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Fundamental group and complete parts of Neutrosophic Quadruple H_v-groups

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Abstract. It is well known that H_v-groups and groups are connected through regular relations. The purpose of this paper is to find a similar connection between neutrosophic H_v-groups and neutrosophic groups by using the concept of fundamental relations on Hv-groups. First, we characterize the complete parts of neutrosophic Hv-groups. Then we study their (strongly) regular relations. Finally, we find their fundamental group.

Keywords: neutrosophic quadruple number; neutrosophic quadruple H_v-group; fundamental group; complete part

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1. **Introduction**

Fuzzy set theory [1] has many real-world applications, but it is not suitable for simulate an indeterminate issue in an abstract situation. By giving indeterminates representation, neutrosophic theory has advanced an important concept. One of the essential aspects in practically all problems in the real world is uncertainty or indeterminacy. Fuzzy theory is used to model uncertainty, whereas neutrosophic theory is employed to represent indeterminacy. In 1995, F. Smarandache created the concept of neutrosophy to represent problems involving indeterminates. For further background on neutrosophy and neutrosophic algebraic structures, see [2–6]. By assuming that the result of "interaction" between two elements of a non-empty set is a non-void set of elements, one is obviously generalizing the definition of a group. (and not only one element, as for groups), A hypergroup was first proposed in 1934 at the eighth congress of Scandinavian mathematicians by the French mathematician Frederic Marty [7]. The rule that describes such a structure is known as a "hyperoperation", and the theory of algebraic structures with at least one multi-valued operation is recognized as the Hyperstructure Theory. Marty's motivation to introduce hypergroups is that the quotient of a group modulo any subgroup (not necessarily normal) is a hypergroup. This theory has been studied in the following decades and nowadays by many mathematicians. As a generalization of algebraic hyperstructures, Vougiouklis, in 1990, at the fourth A.H.A. congress [8,9] introduced the notion of H_v-structures.

Our paper presents a connection between hypergroups, fundamental groups and neutrosophy and it is constructed as follows: After an Introduction, in Section 2, we present some definitions related to (weak) hyperstructures that are used throughout the paper. In Section 3, we use the concept of neutrosophic H_v-group, defined by the authors in [10] and classify its complete parts and its (strongly) regular relations. Finally, in Section 4, we find the fundamental neutrosophic group of neutrosophic H_v-groups.

2. Complete parts and regular relations in neurtosphic H_v-groups

We use the notion of neutrosophic H_v-group, defined by the authors in [10] and classify its complete parts as well as its (strongly) regular relations. For basic definitions about algebraic hyperstructures we refer to [11–14]. In neutrosophy, $< X >$, $< antiX >$, and $< neutX >$ are paired two by two, as well as all three at once, to create the NeutroSynthesis. Neutrosophy lays out the universal relationships between $< X >$, $< antiX >$, and $< neutX >$. $< X >$ is the thesis, $< antiX >$ the antithesis, and $< neutX >$ the neutrothesis (neither $< X >$ nor $< antiX >$, but the neutrality in between them).

Definition 2.1. [4] Let B be a nonempty set. A neutrosophic quadruple B-number is an ordered quadruple (b_1, b_2T, b_3I, b_4F) where $b_1, b_2, b_3, b_4 \in B$ and T, I, F have their usual neutrosophic logic meanings.
The set of all neutrosophic quadruple B-numbers is denoted by NQ(B), that is,

$$NQ(B) = \{(b_1, b_42T, b_3I, b_4F) : b_1, b_2, b_3, b_4 \in B\}.$$

Let $(H\ ,+)$ be an H_v-group with identity " 0" and define "$\oplus$" on $NQ(H\ )$ as follows:

$$\begin{aligned}&(x_1, x_2T, x_3I, x_4F) \oplus (y_1, y_2T, y_3I, y_4F)\\ &= \{(a, bT, cI, dF) :\ a \in x_1 + y_1, b \in x_2 + y_2, c \in x_3 + y_3, d \in x_4 + y_4\}.\end{aligned}$$

Throughout this section, $(H, +)$ is an H_v-group with identity "0" and $0 + 0 = 0$.

Theorem 2.2. *[10] Let H be any set with a hyperoperation $+$. Then $(NQ(H), \oplus)$ is a neutrosophic quadruple H_v-group if and only if $(H, +)$ is an H_v-group with identity "$0 \in H$ " and $0 + 0 = 0$.*

Theorem 2.3. *[10] Let H be any set with a hyperoperation $+$. Then $(NQ(H), \oplus)$ is a neutrosophic quadruple hypergroup if and only if $(H, +)$ is a hypergroup with identity "$0 \in H$" and $0 + 0 = 0$.*

Theorem 2.4. *Let $(H, +)$ be an H_v-group with identity "0", $0 + 0 = 0$ and X be a non empty subset of $NQ(H)$. Then X is a complete part in $NQ(H)$ if and only if there exist $A_1, A_2, A_3, A_4 \subseteq H$ such that $X = \{(a, bT, cI, dF) : a \in A_1, b \in A_2, c \in A_3, d \in A_4\}$ and that A_i is a complete part in H for $i = 1, 2, 3, 4$.*

Proof. Let X be a complete part in $NQ(H)$. Then there exist $A_1, A_2, A_3, A_4 \subseteq H$ such that $X = \{(a, bT, cI, dF) : a \in A_1, b \in A_2, c \in A_3, d \in A_4\}$. We prove that A_1 is a complete part in H and the others are done in a similar manner. Let $x \in A_1 \cap P \neq \emptyset$. Then there exist $x_i \in H$ with $i = 1, 2, \ldots, k$ such that $x \in x_1 + \ldots + x_k$. Let $y \in A_2$, $z \in A_3$ and $w \in A_4$. It is clear that

$$(x, yT, zI, wF) \in X \cap ((x_1, yT, zI, wF) \oplus (x_2, 0T, 0I, 0F) \oplus \ldots \oplus (x_k, 0T, 0I, 0F)).$$

The latter and having X a complete part in $NQ(H)$ imply that $((x_1, yT, zI, wF) \oplus (x_2, 0T, 0I, 0F) \ldots \oplus (x_k, 0T, 0I, 0F) \subseteq X$. Consequently, we get $x_1 + \ldots + x_k \subseteq A_1$.

Conversely, let $A_1, A_2, A_3, A_4 \subseteq H$ be complete parts in H, $X = \{(a, bT, cI, dF) : a \in A_1, b \in A_2, c \in A_3, d \in A_4\}$ and $(a, bT, cI, dF) \in X \cap ((x_1, y_1T, z_1I, w_1F) \oplus \ldots \oplus (x_k, y_kT, z_kI, w_kF))$. Then $a \in A_1 \cap (x_1 + \ldots + x_k)$, $b \in A_2 \cap (y_1 + \ldots + y_k)$, $c \in A_1 \cap (z_1 + \ldots + z_k)$ and $d \in A_1 \cap (w_1 + \ldots + w_k)$. Having A_i a complete part in H for $i = 1, 2, 3, 4$ implies that $x_1 + \ldots + x_k \subseteq A_1$, $y_1 + \ldots + y_k \subseteq A_2$, $z_1 + \ldots + z_k \subseteq A_3$ and $w_1 + \ldots + w_k \subseteq A_4$. The latter implies that $(x_1, y_1T, z_1I, w_1F) \oplus \ldots \oplus (x_k, y_kT, z_kI, w_kF) \subseteq X$. □

Corollary 2.5. *Let $(H, +)$ be an H_v-group with identity "0" and $0+0 = 0$ and A be a complete part in H. Then $NQ(A)$ is a complete part in $NQ(H)$.*

Proof. By setting $A_i = A$ for $i = 1, 2, 3, 4$ and $X = \{(a, bT, cI, dF) : a \in A_1, b \in A_2, c \in A_3, d \in A_4\}$, Theorem 2.4 asserts that $NQ(A) = X$ is a complete part in $NQ(H)$. □

The authors in [15, 16] considered the set of arithmetic functions H and defined a hyperoperation $+$ on it as follows:

$$\alpha + \beta(n) = \{\alpha(d) + \beta(\frac{n}{d}) : d|n\}.$$

Let $0_\star(n) = 0$ for all $n \in \mathbb{N}$. It is clear that $0_\star + 0_\star = 0_\star$. The authors proved that $(H, +)$ is a hypergroup with identity $0_\star$ and characterized all complete parts in H as: $S = \bigcup_{r \in M} A_r$ where M is a non empty subset of the set of complex numbers $\mathbb{C}$ and $A_r = \{\alpha \in H : \alpha(1) = r\}$.

Proposition 2.6. *Let $(H,+)$ be the hypergroup of arithmetic functions under the above hyperoperation. Then $(NQ(H),\oplus)$ is a neutrosophic hypergroup.*

Proof. The proof follows from Theorem 2.3. □

Proposition 2.7. *Let $(H,+)$ be the hypergroup of arithmetic functions defined in [16] and X be a complete part in $NQ(H)$. Then there exist non empty subsets M_i of $\mathbb{C}$ with $i=1,2,3,4$ such that $X=\{(a,bT,cI,dF): a\in S_1, b\in S_2, c\in S_3, d\in S_4\}$ and $S_i=\cup_{r\in M_i}A_r$ for $i=1,2,3,4$.*

Proof. The proof follows from Theorem 2.4. □

Corollary 2.8. *Let $(H,+)$ be the hypergroup of arithmetic functions defined in [16] and r be any complex number. Then $NQ(A_r)$ is a complete part in $NQ(H)$.*

Let $(H,+)$ be an H_v-group with identity "0", $0+0=0$ and R_i be a relation on H for $i=1,2,3,4$. We define ρ on $NQ(H)$ as follows:

$$(a,bT,cI,dF)\rho(a',b'T,c'I,d'F)\Leftrightarrow aR_1a', bR_2b', cR_3c', dR_4d'.$$

Proposition 2.9. *Let $(H,+)$ be an H_v-group with identity "0" and $0+0=0$. Then ρ is an equivalence relation on $NQ(H)$ if and only if R_i is an equivalence relation on H^* for $i=1,2,3,4$.*

Proof. The proof is straightforward. □

Theorem 2.10. *Let $(H,+)$ be an H_v-group with identity "0" and $0+0=0$. Then ρ is a regular relation on $NQ(H)$ if and only if R_i is a regular relation on H for $i=1,2,3,4$.*

Proof. Let ρ be a regular relation on $NQ(H)$ and $a,a'\in H$ with aR_1a'. Then $(a,0T,0I,0F)\rho(a',0T,0I,0F)$ (as $0R_i0$ for $i=2,3,4$). We need to show that $a+x\overline{R_1}a'+x$ and $x+a\overline{R_1}x+a'$ for all $x\in H$. We prove that $a+x\overline{R_1}a'+x$. Let $b\in a+x$. Then $(b,0T,0I,0F)\in(a,0T,0I,0F)\oplus(x,0T,0I,0F)$. Having $(a,0T,0I,0F)\rho(a',0T,0I,0F)$ and ρ a regular relation on $NQ(H)$ imply that

$$(a,0T,0I,0F)\oplus(x,0T,0I,0F)\overline{\rho}(a',0T,0I,0F)\oplus(x,0T,0I,0F).$$

The latter implies that there exist $(y,0T,0I,0F)\in(a',0T,0I,0F)\oplus(x,0T,0I,0F)$ such that $(z,0T,0I,0F)\rho(y,0T,0I,0F)$ for every $(z,0T,0I,0F)\in(a,0T,0I,0F)\oplus(x,0T,0I,0F)$. We get now that for every $z\in a+x$ there exists $y\in a'+x$ such that zR_1y. Thus, R_1 is a regular relation on H . In a similar manner, we can prove that R_i is a regular relation on H for $i = 2, 3, 4$.

Conversely, let R_i be a regular relation on H for $i = 1, 2, 3, 4$,

$$(a, bT, cI, dF)\rho(a', b'T, c'I, d'F) \text{ and } (x, yT, zI, wF) \in NQ(H).$$

We need to show that

$$(a, bT, cI, dF) \oplus (x, yT, zI, wF)\overline{\rho}(a', b'T, c'I, d'F) \oplus (x, yT, zI, wF).$$

Let $(e, fT, gI, hF) \in (a, bT, cI, dF) \oplus (x, yT, zI, wF)$. Then $e \in a+x$, $f \in b+y$, $g \in c+z$ and $h \in d+w$. Having aR_1a', bR_2b', cR_3c', dR_4d' and R_i a regular relation on H for $i = 1, 2, 3, 4$ imply that $a + x\overline{R_1}a' + x$, $b + y\overline{R_2}b' + y$, $c + z\overline{R_3}c' + z$, $d + w\overline{R_4}d' + w$. The latter implies that there exist $e' \in a' + x$, $f' \in b' + y$, $g' \in c' + z$, $h' \in d' + w$ such that eR_1e', fR_2f', gR_3g' and hR_4h'. We get now that $(e', f'T, g'I, h'F) \in (a', b'T, c'I, d'F) \oplus (x, yT, zI, wF)$ with $(e, fT, gI, hF)\rho(e', f'T, g'I, h'F)$. □

Theorem 2.11. *Let $(H, +)$ be an H_v-group with identity "0" and $0 + 0 = 0$. Then ρ is a strongly regular relation on $NQ(H)$ if and only if R_i is a strongly regular relation on H for $i = 1, 2, 3, 4$.*

Proof. Let ρ be a strongly regular relation on $NQ(H)$ and $a, a' \in H$ with aR_1a'. Then $(a, 0T, 0I, 0F)\rho(a', 0T, 0I, 0F)$ (as $0R_i0$ for $i = 2, 3, 4$). We need to show that $a + x\overline{\overline{R_1}}a' + x$ and $x + a\overline{\overline{R_1}}x + a'$ for all $x \in H$. We prove that $a + x\overline{\overline{R_1}}a' + x$ and the other is done in a similar manner. Let $b \in a + x$. Then $(b, 0T, 0I, 0F) \in (a, 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$. Having $(a, 0T, 0I, 0F)\rho(a', 0T, 0I, 0F)$ and ρ a strongly regular relation on $NQ(H)$ imply that $(a, 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)\overline{\overline{\rho}}(a', 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$. The latter implies that for all $(y, 0T, 0I, 0F) \in (a', 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$ and for all $(z, 0T, 0I, 0F) \in (a, 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$, we have $(z, 0T, 0I, 0F)\rho(y, 0T, 0I, 0F)$. We get now that for every $z \in a + x$ and for all $y \in a' + x$ such that zR_1y. Thus, R_1 is a strongly regular relation on H^*. In a similar manner, we can prove that R_i is a strongly regular relation on H for $i = 2, 3, 4$.

Conversely, let R_i be a strongly regular relation on H for $i = 1, 2, 3, 4$, $(a, bT, cI, dF)\rho(a', b'T, c'I, d'F)$ and $(x, yT, zI, wF) \in NQ(H)$. We need to show that $(a, bT, cI, dF) \oplus (x, yT, zI, wF)\overline{\overline{\rho}}(a', b'T, c'I, d'F) \oplus (x, yT, zI, wF)$. Let $(e, fT, gI, hF) \in (a, bT, cI, dF) \oplus (x, yT, zI, wF)$. Then $e \in a + x$, $f \in b + y$, $g \in c + z$ and $h \in d + w$. Having aR_1a', bR_2b', cR_3c', dR_4d' and R_i a strongly regular relation on H for $i = 1, 2, 3, 4$ imply that $a + x\overline{\overline{R_1}}a' + x$, $b + y\overline{\overline{R_2}}b' + y$, $c + z\overline{\overline{R_3}}c' + z$, $d + w\overline{\overline{R_4}}d' + w$. The latter implies that for all $e' \in a' + x$, $f' \in b' + y$, $g' \in c' + z$, $h' \in d' + w$, we have eR_1e', fR_2f', gR_3g' and hR_4h'. We get now that $(e', f'T, g'I, h'F) \in (a', b'T, c'I, d'F) \oplus (x, yT, zI, wF)$ with $(e, fT, gI, hF)\rho(e', f'T, g'I, h'F)$. □

Example 2.12. Let $\mathbb{Q}$ be the set of rational numbers, $(H, +)$ be the hypergroup of arithmetic functions and define the strongly regular equivalence relation R_i for $i = 1, 2, 3, 4$ on H as follows:

$$\alpha R_1 \gamma \Leftrightarrow \alpha(1) = \gamma(1) + q; q \in \mathbb{Q},$$

and for $i = 2, 3, 4$

$$\alpha R_i \gamma \Leftrightarrow \alpha(1) = \gamma(1).$$

Applying Theorem 2.11, we get ρ a strongly regular equivalence relation on $NQ(H)$, where ρ is defined as follows:

$$(\alpha, \beta T, \gamma I, \lambda F)\rho(\alpha', \beta' T, \gamma' I, \lambda' F)$$
$$\Leftrightarrow \alpha(1) = \alpha'(1) + q; q \in \mathbb{Q}, \beta(1) = \beta'(1), \gamma(1) = \gamma'(1), \lambda(1) = \lambda'(1).$$

3. Fundamental group of neutrosophic quadruple H_v-groups

In this part, we investigate the fundamental relation on neutrosophic H v-groups and determine its fundamental neutrosophic group..

In [3], Akinleye et. al. conducted their investigation of neutrosophic quadruple algebraic structures on quadruple numbers based on the set of real numbers. And they proved that if G is a group of real numbers then $NQ(G) = \{(g_1, g_2T, g_3I, g_4F) : g_1, g_2, g_3, g_4 \in G\}$ is a neutrosophic group under the operation "$\oplus$" given by:

$$g_1, g_2T, g_3I, g_4F) \oplus (g_1{}', g_2{}'T, g_3{}'I, g_4{}'F) = (g_1 + g_1', (g_2 + g_2{}')T, (g_3 + g_3{}')I, (g_4 + g_4{}')F).$$

The following theorem generalizes their result to any group.

Theorem 3.1. *Let G be a set with operation $+$ and $0 \in G$. Then $(NQ(G), \oplus)$ is a group if and only if $(G, +)$ is a group.*

Proof. The proof is similar to that of Theorem 2.1 in [3]. □

Proposition 3.2. *Let $(G, +)$ and $(G', +')$ be isomorphic groups. Then $(NQ(G), \oplus)$ and $(NQ(G'), \oplus')$ are isomorphic neutrosophic groups.*

Proof. Let $(G, +)$ and $(G', +')$ be isomorphic groups. Then there exists a group isomorphism $f : G \to G'$. Let $\phi : NQ(G) \to NQ(G')$ be defined as follows:

$$\phi((a, bT, cI, dF)) = (f(a), f(b)T, f(c)I, f(d)F).$$

One can easily see that ϕ is a group isomorphism. □

Definition 3.3. [13] For all $n > 1$, we define the relation β_n on a semihypergroup $(H, \circ)$ as follows:

$$x\beta_n y \text{ if there exist } a_1, \ldots, a_n \text{ in } H \text{ such that } \{x, y\} \subseteq \prod_{i=1}^{n} a_i$$

and we set $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H^*.

This relation was established by Koskas [17] and studied by many authors. Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure of β.

Throughout this section, $\beta, \beta^\star$ are the relations on H and $\beta_N, \beta_N^\star$ are the relations on $NQ(H)$.

Theorem 3.4. *Let $(H, +)$ be an H_v-group with identity "0" and $0 + 0 = 0$ and let $a, a', b, b', c, c', d, d' \in H$. Then $(a, bT, cI, dF)\beta_N(a', b'T, c'I, d'F)$ if and only if $a\beta a'$, $b\beta b'$, $c\beta c'$ and $d\beta d'$.*

Proof. Let $(a, bT, cI, dF)\beta_N(a', b'T, c'I, d'F)$. Then there exist (x_i, y_iT, z_iI, w_iF) with $i = 1, \ldots, k$ such that $\{(a, bT, cI, dF), (a', b'T, c'I, d'F)\} \subseteq (x_1, y_1T, z_1I, w_1F) \oplus \ldots \oplus (x_k, y_kT, z_kI, w_kF)$. The latter implies that $a, a' \in x_1 + \ldots + x_k$, $b, b' \in y_1 + \ldots + y_k$, $c, c' \in z_1 + \ldots + z_k$ and $d \in w_1 + \ldots + w_k$. Thus, $a\beta a'$, $b\beta b'$, $c\beta c'$ and $d, d'\beta d'$.

Conversely, let $a\beta a'$, $b\beta b'$, $c\beta c'$ and $d\beta d'$. Then there exist $k_1, k_2, k_3, k_4 \in \mathbb{N}$ and $x_1, \ldots, x_{k_1}$, $y_1, \ldots, y_{k_2}$, $z_1, \ldots, z_{k_3}$, $w_1, \ldots, w_{k_4} \in H$ such that $a, a' \in x_1 + \ldots + x_{k_1}$, $b, b' \in y_1 + \ldots + y_{k_2}$, $c, c' \in z_1 + \ldots + z_{k_3}$ and $d, d' \in w_1 + \ldots + w_{k_4}$. By setting $k = \max\{k_1, k_2, k_3, k_4\}$ and $x_i = 0$ for $k_1 < i \leq k$, $y_i = 0$ for $k_2 < i \leq k$, $z_i = 0$ for $k_3 < i \leq k$ and $w_i = 0$ for $k_4 < i \leq k$ and using the fact that $x \in 0 + x \cap x + 0$ for all $x \in H$, we get $a, a' \in x_1 + \ldots + x_k$, $b, b' \in y_1 + \ldots + y_k$, $c, c' \in z_1 + \ldots + z_k$ and $d, d' \in w_1 + \ldots + w_k$. The latter implies that $\{(a, bT, cI, dF), (a', b'T, c'I, d'F)\} \subseteq (x_1, y_1T, z_1I, w_1F) \oplus \ldots \oplus (x_k, y_kT, z_kI, w_kF)$. Thus, $(a, bT, cI, dF)\beta_N(a', b'T, c'I, d'F)$. □

Theorem 3.5. *Let $(H\ , +)$ be an H_v^*-group with identity "0" and $0 + 0 = 0$ and let $a, a', b, b', c, c', d, d' \in H$. Then $(a, bT, cI, dF)\beta_N^\star(a', b'T, c'I, d'F)$ if and only if $a\beta^\star a'$, $b\beta^\star b'$, $c\beta^\star c'$ and $d\beta^\star d'$.*

Proof. Let $(a, bT, cI, dF)\beta_N^\star(a', b'T, c'I, d'F)$. Then there exist $(x_i, y_iT, z_iI, w_iF) \in NQ(H)$ with $i = 1, \ldots, k$ such that
$(a, bT, cI, dF)\beta_N(x_1, y_1T, z_1I, w_1F)$, $(x_i, y_iT, z_iI, w_iF)\beta_N(x_{i+1}, y_{i+1}T, z_{i+1}I, w_{i+1}F)$ for $i = 1, \ldots, k-1$ and $(x_k, y_kT, z_kI, w_kF)\beta_N(a', b'T, c'I, d'F)$. Theorem 3.4 implies that $a\beta x_1$, $x_i\beta x_{i+1}$ for $i = 1, \ldots, k-1$, $x_k\beta a'$, $b\beta y_1$, $y_i\beta y_{i+1}$ for $i = 1, \ldots, k-1$, $y_k\beta b'$, $c\beta z_1$, $z_i\beta z_{i+1}$ for $i = 1, \ldots, k-1$, $z_k\beta c'$, $d\beta w_1$, $w_i\beta w_{i+1}$ for $i = 1, \ldots, k-1$, $w_k\beta d'$. Thus, $a\beta^\star a'$, $b\beta^\star b'$, $c\beta^\star c'$ and $d\beta^\star d'$.

Conversely, let $a\beta^\star a'$, $b\beta^\star b'$, $c\beta^\star c'$ and $d\beta^\star d'$. Then there exist $x_i, y_i, z_i, w_i \in H$ such that $a\beta x_1$, $x_i\beta x_{i+1}$ for $i = 1, \ldots, k-1$, $x_k\beta a'$, $b\beta y_1$, $y_i\beta y_{i+1}$ for $i = 1, \ldots, l-1$, $y_l\beta b'$, $c\beta z_1$, $z_i\beta z_{i+1}$ for $i = 1, \ldots, m-1$, $z_m\beta c'$, $d\beta w_1$, $w_i\beta w_{i+1}$ for $i = 1, \ldots, s-1$, $w_s\beta d'$. By setting $t = \max\{k, l, m, s\}$ and $x_i = a'$ for $k < i \leq t$, $y_i = b'$ for $t < i \leq t$, $z_i = c'$ for $m < i \leq t$ and $w_i = d'$ for $s < i \leq t$. The latter implies that

$$(a, bT, cI, dF)\beta_N(x_1, y_1T, z_1I, w_1F), \quad (x_i, y_iT, z_iI, w_iF)\beta_N(x_{i+1}, y_{i+1}T, z_{i+1}I, w_{i+1}F),$$

for $i = 1, \ldots, t-1$ and

$$(x_t, y_tT, z_tI, w_tF)\beta_N(a', b'T, c'I, d'F).$$

Thus, $(a, bT, cI, dF)\beta_N^\star(a', b'T, c'I, d'F)$. □

Theorem 3.6. *Let $(H\ ,+)$ be an H_v^*-group with identity "0" and $0 + 0 = 0$. Then $NQ(H)/\beta_N^\star \cong NQ(H/\beta^\star)$.*

Proof. Let $\phi : NQ(H)/\beta_N^\star \to NQ(H/\beta^\star)$ be defined as

$$\phi(\beta_N^\star((a, bT, cI, dF))) = (\beta^\star(a), \beta^\star(b)T, \beta^\star(c)I, \beta^\star(d)F).$$

Theorem 3.5 implies that ϕ is well-defined and one-to-one. Also, it is clear that ϕ is onto. We need to show that ϕ is a group homomorphism. Since

$$\beta_N^\star((a, bT, cI, dF)) \boxplus' \beta_N^\star((a', b'T, c'I, d'F)) = \beta_N^\star((x, yT, zI, wF))$$

where $(x, yT, zI, wF) \in (a, bT, cI, dF) \oplus (a', b'T, c'I, d'F) = (a + a', (b + b')T, (c + c')I, (d + d')F)$, it follows that $\phi(\beta_N^\star((a, bT, cI, dF)) \boxplus \beta_N^\star((a', b'T, c'I, d'F))) = \phi((x, yT, zI, wF)) = (\beta^\star(x), \beta^\star(y)T, \beta^\star(z)I, \beta^\star(w)F)$. Having $\beta^\star(x) = \beta^\star(a) \boxplus \beta^\star(a')$, $\beta^\star(y) = \beta^\star(b) \boxplus \beta^\star(b')$, $\beta^\star(z) = \beta^\star(c) \boxplus \beta^\star(c')$ and $\beta^\star(w) = \beta^\star(d) \boxplus \beta^\star(d')$ imply that $\phi(\beta_N^\star((a, bT, cI, dF)) \boxplus' \beta_N^\star((a', b'T, c'I, d'F))) = \phi(\beta_N^\star((a, bT, cI, dF)) \oplus' \phi(\beta_N^\star((a', b'T, c'I, d'F)))$. □

Theorem 3.7. *Let $(H, +)$ be an H_v^*-group with identity "0" and $0 + 0 = 0$. If G is the fundamental group of H^* (up to isomorphism) then $NQ(G)$ is the fundamental group of $NQ(H)$ (up to isomorphism).*

Proof. Since G is the fundamental group of H (up to isomorphism), it follows that $H/\beta^\star \cong G$. The latter and Proposition 3.2 imply that $NQ(H/\beta^\star) \cong NQ(G)$. Theorem 3.6 completes the proof. □

Example 3.8. Let $(H, +)$ be the hypergroup of arithmetic functions defined in [15] with the group of complex numbers $(\mathbb{C}, +)$ under standard addition as a fundamental group (up to isomorphism). Then $(NQ(\mathbb{C}), +)$ is the fundamental group of $NQ(H)$ up to isomorphism.

Corollary 3.9. *Let $(H,+)$ be an H_v-group with identity "0" and $0+0=0$. If H has a trivial fundamental group then $NQ(H)$ has a trivial fundamental group.*

Proof. The proof is straightforward by applying Corollary 3.7. □

Example 3.10. Let $H_1=\{0,1\}$ and define $(H_1,+_1)$ as follows:

$+_1$	0	1
0	0	1
1	1	H_1

Then $(NQ(H_1),\oplus)$ is a quadruple H_v-group. Since $0,1\in 1+_1 1$, it follows that $x\beta y$ for all $x,y\in H_1$. Thus, H_1 has trivial fundamental group. Corollary 3.9 asserts that $(NQ(H_1),\oplus)$ has trivial fundamental group.

Theorem 3.11. *Let $(H,+)$ be an H_v-group with identity "0" and $0+0=0$ and w_H be the heart of H. Then $NQ(w_H)$ is the heart of $NQ(H)$.*

Proof. Let $w_H=\{x\in H:\beta^\star(x)=\beta^\star(0)\}$ be the heart of H. Having $w_{NQ(H)}=\{(a,bT,cI,dF)\in NQ(H):\beta^\star_N((a,bT,cI,dF))=\beta^\star_N(0,0T,0I,0F)\}$ and applying Theorem 3.5, we get that $\beta^\star(a)=\beta^\star(b)=\beta^\star(c)=\beta^\star(d)=\beta^\star(0)$. Thus, $w_{NQ(H)}=\{(a,bT,cI,dF)\in NQ(H):a,b,c,d\in w_H\}=NQ(w_H)$. □

Example 3.12. Let $(H,+)$ be the hypergroup of arithmetic functions defined in [15]. Then $NQ(A_0)$ is the heart of $NQ(H)$.

4. **Conclusion**

This paper connected neutrosophic H_v-groups and neutrosophic groups by means of complete parts and regular relations. More precisely, it used the complete parts and the fundamental relation of H_v-groupls to reach its main results that are summarized in Theorems 4.6 and 4.7. The results were supported by non-trivial illustrative examples. For future research, it is interesting to find a connection between other types of neutrosophic H_v-structures and neutrosophic algebraic structures. One can investigate the connection between neutrosophic H_v-rings and neutrosophic rings or the connection between neutrosophic H_v-modules and neutrosophic modules.

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