha \cdot L_b)$.

- The *Unitary Element* for vector multiplication is L_{m+1}, since

  Axiom D1:

$$L_a \times L_{m+1} = L_{m+1} \times L_a = L_{(a(m+1))/(m+1)} = L_a, \forall a \in \mathbb{R}. \tag{2.18}$$

- All $L_a \neq L_0$ are *invertible* with respect to vector multiplication and the inverse of L_a is $(L_a)^{-1}$ with:

  Axiom E1:

$$(L_a)^{-1} = L_{(m+1)^2/a} = \frac{1}{L_a} \tag{2.19}$$

  since $L_a \times (L_a)^{-1} = L_a \times L_{(m+1)^2/a} = L_{(a\cdot(m+1)^2/a)/(m+1)} = L_{m+1}$.

Therefore the DSm linear algebra is a *Division Algebra*. DSm Linear Algebra is also a trivial Lie Algebra since we can define a law:

$$(L_a, L_b) \to [L_a, L_b] = L_a \times L_b - L_b \times L_a = L_0$$

such that

$$[L_a, L_a] = L_0 \tag{2.20}$$

and Jacobi identity is satisfied:

$$[L_a, [L_b, L_c]] + [L_b, [L_c, L_a]] + [L_c, [L_a, L_b]] = L_0 \tag{2.21}$$

Actually $(L_{\mathbb{R}}, +, \times, \cdot)$ is a field, and therefore in particular a ring, and any ring with the law: $[x, y] = xy - yx$ is a Lie Algebra.

We can extend the field isomorphism $f_{\mathbb{R}}$ to a linear algebra isomorphism by defining[1]: $f_{\mathbb{R}} : \mathbb{R} \cdot L_{\mathbb{R}} \to \mathbb{R} \cdot \mathbb{R}$ with $f_{\mathbb{R}}(\alpha \cdot L_{r_1}) = \alpha \cdot f_{\mathbb{R}}(L_{r_1})$ since $f_{\mathbb{R}}(\alpha \cdot L_{r_1}) = f_{\mathbb{R}}(L_{(\alpha \cdot r_1)}) = \alpha \cdot r_1/(m+1) = \alpha \cdot \frac{r_1}{m+1} = \alpha \cdot f_{\mathbb{R}}(L_{r_1})$. Since $(\mathbb{R}, +, \cdot)$ is a trivial linear algebra over the field of reals $\mathbb{R}$, and because $(L_{\mathbb{R}}, +, \cdot)$ is isomorphic with it through the above $f_{\mathbb{R}}$ linear algebra isomorphism, it results that $(L_{\mathbb{R}}, +, \cdot)$ is also a linear algebra which is associative and commutative.

2.3 New operators

Let's now define new operators, such as scalar-vector (mixed) addition, scalar-vector (mixed) subtraction, scalar-vector (mixed) division, vector power, and vector root.

They might be surprizing since such "strange" operators have not been defined in science, but for DSm linear Algebra they make perfect sense since $(L_{\mathbb{R}}, +, \times)$ is isomorphic to $(\mathbb{R}, +, \times)$ and a label is equivalent to a real number, since for a fixed $m \geq 1$ we have

$$\forall L_a \in L_{\mathbb{R}}, \exists! r \in \mathbb{R}, r = \frac{a}{m+1} \quad \text{such that} \quad L_a = r$$

and reciprocally

$$\forall r \in \mathbb{R}, \exists! L_a \in L_{\mathbb{R}}, L_a = L_{r(m+1)} \quad \text{such that} \quad r = L_a$$

In consequence, we can substitute a real number by a label, and reciprocally.

[1] where $\cdot$ denotes the scalar multiplication.

- **Scalar-vector (mixed) addition**:

$$L_a + \alpha = \alpha + L_a = L_{a+\alpha(m+1)} \tag{2.22}$$

since $L_a + \alpha = L_a + \frac{\alpha(m+1)}{(m+1)} = L_a + L_{\alpha(m+1)} = L_{a+\alpha(m+1)}$.

- **Scalar-vector (mixed) subtractions**:

$$L_a - \alpha = L_{a-\alpha(m+1)} \tag{2.23}$$

since $L_a - \alpha = L_a - \frac{\alpha(m+1)}{(m+1)} = L_a - L_{\alpha(m+1)} = L_{a-\alpha(m+1)}$.

$$\alpha - L_a = L_{\alpha(m+1)-a} \tag{2.24}$$

since $\alpha - L_a = \frac{\alpha(m+1)}{(m+1)} - L_a = L_{\alpha(m+1)} - L_a = L_{\alpha(m+1)-a}$.

- **Scalar-vector (mixed) divisions**:

$$L_a \div \alpha = \frac{L_a}{\alpha} = \frac{1}{\alpha} \cdot L_a = L_{\frac{a}{\alpha}}, \text{ for } \alpha \neq 0, \tag{2.25}$$

which is equivalent to the scalar multiplication $(\frac{1}{\alpha}) \cdot L_a$ where $\frac{1}{\alpha} \in \mathbb{R}$.

$$\alpha \div L_a = L_{\frac{\alpha(m+1)^2}{a}} \tag{2.26}$$

since $\alpha \div L_a = \frac{\alpha(m+1)}{m+1} \div L_a = L_{\alpha(m+1)} \div L_a = L_{\alpha(m+1)/a)\cdot(m+1)} = L_{\frac{\alpha(m+1)^2}{a}}$.

- **Vector power**:

$$(L_a)^{L_b} = L_{a^{\frac{b}{m+1}}/(m+1)^{\frac{b-m-1}{m+1}}} \tag{2.27}$$

since $(L_a)^{L_b} = (L_a)^{\frac{b}{m+1}} = L_{a^{\frac{b}{m+1}}/(m+1)^{\frac{b}{m+1}-1}} = L_{a^{\frac{b}{m+1}}/(m+1)^{\frac{b-m-1}{m+1}}}$ where we replaced $p = \frac{b}{m+1}$ in the scalar product formula.

- **Vector root**:

$$\sqrt[L_b]{L_a} = L_{a^{\frac{m+1}{b}}/(m+1)^{\frac{m-b+1}{b}}} \tag{2.28}$$

since $\sqrt[L_b]{L_a} = (L_a)^{\frac{1}{L_b}} = (L_a)^{\frac{1}{b/(m+1)}} = (L_a)^{\frac{m+1}{b}} = L_{a^{\frac{m+1}{b}}/(m+1)^{\frac{m+1}{b}-1}} = L_{a^{\frac{m+1}{b}}/(m+1)^{\frac{m-b+1}{b}}}$.

$L_{\mathbb{R}}$ endowed with all these scalar and vector (addition, subtraction, multiplication, division, power, and root) operators becomes a powerful mathematical tool in the DSm field and simultaneously linear algebra of refined labels.

Therefore, if we want to work with only 1-Tuple labels (ordinary labels), in all these operators we set the restrictions that indexes are integers belonging to $\{0, 1, 2, \ldots, m, m+1\}$; if an index is less than 0 then we force it to be 0, and if greater than $m+1$ we force it to $m+1$.

2.4 Working with 2-tuple labels

For 2-Tuple labels defined by Herrera and Martinez [3], that have the form (L_i, σ_i^h) where i is an integer and σ_i^h is a remainder in $[-\frac{0.5}{m+1}, \frac{0.5}{m+1})$, we use the scalar addition (when $\sigma_i^h \geq 0$) or scalar subtraction (when $\sigma_i^h < 0$) as defined previously in order to transform a 2-Tuple label into a refined label and then we use all previous twelve operators defined in FLARL. Actually, $(L_i, \sigma_i^h) = L_i + \sigma_i^h$ and it doesn't matter if σ_i^h is positive, zero, or negative.

2.5 Working with interval of labels

Interval of labels (i.e. imprecise labels) in the DSm Linear Algebra are intervals of the form $[L_{r_1}, L_{r_2}]$, $[L_{r_1}, L_{r_2})$, $(L_{r_1}, L_{r_2}]$, (L_{r_1}, L_{r_2}) where $r_1, r_2 \in \mathbb{R}$ and $r_1 < r_2$. To observe that r_1 and r_2 can be positive, negative, zero, decimals, etc. For $r_1 = r_2$, the closed labeled interval $[L_{r_1}, L_{r_2}] \equiv L_{r_1} \equiv L_{r_2}$, while the other intervals are empty.

For intervals of labels or, more general, for sets of labels, we use the operations on sets (addition, subtraction, multiplication, division, power, root of sets) employed in working with imprecise information as proposed in [5].

2.6 Concluding remark

All previous four categories of labels: 1-Tuple labels, 2-Tuple labels, Imprecise labels, and specially refined labels can be enriched. Enrichment of a category of labels means that one take into account also the degree of confidence in each label (or in each interval of labels), as in statistics. For example, the refined labels $L_i(c_i)$ means that we are c_i percent confident in label L_i, where $c_i \in [0, 1]$. L_i and c_i are independent, which means that we apply all previous twelve qualitative operators on L_i's, while for the percentage c_i we can use quantitative operators such as min, max, average, etc. To remark that σ_i^h from Herrera-Martinez 2-Tuple labels is not independent from L_i that is associated and σ_i^h

can be interpreted as a refinement factor of L_i whereas c_i is interpreted as a confidence factor for L_i. Therefore, c_i from enriched labels is different from σ_i^h from 2-Tuple labels and they have totally different meanings. From the refined label model of qualitative beliefs and the previous operators, we are able to extend the DSm classic (DSmC) and the PCR5 numerical fusion rules proposed in Dezert-Smarandache Theory (DSmT) and all other numerical fusion rules from any fusion theory (DST, TBM, etc.) in the qualitative domain.

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