A Generalization of Certain Remarkable Points of the Triangle Geometry

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In this article we prove a theorem that will generalize the concurrence theorems that are leading to the Franke’s point, Kariya’s point, and to other remarkable points from the triangle geometry.

Theorem 1:

Let $P(\alpha,\beta,\gamma)$ and A',B',C' its projections on the sides BC, CA respectively AB of the triangle ABC.

We consider the points A'', B'', C'' such that $\overrightarrow{PA''}=k\overrightarrow{PA'}$, $\overrightarrow{PB''}=k\overrightarrow{PB'}$, $\overrightarrow{PC''}=k\overrightarrow{PC'}$, where $k\in R^*$. Also we suppose that AA', BB', CC' are concurrent. Then the lines AA'', BB'', CC'' are concurrent if and only if are satisfied simultaneously the following conditions:

$$\alpha\beta c\left(\frac{\beta}{b}\cos A-\frac{\alpha}{a}\cos B\right)+\beta\gamma a\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\gamma\alpha b\left(\frac{\alpha}{a}\cos C-\frac{\gamma}{c}\cos A\right)=0$$

$$\frac{\alpha^2}{a^2}\cos A\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\frac{\beta^2}{b^2}\cos B\left(\frac{\alpha}{a}\cos C-\frac{\gamma}{c}\cos A\right)+\frac{\gamma^2}{c^2}\cos C\left(\frac{\beta}{b}\cos A-\frac{\alpha}{a}\cos B\right)=0$$

Proof:

We find that

$$A'\left(0,\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta,\ \frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma\right)$$

$$\overrightarrow{PA''}=k\overrightarrow{PA'}=k\left[-\alpha\overrightarrow{r_A}+\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)\overrightarrow{r_B}+\frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)\overrightarrow{r_C}\right]$$

$$\overrightarrow{PA''}=\left(\alpha''-\alpha\right)\overrightarrow{r_A}+\left(\beta''-\beta\right)\overrightarrow{r_B}+\left(\gamma''-\gamma\right)\overrightarrow{r_C}$$

We have:

$$\begin{cases}\alpha''-\alpha=-k\alpha\\ \beta''-\beta=\dfrac{k\alpha}{2a^2}\left(a^2+b^2-c^2\right),\\ \gamma''-\gamma=\dfrac{k\alpha}{2a^2}\left(a^2-b^2+c^2\right)\end{cases}$$

Therefore:

$$\begin{cases} \alpha'' = (1-k)\alpha \\ \beta'' = \dfrac{k\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta \\ \gamma'' = \dfrac{k\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma \end{cases}$$

Hence:

$$A''\left((1-k)\alpha,\ \frac{k\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta,\ \frac{k\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma\right)$$

Similarly:

$$B'\left(-\frac{\beta}{2b^2}\left(-a^2-b^2+c^2\right)+\alpha,\ 0,\ -\frac{\beta}{2b^2}\left(a^2-b^2-c^2\right)+\gamma\right)$$

$$B''\left(-\frac{k\beta}{2b^2}\left(-a^2-b^2+c^2\right)+\alpha,\ (1\text{-}k)\beta,\ -\frac{k\beta}{2b^2}\left(a^2-b^2-c^2\right)+\gamma\right)$$

$$C'\left(-\frac{\gamma}{2c^2}\left(-a^2+b^2-c^2\right)+\alpha,\ -\frac{\gamma}{2c^2}\left(a^2-b^2-c^2\right)+\beta,\ 0\right)$$

$$C''\left(-\frac{k\gamma}{2c^2}\left(-a^2+b^2-c^2\right)+\alpha,\ -\frac{k\gamma}{2c^2}\left(a^2-b^2-c^2\right)+\beta,\ (1\text{-}k)\gamma\right)$$

Because $AA',\ BB',\ CC'$ are concurrent, we have:

$$\frac{-\dfrac{\alpha}{2a^2}\left(-a^2-b^2+c^2\right)+\beta}{-\dfrac{\alpha}{2a^2}\left(-a^2+b^2-c^2\right)+\gamma}\cdot\frac{-\dfrac{\beta}{2b^2}\left(-a^2-b^2-c^2\right)+\gamma}{-\dfrac{\beta}{2b^2}\left(-a^2-b^2+c^2\right)+\alpha}\cdot\frac{-\dfrac{\gamma}{2c^2}\left(-a^2+b^2-c^2\right)+\alpha}{-\dfrac{\gamma}{2c^2}\left(a^2-b^2-c^2\right)+\beta}=1$$

We note

$$M=\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)=\frac{\alpha}{a}\cdot b\cos C$$

$$N=\frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)=\frac{\alpha}{a}\cdot c\cos B$$

$$P=\frac{\beta}{2b^2}\left(-a^2+b^2+c^2\right)=\frac{\beta}{b}\cdot c\cos A$$

$$Q=\frac{\beta}{2b^2}\left(a^2+b^2-c^2\right)=\frac{\beta}{b}\cdot a\cos C$$

$$R=\frac{\gamma}{2c^2}\left(a^2-b^2+c^2\right)=\frac{\gamma}{c}\cdot a\cos B$$

$$S=\frac{\gamma}{2c^2}\left(-a^2+b^2+c^2\right)=\frac{\gamma}{c}\cdot a\cos A$$

The precedent relation becomes

$$\frac{M+\beta}{N+\gamma}\cdot\frac{P+\gamma}{Q+\alpha}\cdot\frac{R+\alpha}{S+\beta}=1$$

The coefficients M, N, P, Q, R, S verify the following relations:

$$M+N=\alpha$$
$$P+Q=\beta$$
$$R+S=\gamma$$

$$\frac{M}{Q}=\frac{\alpha}{\beta}\cdot\frac{b^2}{a^2}=\frac{\frac{\alpha}{a^2}}{\frac{\beta}{b^2}}$$

$$\frac{P}{S}=\frac{\beta}{\gamma}\cdot\frac{c^2}{b^2}=\frac{\frac{\beta}{b^2}}{\frac{\gamma}{c^2}}$$

$$\frac{R}{N}=\frac{\gamma}{\alpha}\cdot\frac{a^2}{c^2}=\frac{\frac{\gamma}{c^2}}{\frac{\alpha}{a^2}}$$

Therefore $\frac{M}{Q}\cdot\frac{P}{S}\cdot\frac{R}{N}=1$

$$(M+\beta)(P+\gamma)(R+\alpha)=\alpha\beta\gamma+\alpha\beta P+\beta\gamma R+\gamma\alpha M+\alpha MP+\beta PR+\gamma RM+MPR$$
$$(N+\gamma)(Q+\alpha)(S+\beta)=\alpha\beta\gamma+\alpha\beta N+\beta\gamma Q+\gamma\alpha S+\alpha NS+\beta NQ+\gamma QS+NQS\ .$$

We deduct that:

$$\alpha\beta P+\beta\gamma R+\gamma\alpha M+\alpha MP+\beta PR+\gamma RM=\alpha\beta N+\beta\gamma Q+\gamma\alpha S+\alpha NS+\beta NQ+\gamma QS+NQS \quad (1)$$

We apply the theorem:

Given the points $Q_i(a_i,b_i,c_i),\ i=\overline{1,3}$ in the plane of the triangle ABC, the lines AQ_1, BQ_2, CQ_3 are concurrent if and only if $\frac{b_1}{c_1}\cdot\frac{c_2}{a_2}\cdot\frac{a_3}{b_3}=1$.

For the lines $AA", BB", CC"$ we obtain

$$\frac{kM+\beta}{kN+\gamma}\cdot\frac{kP+\alpha}{kS+\beta}\cdot\frac{kR+\alpha}{kS+\beta}=1\ .$$

It result that

$$k^2(\alpha\beta P+\beta\gamma R+\gamma\alpha M)+k(\alpha MP+\beta PR+\gamma RM)=$$
$$=k^2(\alpha\beta N+\beta\gamma Q+\gamma\alpha S)+k(\alpha NS+\beta NQ+\gamma QS) \quad (2)$$

For relation (1) to imply relation (2) it is necessary that

$$\alpha\beta P+\beta\gamma R+\gamma\alpha M=\alpha\beta N+\beta\gamma Q+\gamma\alpha S$$

and

$$\alpha NS+\beta NQ+\gamma QS=\alpha MP+\beta PR+\gamma RM$$

or

$$\begin{cases} \alpha\beta c\left(\frac{\beta}{b}\cos A-\frac{\alpha}{a}\cos B\right)+\beta\gamma a\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\gamma\alpha b\left(\frac{\alpha}{a}\cos C-\frac{\gamma}{c}\cos A\right)=0 \\ \frac{\alpha^2}{a^2}\cos A\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\frac{\beta^2}{b^2}\cos B\left(\frac{\gamma}{c}\cos B-\frac{\beta}{b}\cos C\right)+\frac{\gamma^2}{c^2}\cos C\left(\frac{\beta}{b}\cos A-\frac{\alpha}{a}\cos B\right)=0 \end{cases}$$

As an open problem, we need to determine the set of the points from the plane of the triangle ABC that verify the precedent relations.

We will show that the points I and O verify these relations, proving two theorems that lead to Kariya's point and Franke's point.

Theorem 2 (Kariya -1904)

Let I be the center of the circumscribe circle to triangle ABC and A', B', C' its projections on the sides BC, CA, AB. We consider the points $A", B", C"$ such that:

$$\overrightarrow{IA"}=k\overrightarrow{IA'},\ \overrightarrow{IB"}=k\overrightarrow{IB'},\ \overrightarrow{IC"}=k\overrightarrow{IC'},\ k\in R^*.$$

Then $AA"$, $BB"$, $CC"$ are concurrent (the Kariya's point)

Proof:

The barycentric coordinates of the point I are $I\left(\frac{a}{2p},\frac{b}{2p},\frac{c}{2p}\right)$.

Evidently:

$$abc(\cos A-\cos B)+abc(\cos B-\cos C)+abc(\cos C-\cos A)=0$$

and

$$\cos A(\cos B-\cos C)+\cos B(\cos C-\cos A)+\cos C(\cos A-\cos B)=0.$$

In conclusion $AA"$, $BB"$, $CC"$ are concurrent.

Theorem 3 (de Boutin - 1890)

Let O be the center of the circumscribed circle to the triangle ABC and A', B', C' its projections on the sides BC, CA, AB. Consider the points $A"$, $B"$, $C"$ such that $\frac{OA'}{OA"}=\frac{OB'}{OB"}=\frac{OC'}{OC"}=k,\ k\in R^*$. Then the lines $AA"$, $BB"$, $CC"$ are concurrent (The point of Franke – 1904).

Proof:

$O\left(\frac{R^2}{2S}\sin 2A,\frac{R^2}{2S}\sin 2B,\frac{R^2}{2S}\sin 2C\right)$, $P=N$, because $\frac{\sin 2B\cos A}{\sin B}-\frac{\sin 2A\cos B}{\sin A}=0$.

Similarly we find that $R=Q$ and $M=S$.

Also $\alpha MP=\alpha NS$, $\beta PR=\beta NQ$, $\gamma RM=\gamma QS$. It is also verified the second relation from the theorem hypothesis. Therefore the lines $AA"$, $BB"$, $CC"$ are concurrent in a point called the Franke's point.

Remark 1:

It is possible to prove that the Franke's points belong to Euler's line of the triangle ABC.

Theorem 4:

Let I_a be the center of the circumscribed circle to the triangle ABC (tangent to the side BC) and A', B', C' its projections on the sites BC, CA, AB. We consider the points $A", B", C"$ such that $\overrightarrow{IA"} = k\overrightarrow{IA'}, \overrightarrow{IB"} = k\overrightarrow{IB'}, \overrightarrow{IC"} = k\overrightarrow{IC'}, k \in R^*$. Then the lines $AA", BB", CC"$ are concurrent.

Proof

$$I_a\left(\frac{-a}{2(p-a)}, \frac{b}{2(p-a)}, \frac{c}{2(p-a)}\right);$$

The first condition becomes:

$$-abc(\cos A + \cos B) + abc(\cos B - \cos C) - abc(-\cos C - \cos A) = 0,$$ and the second condition:

$$\cos A(\cos B - \cos C) + \cos B(-\cos C - \cos A) + \cos C(\cos A + \cos B) = 0$$

Is also verified.

From this theorem it results that the lines $AA", BB", CC"$ are concurrent.

Observation 1:

Similarly, this theorem is proven for the case of I_b and I_c as centers of the ex-inscribed circles.

References

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