ON TWO OF ERDÖS'S OPEN PROBLEMS

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Abstract.
This short note presents some remarks and conjectures on two open problems proposed by P. Erdös.

First Problem.

In one of his books ("Analysis…") Mr. Paul Erdös proposed the following problem:

"The integer n is called a barrier for an arithmetic function f if $m + f(m) \le n$ for all $m < n$.

Question: Are there infinitely many barriers for $\varepsilon v(n)$, for some $\varepsilon > 0$? Here $v(n)$ denotes the number of distinct prime factors of n."

We found some results regarding this question, which results make us to conjecture that there is a finite number of barriers, for all $\varepsilon > 0$.

Let $R(n)$ be the relation: $m + \varepsilon v(m) \le n,\ \forall m < n$.

Lemma 1.1. If $\varepsilon > 1$ there are two barriers only: $n = 1$ and $n = 2$ (which we call trivial barriers).

Proof. It is clear for $n = 1$ and $, n = 2$ because $v(0) = v(1) = 0$.

Let's consider $n \ge 3$. Then, if $m = n - 1$ we have $m + \varepsilon v(m) \ge n - 1 + \varepsilon > n$, contradiction.

Lemma 1.2. There is an infinity of numbers which cannot be barriers for $\varepsilon v(n)$, $\forall \varepsilon > 0$.

Proof. Let's consider $s, k \in \mathbb{N}^*$ such that $s \cdot \varepsilon > k$. We write n in the form $n = p_{i_1}^{\alpha_{i_1}} \cdots p_{i_s}^{\alpha_{i_s}} + k$, where for all j, $\alpha_{i_j} \in \mathbb{N}^*$ and p_{i_j} are positive distinct primes.

Taking $m = n - k$ we have $m + \varepsilon v(m) = n - k + \varepsilon \cdot s > n$.

But there exists an infinity of n's because the parameters $\alpha_{i_1}, ..., \alpha_{i_s}$ are arbitrary in $\mathbb{N}^*$ and $p_{i_1}, ..., p_{i_s}$ are arbitrary positive distinct primes, also there is an infinity of couples (s, k) for an $\varepsilon > 0$, fixed , with the property $s \cdot \varepsilon > k$.

Lemma 1.3. For all $\varepsilon \in (0,1]$ there are nontrivial barriers for $\varepsilon v(n)$.

Proof. Let t be the greatest natural number such that $t\varepsilon \le 1$ (always there is such t).

Let n be from $[3,..., p_1 \cdots p_t p_{t+1})$, where $\{p_i\}$ is the sequence of the positive primes. Then $1 \le v(n) \le t$.

All $n \in [1,..., p_1 \cdots p_t p_{t+1}]$ is a barrier, because: $\forall\ 1 \le k \le n-1$, if $m = n-k$ we have $m + \varepsilon v(m) \le n - k + \varepsilon \cdot t \le n$.

Hence, there are at list $p_1 \cdots p_t p_{t+1}$ barriers.

Corollary. If $\varepsilon \to 0$ then n (the number of barriers) $\to \infty$.

Lemma 1.4. Let's consider $n \in [1,..., p_1 \cdots p_r p_{r+1}]$ and $\varepsilon \in (0,1]$. Then: n is a barrier if and only if $R(n)$ is verified for $m \in \{n-1, n-2,..., n-r+1\}$.

Proof. It is sufficient to prove that $R(n)$ is always verified for $m \le n - r$.

Let's consider $m = n - r - u$, $u \ge 0$. Then $m + \varepsilon v(m) \le n - r - u + \varepsilon \cdot r \le n$.

Conjecture.

We note $I_r \in [p_1 \cdots p_r, ..., \cdot p_1 \cdots p_r p_{r+1})$. Of course $\bigcup_{r \ge 1} I_r = \mathbb{N} \setminus \{0,1\}$, and $I_{r_1} \cap I_{r_2} = \Phi$ for $r_1 \ne r_2$.

Let $\mathcal{N}_r(1+t)$ be the number of all numbers n from I_r such that $1 \le v(n) \le t$.

We conjecture that there is a finite number of barriers for $\varepsilon v(n)$, $\forall \varepsilon > 0$; because

$$\lim_{r \to \infty} \frac{\mathcal{N}_r(1+t)}{p_1 \cdots p_{r+1} - p_1 \cdots p_r} = 0$$

and the probability (of finding of $r-1$ consecutive values for m, which verify the relation $R(n)$) approaches zero.

Second Problem.

Paul Erdös has proposed another problem:

(1) "Is it true that $\lim_{n \to \infty} \max_{m<n}(m + d(m)) - n = \infty$?, where $d(m)$ represents the number of all positive divisors of m."

We clearly have :

Lemma 2.1. $(\forall) n \in \mathbb{N} \setminus \{0,1,2\}$, $(\exists)! s \in \mathbb{N}^*$, $(\exists)! \alpha_1, ..., \alpha_s \in \mathbb{N}$, $\alpha_s \ne 0$, such that $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s} + 1$, where $p_1, p_2, ...$ constitute the increasing sequence of all positive primes.

Lemma 2.2. Let $s \in \mathbb{N}^*$. We define the subsequence $n_s(i) = p_1^{\alpha_1} \cdots p_s^{\alpha_s} + 1$, where $\alpha_1, ..., \alpha_s$ are arbitrary elements of $\mathbb{N}$, such that $\alpha_s \ne 0$ and $\alpha_1 + ... + \alpha_s \to \infty$ and we order it such that $n_s(1) < n_s(2) < ...$ (increasing sequence).

We find an infinite number of subsequences $\{n_s(i)\}$, when s traverses $\mathbb{N}^*$, with the properties:

a) $\lim_{i\to\infty} n_s(i) = \infty$ for all $s \in \mathbb{N}^*$.

b) $\{n_{s_1}(i),\ i \in \mathbb{N}^*\} \cap \{n_{s_2}(j),\ j \in \mathbb{N}^*\} = \Phi$, for $s_1 \neq s_2$ (distinct subsequences).

c) $\mathbb{N} \setminus \{0,1,2\} = \bigcup_{s\in\mathbb{N}^*} \{n_s(i),\ i \in \mathbb{N}^*\}$

Then:

Lemma 2.3. If in (1) we calculate the limit for each subsequence $\{n_s(i)\}$ we obtain:

$$\lim_{n\to\infty}\left(\max_{m<p_1^{\alpha_1}\cdots p_s^{\alpha_s}} (m+d(m)) - p_1^{\alpha_1}\cdots p_s^{\alpha_s} - 1 \right) \geq \lim_{n\to\infty}\left(p_1^{\alpha_1}\cdots p_s^{\alpha_s} + (\alpha_1+1)\ldots(\alpha_s+1) - p_1^{\alpha_1}\cdots p_s^{\alpha_s} - 1 \right) =$$

$$= \lim_{n\to\infty}\left((\alpha_1+1)\ldots(\alpha_s+1) - 1 \right) > \lim_{n\to\infty}\left(\alpha_1 + \ldots + \alpha_s \right) = \infty$$

From these lemmas it results the following:

Theorem: We have $\overline{\lim_{n\to\infty}} \max_{m<n}(m+d(m)) - n = \infty$.

REFERENCES

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[2] P. Erdös - Letter to the Author - 1986: 01: 12.

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