Generalization of a Remarkable Theorem

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In [1] Professor Claudiu Coandă proved, using the barycentric coordinates, the following remarkable theorem:

Theorem (C. Coandă)

Let ABC be a triangle, where $m(\sphericalangle A) \neq 90°$ and Q_1, Q_2, Q_3 are three points on the circumscribed circle of the triangle ABC. We'll note $BQ_i \cap AC = \{B_i\}$, $i = \overline{1,3}$. Then the lines B_1C_1, B_2C_2, B_3C_3 are concurrent.

We will generalize this theorem using some results from projective geometry relative to the pole and polar notions.

Theorem (Generalization of C. Coandă theorem)

Let ABC be a triangle where $m(\sphericalangle A) \neq 90°$ and $Q_1, Q_2, ..., Q_n$ points on its circumscribed circle $(n \in N,\ n \geq 3)$, $i = \overline{1,n}$. Then the lines B_1C_1, $B_2C_2, ..., B_nC_n$ are concurrent in fixed point.

To prove this theorem we'll utilize the following lemmas:

Lemma 1

If $ABCD$ is an inscribed quadrilateral in a circle and $\{P\} = AB \cap CD$, then the polar of the point P in rapport with the circle is the line EF, where $\{E\} = AC \cap BD$ and $\{F\} = BC \cap AD$

Lemma 2

The pole of a line is the intersection of the corresponding polar to any two points of the line.

The pols of concurrent lines in rapport to a given circle are collinear points and the reciprocal is also true: the polar of collinear points, in rappoer with a given circle, are concurrent lines.

Lemma 3

If $ABCD$ is an inscribed quadrilateral in a circle and $\{P\} = AB \cap CD$, $\{E\} = AC \cap BD$ and $\{F\} = BC \cap AD$, then the polar of point E in rapport to the circle is the line PF.

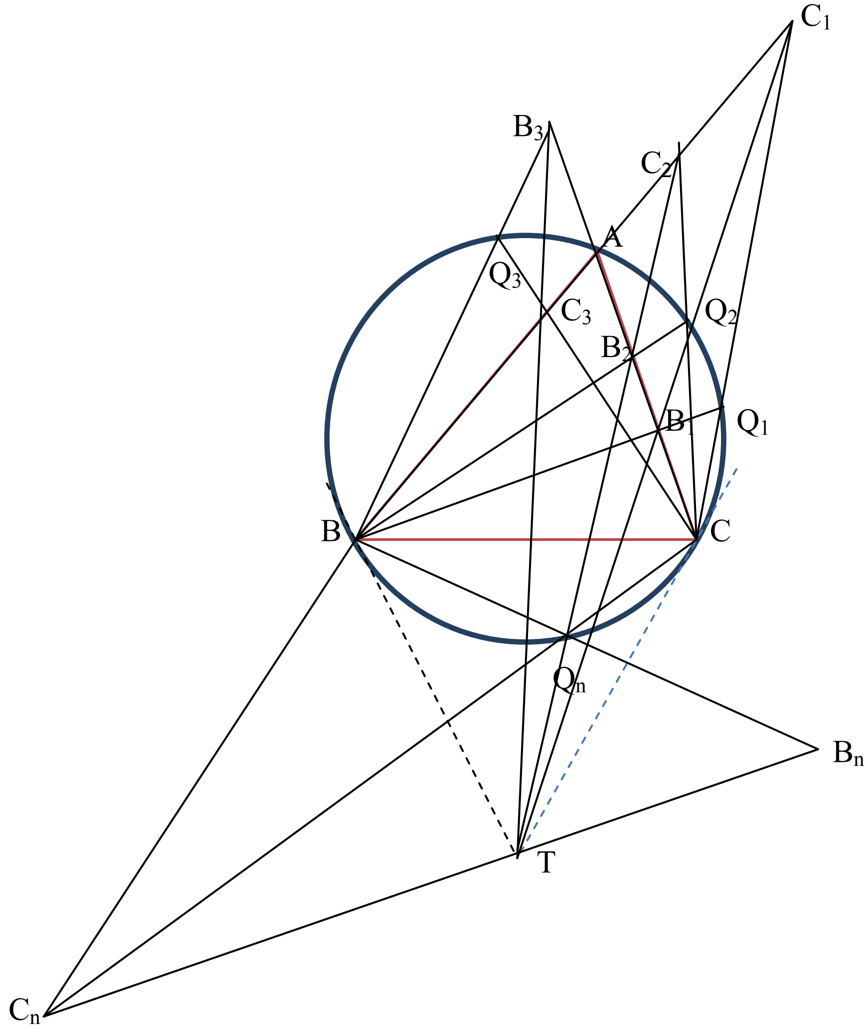
The proof for the Lemmas 1 - 3 and other information regarding the notions of pole and polar in rapport to a circle can be found in [2] or [3].

Proof of the generalized theorem of C. Coandă

Let $Q_1, Q_2, ..., Q_n$ points on the circumscribed circle to the triangle ABC (see the figure)

We'll consider the inscribed quadrilaterals $ABCQ_n$, $i = \overline{1,n}$ and we'll note $\{T_i\} = AQ_i \cap BC$.

In accordance to Lemma 1 and Lemma 3, the lines B_iC_i are the respectively polar



(in rapport with the circumscribed circle to the triangle ABC) to the points T_i.

Because the points T_i are collinear (belonging to the line BC), from Lemma 2 we'll obtain that their polar, that is the lines B_iC_i, are concurrent in a point T.

Remark

The concurrency point T is the harmonic conjugate in rapport with the circle of the symmedian center K of the given triangle.

References

[1] Claudiu Coandă – Geometrie analitică în coordonate baricentrice – Editura Reprograph, Craiova, 2005.

[2] Ion Pătraşcu – O aplicație practică a unei teoreme de geometrie proiectivă – Journal: Sfera matematică, m 1b (2/2009-2010). Editura Reprograph.

[3] Roger A. Johnson – Advanced Euclidean Geometry – Dover Publications, Inc. Mineola, New York, 2007.