Eight Solved and Eight Open Problems in Elementary Geometry

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In this paper we review eight previous proposed and solved problems of elementary 2D geometry [4], and we extend them either from triangles to polygons or from 2D-space to 3D-space and make some comments about them.

Problem 1.

We draw the projections M_i of a point M on the sides A_iA_{i+1} of the polygon $A_1...A_n$. Prove that:

$$\|M_1A_1\|^2 + ... + \|M_nA_n\|^2 = \|M_1A_2\|^2 + ... + \|M_{n-1}A_n\|^2 + \|M_nA_1\|^2$$

Solution 1.

For all i we have:

$$\|MM_i\|^2 = \|MA_i\|^2 - \|A_iM_i\|^2 = \|MA_{i+1}\|^2 - \|A_{i+1}M_i\|^2$$

It results that:

$$\|M_iA_i\|^2 - \|M_iA_{i+1}\|^2 = \|MA_i\|^2 - \|MA_{i+1}\|^2$$

From where:

$$\sum_i \left(\|M_iA_i\|^2 - \|M_iA_{i+1}\|^2 \right) = \sum_i \left(\|MA_i\|^2 - \|MA_{i+1}\|^2 \right) = 0$$

Open Problem 1.

1.1. If we consider in a 3D-space the projections M_i of a point M on the *edges* A_iA_{i+1} of a polyhedron $A_1...A_n$ then what kind of relationship (similarly to the above) can we find?

1.2. But if we consider in a 3D-space the projections M_i of a point M on the *faces* F_i of a polyhedron $A_1...A_n$ with k≥4 faces, then what kind of relationship (similarly to the above) can we find?

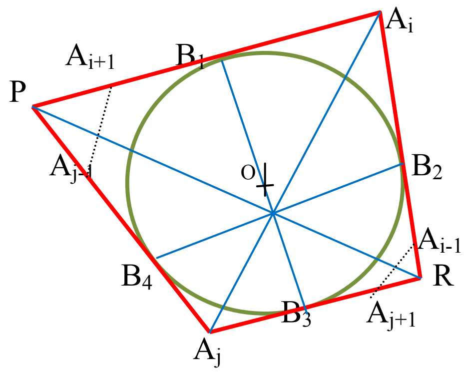
Problem 2.

Let's consider a polygon (which has at least 4 sides) circumscribed to a circle, and D the set of its diagonals and the lines joining the points of contact of two non-adjacent sides. Then D contains at least 3 concurrent lines.

Solution 2.

Let n be the number of sides. If $n = 4$, then the two diagonals and the two lines joining the points of contact of two adjacent sides are concurrent (according to Newton's Theorem).

The case $n > 4$ is reduced to the previous case: we consider any polygon $A_1 \ldots A_n$ (see the figure)



circumscribed to the circle and we choose two vertices A_i, A_j ($i \neq j$) such that

$$A_j\ A_{j-1} \cap A_i\ A_{i+1} = P$$

and

$$A_j\ A_{j+1} \cap A_i\ A_{i-1} = R.$$

Let B_h, $h \in \{1, 2, 3, 4\}$ the contact points of the quadrilateral PA_jRA_i with the circle of center O. Because of the Newton's theorem, the lines $A_i\ A_j$, B_1B_3 and B_2B_4 are concurrent.

Open Problem 2.

2.1. In what conditions there are more than three concurrent lines?

2.2. What is the maximum number of concurrent lines that can exist (and in what conditions)?

2.3. What about an alternative of this problem: to consider instead of a circle an ellipse, and then a polygon *ellipsoscribed* (let's invent this word, *ellipso-scribed*, meaning a polygon whose all sides are tangent to an ellipse which inside of it): how many concurrent lines we can find among its diagonals and the lines connecting the point of contact of two non-adjacent sides?

2.4. What about generalizing this problem in a 3D-space: a sphere and a polyhedron circumscribed to it?

2.5. Or instead of a sphere to consider an ellipsoid and a polyhedron *ellipsoido-scribed* to it?

Of course, we can go by construction reversely: take a point inside a circle (similarly for an ellipse, a sphere, or ellipsoid), then draw secants passing through this point that intersect the circle (ellipse, sphere, ellipsoid) into two points, and then draw tangents to the circle (or ellipse), or tangent planes to the sphere or ellipsoid) and try to construct a polygon (or polyhedron) from the intersections of the tangent lines (or of tangent planes) if possible.

For example, a regular polygon (or polyhedron) has a higher chance to have more concurrent such lines.

In the 3D space, we may consider, as alternative to this problem, the intersection of planes (instead of lines).

Problem 3.

In a triangle ABC let's consider the Cevians AA', BB' and CC' that intersect in P. Calculate the minimum value of the expressions:

$$E(P)=\frac{\|PA\|}{\|PA'\|}+\frac{\|PB\|}{\|PB'\|}+\frac{\|PC\|}{\|PC'\|}$$

and

$$F(P)=\frac{\|PA\|}{\|PA'\|}\cdot\frac{\|PB\|}{\|PB'\|}\cdot\frac{\|PC\|}{\|PC'\|}$$

where $A'\in[BC]$, $B'\in[CA]$, $C'\in[AB]$.

Solution 3.

We'll apply the theorem of Van Aubel three times for the triangle ABC, and it results:

$$\frac{\|PA\|}{\|PA'\|}=\frac{\|AC'\|}{\|C'B\|}+\frac{\|AB'\|}{\|B'C\|}$$
$$\frac{\|PB\|}{\|PB'\|}=\frac{\|BA'\|}{\|A'C\|}+\frac{\|BC'\|}{\|C'A\|}$$
$$\frac{\|PC\|}{\|PC'\|}=\frac{\|CA'\|}{\|A'B\|}+\frac{\|CB'\|}{\|B'A\|}$$

If we add these three relations and we use the notation

$$\frac{\|AC'\|}{\|C'B\|}=x>0,\ \frac{\|AB'\|}{\|B'C\|}=y>0,\ \frac{\|BA'\|}{\|A'C\|}=z>0$$

then we obtain:

$$E(P)=\left(x+\frac{1}{y}\right)+\left(x+\frac{1}{y}\right)+\left(z+\frac{1}{z}\right)\geq 2+2+2=6$$

The minimum value will be obtained when $x=y=z=1$, therefore when P will be the gravitation center of the triangle.

When we multiply the three relations we obtain

$$F(P)=\left(x+\frac{1}{y}\right)\cdot\left(x+\frac{1}{y}\right)\cdot\left(z+\frac{1}{z}\right)\geq 8$$

Open Problem 3.

3.1. Instead of a triangle we may consider a polygon $A_1A_2\ldots A_n$ and the lines A_1A_1', A_2A_2', …, A_nA_n' that intersect in a point P.
Calculate the minimum value of the expressions:

$$E(P)=\frac{\|PA_1\|}{\|PA_1'\|}+\frac{\|PA_2\|}{\|PA_2'\|}+\ldots+\frac{\|PA_n\|}{\|PA_n'\|}$$

$$F(P)=\frac{\|PA_1\|}{\|PA_1'\|}\cdot\frac{\|PA_2\|}{\|PA_2'\|}\cdot\ldots\cdot\frac{\|PA_n\|}{\|PA_n'\|}$$

3.2. Then let's generalize the problem in the 3D space, and consider the polyhedron $A_1A_2\ldots A_n$ and the lines A_1A_1', A_2A_2', …, A_nA_n' that intersect in a point P. Similarly, calculate the minimum of the expressions E(P) and F(P).

Problem 4.

If the points A_1, B_1, C_1 divide the sides BC, CA respectively AB of a triangle in the same ratio $k>0$, determine the minimum of the following expression:

$$\|AA_1\|^2+\|BB_1\|^2+\|CC_1\|^2.$$

Solution 4.

Suppose $k>0$ because we work with distances.

$$\|BA_1\|=k\|BC\|,\ \|CB_1\|=k\|CA\|,\ \|AC_1\|=k\|AB\|$$

We'll apply tree times Stewart's theorem in the triangle ABC, with the segments AA_1, BB_1, respectively CC_1:

$$\|AB\|^2\cdot\|BC\|(1-k)+\|AC\|^2\cdot\|BC\|k-\|AA_1\|^2\cdot\|BC\|=\|BC\|^3(1-k)k$$

where

$$\|AA_1\|^2=(1-k)\|AB\|^2+k\|AC\|^2-(1-k)k\|BC\|^2$$

similarly,

$$\|BB_1\|^2=(1-k)\|BC\|^2+k\|BA\|^2-(1-k)k\|AC\|^2$$

$$\|CC_1\|^2=(1-k)\|CA\|^2+k\|CB\|^2-(1-k)k\|AB\|^2$$

By adding these three equalities we obtain:

$$\|AA_1\|^2+\|BB_1\|^2+\|CC_1\|^2=\left(k^2-k+1\right)\left(\|AB\|^2+\|BC\|^2+\|CA\|^2\right),$$

which takes the minimum value when $k = \frac{1}{2}$, which is the case when the three lines from the enouncement are the medians of the triangle.

The minimum is $\frac{3}{4}\left(\|AB\|^2 + \|BC\|^2 + \|CA\|^2\right)$.

Open Problem 4.

4.1. If the points A_1', A_2', …, A_n' divide the sides A_1A_2, A_2A_3, …, A_nA_1 of a polygon in the same ratio k>0, determine the minimum of the expression:

$$\|A_1A_1'\|^2 + \|A_2A_2'\|^2 + \ldots + \|A_nA_n'\|^2 .$$

4.2. Similarly question if the points A_1', A_2', …, A_n' divide the sides A_1A_2, A_2A_3, …, A_nA_1 in the positive ratios k_1, k_2, …, k_n respectively.

4.3. Generalize this problem for polyhedrons.

Problem 5.

In the triangle ABC we draw the lines AA_1, BB_1, CC_1 such that

$$\|A_1B\|^2 + \|B_1C\|^2 + \|C_1A\|^2 = \|AB_1\|^2 + \|BC_1\|^2 + \|CA_1\|^2 .$$

In what conditions these three Cevians are concurrent?

Partial Solution 5.

They are concurrent for example when A_1, B_1, C_1 are the legs of the medians of the triangle BCA. Or, as Prof. Ion Pătrașcu remarked, when they are the legs of the heights in an acute angle triangle BCA .

More general.

The relation from the problem can be written also as:

$$a\left(\|A_1B\| - \|A_1C\|\right) + b\left(\|B_1C\| - \|C_1A\|\right) + c\left(\|C_1A\| - \|C_1B\|\right) = 0 ,$$

where a, b, c are the sides of the triangle.

We'll denote the three above terms as α, β, and respective γ, such that $\alpha + \beta + \gamma = 0$.

$$\alpha = a\left(\|A_1B\| - \|A_1C\|\right) \Leftrightarrow \frac{\alpha}{a} = \|A_1B\| - \|A_1C\| - 2\|A_1C\|$$

where

$$\frac{\alpha}{a^2} = \frac{a - 2\|A_1C\|}{a} \Leftrightarrow \frac{a^2}{a^2 - \alpha} = \frac{a}{2\|A_1C\|} \Leftrightarrow \frac{a}{2\|A_1C\|} = \frac{2a^2}{a^2 - \alpha} \Leftrightarrow \frac{2a^2 - a^2 + \alpha}{a^2 - \alpha} = \frac{a - \|A_1C\|}{\|A_1C\|}$$

Then

$$\frac{\|A_1B\|}{\|A_1C\|}=\frac{a^2+\alpha}{a^2-\alpha}.$$

Similarly:

$$\frac{\|B_1C\|}{\|B_1A\|}=\frac{b^2+\beta}{b^2-\beta} \text{ and } \frac{\|C_1A\|}{\|C_1B\|}=\frac{c^2+\gamma}{c^2-\gamma}$$

In conformity with Ceva's theorem, the three lines from the problem are concurrent if and only if:

$$\frac{\|A_1B\|}{\|A_1C\|}\cdot\frac{\|B_1C\|}{\|B_1A\|}\cdot\frac{\|C_1A\|}{\|C_1B\|}=1 \Leftrightarrow \left(a^2+\alpha\right)\left(b^2+\beta\right)\left(c^2+\gamma\right)=\left(a^2-\alpha\right)\left(b^2-\beta\right)\left(c^2-\gamma\right)$$

Unsolved Problem 5.

Generalize this problem for a polygon.

Problem 6.

In a triangle we draw the Cevians $AA_1,\ BB_1,\ CC_1$ that intersect in P. Prove that

$$\frac{PA}{PA_1}\cdot\frac{PB}{PB_1}\cdot\frac{PC}{PC_1}=\frac{AB\cdot BC\cdot CA}{A_1B\cdot B_1C\cdot C_1A}$$

Solution 6.

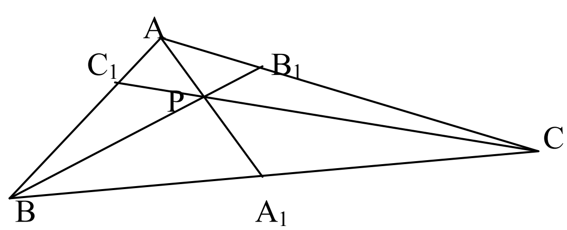
In the triangle ABC we apply the Ceva's theorem:

$$AC_1\cdot BA_1\cdot CB_1=-AB_1\cdot CA_1\cdot BC_1 \qquad (1)$$

In the triangle AA_1B, cut by the transversal CC_1, we'll apply the Menelaus' theorem:

$$AC_1\cdot BC\cdot A_1P=AP\cdot A_1C\cdot BC_1 \qquad (2)$$

In the triangle BB_1C, cut by the transversal AA_1, we apply again the Menelaus' theorem:



$$BA_1\cdot CA\cdot B_1P=BP\cdot B_1A\cdot CA_1 \qquad (3)$$

We apply one more time the Menelaus' theorem in the triangle CC_1A cut by the transversal BB_1:

$$AB\cdot C_1P\cdot CB_1=AB_1\cdot CP\cdot C_1B \qquad (4)$$

We divide each relation (2), (3), and (4) by relation (1), and we obtain:

$$\frac{PA}{PA_1} = \frac{BC}{BA_1} \cdot \frac{B_1A}{B_1C} \quad (5)$$

$$\frac{PB}{PB_1} = \frac{CA}{CB_1} \cdot \frac{C_1B}{C_1A} \quad (6)$$

$$\frac{PC}{PC_1} = \frac{AB}{AC_1} \cdot \frac{A_1C}{A_1B} \quad (7)$$

Multiplying (5) by (6) and by (7), we have:

$$\frac{PA}{PA_1} \cdot \frac{PB}{PB_1} \cdot \frac{PC}{PC_1} = \frac{AB \cdot BC \cdot CA}{A_1B \cdot B_1C \cdot C_1A} \cdot \frac{AB_1 \cdot BC_1 \cdot CA_1}{A_1B \cdot B_1C \cdot C_1A}$$

but the last fraction is equal to 1 in conformity to Ceva's theorem.

Unsolved Problem 6.

Generalize this problem for a polygon.

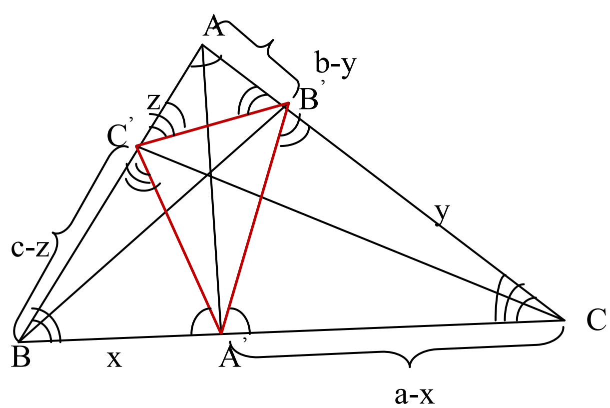
Problem 7.

Given a triangle ABC whose angles are all acute (acute triangle), we consider $A'B'C'$, the triangle formed by the legs of its altitudes.

In which conditions the expression:

$$\|A'B'\| \cdot \|B'C'\| + \|B'C'\| \cdot \|C'A'\| + \|C'A'\| \cdot \|A'B'\|$$

is maximum?



Solution 7.

We have

$$\Delta ABC \sim \Delta A'B'C' \sim \Delta AB'C \sim \Delta A'BC' \quad (1)$$

We note

$$\|BA'\| = x, \ \|CB'\| = y, \ \|AC'\| = z.$$

It results that

$$\|A'C\| = a - x, \ \|B'A\| = b - y, \ \|C'B\| = c - z$$

$$\widehat{BAC} = \widehat{B'A'C} = \widehat{BA'C'}; \ \widehat{ABC} = \widehat{AB'C'} = \widehat{A'B'C}; \ \widehat{BCA} = \widehat{BC'A'} = \widehat{B'C'A}$$

From these equalities it results the relation (1)

$$\Delta A'BC' \sim \Delta A'B'C \Rightarrow \frac{\|A'C'\|}{a-x} = \frac{x}{\|A'B'\|} \qquad (2)$$

$$\Delta A'B'C \sim \Delta AB'C' \Rightarrow \frac{\|A'C'\|}{z} = \frac{c-z}{\|B'C'\|} \qquad (3)$$

$$\Delta AB'C' \sim \Delta A'B'C \Rightarrow \frac{\|B'C'\|}{y} = \frac{b-y}{\|A'B'\|} \qquad (4)$$

From (2), (3) and (4) we observe that the sum of the products from the problem is equal to:

$$x(a-x) + y(b-y) + z(c-z) = \frac{1}{4}\left(a^2 + b^2 + c^2\right) - \left(x - \frac{a}{2}\right)^2 - \left(y - \frac{b}{2}\right)^2 - \left(z - \frac{c}{2}\right)^2$$

which will reach its maximum as long as $x = \frac{a}{2}$, $y = \frac{b}{2}$, $z = \frac{c}{2}$, that is when the altitudes' legs are in the middle of the sides, therefore when the ΔABC is equilateral. The maximum of the expression is $\frac{1}{4}\left(a^2 + b^2 + c^2\right)$.

Conclusion[1]: If we note the lengths of the sides of the triangle ΔABC by ||AB|| = *c*, ||BC|| = *a*, ||CA|| = *b*, and the lengths of the sides of its orthic triangle ΔA\`B\`C\` by ||A\`B\`|| = *c*\`, ||B\`C\`|| = *a*\`, ||C\`A\`|| = *b*\`, then we proved that:

$$4(a`b` + b`c` + c`a`) \le a^2 + b^2 + c^2.$$

Unsolved Problems 7.

7.1. Generalize this problem to polygons. Let $A_1A_2\ldots A_m$ be a polygon and P a point inside it. From P, which is called a pedal point, we draw perpendiculars on each side A_iA_{i+1} of the polygon and we note by A_i' the intersection between the perpendicular and the side A_iA_{i+1}. Let's extend the definition of pedal triangle to a **pedal polygon** in a straight way: i.e. the polygon formed by the orthogonal projections of a pedal point on the sides of the polygon. The pedal polygon A_1'A_2'…A_m' is formed. What properties does this pedal polygon have?

7.2. Generalize this problem to polyhedrons. Let $A_1A_2\ldots A_n$ be a polyhedron and P a point inside it. From P we draw perpendiculars on each edge A_iA_j of the polyhedron and we note by A_{ij}' the intersection between the perpendicular and the side A_iA_{ij}. Let's name the

[1] This is called the **Smarandache's Orthic Theorem** (see [2], [3]).

new formed polyhedron an **edge pedal polyhedron**, $A_1'A_2'\ldots A_n'$. What properties does this edge pedal polyhedron have?

7.3. Generalize this problem to polyhedrons in a different way. Let $A_1A_2\ldots A_n$ be a poliyhedron and P a point inside it. From P we draw perpendiculars on each polyhedron face F_i and we note by A_i' the intersection between the perpendicular and the side F_i. Let's call the new formed polyhedron a **face pedal polyhedron**, which is $A_1'A_2'\ldots A_p'$, where p is the number of polyhedron's faces. What properties does this face pedal polyhedron have?

Problem 8.

Given the distinct points $A_1,\ldots,A_n$ on the circumference of a circle with the center in O and of ray R.

Prove that there exist two points A_i, A_j such that $\left\|\overrightarrow{OA_i}+\overrightarrow{OA_j}\right\| \geq 2R\cos\dfrac{180^o}{n}$

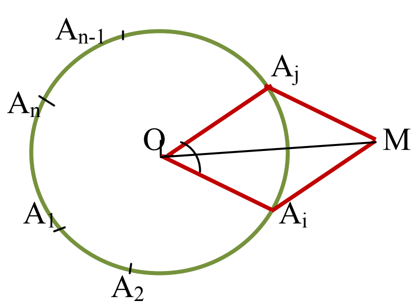
Solution 8.

Because

$$\sphericalangle A_1OA_2 + \sphericalangle A_2OA_3 + \ldots + \sphericalangle A_{n-1}OA_n + \sphericalangle A_nOA_1 = 360^o$$

and $\forall i \in \{1,2,\ldots,n\}$, $\sphericalangle A_iOA_{i+2} > 0^o$, it result that it exist at least one angle $\sphericalangle A_iOA_j \leq \dfrac{360^o}{n}$

(otherwise it follows that $S > \dfrac{360^0}{n}\cdot n = 360^o$).



$$\overrightarrow{OA_i}+\overrightarrow{OA_j}=\overrightarrow{OM} \Rightarrow \left\|\overrightarrow{OA_i}+\overrightarrow{OA_j}\right\| = \left\|\overrightarrow{OM}\right\|$$

The quadrilateral OA_iMA_j is a rhombus. When α is smaller, $\left\|\overrightarrow{OM}\right\|$ is greater. As $\alpha \leq \dfrac{360^o}{n}$, it results that: $\left\|\overrightarrow{OM}\right\| = 2R\cos\dfrac{\alpha}{2} \geq 2R\cos\dfrac{180^o}{n}$.

Open Problem 8:

Is it possible to find a similar relationship in an ellipse? (Of course, instead of the circle's radius *R* one should consider the ellipse's axes *a* and *b*.)

References:

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