Examples where the Conjunctive and Dempster's Rules are Insensitive

Florentin Smarandache
Jean Dezert
Valeri Kroumov

Originally published as *Smarandache F., Kroumov V., Dezert J., Examples where the conjunctive and Dempster's rules are insensitive,* Proc. of 2013 International Conference on Advanced Mechatronic Systems, Luoyang, China, Sept. 25-27, 2013, and reprinted with permission.

***Abstract*—In this paper we present several counter-examples to the Conjunctive rule and to Dempster rule of combinations in information fusion.**

***Keywords*— conjunctive rule, Dempster rule, DSmT, counter-examples to Conjunctive rule, counter-examples to Dempster rule, information fusion**

I. INTRODUCTION

In *Counter-Examples to Dempster's Rule of Combination* {Ch. 5 of *Advances and Applications to DSmT on Information Fusion,* Vol. I, pp. 105-121, 2004} [1], J. Dezert, F. Smarandache, and M. Khoshnevisan have presented several classes of fusion problems which could not be directly approached by the classical mathematical theory of evidence, also known as Dempster-Shafer Theory (DST), either because Shafer's model for the frame of discernment was impossible to obtain, or just because Dempster's rule of combination failed to provide coherent results (or no result at all). We have showed and discussed the potentiality of the DSmT combined with its classical (or hybrid) rule of combination to attack these infinite classes of fusion problems.
We have given general and concrete counter-examples for Bayesian and non-Bayesian cases.

In this article we construct new classes where both the conjunctive and Dempster's rule are insensitive.

II. DEZERT-TCHAMOVA COUNTER-EXAMPLE

In [2], J. Dezert and A. Tchamova have introduced for the first time the following counter-example with some generalizations. This first type of example has then been discussed in details in [3,4] to question the validity of foundations of Dempster-Shafer Theory (DST). In the next sections of this short paper, we provide more counter-examples extending this idea. Let the frame of discernment Θ = {A, B, C}, under Shafer's model (i.e. all intersections are empty), and $m_1(.)$ and $m_2(.)$ be two independent sources of information that give the below masses:

Focal Elements	A	C	$A \cup B$	$A \cup B \cup C$
m_1	a	0	$1-a$	0
m_2	0	$1-b_1-b_2$	b_1	b_2

Table 1

where the parameters $a, b_1, b_2 \in [0,1]$, and $b_1+b_2 \leq 1$.

Applying the conjunctive rule, in order to combine $m_1 \oplus m_2 = m_{12}$, one gets:

$$m_{12}(A) = a(b_1+b_2) \tag{1}$$

$$m_{12}(C) = 0 \tag{2}$$

$$m_{12}(A \cup B) = (1-a)(b_1+b_2) \tag{3}$$

$$m_{12}(A \cup B \cup C) = 0 \tag{4}$$

and the conflicting mass

$$m_{12}(\phi) = 1-b_1-b_2 = K_{12}. \tag{5}$$

After normalizing by diving by $1-K_{12} = b_1+b_2$ one gets Demspter's rule result $m_{DS}(.)$:

$$m_{DS}(A) = \frac{m_{12}(A)}{1 \;\; K_{12}} = \frac{a(b_1+b_2)}{b_1+b_2} = a = m_1(A)$$

$$m_{DS}(A \cup B) = \frac{m_{12}(A \cup B)}{1 \;\; K_{12}} = \frac{(1-a)(b_1+b_2)}{b_1+b_2} = 1-a = m_1(A \cup B) \tag{6}$$

Counter-intuitively after combining two sources of information, $m_1(.)$ and $m_2(.)$, with Dempster's rule, the result does not depend at all on $m_2(.)$. Therefore Dempster's rule is insensitive to $m_2(.)$ no matter what the parameters a, b_1, b_2 are equal to.

III. FUSION SPACE

In order to generalize this counter-example, let's start by defining the fusion space.

Let Θ be a frame of discernment formed by *n* singletons Ai, defined as:

$$\Theta = \{\phi_1, \phi_2, \ldots, \phi_n\}, n \geq 2, \qquad (7)$$

and its Super-Power Set (or fusion space):

$$S^\Theta = (\Theta, \cup, \cap, \complement) \qquad (8)$$

which means the set Θ closed under union $\cup$, intersection $\cap$, and respectively complement $\complement$.

IV. ANOTHER CLASS OF COUNTER-EXAMPLES TO DEMPSTER'S RULE

Let $A_1, A_2, \ldots, A_p \in S^\Theta \setminus \{I_t, \phi\}$, for $p \geq 1$, such that $Ai \cap Aj = \phi$ for $i \neq j$, where I_t is the total ignorance ($A_1 \cup A_2 \cup \ldots \cup A_n$), and ϕ is the empty set.

Therefore each A_i, for $i \in \{1, 2, \ldots, p\}$, can be either a singleton, or a partial ignorance (union of singletons), or an intersection of singletons, or any element from the Super-Power Set S^Θ (except the total ignorance or the empty set), i.e. a general element in the set theory that is formed by the operators $\cup, \cap, \complement$.

Let's consider two sources $m_1(.)$ and $m_2(.)$ defined on S^Θ:

	A_1	A_2	…	A_p	I_t
m_1	a_1	a_2	…	a_p	0
m_2	b	b	…	b	$1-p\cdot b$

where of course all $a_i \in [0, 1]$ and $a_1 + a_2 + \ldots + a_p = 1$,
also b and $1-p\cdot b \in [0, 1]$.

$m_1(.)$ can be Bayesian or non-Bayesian depending on the way we choose the focal elements $A_1, A_2, \ldots, A_p$.

We can make sure $m_2(.)$ is not the uniform basic believe assignment by setting $b \neq 1-p\cdot b$.

Let's use the conjunctive rule for $m_1(.)$ and $m_2(.)$:

$m_{12}(A_i) = m_1(A_i)m_2(A_i) + [m_1(A_i)m_2(I_t) + m_1(I_t)m_2(A_i)] = a_i\cdot b + [a_i\cdot(1-p\cdot b) + 0\cdot b] = a_i\cdot(1-p\cdot b+b)$,

for all $i \in \{1, 2, \ldots, p\}$. (9)

It is interesting to finding out, according to the Conjunctive Rule, that the conflict of the above two sources does not depend on $m_1(.)$ at all, but only on $m_2(.)$, which is abnormal:

$$K_{12} = \sum_{i=1}^{p}\sum_{\substack{j=1\\ j\neq i}}^{p} m_1(A_i)m_2(A_j) = \sum_{i=1}^{p}\sum_{\substack{j=1\\ j\neq i}}^{p} a_i\cdot b = \sum_{i=1}^{p}(p-1)a_i\cdot b = (p-1)b\sum_{i=1}^{p} a_i = (p-1)b. \qquad (10)$$

Therefore even the feasibility of the Conjunctive Rule is questioned.

When we normalize, as in Dempster's Rule, by dividing all $m_{12}(.)$ masses by the common factor $1-K = 1-p\cdot b+b$, we actually get: $m_1 \oplus m_2 = m_1$! So, $m_2(.)$ makes no impact on the fusion result according to Dempster's Rule, which is not normal.

V. MORE GENERAL CLASS OF COUNTER-EXAMPLES TO DEMPSTER'S RULE

Let's consider $r+1$ sources: the previous $m_1(.)$ and respectively various versions of the previous $m_2(.)$:

	A_1	A_2	…	A_p	I_t
m_1	a_1	a_2	…	a_p	0
m_{21}	b_1	b_1	…	$b1$	$1-p\cdot b_1$
m_{22}	b_2	b_2	…	b_2	$1-p\cdot b_2$
.					
.					
.					
m_{2r}	b_r	b_r	…	b_r	$1-p\cdot b_r$

where of course all $a_i \in [0, 1]$ and $a_1 + a_2 + \ldots + a_p = 1$,
also all b_j and $1-p\cdot b_j \in [0, 1]$, for $j \in \{1, 2, \ldots, r\}$. *(11)*

Now, if we combine $m_1 \oplus m_{21} \oplus m_{22} \oplus \ldots \oplus m_{2r} = m_1$. Therefore all r sources $m_{21}(.), m_{22}(.), \ldots, m_{2r}(.)$ have no impact on the fusion result!

Interesting particular examples can be found in this case.

VI. SHORT GENERALIZATION OF DEZERT-TCHAMOVA COUNTER-EXAMPLE

Let's consider four focal elements A, B_1, B_2, B_3, such that $A \cap B_i = \phi$ for $i \in \{1,2,3\}$, and B_1, B_2, B_3 are nested, i.e. $B_1 \subset B_2 \subset B_3$, and two masses, where of course $b_1+b_2 = 1$ and $c_1+c_2+c_3 = 1$, and all $b_1, b_2, c_1, c_2, c_3 \in [0, 1]$:

	A	B_1	B_2	B_3
m_1	0	b_1	b_2	0
m_2	c_1	0	c_2	c_3
m_{12}	0	$b_1(1-c_1)$	$b_2(1-c_1)$	0

and the conflict $K_{12} = c_1(b_1+b_2)=c_1$

m_D	0	b_1	b_2	0

a) This generalization permits the usefulness of hybrid models, for example one may have the frame of discernment of exclusive elements {A, B, C}, where $B_1 = B \cap C$, $B_2 = B$, and $B_3 = B \cup C$.

b) Other interesting particular cases may be derived from this short generalization.

VII. PARTICULAR COUNTER-EXAMPLE TO THE CONJUNCTIVE RULE AND DEMPSTER'S RULE

For example let $\Theta = \{A, B, C\}$, in Shafer's model. We show that the conflicts between sources are not correctly reflected by the conjunctive rule, and that a certain non-vacuous non-uniform source is ignored by Dempster's rule.

Let's consider the masses:

	A	B	C	$A \cup B \cup C$	
m_1	1	0	0	0	*(the most specific mass)*
m_2	1/3	1/3	1/3	0	*(very unspecific mass)*
m_3	0.6	0.4	0	0	*(mass between the very unspecific and the most specific masses)*
m_0	0.2	0.2	0.2	0.4	*(not vacuous mass, not uniform mass)*

Then the conflict $K_{10} = 0.4$ between $m_1(.)$ and $m_0(.)$ is the same as the conflict K_{20} between $m_2(.)$ and $m_0(.)$, and similarly the same as the conflict K_{30} between $m_3(.)$ and $m_0(.)$, which is not normal, since $m_1(.)$ is the most specific mass while $m_2(.)$ is the most unspecific mass.

Let's check other thing combining two sources using Dempster's rule:

$m_1 \oplus m_o = m_1$, $m_2 \oplus m_o = m_2$, $m_3 \oplus m_0 = m_3$,

which is not normal.

In order to get the "normal behavior" we combine $m_1(.)$ and $m_0(.)$ with *PCR5*, and similarly for others: $m_2(.)$ combined with $m_0(.)$, and $m_3(.)$ combined with $m_o(.)$.

In order to know what should have been the "normal behavior" for the conflict (the initial conflict was $K_{10} = 0.4$), let's make a small change to $m_0(.)$ as below:

	A	B	C	$A \cup B \cup C$	
m_1	1	0	0	0	*(the most specific mass)*
m_2	1/3	1/3	1/3	0	*(very unspecific mass)*
m_3	0.6	0.4	0	0	*(mass between the very unspecific and the most specific masses)*
m_0	0.3	0.2	0.1	0.4	*(not vacuous mass, not uniform mass)*

$K_{10} = 0.30$
$K_{20} = 0.40$
$K_{30} = 0.34$

Now, the conflicts are different.

VIII. Conclusion

We showed in this paper that: first the conflict was the same, no matter what was one of the sources (and it is abnormal that a non-vacuous non-uniform source has no impact on the conflict), and second that the result using Dempster's rule is not all affected by a non-vacuous non-uniform source of information.

Normally, the most specific mass (*bba*) should dominate the fusion result.

Therefore, the conflicts between sources are not correctly reflected by the conjunctive rule, and certain non-vacuous non-uniform sources are ignored by Dempster's rule in the fusion process.

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