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Deducibility Theorems in Boolean Logic

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DEDUCIBILITY THEOREMS IN BOOLEAN LOGIC

ABSTRACT

In this paper we give two theorems from the Propositional Calculus of the Boolean Logic with their consequences and applications and we prove them axiomatically.

§1. THEOREMS, CONSEQUENCES

In the beginning I shall put forward the axioms of the Propositional Calculus.

I. a) $\vdash A \supset (B \supset A)$,

b) $\vdash (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$.

II. a) $\vdash A \wedge B \supset A$,

b) $\vdash A \wedge B \supset B$,

c) $\vdash (A \supset B) \supset ((A \supset C) \supset (A \supset B \wedge C))$.

III. a) $\vdash A \supset A \vee B$,

b) $\vdash B \supset A \vee B$,

c) $\vdash (A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$.

IV. a) $\vdash (A \supset B) \supset (\overline{B} \supset \overline{A})$,

b) $\vdash A \supset \overline{\overline{A}}$,

c) $\vdash \overline{\overline{A}} \supset A$.

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THEOREMS. If: $\vdash A_i \supset B_i, i = \overline{1,n}$**,** then

1) $\vdash A_1 \wedge A_2 \wedge ... \wedge A_n \supset B_1 \wedge B_2 \wedge ... \wedge B_n$,

2) $\vdash A_1 \vee A_2 \vee ... \vee A_n \supset B_1 \vee B_2 \vee ... \vee B_n$.

Proof:

It is made by complete induction. For $n = 1$: $\vdash A_1 \supset B_1$, which is true from the given hypothesis. For $n = 2$: hypotheses $\vdash A_1 \supset B_1$, $\vdash A_2 \supset B_2$; let's show that $\vdash A_1 \wedge A_2 \supset B_1 \wedge B_2$. We use the axiom II, c) replacing $A \to A_1 \wedge A_2$, $B \to B_1$, $C \to B_2$, it results:

(1) $\vdash (A_1 \wedge A_2 \supset B_1) \supset ((A_1 \wedge A_2 \supset B_2) \supset (A_1 \wedge A_2 \supset B_1 \wedge B_2))$.

We use the axiom II, a) replacing $A \to A_1$, $B \to A_2$; we have $\vdash A_1 \wedge A_2 \supset A_1$. But $\vdash A_1 \supset B_1$ (hypothesis) applying the syllogism rule, it results $\vdash A_1 \wedge A_2 \supset B_1$. Analogously, using the axiom II, b), we have $\vdash A_1 \wedge A_2 \supset B_2$. We know that $\vdash A_1 \wedge A_2 \supset B_i$, $i = 1,2$, are deducible, then applying in (I) inference rule twice, we have $\vdash A_1 \wedge A_2 \supset B_1 \wedge B_2$.

We suppose it's true for n; let's prove that for $n+1$ it is true. In $\vdash A_1 \wedge A_2 \supset B_1 \wedge B_2$ replacing $A_1 \to A_1 \wedge ... \wedge A_n$, $A_2 \to A_{n+1}$, $B_1 \to B_1 \wedge ... \wedge B_n$, $B_2 \to B_{n+1}$ and using induction hypothesis it results $\vdash A_1 \wedge ... \wedge A_n \wedge A_{n+1} \supset B_1 \wedge ... \wedge B_n \wedge B_{n+1}$ and item 1) from the Theorem is proved.

2) It is made by induction. For $n=1$; if $\vdash A_1 \supset B_1$, then of course $\vdash A_1 \supset B_1$. For $n=2$: if $\vdash A_1 \supset B_1$ and $\vdash A_2 \supset B_2$, then $\vdash A_1 \vee A_2 \supset B_1 \vee B_2$.

We use axiom III, c) replacing $A \to A_1$, $B \to A_2$, $C \to B_1 \vee B_2$ we get

(2) $\vdash (A_1 \supset B_1 \vee B_2) \supset ((A_2 \supset B_1 \vee B_2) \supset (A_1 \vee A_2 \supset B_1 \vee B_2))$.

Let's show that $\vdash A_1 \supset B_1 \vee B_2$. We use the axiom III, a) replacing $A \to B_1$, $B \to B_2$ we get $\vdash B_1 \supset B_1 \vee B_2$ and we know from the hypothesis $A_1 \quad B_1$. Applying the syllogism we get $\vdash A_1 \supset B_1 \vee B_2$.

In the axiom III, b) replacing $A \to B_1$, $B \to B_2$, we get $\vdash B_2 \supset B_1 \vee B_2$. But $\vdash A_2 \supset B_2$ (from the hypothesis), applying the syllogism we get $\vdash A_2 \supset B_1 \vee B_2$. Applying the inference rule twice in (2) we get $\vdash A_1 \vee A_2 \supset B_1 \vee B_2$.

Suppose it's true for n and let's show that for $n+1$ it is true. Replace in $\vdash A_1 \vee A_2 \supset B_1 \vee B_2$ (true formula if $\vdash A_1 \supset B_1$ and $\vdash A_2 \supset B_2$) $A_1 \to A_1 \vee ... \vee A_n$, $A_2 \to A_{n+1}$, $B_1 \to B_1 \vee ... \vee B_n$, $B_2 \to B_{n+1}$. From induction hypothesis it results $\vdash A_1 \vee ... \vee A_n \vee A_{n+1} \supset B_1 \vee ... \vee B_n \vee B_{n+1}$ and the theorem is proved.

CONSEQUENCES.

1°) If $\vdash A_i \supset B$, $i = \overline{1,n}$ then $\vdash A_1 \wedge ... \wedge A_n \supset B$.

2°) If $\vdash A_i \supset B$, $i = \overline{1,n}$, then $\vdash A_1 \vee ... \vee A_n \supset B$.

Proof: 1°) Using 1) from the theorem, we get

(3) $\vdash A_1 \wedge ... \wedge A_n \supset B \wedge ... \wedge B$ (n times).

In axiom II, a) we replace $A \to B$, $B \to B \wedge ... \wedge B$ ($n-1$ times), and we get

(4) $\vdash B \wedge ... \wedge B \supset B$ (n times).

From (3) and (4) by means of the syllogism rule we get $\vdash A_1 \wedge ... \wedge A_n \supset B$.

2°) Using 2) from theorem, we get $\vdash A_1 \vee ... \vee A_n \supset B \vee ... \vee B$ (n times).

LEMMA. $\vdash B \vee ... \vee B \supset B$ (n times), $n \geq 1$.

Proof:

It is made by induction. For $n=1$, obvious. For $n=2$: in axiom III, c) we replace $A \to B$, $C \to B$ and we get $\vdash (B \supset B) \supset ((B \supset B) \supset (B \vee B \supset B))$. Applying the inference rule twice we get $\vdash B \vee B \supset B$.

Suppose for n that the formula is deducible, let's prove that is for $n+1$.

We proved that $\vdash B \supset B$. In axiom III, c) we replace $A \to B \vee ... \vee B$ (n times), $C \to B$, and we get $\vdash (B \vee ... \vee B \supset B) \supset ((B \supset B) \supset (B \vee ... \vee B \supset B))$ (n times). Applying two times the interference rule, we get $\vdash B \vee ... \vee B \supset B$ ($n+1$ times) so lemma is proved.

From $\vdash A_1 \vee ... \vee A_n \supset B \vee ... \vee B$ (n times) and applying the syllogism rule, from lemma we get $\vdash A_1 \vee ... \vee A_n \supset B$.

3°) $\vdash A \wedge ... \wedge A \supset A$ (n times)

4°) $\vdash A \vee ... \vee A \supset A$ (n times).

Previously we proved, replacing in Consequence 1°) and 2°), $B \to A$. Analogously, the consequences are proven:

5°) If $\vdash A \supset B_i, i = \overline{1,n}$, then $\vdash A \supset B_1 \wedge ... \wedge B_n$.

6°) If $\vdash A \supset B_i, i = \overline{1,n}$, then $\vdash A \supset B_1 \vee ... \vee B_n$.

Analogously,

7°) $\vdash A \supset A \wedge ... \wedge A$ (n times)

8°) $\vdash A \supset A \vee ... \vee A$ (n times)

9°) $\vdash A_1 \wedge ... \wedge A_n \supset A_1 \vee ... \vee A_n$.

Proof:

Method I. It is initially proved by induction: $\vdash A_1 \wedge ... \wedge A_n \supset A_i$, $i = \overline{1,n}$ and 2) is applied from the Theorem.

Method II. It is proven by induction that: $\vdash A_\iota \supset A_1 \wedge ... \wedge A_n$, $i = \overline{1,n}$ and then 1) is applied from the Theorem.

10°) If $\vdash A_\iota \supset B_i$, $i = \overline{1,n}$, then $\vdash A_1 \wedge ... \wedge A_n \supset B_1 \vee ... \vee B_n$.

Proof:

Method I. Using 1) from the Theorem, it results:

(5) $\vdash A_1 \wedge ... \wedge A_n \supset B_1 \wedge ... \wedge B_n$

We apply the Consequence 9°) where we replace $A_i \to B_i$, $i = \overline{1,n}$ and results:

(6) $\vdash B_1 \wedge ... \wedge B_n \supset B_1 \vee ... \vee B_n$.

From (5) and (6), applying the syllogism rule we get 10°).

Method II. We firstly use the Consequence 9°) and then 2) from the Theorem and so we obtain the Consequence 10°).

§2. APPLICATIONS AND REMARKS ON THEOREMS

The theorems are used in order to prove the formulae of the shape:

$\vdash A_1 \wedge ... \wedge A_p \supset B_1 \wedge ... \wedge B_r$

$\vdash A_1 \vee ... \vee A_p \supset B_1 \vee ... \vee B_r$, where $p, r \in \mathbb{N}^*$

It is proven that $\vdash A_\iota \supset B_j$, i.e.

$\forall i \in \overline{1,p}$, $\exists j_0 \in \overline{1,r}$, $j_0 = j_0(i)$, $\vdash A_\iota \supset B_{j_0}$

and

$\forall j \in \overline{1,r}$, $\exists i_0 \in \overline{1,p}$, $i_0 = i_0(j)$, $\vdash A_{\iota_0} \supset B_j$.

EXAMPLES: The following formulas are deducible:

(i) $\vdash A \supset (A \vee B) \wedge (B \supset A)$,

(ii) $\vdash (A \wedge B) \vee C \supset A \vee B \vee C$,

(iii) $\vdash A \wedge C \supset A \vee C$.

Solution:

(i) We have $\vdash A \supset A \vee B$ and $\vdash A \supset (B \supset A)$ (axiom III, a) and I, a)) and according to 1) from Theorem it results (i).

(ii) From $\vdash A \supset (B \supset A)$, $\vdash A \wedge B \supset B$, $\vdash C \supset C$ and Theorem 1), we have (ii).

(iii) Method I. From $\vdash A \wedge C \supset A$, $\vdash A \wedge C \supset C$ and Theorem 2).
Method II. From $\vdash A \supset A \vee C$, $\vdash C \supset A \vee C$ and using Theorem 1).

REMARKS. 1) The reciprocals of Theorem 1) and 2) are not always true.

a) Counter-example for Theorem 1). The formula $\vdash A \wedge B \supset A \wedge A$ is deducible from axiom II, a), $\vdash A \wedge A \supset A$ (Consequence 7°) and the syllogism rule. But $\vdash A \supset A$ for all A, that the formula $B \supset A$ is not deducible, so the reciprocal of the Theorem 1) is false.

Counter-example for Theorem 2). The formula $\vdash A \vee A \supset A \vee B$ is deducible from Lemma, axiom III, a) and applying the syllogism rule. But $\vdash A \supset A$ for all A, that the formula $A \supset B$ is not deducible, so the reciprocal of Theorem 2) is false.

2) The reciprocals of Theorem 1) and 2) are not always true.

Counter-examples:

a) for Theorem 1): $\vdash A \supset A$ and $B \not\supset A$ results that $\vdash A \wedge B \supset A \wedge A$ so the reciprocal of Theorem 1) is false.
b) for Theorem 2): $\vdash A \supset A$ and $A \not\supset B$ results that $\vdash A \vee A \supset A \vee B$ so the reciprocal of Theorem 2) is false.

REFERENCES:

[1] P. S. NOVOKOV. Elemente de logică matematică, Editura Ştiinţifică, Bucureşti, 1966.
[2] H. FREUDENTHAL, Limbajul logicii matematice, Editura Tehnică, Bucureşti, 1973.

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