

A GENERALIZATION OF THE INEQUALITY CAUCHY-BOUNIAKOVSKI-SCHWARZ

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Statement: Let us consider the real numbers $a_i^{(k)}$, $i \in \{1, 2, \dots, n\}$, $k \in \{1, 2, \dots, m\}$, with $m \geq 2$. Then:

$$\left(\sum_{i=1}^n \prod_{k=1}^m a_i^{(k)} \right)^2 \leq \prod_{k=1}^m \sum_{i=1}^n (a_i^{(k)})^2.$$

Proof:

One notes A the left member of the inequality and B the right member. One has:

$$A = \sum_{i=1}^n (a_i^{(1)} \dots a_i^{(m)})^2 + 2 \sum_{i=1}^{n-1} \sum_{k=i+1}^n (a_i^{(1)} \dots a_i^{(m)}) (a_k^{(1)} \dots a_k^{(m)})$$

and

$$B = \sum_{(i_1, \dots, i_m) \in E} (a_{i_1}^{(1)} \dots a_{i_m}^{(m)})^2,$$

where

$$E = \{(i_1, \dots, i_m) / i_k \in \{1, 2, \dots, n\}, 1 \leq k \leq m\}.$$

From where:

$$B = \sum_{i=1}^n (a_i^{(1)} \dots a_i^{(m)})^2 + \sum_{i=1}^{n-1} \sum_{k=i+1}^n \left[(a_i^{(1)} \dots a_i^{(m-1)} a_k^{(m)})^2 + (a_k^{(1)} \dots a_k^{(m-1)} a_i^{(m)})^2 \right] + \sum_{(i_1, \dots, i_m) \in E - (\Delta_E \cup L^m)} (a_{i_1}^{(1)} \dots a_{i_m}^{(m)})^2$$

with

$$\Delta_E = \left\{ \underbrace{(\gamma, \dots, \gamma)}_{m \text{ times}} / \gamma \in \{1, 2, \dots, n\} \right\}$$

and

$$L = \left\{ \underbrace{(\alpha, \dots, \alpha)}_{m-1 \text{ times}}, \underbrace{(\beta, \dots, \beta)}_{m-1 \text{ times}}, (\alpha, \beta) / (\alpha, \beta) \in \{1, 2, \dots, n\}^2 \text{ and } \alpha < \beta \right\}$$

Then

$$\begin{aligned}
A - B &= \sum_{i=1}^{n-1} \sum_{k=i+1}^n \left[-\left(a_i^{(1)} \dots a_i^{(m-1)} a_k^{(m)}\right)^2 - \left(a_k^{(1)} \dots a_k^{(m-1)} a_i^{(m)}\right)^2 \right] - \\
&- \sum_{(i_1, \dots, i_m) \in E - (\Delta_E \cup L)} \left(a_{i_1}^{(1)} \dots a_{i_m}^{(m)}\right)^2 \leq 0
\end{aligned}$$

Note: for $m = 2$ one obtains the inequality of Cauchy-Bouniakovski-Schwarz.