The Dual of a Theorem relative to the Orthocenter of a Triangle

Professor Ion Patrascu, National College "Buzeşti Brothers" Craiova - Romania
Professor Florentin Smarandache, University of New Mexico, Gallup, USA

In [1] we introduced the notion of Bobillier transversal relative to a point O in the plane of a triangle ABC; we use this notion in what follows.

We transform by duality with respect to a circle $\mathbb{C}(o, r)$ the following theorem relative to the orthocenter of a triangle.

Theorem 1. If ABC is a nonisosceles triangle, H its orthocenter, and $\mathrm{AA_1}, \mathrm{BB_1}, \mathrm{CC_1}$ are cevians of a triangle concurrent at point Q different from H, and M, N, P are the intersections of the perpendiculars taken from H on given cevians respectively, with BC, CA, AB, then the points M, N, P are collinear.

Proof. We note with $\alpha = m(\sphericalangle \mathrm{BAA_1})$; $\beta = m(\sphericalangle \mathrm{CBB_1})$, $\gamma = m(\sphericalangle \mathrm{ACC_1})$, see *Figure 1*. According to Ceva's theorem, trigonometric form, we have the relation:

$$\frac{\sin\alpha}{\sin(A-\alpha)} \cdot \frac{\sin\beta}{\sin(B-\beta)} \cdot \frac{\sin\gamma}{\sin(C-\gamma)} = 1. \quad (1)$$

We notice that:

$$\frac{MB}{MC} = \frac{\mathrm{Arie}(MHB)}{\mathrm{Arie}(MHC)} = \frac{MH \cdot HB \cdot \sin\sin(MHB)}{MH \cdot HC \cdot \sin(MHC)}.$$

Because: $\sphericalangle MHB \equiv \sphericalangle A_1AC$ as angles of perpendicular sides, it follows that $m(\sphericalangle MHB) = m(\hat{A}) - \alpha$.

Therewith $m \sphericalangle MHC = m\ MHB + m\ BHC = 180^0 \alpha$.

We thus get that:

$$\frac{MB}{MC} = \frac{\sin\ A - \alpha}{\sin\alpha} \cdot \frac{HB}{HC}.$$

Analogously, we find that:

$$\frac{NC}{NA} = \frac{\sin\ B - \beta}{\sin\beta} \cdot \frac{HC}{HA};$$

$$\frac{PA}{PB} = \frac{\sin\ C - \gamma}{\sin\gamma} \cdot \frac{HA}{HB}.$$

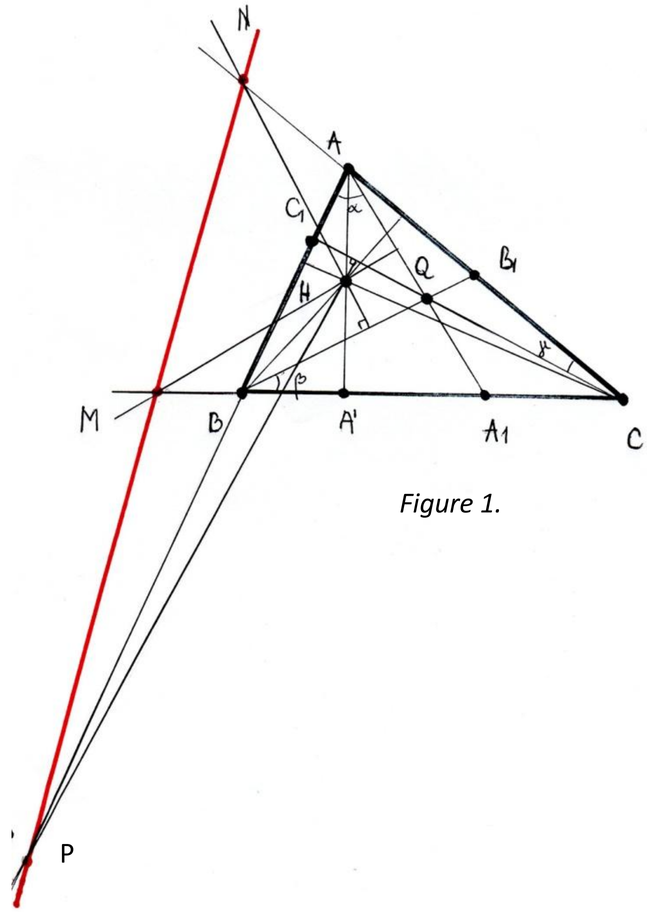

Figure 1.

Applying the reciprocal of Menelaus' theorem, we find, in view of (1), that:

$$\frac{MB}{MC} \cdot \frac{HC}{HA} \cdot \frac{PA}{PB} = 1.$$

This shows that M, N, P are collinear.

Note. Theorem 1 is true even if ABC is an obtuse, nonisosceles triangle. The proof is adapted analogously.

Theorem 2 (The Dual of the Theorem 1). If ABC is a triangle, O a certain point in his plan, and A_1, B_1, C_1 Bobillier transversals relative to O of ABC triangle, as well as $A_2 - B_2 - C_2$ a certain transversal in ABC, and the perpendiculars in O, and on OA_2, OB_2, OC_2 respectively, intersect the Bobillier transversals in the points A_3, B_3, C_3, then the ceviens AA_3, BB_3, CC_3 are concurrent.

Proof. We convert by duality with respect to a circle $\mathbb{C}\ o, r$ the figure indicated by the statement of this theorem, i.e. *Figure 2*. Let a, b, c be the polars of the points A, B, C with respect to the circle $\mathbb{C}\ o, r$. To the lines BC, CA, AB will correspond their poles $A'4 = bnc$; $B'4 = cna$; $C'4 = anb$.

To the points A_1, B_1, C_1 will respectively correspond their polars a_1, b_1, c_1 concurrent in transversal's pole $A_1 - B_1 - C_1$.

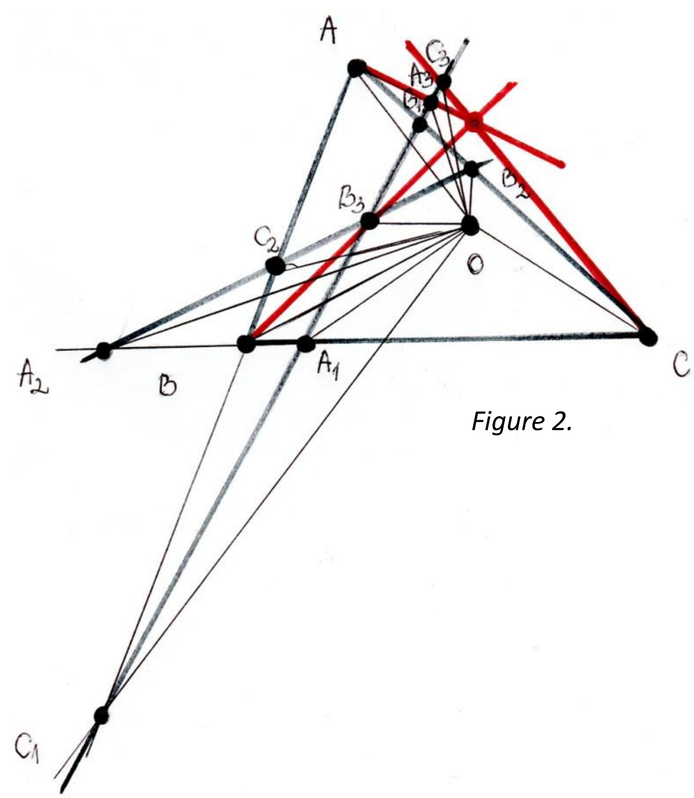

Figure 2.

Since $OA_1 \perp OA$, it means that the polars a and a_1 are perpendicular, so $a_1 \perp B'C'$, but a_1 pass through A', which means that Q' contains the height from A' of $A'B'C'$ triangle and similarly b_1 contains the height from B' and c_1 contains the height from C' of $A'B'C'$ triangle. Consequently, the pole of $A_1 - B_1 - C_1$ transversal is the orthocenter H' of $A'B'C'$ triangle. In the same way, to the points A_2, B_2, C_2 will correspond the polars to a_2, b_2, c_2 which pass respectively through A', B', C' and are concurrent in a point Q', the pole of the line $A_2 - B_2 - C_2$ with respect to the circle $\mathbb{C}\ o, r$. Given $OA_2 \perp OA_3$, it means that the polara a_2 and a_3 are perpendicular, a_2 correspond to the cevian $A'Q'$, also a_3 passes through the the pole of the transversal $A_1 - B_1 - C_1$, so through

H', in other words Q_3 is perpendicular taken from H' on $A'Q'$; similarly, $b_2 \perp b_3, c_2 \perp c_3$, so b_3 is perpendicular taken from H' on $C'Q'$. To the cevian AA_3 will correspond by duality considered to its pole, which is the intersection of the polars of A and A_3, i.e. the intersection of lines a and a_3, namely the intersection of $B'C'$ with the perpendicular taken from H' on $A'Q'$; we denote this point by M'. Analogously, we get the points N' and P'. Thereby, we got the configuration from **Theorem 1** and *Figure 1*, written for triangle $A'B'C'$ of orthocenter H'. Since from **Theorem 1** we know that M', N', P' are collinear, we get the the cevians AA_3, BB_3, CC_3 are concurrent in the pole of transversal $M' - N' - P'$ with respect to the circle $\mathbb{C}(o, r)$, and **Theorem 2** is proved.

References

[1] Ion Patrascu, Florentin Smarandache: „The Dual Theorem concerning Aubert Line".

[2] Florentin Smarandache, Ion Patrascu: „The Geometry of Homological Triangles". The Education Publisher Inc., Columbus, Ohio, USA – 2012.