The Duality and the Euler's Line

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In this article we'll discuss about a theorem which results from a duality transformation of a theorem and the configuration in relation to the Euler's line.

Theorem

Let ABC a given random triangle, I the center of its inscribed circle, and $A'B'C'$ its triangle of contact. The perpendiculars constructed in I on AI, BI, CI intersect BC, CA, AB respectively in the points A_1, B_1, C_1. The medians of the triangle of contact intersect the second time the inscribed circle in the points A_1', B_1', C_1', and the tangents in these points to the inscribed circle intersect the lines BC, CA, AB in the points A_2, B_2, C_2 respectively.

Then:

i) The points A_1, B_1, C_1 are collinear;

ii) The points A_2, B_2, C_2 are collinear;

iii) The lines A_1B_1, A_2B_2 are parallel.

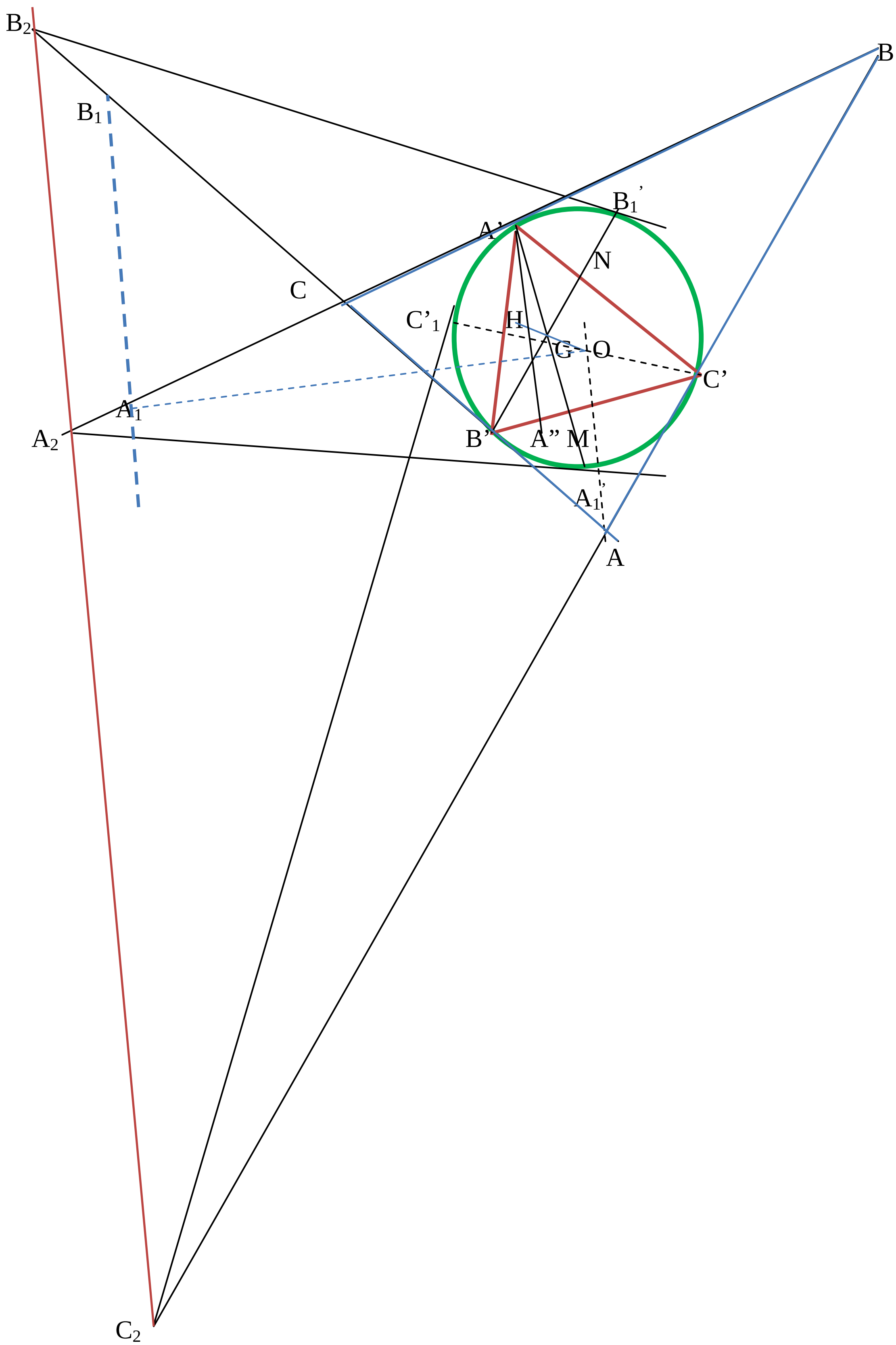
Proof

We'll consider a triangle $A'B'C'$ circumscribed to the circle of center O. Let $A'A'', B'B'', C'C''$ its heights concurrent in a point H and $A'M, B'N, C'P$ its medians concurrent in the weight center G. It is known that the points O, H, G are collinear; these are situated on the Euler's line of the triangle $A'B'C'$.

We'll transform this configuration (see the figure) through a duality in rapport to the circumscribed circle to the triangle $A'B'C'$.

To the points A', B', C' correspond the tangents in A', B', C' to the given circle, we'll note A, B, C the points of intersection of these tangents. For triangle ABC the circle $A'B'C'$ becomes inscribed circle, and $A'B'C'$ is the triangle of contact of ABC.

To the mediators $A'M, B'N, C'P$ will correspond through the considered duality, their pols, that is the points A_2, B_2, C_2 obtained as the intersections of the lines BC, CA, AB with the tangents in the points A_1', B_1', C_1' respectively to the circle $A'B'C'$ (A_1', B_1', C_1' are the intersection points with the circle $A'B'C'$ of the lines $(A'M, (B'N, (C'P)$. To the height $A'M$ corresponds its pole noted A_1 situated on BC such that $m\left(\widehat{AOA_1}\right) = 90°$ (indeed the pole of $B'C'$ is the point A and because $A'M \perp B'C'$ we have $m\left(\widehat{AOA_1}\right) = 90°$), similarly to the height $B'N$ we'll correspond the point B_1 on AC such that $m\left(\widehat{BOB_1}\right) = 90°$, and to the height $C'N$ will correspond the point C_1 on AB such that $m\left(\widehat{COC_1}\right) = 90°$.


B2
B
B1
B1'
A'
N
C
C'1
H
G
O
C'
A1
A2
B"
A"
M
A1'
A
C2

Because the heights are concurrent in H it means that their poles, that is the points A_1, B_1, C_1 are collinear.

Because the medians are concurrent in the point G it means that their poles, that is the points A_2, B_2, C_2 are collinear.

The lines $A_1B_1C_1$ and $A_2B_2C_2$ are respectively the poles of the points H and G, because H, G are collinear with the point O; this means that these poles are perpendicular lines on OG respectively on OH; consequently these are parallel lines.

By re-denoting the point O with I we will be in the conditions of the propose theorem and therefore the proof is completed.

Note

This theorem can be proven also using an elementary method. We'll leave this task for the readers.