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Refined Labels for Qualitative Information Fusion in Decision-Making Support System

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Abstract – *This paper introduces the Dezert-Smarandache (DSm) Field and Linear Algebra of Refined Labels (FLARL) useful for dealing accurately with qualitative information expressed in terms of qualitative belief functions. This work extends substantially our previous works done in DSmT framework which were mainly based on approximate qualitative operators. Here, new well justified accurate basic operators on qualitative labels (addition, subtraction, multiplication, division, root, power, etc) are presented. The end of this paper is devoted to an exemples of qualitative fusion rules based on this new FLARL approach for decision-making support.*

Keywords: DSmT, Refined label, 2-Tuples, Qualitative information fusion, Decision-making.

1 Introduction

Efficient qualitative information processing methods for reasoning under uncertainty are of crucial importance for the Information Fusion community, specially for the researchers and system designers working for the development of sophisticated semi-automatic[1] multi-source systems for information retrieval, fusion and management in defense, in robotics and so on. This is because traditional methods based only on quantitative representation and analysis are not able to adequately satisfy the needs of the development of science and technology that integrate at higher fusion levels human beliefs and reports in complex systems. Therefore qualitative knowledge representation and analysis becomes more and more important and necessary in next generations of decision-making support systems.

In the past, many mathematicians and researchers did work on this exciting topic. For example, In 1954, Polya was one of the pioneers to characterize formally the qualitative human reports [9]. Then Zadeh [15–19] made important contributions in this field in proposing a fuzzy linguistic approach to model and to combine qualitative/vague information expressed in natural language. However, since the combination process highly depends on the fuzzy operators chosen, a possible issue has been pointed out by Yager in [14]. Dubois and Prade proposed a Qualitative Possibility Theory (QPT) in Decision Analysis (DA) for the representation and the aggregation of preferences. QPT was driven by the principle of minimal specificity [3]. They use refined linguistic quantifiers to represent either the possibility distributions which encode a piece of imprecise knowledge about a situation, or to represent the qualitative belief masses over the elements in 2^{Θ}. In most of previous works, the 1-Tuple label representation label was the basis for the definition of (approximate) qualitative operations on linguistic labels. In 2000, Herrera-Martínez have proposed a 2-Tuple representation model for linguistic label in order to keep preserving the precision in qualitative operations. Herrera-Martínez 2-Tuple model can be interpreted as the standard 1-Tuple model of the label extended with a remainder [5]. Later, they proposed to deal with unbalanced labels with Multi-granular Hierarchical Linguistic Contexts in [6, 7], whose approach seems too complex in our opinions. Very recently, Wang and Hao [12, 13] have proposed another version of 2-Tuple fuzzy linguistic representation model for computing with words by considering a proportional factor as 2 order component, which can be transformed to Herrera-Martínez' 2-Tuple linguistic representation model. Obviously, These 2-Tuple models are complex and in our opinion not sufficiently well justified, because of the need of endless transformation in the course of processing of linguistic labels through algebraic operations. In this paper, were propose a new refined label model based of DSm Field and Linear Algebra of Refined Labels (FLARL) to overcome the shortcomings of previous 2-Tuple model approaches encountered in literature. The great advantage of our FLARL approach

[1] Such systems need to include human experts feedbacks in the loop of processing for a better decision-making support.

is its full theoretical justification, its simplicity and of course its ability to preserve accuracy in any derivations dealing with linguistic labels.

This paper is organized as follows: In section 2, we review the ordinary labels and then we present the model of refined labels. In section 3, we introduce the DSm Field and Linear Algebra of Refined Labels (FLARL) and we present new accurate operations on qualitative labels. In section 4, the PCR5[2] fusion rule based on this new model and on FLARL is directly extended in the qualitative domain from its original quantitative version. A simple interesting example for decision-making support is presented in section 5 to show how to work efficiently with these types of new qualitative operators. Concluding remarks are finally given in section 6.

2 The model of refined labels

Definitions of group, field, algebra, vector space, and linear algebra used in this paper can be found in [1,4,8]. Let L_1, L_2, $\dots$, L_m be labels, where $m \geq 1$ is an integer. Let's extend this set of labels with a minimum label L_0, and a maximum label L_{m+1}. In the case when the labels are equidistant, i.e. the qualitative distance between any two consecutive labels is the same, we get an exact qualitative result, and a qualitative basic belief assignment (bba) is considered normalized if the sum of all its qualitative masses is equal to $L_{\max} = L_{m+1}$. If the labels are not equidistant, we still can use all qualitative operators defined in the FLARL, but the qualitative result is approximate, and a qualitative bba is considered quasi-normalized if the sum of all its masses is equal to $L_{\max}$. We consider a relation of order defined on these labels which can be "smaller", "less in quality", "lower", etc., $L_1 < L_2 <, \dots, < L_m$. Connecting them to the classical interval $[0, 1]$, we have:

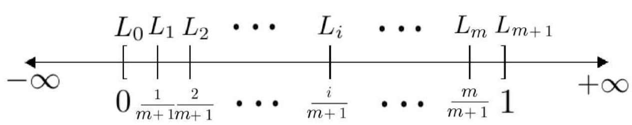

Figure 1: Ordered set of labels in $[0, 1]$.

So, $0 \equiv L_0 < L_1 < L_2 < \dots < L_i < \dots < L_m < L_{m+1} \equiv 1$, and $L_i = \frac{i}{m+1}$ for $i \in \{0, 1, 2, \dots, m, m+1\}$.

1. **Ordinary labels**: The set of labels $\tilde{L} \triangleq \{L_0, L_1, L_2, \dots, L_i, \dots, L_m, L_{m+1}\}$ whose indexes are positive integers between 0 and $m+1$, is called the set of *1-Tuple labels*. We call a set of labels to be *equidistant labels*, if the geometric distance between any two consecutive labels is the same, i.e. $L_{i+1} - L_i =$ Constant for any i.

   And, the opposite definition: a set of labels is of *non-equidistant labels* if the distances between consecutive labels is not the same, i.e. there exists $i \neq j$ such that $L_{i+1} - L_i \neq L_{j+1} - L_j$.

   For simplicity and symmetry of the calculations, we further consider the case of equidistant labels. But the same procedures can *approximately* work for non-equidistant labels.

   This set of 1-Tuple labels is isomorphic with the numerical set $\{\frac{i}{m+1}, i = 0, 1, \dots, m+1\}$ through the isomorphism $f_{\tilde{L}}(L_i) = \frac{i}{m+1}$.

2. **Refined labels**: We theoretically extend the set of labels $\tilde{L}$ to the left and right sides of the interval $[0, 1]$ towards $-\infty$ and respectively $+\infty$. So, we define:
$$L_{\mathbb{Z}} \triangleq \{\frac{j}{m+1}, j \in \mathbb{Z}\}$$
   where $\mathbb{Z}$ is the set of all positive and negative integers (zero included).

   Thus:
$$L_{\mathbb{Z}} = \{\dots, L_{-j}, \dots, L_{-1}, L_0, L_1, \dots, L_j, \dots\}$$
   $= \{L_j, j \in \mathbb{Z}\}$, i.e. the set of extended labels with positive and negative indexes.

   Similarly, one defines $L_{\mathbb{Q}} \triangleq \{L_q, q \in \mathbb{Q}\}$ as the set of labels whose indexes are fractions. $L_{\mathbb{Q}}$ is isomorphic with $\mathbb{Q}$ through the isomorphism $f_{\mathbb{Q}}(L_q) = \frac{q}{m+1}$ for any $q \in \mathbb{Q}$.

   Even more general, we can define:
$$L_{\mathbb{R}} \triangleq \{\frac{r}{m+1}, r \in \mathbb{R}\}$$
   where $\mathbb{R}$ is the set of all real numbers. $L_{\mathbb{R}}$ is isomorphic with $\mathbb{R}$ through the isomorphism $f_{\mathbb{R}}(L_r) = \frac{r}{m+1}$ for any $r \in \mathbb{R}$.

3 DSm Field and Linear Algebra of Refined labels (FLARL)

We will prove that $(L_{\mathbb{R}}, +, \times)$ is a field, where $+$ is the vector addition of labels, and $\times$ is the vector multiplication of labels which is called the *DSm field of refined labels*. Therefore, for the first time we introduce decimal or refined labels, i.e. labels whose index is decimal. For example: $L_{\frac{3}{2}}$ which is $L_{1.5}$ means a label in the middle of the label interval $[L_1, L_2]$. We also theoretically introduce *negative* labels, L_{-i} which is equal to $-L_i$, that occur in qualitative calculations.

Even more, $(L_{\mathbb{R}}, +, \times, \cdot)$, where $\cdot$ means scalar product, is a commutative linear algebra over the field of real numbers $\mathbb{R}$, with unit element, and whose each non-null element is invertible with respect to the multiplication of labels. This is called the DSm *Field and Linear Algebra of Refined labels* (FLARL for short).

[2] i.e. the proportional conflict redistribution rule no. 5 introduced in DSmT [11].

3.1 Qualitative operators on FLARL

Let's define the *qualitative operators* on this linear algebra. Let a, b, c in $\mathbb{R}$, and the labels $L_a = \frac{a}{m+1}$, $L_b = \frac{b}{m+1}$ and $L_c = \frac{c}{m+1}$. Let the scalars α, β in $\mathbb{R}$.

- **Vector Addition (addition of labels)**:

$$L_a + L_b = L_{a+b} \tag{1}$$

since $\frac{a}{m+1} + \frac{b}{m+1} = \frac{a+b}{m+1}$.

- **Vector Subtraction (subtraction of labels)**:

$$L_a - L_b = L_{a-b} \tag{2}$$

since $\frac{a}{m+1} - \frac{b}{m+1} = \frac{a-b}{m+1}$.

- **Vector Multiplication (multiplication of labels)**:

$$L_a \times L_b = L_{(ab)/(m+1)} \tag{3}$$

since $\frac{a}{m+1} \cdot \frac{b}{m+1} = \frac{(ab)/(m+1)}{m+1}$.

- **Scalar Multiplication (number times label)**:

$$\alpha \cdot L_a = L_a \cdot \alpha = L_{\alpha a} \tag{4}$$

since $\alpha \cdot \frac{a}{m+1} = \frac{\alpha a}{m+1}$.
As a particular case, for $\alpha = -1$, we get: $-L_a = L_{-a}$.
Also, $\frac{L_a}{\beta} = L_a \div \beta = \frac{1}{\beta} \cdot L_a = L_{\frac{a}{\beta}}$.

- **Vector Division (division of labels)**:

$$L_a \div L_b = L_{(a/b)(m+1)} \tag{5}$$

since $\frac{a}{m+1} \div \frac{b}{m+1} = \frac{a}{b} = \frac{(a/b)(m+1)}{m+1}$.

- **Scalar Power**:

$$(L_a)^p = L_{a^p/(m+1)^{p-1}} \tag{6}$$

since $(\frac{a}{m+1})^p = \frac{a^p/(m+1)^{p-1}}{m+1}$, $\forall p \in \mathbb{R}$.

- **Scalar Root**:

$$\sqrt[k]{L_a} = (L_a)^{\frac{1}{k}} = L_{a^{\frac{1}{k}}/(m+1)^{\frac{1}{k}-1}} \tag{7}$$

which results from replacing $p = \frac{1}{k}$ in the power formula (6), $\forall k$ integer ≥ 2.

3.2 The DSm field of refined labels

Since $(L_{\mathbb{R}}, +, \times)$ is isomorphic with the set of real numbers $(\mathbb{R}, +, \times)$, it results that $(L_{\mathbb{R}}, +, \times)$ is a field, called *DSm field of refined labels*. The field isomorphism: $f_{\mathbb{R}} : L_{\mathbb{R}} \rightarrow \mathbb{R}$, $f_{\mathbb{R}}(L_r) = \frac{r}{m+1}$ satisfies the axioms:

Axiom A1:

$$f_{\mathbb{R}}(L_a + L_b) = f_{\mathbb{R}}(L_a) + f_{\mathbb{R}}(L_b) \tag{8}$$

since $f_{\mathbb{R}}(L_a+L_b) = f_{\mathbb{R}}(L_{a+b}) = \frac{a+b}{m+1}$ and $f_{\mathbb{R}}(L_a) + f_{\mathbb{R}}(L_b) = \frac{a}{m+1} + \frac{b}{m+1} = \frac{a+b}{m+1}$.

Axiom A2:

$$f_{\mathbb{R}}(L_a \times L_b) = f_{\mathbb{R}}(L_a) \cdot f_{\mathbb{R}}(L_b) \tag{9}$$

since $f_{\mathbb{R}}(L_a \times L_b) = f_{\mathbb{R}}(L_{(ab)/(m+1)}) = \frac{ab}{m+1}$ and $f_{\mathbb{R}}(L_a) \cdot f_{\mathbb{R}}(L_b) = \frac{a}{m+1} \cdot \frac{b}{m+1} = \frac{ab}{(m+1)^2}$.

$(L_{\mathbb{R}}, +, \cdot)$ is a *vector space of refined labels* over the field of real numbers $\mathbb{R}$, since $(L_{\mathbb{R}}, +)$ is a commutative group, and the scalar multiplication (which is an external operation) verifies the axioms:

Axiom B1:

$$1 \cdot L_a = L_{1 \cdot a} = L_a \tag{10}$$

Axiom B2:

$$(\alpha \cdot \beta) \cdot L_a = \alpha \cdot (\beta \cdot L_a) \tag{11}$$

since both, left and right sides, are equal to $L_{\alpha\beta a}$

Axiom B3:

$$\alpha \cdot\ (L_a + L_b) = \alpha \cdot\ L_a + \alpha \cdot\ L_b \tag{12}$$

since $\alpha \cdot (L_a + L_b) = \alpha \cdot L_{a+b} = L_{\alpha(a+b)} = L_{\alpha a + \alpha b} = L_{\alpha a} + L_{\alpha b} = \alpha \cdot L_a + \alpha \cdot L_b$.

Axiom B4:

$$(\alpha + \beta) \cdot\ L_a = \alpha \cdot\ L_a + \beta \cdot\ L_a \tag{13}$$

since $(\alpha + \beta) \cdot\ L_a = L_{(\alpha+\beta)a} = L_{\alpha a + \beta a} = L_{\alpha a} + L_{\beta a} = \alpha \cdot\ L_a + \beta \cdot\ L_a$.

$(L_{\mathbb{R}}, +, \times, \cdot)$ is a *a Linear Algebra of Refined Labels* over the field $\mathbb{R}$ of real numbers, called *DSm Linear Algebra of Refined Labels* (DSm-LARL for short), which is commutative, with identity element (which is L_{m+1}) for vector multiplication, and whose non-null elements (labels) are invertible with respect to the vector multiplication. This occurs since $(L_{\mathbb{R}}, +, \cdot)$ is a vector space, $(L_{\mathbb{R}}, \times)$ is a commutative group, the set of scalars $\mathbb{R}$ is well-known as a field, and also one has:

- The vector multiplication is associative:

Axiom C1 (Associativity of vector multiplication):

$$L_a \times (L_b \times L_c) = (L_a \times L_b) \times L_c \tag{14}$$

since $L_a \times (L_b \times L_c) = L_a \times L_{(b \cdot c)/(m+1)} = L_{(a \cdot b \cdot c)/(m+1)^2}$ while $(L_a \times L_b) \times L_c = L_{(ab)/(m+1)} \times L_c = L_{(a \cdot b \cdot c)/(m+1)^2}$ as well.

- The vector multiplication is distributive with respect to addition:

  Axiom C2:

$$L_a \times (L_b + L_c) = L_a \times L_b + L_a \times L_c \quad (15)$$

  since $L_a \times (L_b + L_c) = L_a \times L_{b+c} = L_{(a\cdot(b+c))/(m+1)}$ and $L_a \times L_b + L_a \times L_c = L_{(ab)/(m+1)} + L_{(ac)/(m+1)} = L_{(ab+ac)/(m+1)} = L_{(a(b+c))/(m+1)}$.

  Axiom C3:

$$(L_a + L_b) \times L_c = L_a \times L_c + L_b \times L_c \quad (16)$$

  since $(L_a + L_b) \times L_c = L_{a+b} \times L_c = L_{((a+b)c))/(m+1)} = L_{(ac+bc)/(m+1)} = L_{(ac)/(m+1)+(bc)/(m+1)} = L_{(ac)/(m+1)} + L_{(bc)/(m+1)} = L_a \times L_c + L_b \times L_c$.

  Axiom C4:

$$\alpha\cdot(L_a \times L_b) = (\alpha\cdot L_a)\times L_b = L_a \times(\alpha\cdot L_b) \quad (17)$$

  since $\alpha \cdot (L_a \times L_b) = \alpha \cdot L_{(ab)/(m+1)} = L_{(\alpha ab)/(m+1)} = L_{((\alpha a)b)/(m+1)} = L_{\alpha a} \times L_b = (\alpha \cdot L_a) \times L_b$; but also $L_{(\alpha ab)/(m+1)} = L_{(a(\alpha b))/(m+1)} = L_a \times L_{\alpha b} = L_a \times (\alpha \cdot L_b)$.

- The *Unitary Element* for vector multiplication is L_{m+1}, since

  Axiom D1:
  $\forall a \in \mathbb{R}$,

$$L_a \times L_{m+1} = L_{m+1} \times L_a = L_{(a(m+1))/(m+1)} = L_a \quad (18)$$

- All $L_a \neq L_0$ are *invertible* with respect to vector multiplication and the inverse of L_a is $(L_a)^{-1}$ with:

  Axiom E1:

$$(L_a)^{-1} = L_{(m+1)^2/a} = \frac{1}{L_a} \quad (19)$$

  since $L_a \times (L_a)^{-1} = L_a \times L_{(m+1)^2/a} = L_{(a\cdot(m+1)^2/a)/(m+1)} = L_{m+1}$.

Therefore the DSm linear algebra is a *Division Algebra*. DSm Linear Algebra is also a trivial Lie Algebra since we can define a law:

$$(L_a, L_b) \to [L_a, L_b] = L_a \times L_b - L_b \times L_a = L_0$$

such that

$$[L_a, L_a] = L_0 \quad (20)$$

and Jacobi identity is satisfied:

$$[L_a, [L_b, L_c]] + [L_b, [L_c, L_a]] + [L_c, [L_a, L_b]] = L_0 \quad (21)$$

Actually $(L_\mathbb{R}, +, \times, \cdot)$ is a field, and therefore in particular a ring, and any ring with the law: $[x, y] = xy - yx$ is a Lie Algebra.

We can extend the field isomorphism $f_\mathbb{R}$ to a linear algebra isomorphism by defining[3]: $f_\mathbb{R} : \mathbb{R} \cdot L_\mathbb{R} \to \mathbb{R} \cdot \mathbb{R}$ with $f_\mathbb{R}(\alpha \cdot L_{r_1}) = \alpha \cdot f_\mathbb{R}(L_{r_1})$ since $f_\mathbb{R}(\alpha \cdot L_{r_1}) = f_\mathbb{R}(L_{(\alpha\cdot r_1)}) = \alpha \cdot r_1/(m+1) = \alpha \cdot \frac{r_1}{m+1} = \alpha \cdot f_\mathbb{R}(L_{r_1})$. Since $(\mathbb{R}, +, \cdot)$ is a trivial linear algebra over the field of reals $\mathbb{R}$, and because $(L_\mathbb{R}, +, \cdot)$ is isomorphic with it through the above $f_\mathbb{R}$ linear algebra isomorphism, it results that $(L_\mathbb{R}, +, \cdot)$ is also a linear algebra which is associative and commutative.

3.3 More operators

Let's now define more new operators, such as scalar-vector (mixed) addition, scalar-vector (mixed) subtraction, scalar-vector (mixed) division, vector power, and vector root.

They might be surprising since such *strange* hybrid operators have not been already defined, but for DSm Linear Algebra they make perfect sense since $(L_\mathbb{R}, +, \times)$ is isomorphic to $(\mathbb{R}, +, \times)$ and a label is equivalent to a real number, since for a fixed $m \geq 1$ we have:

$$\forall L_a \in L_\mathbb{R}, \exists! r \in \mathbb{R}, r = \frac{a}{m+1} \quad \text{such that} \quad L_a = r$$

and reciprocally

$$\forall r \in \mathbb{R}, \exists! L_a \in L_\mathbb{R}, L_a = L_{r(m+1)} \quad \text{such that} \quad r = L_a$$

In consequence, we can substitute a real number by a label, and reciprocally.

- **Scalar-vector (mixed) addition**:

$$L_a + \alpha = \alpha + L_a = L_{a+\alpha(m+1)} \quad (22)$$

  since $L_a + \alpha = L_a + \frac{\alpha(m+1)}{(m+1)} = L_a + L_{\alpha(m+1)} = L_{a+\alpha(m+1)}$.

- **Scalar-vector (mixed) subtractions**:

$$L_a - \alpha = L_{a-\alpha(m+1)} \quad (23)$$

  since $L_a - \alpha = L_a - \frac{\alpha(m+1)}{(m+1)} = L_a - L_{\alpha(m+1)} = L_{a-\alpha(m+1)}$.

$$\alpha - L_a = L_{\alpha(m+1)-a} \quad (24)$$

  since $\alpha - L_a = \frac{\alpha(m+1)}{(m+1)} - L_a = L_{\alpha(m+1)} - L_a = L_{\alpha(m+1)-a}$.

[3] where $\cdot$ denotes the scalar multiplication.

- **Scalar-vector (mixed) divisions**:

$$L_a \div \alpha = \frac{L_a}{\alpha} = \frac{1}{\alpha} \cdot L_a = L_{\frac{a}{\alpha}}, \text{ for } \alpha \neq 0, \quad (25)$$

which is equivalent to the scalar multiplication $(\frac{1}{\alpha}) \cdot L_a$ where $\frac{1}{\alpha} \in \mathbb{R}$.

$$\alpha \div L_a = L_{\frac{\alpha(m+1)^2}{a}} \quad (26)$$

since $\alpha \div L_a = \frac{\alpha(m+1)}{m+1} \div L_a = L_{\alpha(m+1)} \div L_a = L_{\alpha(m+1)/a)\cdot(m+1)} = L_{\frac{\alpha(m+1)^2}{a}}$.

- **Vector power**:

$$(L_a)^{L_b} = L_{a^{\frac{b}{m+1}}/(m+1)^{\frac{b-m-1}{m+1}}} \quad (27)$$

since $(L_a)^{L_b} = (L_a)^{\frac{b}{m+1}} = L_{a^{\frac{b}{m+1}}/(m+1)^{\frac{b}{m+1}-1}} = L_{a^{\frac{b}{m+1}}/(m+1)^{\frac{b-m-1}{m+1}}}$ where we replaced $p = \frac{b}{m+1}$ in the scalar product formula.

- **Vector root**:

$$\sqrt[L_b]{L_a} = L_{a^{\frac{m+1}{b}}/(m+1)^{\frac{m-b+1}{b}}} \quad (28)$$

since $\sqrt[L_b]{L_a} = (L_a)^{\frac{1}{L_b}} = (L_a)^{\frac{1}{b/(m+1)}} = (L_a)^{\frac{m+1}{b}} = L_{a^{\frac{m+1}{b}}/(m+1)^{\frac{m+1}{b}-1}} = L_{a^{\frac{m+1}{b}}/(m+1)^{\frac{m-b+1}{b}}}$.

$L_{\mathbb{R}}$ endowed with all these scalar and vector (addition, subtraction, multiplication, division, power, and root) operators becomes a powerful mathematical tool in the DSm field and simultaneously linear algebra of refined labels.

Therefore, if we want to work with only 1-Tuple labels (ordinary labels), in all these operators we set the restrictions that indexes are integers belonging to $\{0, 1, 2, \ldots, m, m+1\}$; if an index is less than 0 then we force it to be 0, and if greater than $m+1$ we force it to $m+1$.

For 2-Tuple labels defined by Herrera and Martinez [5], that have the form (L_i, σ_i^h) where i is an integer and σ_i^h is a remainder

$$\sigma_i^h \in [-\frac{0.5}{m+1}, \frac{0.5}{m+1})$$

we use the scalar addition (when $\sigma_i^h \geq 0$) or scalar subtraction (when $\sigma_i^h < 0$) as defined previously in order to transform a 2-Tuple label into a refined label and then use all previous twelve operators defined in DSm Linear Algebra. Actually, $(L_i, \sigma_i^h) = L_i + \sigma_i^h$ and it doesn't matter if σ_i^h is positive, zero, or negative. Of course, other 2-Tuple models such those defined by Wang and Hao can also be represented by our refined labels model, because there is a linear relation between both 2-Tuple models. That is, our refined label model is a generalized accurate model for 1-Tuple and 2-Tuple models.

4 Qualitative DSm fusion rules

From the refined label model of qualitative beliefs and the previous operators, we are able to extend the DSm classic (DSmC) and the PCR5 numerical fusion rules proposed in Dezert-Smarandache Theory (DSmT) [10, 11] and all other numerical fusion rules from any fusion theory (DST, TBM, etc.) in the qualitative domain. We will not go in deep presentation of these rules since they have been already widely presented in the literature, tutorials and conferences. The qualitative belief mass/assignment (qba) $q_r m(\cdot)$ based on Refined labels representation is defined as $qm(\cdot)$: $G^\Theta \to L_r$ such that $qm(\emptyset) = L_0$ and $\sum_{A \in G^\Theta} qm(A) = L_{m+1}$. The q-extensions of $DSmC$ and $PCR5$ fusion rules for two sources[4] on a frame Θ based on the Refined label operators are then given by (the extension for $N > 2$ sources is possible):

- q-extension of $DSmC$ fusion rule ($qDSmC$): $qm_{DSmC}(\emptyset) = L_0$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$

$$qm_{DSmC}(X) = \sum_{\substack{X_1, X_2, \cdots, X_k \in D^\Theta \\ X_1 \cap X_2, \cdots, \cap X_k = X}} \prod_{i=1}^{k} qm_i(X_i) \quad (29)$$

- q-extension of PCR5 fusion rule ($qPCR5$): $qm_{PCR5}(\emptyset) = L_0$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$

$$qm_{PCR5}(X) = qm_{12}(X) + \sum_{\substack{Y \in G^\Theta \setminus \{X\} \\ X \cap Y = \emptyset}} [\frac{qm_1(X)^2 qm_2(Y)}{qm_1(X) + qm_2(Y)} + \frac{qm_2(X)^2 qm_1(Y)}{qm_2(X) + qm_1(Y)}] \quad (30)$$

where $qm_{12}(X)$ corresponds to the qualitative q-extension of the conjunctive consensus and G^Θ is the generic notation for the fusion space on which the masses of beliefs have been defined. More precisely, $G^\Theta = 2^\Theta$ when working with Shafer's model for the frame Θ (i.e. all elements of Θ are truly exclusive), $G^\Theta = D^\Theta$ when working with a free DSm model (i.e. none of elements of Θ are disjoint), $G^\Theta = S^\Theta$ when working with a minimal refinement of Θ when the refinement is possible and makes sense, or G^Θ can be any subsets of D^Θ or of S^Θ if some integrity constraints must be taken into account (see [11] for details, discussions and examples of power-set 2^Θ, hyper-power set D^Θ and super-power set S^Θ).

[4] The extension of these fusion rules for the fusion of $s > 2$ sources can be found in [11].

5 Examples of decision-making

5.1 A Bayesian example

Let's consider an investment corporation which must choose one of three projects in $\Theta = \{\theta_1, \theta_2, \theta_3\}$ (assume here that Shafer's model holds for simplicity) to invest through two consulting departments. A set of qualitative values are used to describe the opinions of two consulting companies, i.e. I $\mapsto$ Impossible, EU $\mapsto$ Extremely-Unlikely, VLC $\mapsto$ Very-Low-Chance, LLC $\mapsto$ Little-Low-Chance, SC $\mapsto$ Small-Chance, IM $\mapsto$ IT-May, MC $\mapsto$ Meanful-Chance, LBC $\mapsto$ Little-Big-Chance, BC $\mapsto$ Big-Chance, ML $\mapsto$ Most-likely, C $\mapsto$ Certain. So, we consider the set of ordered linguistic labels $L = \{L_0 \equiv I, L_1 \equiv EU, L_2 \equiv VLC, L_3 \equiv LLC, L_4 \equiv SC, L_5 \equiv IM, L_6 \equiv MC, L_7 \equiv LBC, L_8 \equiv BC, L_9 \equiv ML, L_{10} \equiv C\}$ and in this example $m = 9$. The opinions of the two consulting companies/sources are given in Table 1 according to Refined label model.

$qm(\cdot)$	θ_1	θ_2	θ_3
Source no 1	$L_{4.3}$	$L_{2.7}$	L_3
Source no 2	L_5	$L_{2.1}$	$L_{2.9}$

Table 1: Qualitative bba's to combine.

When working in Shafer model, according to the $qDSmC$ combinational rule (29), the fusion of the two qualitative sources of evidences gives:

$$qm_{DSmC}(\theta_1) = L_{(\frac{4.3\times5.0}{10})} = L_{2.15}$$
$$qm_{DSmC}(\theta_2) = L_{(\frac{2.7\times2.1}{10})} = L_{0.567}$$
$$qm_{DSmC}(\theta_3) = L_{(\frac{3.0\times2.9}{10})} = L_{0.87}$$
$$qm_{DSmC}(\theta_1\cap\theta_2) = L_{(\frac{4.3\times2.1}{10}+\frac{5.0\times2.7}{10})} = L_{2.253}$$
$$qm_{DSmC}(\theta_1\cap\theta_3) = L_{(\frac{4.3\times2.9}{10}+\frac{5.0\times3.0}{10})} = L_{2.747}$$
$$qm_{DSmC}(\theta_2\cap\theta_3) = L_{(\frac{2.7\times2.9}{10}+\frac{2.1\times3.0}{10})} = L_{1.413}$$

Note that $qm_{DSmC}(.)$ is normalized since $L_{2.15} + L_{0.567} + L_{0.87} + L_{2.253} + L_{2.747} + L_{1.413} = L_{m+1} = L_{10}$ (here $m = 9$ interior labels).

Based on $qPCR5$, the masses of the partial conflicts $\theta_1\cap\theta_2$, $\theta_1\cap\theta_3$ and $\theta_2\cap\theta_3$ are redistributed to those belief masses involved in these conflicts according to (30). One gets:

$$qm_{xA1}(\theta_1) = \frac{L_{4.3}\times L_{0.903}}{L_{6.4}} \approx L_{0.607}$$
$$qm_{yA1}(\theta_2) = \frac{L_{2.1}\times L_{0.903}}{L_{6.4}} \approx L_{0.704}$$
$$qm_{xB1}(\theta_1) = \frac{L_5\times L_{1.35}}{L_{7.7}} \approx L_{0.877}$$
$$qm_{yB1}(\theta_2) = \frac{L_{2.7}\times L_{1.35}}{L_{7.7}} \approx L_{0.473}$$
$$qm_{xA2}(\theta_1) = \frac{L_{4.3}\times L_{1.247}}{L_{7.02}} \approx L_{0.745}$$
$$qm_{yA2}(\theta_3) = \frac{L_{2.9}\times L_{1.247}}{L_{7.02}} \approx L_{0.502}$$

and similarly, one has $qm_{xB2}(\theta_1) \approx L_{0.938}$, $qm_{yB2}(\theta_3) \approx L_{0.563}$, $qm_{xA3}(\theta_2) \approx L_{0.377}$, $qm_{yA3}(\theta_3) \approx L_{0.405}$, $qm_{xB3}(\theta_2) \approx L_{0.259}$ and $qm_{yB3}(\theta_3) \approx L_{0.370}$. Thus, one finally gets:

$$\begin{aligned} qm_{PCR5}(\theta_1) &= qm_{12}(\theta_1) + qm_{xA1}(\theta_1) + qm_{xB1}(\theta_1) \\ &\quad + qm_{xA2}(\theta_1) + qm_{xB2}(\theta_1) \approx L_{5.315} \\ qm_{PCR5}(\theta_2) &= qm_{12}(\theta_2) + qm_{yA1}(\theta_2) + qm_{yB1}(\theta_2) \\ &\quad + qm_{xA3}(\theta_2) + qm_{xB3}(\theta_2) \approx L_{1.974} \\ qm_{PCR5}(\theta_3) &= qm_{12}(\theta_3) + qm_{yA2}(\theta_3) + qm_{yB2}(\theta_3) \\ &\quad + qm_{yA3}(\theta_3) + qm_{yB3}(\theta_3) \approx L_{2.711} \end{aligned}$$

One can easily verified that $qm_{PCR5}(.)$ is also normalized since $L_{5.315} + L_{1.974} + L_{2.711} = L_{m+1} = L_{10}$.

If the decision-making on elements of Θ is based on the criterion of the maximum of qualitative belief usually adopted in the literature, one will decide θ_1 since one has in this Bayesian fusion result case:

$$(qBel(\theta_1) = qm_{PCR5}(\theta_1)) > (qBel(\theta_2) = qm_{PCR5}(\theta_2))$$
$$(qBel(\theta_1) = qm_{PCR5}(\theta_1)) > (qBel(\theta_3) = qm_{PCR5}(\theta_3))$$

Therefore, the investment corporation should normally invest in the project θ_1 based on the qualitative sources of evidences available.

5.2 A non Bayesian example

Let's consider a more general example with two non Bayesian qualitative sources of evidences. As previously, we consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and the set of linguistic labels $L = \{L_0, L_1, \dots, L_{10}\}$ ($m = 9$ interior labels). We consider the DSm hybrid model with the exclusivity constraints $\theta_1\cap\theta_3 = \emptyset$ and $\theta_2\cap\theta_3 = \emptyset$. We consider the following qualitative bba's to combine:

$qm(\cdot)$	θ_1	θ_2	θ_3
Source no 1	L_2	L_3	L_1
Source no 2	L_2	L_1	L_2

$qm(\cdot)$	$\theta_2\cup\theta_3$	$\theta_1\cup\theta_2\cup\theta_3$
Source no 1	L_1	L_3
Source no 2	L_3	L_2

Table 2: Qualitative bba's to combine.

Using qualitative DSmC rule based on free DSm model, one obtains[5]:

[5] The verification is left to the reader.

$$qm_{DSmC}(\theta_1) = L_{1.4}$$
$$qm_{DSmC}(\theta_2) = L_{2.2}$$
$$qm_{DSmC}(\theta_3) = L_{1.5}$$
$$qm_{DSmC}(\theta_2 \cup \theta_3) = L_{1.4}$$
$$qm_{DSmC}(\theta_1 \cup \theta_2 \cup \theta_3) = L_{0.6}$$
$$qm_{DSmC}(\theta_1 \cap \theta_2) = L_{0.8} + L_{0.8} = L_{1.6}$$
$$qm_{DSmC}(\theta_1 \cap \theta_3) = L_{0.6}$$
$$qm_{DSmC}(\theta_2 \cap \theta_3) = L_{0.7}$$

Working now with the hybrid DSm model, one uses the qualitative PCR5 rule to transfer the conflicting mass

$$\begin{aligned} qm_{DSmC}(\theta_1 \cap \theta_3) &= qm_1(\theta_1)qm_2(\theta_3) + qm_2(\theta_1)qm_1(\theta_3) \\ &= L_{0.4} + L_{0.2} = L_{0.6} \end{aligned}$$

to θ_1 and θ_3 according to:

$$\frac{x_{\theta_1}}{L_2} = \frac{z_{\theta_3}}{L_2} = \frac{L_2 \times L_2}{L_2 + L_2} = \frac{L_{\frac{2\cdot 2}{10}}}{L_{2+2}} = \frac{L_{0.4}}{L_4} = L_{\frac{0.4}{4}\cdot 10} = L_1$$

whence $x_{\theta_1} = z_{\theta_3} = L_2 \times L_1 = L_{\frac{2\cdot 1}{10}} = L_{0.2}$.

Similarly, the partial conflicting qualitative mass $qm_2(\theta_1)qm_1(\theta_3) = L_{0.2}$ involved in $qm_{DSmC}(\theta_1 \cap \theta_3)$ is redistributed to θ_1 and θ_3 according to:

$$\frac{x'_{\theta_1}}{L_2} = \frac{z'_{\theta_3}}{L_1} = \frac{L_2 \times L_1}{L_2 + L_1} = \frac{L_{\frac{2\cdot 1}{10}}}{L_{2+1}} = \frac{L_{0.2}}{L_3} = L_{\frac{0.2}{3}\cdot 10} = L_{\frac{2}{3}}$$

whence $x'_{\theta_1} = L_2 \times L_{\frac{2}{3}} = L_{(2\cdot\frac{2}{3})/10} = L_{\frac{4}{30}} \approx L_{0.13}$ and $z'_{\theta_3} = L_1 \times L_{\frac{2}{3}} = L_{(1\cdot\frac{2}{3})/10} = L_{\frac{20}{30}} \approx L_{0.07}$.
Using again PCR5 to transfer

$$\begin{aligned} qm_{DSmC}(\theta_2 \cap \theta_3) &= qm_1(\theta_2)qm_2(\theta_3) + qm_2(\theta_2)qm_1(\theta_3) \\ &= L_{0.6} + L_{0.1} = L_{0.7} \end{aligned}$$

to θ_2 and θ_3 according to:

$$\frac{y_{\theta_2}}{L_3} = \frac{z''_{\theta_3}}{L_2} = \frac{L_3 \times L_2}{L_3 + L_2} = \frac{L_{\frac{3\cdot 2}{10}}}{L_{3+2}} = \frac{L_{0.6}}{L_5} = L_{\frac{0.6}{5}\cdot 10} = L_{1.2}$$

whence $y_{\theta_2} = L_3 \times L_{1.2} = L_{\frac{3\cdot 1.2}{10}} = L_{0.36}$ and $z''_{\theta_3} = L_2 \times L_{1.2} = L_{\frac{2\cdot 1.2}{10}} = L_{0.24}$. Also,

$$\frac{y'_{\theta_2}}{L_1} = \frac{z'''_{\theta_3}}{L_1} = \frac{L_1 \times L_1}{L_1 + L_1} = \frac{L_{\frac{1\cdot 1}{10}}}{L_{1+1}} = \frac{L_{0.1}}{L_2} = L_{\frac{0.1}{2}\cdot 10} = L_{0.5}$$

whence $y'_{\theta_2} = z'''_{\theta_3} = L_1 \times L_{0.5} = L_{\frac{1\cdot 0.5}{10}} = L_{0.05}$.

Adding the masses of x's to the mass of θ_1, y's to the mass of θ_2, and z's to the mass of θ_3 one finally gets:

$$qm_{PCR5}(\theta_1) = L_{1.4} + L_{0.2} + L_{0.13} = L_{1.73}$$
$$qm_{PCR5}(\theta_2) = L_{2.2} + L_{0.36} + L_{0.05} = L_{2.61}$$
$$qm_{PCR5}(\theta_3) = L_{1.5} + L_{0.2} + L_{0.07} + L_{0.24} + L_{0.05} = L_{2.06}$$
$$qm_{PCR5}(\theta_2 \cup \theta_3) = L_{1.4}$$
$$qm_{PCR5}(\theta_1 \cup \theta_2 \cup \theta_3) = L_{0.6}$$
$$qm_{PCR5}(\theta_1 \cap \theta_2) = L_{1.6}$$

- If the decision-making is based on the criteria of the maximum of qualitative belief, one will decide θ_2 since $qBel(\theta_2) = L_{4.21}$ takes the maximum value. Indeed,

$$\begin{aligned} qBel(\theta_1) &= qm_{PCR5}(\theta_1) + qm_{PCR5}(\theta_1 \cap \theta_2) \\ &= L_{1.73} + L_{1.6} = L_{3.33} \\ qBel(\theta_2) &= qm_{PCR5}(\theta_2) + qm_{PCR5}(\theta_1 \cap \theta_2) \\ &= L_{2.61} + L_{1.6} = L_{4.21} \\ qBel(\theta_3) &= qm_{PCR5}(\theta_3) = L_{2.06} \end{aligned}$$

- If the decision-making is based on the criterion of the maximum of qualitative subjective probability defined by[6] $qDSmP_\epsilon(\emptyset) = L_0$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$ by:

$$qDSmP_\epsilon(X) = \sum_{Y \in G^\Theta} \frac{\sum\limits_{\substack{Z \subseteq X \cap Y \\ \mathcal{C}(Z)=1}} qm(Z) + \epsilon \cdot \mathcal{C}(X \cap Y)}{\sum\limits_{\substack{Z \subseteq Y \\ \mathcal{C}(Z)=1}} qm(Z) + \epsilon \cdot \mathcal{C}(Y)} qm(Y) \quad (31)$$

where $\epsilon \geq 0$ is a tuning parameter and G^Θ corresponds to the hyper-power set including eventually all the integrity constraints (if any) of the frame; $\mathcal{C}(X \cap Y)$ and $\mathcal{C}(Y)$ denote the DSm cardinals of the sets $X \cap Y$ and Y respectively. When at least a mass of the subsets that we transfer to is zero, we always take $\epsilon > 0$ (which is the case in this example).

Applying (31) (which involves FLARL operators) with an arbirary small tuning parameter (here we take $\epsilon = 0.01$), one obtains (the verification is left to the reader):

$$qDSmP_{\epsilon=0.01}(\theta_1 \cap \theta_2) = L_{6.55}$$
$$qDSmP_{\epsilon=0.01}(\theta_1) = L_{6.665}$$
$$qDSmP_{\epsilon=0.01}(\theta_2) = L_{6.745}$$
$$qDSmP_{\epsilon=0.01}(\theta_3) = L_{3.14}$$

Therefore, one should decide θ_2 since it has the maximum of subjective probability. It is worth to note that the difference between $qDSmP_{\epsilon=0.01}(\theta_1)$ and $qDSmP_{\epsilon=0.01}(\theta_2)$ is very small. This indicates a possible risk of error in deciding θ_2 against θ_1 if this criterion is chosen. When the max of belief criterion is used, one sees a larger difference between $qBel(\theta_1)$ and $qBel(\theta_2)$ which makes the decision θ_2 more obvious to take.

[6]This is a direct extension in the qualitative domain of the quantitative DSmP formula presented in details with examples in [2].

6 Conclusions

In this paper we have introduced the DSm Field and Linear Algebra of Refined Labels to palliate the limitations (complexity and inherent approximations) of the 2-Tuple models of representation of linguistic labels proposed in the literature until very recently. Based on our new theoretical approach, we are able to deal accurately in derivations with linguistic labels and so maintain high precision in the results. The linguistic refined labels model that we have proposed in this paper can be seen as a generalization of 2-Tuple and 1-Tuple linguistic labels generally used by people working in artificial intelligence with fuzzy sets and dealing with qualitative information. This work is very valuable to information retrieval, fusion and management, especially to decision-making support system from qualitative belief functions coming from human experts reports.

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