

GENERALIZATIONS OF THE THEOREM OF CEVA AND THEIR APPLICATIONS

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In these paragraphs one presents three generalizations of the famous theorem of Ceva, which states:

“If in a triangle ABC one draws the concurrent straight lines

$$AA_1, BB_1, CC_1, \text{ then } \frac{\overline{A_1B}}{\overline{A_1C}} \cdot \frac{\overline{B_1C}}{\overline{B_1A}} \cdot \frac{\overline{C_1A}}{\overline{C_1B}} = -1^{\ast}.$$

Theorem 1: Let us have the polygon $A_1A_2\dots A_n$, a point M in its plane, and a circular permutation

$$p = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 2 & 3 & \dots & n & 1 \end{pmatrix}. \text{ One notes } M_{ij} \text{ the intersections of the line } A_iM \text{ with the lines}$$

$A_{i+s}A_{i+s+1}, \dots, A_{i+s+t-1}A_{i+s+t}$ (for all i and j , $j \in \{i+s, \dots, i+s+t-1\}$).

If $M_{ij} \neq A_n$ for all respective indices, and if $2s+t=n$, one has:

$$\prod_{i,j=1,i+s}^{n,i+s+t-1} \frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_p(j)}} = (-1)^n \text{ (} s \text{ and } t \text{ are natural non-zero numbers).}$$

Analytical demonstration: Let M be a point in the plain of the triangle ABC , such that it satisfies the conditions of the theorem. One chooses a Cartesian system of axes, such that the two parallels with the axes which pass through M do not pass by any point A_i (this is possible).

One considers $M(a,b)$, where a and b are real variables, and $A_i(X_i, Y_i)$ where X_i and Y_i are known, $i \in \{1, 2, \dots, n\}$.

The former choices ensure us the following relations:

$$X_i - a \neq 0 \text{ and } Y_i - b \neq 0 \text{ for all } i \in \{1, 2, \dots, n\}.$$

The equation of the line A_iM ($1 \leq i \leq n$) is:

$$\frac{x-a}{X_i-a} - \frac{y-b}{Y_i-b}. \text{ One notes that } d(x, y; X_i, Y_i) = 0.$$

One has

$$\frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_{p(j)}}} = \frac{\delta(A_j, A_iM)}{\delta(A_{p(j)}, A_iM)} = \frac{d(X_j, Y_j; X_i, Y_i)}{d(X_{p(j)}, Y_{p(j)}; X_i, Y_i)} = \frac{D(j, i)}{D(p(j), i)}$$

where $\delta(A, ST)$ is the distance from A to the line ST , and where one notes with $D(a, b)$ for $d(X_a, Y_a; X_b, Y_b)$.

Let's calculate the product, where we will use the following convention: $a + b$ will mean $\underbrace{p(p(\dots p(a)\dots))}_{b \text{ times}}$, and $a - b$ will mean $\underbrace{p^{-1}(p^{-1}(\dots p^{-1}(a)\dots))}_{b \text{ times}}$

$$\begin{aligned} \prod_{j=i+s}^{i+s+t-1} \frac{\overline{M_{ij}A_j}}{M_{ij}A_{j+1}} &= \prod_{j=i+s}^{i+s+t-1} \frac{D(j, i)}{D(j+1, i)} = \\ &= \frac{D(i+s, i)}{D(i+s+1, i)} \cdot \frac{D(i+s+1, i)}{D(i+s+2, i)} \dots \frac{D(i+s+t-1, i)}{D(i+s+t, i)} = \\ &= \frac{D(i+s, i)}{D(i+s+t, i)} = \frac{D(i+s, i)}{D(i-s, i)} \end{aligned}$$

The initial product is equal to:

$$\begin{aligned} \prod_{i=1}^n \frac{D(i+s, i)}{D(i-s, i)} &= \frac{D(1+s, 1)}{D(1-s, 1)} \cdot \frac{D(2+s, 2)}{D(2-s, 2)} \dots \frac{D(2s, s)}{D(n, s)} \cdot \\ &\cdot \frac{D(2s+2, s+2)}{D(2, s+2)} \dots \frac{D(2s+t, s+t)}{D(t, s+t)} \cdot \frac{D(2s+t+1, s+t+1)}{D(t+1, s+t+1)} \cdot \\ &\cdot \frac{D(2s+t+2, s+t+2)}{D(t+2, s+t+2)} \dots \frac{D(2s+t+s, s+t+s)}{D(t+s, s+t+s)} = \\ &= \frac{D(1+s, 1)}{D(1, 1+s)} \cdot \frac{D(2+s, 2)}{D(2, 2+s)} \dots \frac{D(2s+t, s+t)}{D(s+t, 2s+t)} \dots \frac{D(s, n)}{D(n, s)} = \\ &= \prod_{i=1}^n \frac{D(i+s, i)}{D(i, i+s)} = \prod_{i=1}^n \left(-\frac{P(i+s)}{P(i)} \right) = (-1)^n \end{aligned}$$

because:

$$\frac{D(r, p)}{D(p, r)} = \frac{\frac{X_r - a}{X_p - a} - \frac{Y_r - b}{Y_p - b}}{\frac{X_p - a}{X_r - a} - \frac{Y_p - b}{Y_r - b}} = -\frac{(X_r - a)(Y_r - b)}{(X_p - a)(Y_p - b)} = -\frac{P(r)}{P(p)},$$

The last equality resulting from what one notes: $(X_t - a)(Y_t - b) = P(t)$. From (1) it results that $P(t) \neq 0$ for all t from $\{1, 2, \dots, n\}$. The proof is completed.

Comments regarding Theorem 1:

t represents the number of lines of a polygon which are intersected by a line $A_i M$; if one notes the sides $A_i A_{i+1}$ of the polygon, by a_i , then $s+1$ represents the order of the first line intersected by the line $A_1 M$ (that is a_{s+1} the first line intersected by $A_1 M$).

Example: If $s = 5$ and $t = 3$, the theorem says that :

- the line $A_1 M$ intersects the sides $A_6 A_7, A_7 A_8, A_8 A_9$.
- the line $A_2 M$ intersects the sides $A_7 A_8, A_8 A_9, A_9 A_{10}$.
- the line $A_3 M$ intersects the sides $A_8 A_9, A_9 A_{10}, A_{10} A_{11}$, etc.

Observation: The restrictive condition of the theorem is necessary for the existence of the ratios $\frac{\overline{M_{ij} A_j}}{\overline{M_{ij} A_{p(j)}}}$.

Consequence 1.1: Let's have a polygon $A_1 A_2 \dots A_{2k+1}$ and a point M in its plan. For all i from $\{1, 2, \dots, 2k+1\}$, one notes M_i the intersection of the line $A_i A_{p(i)}$ with the line which passes through M and by the vertex which is opposed to this line. If

$M_i \notin \{A_i, A_{p(i)}\}$ then one has: $\prod_{i=1}^n \frac{\overline{M_i A_i}}{\overline{M_i A_{p(i)}}} = -1$.

The demonstration results immediately from the theorem, since one has $s = k$ and $t = 1$, that is $n = 2k + 1$.

The reciprocal of this consequence is not true.

From where it results immediately that the reciprocal of the theorem is not true either.

Counterexample:

Let us consider a polygon of 5 sides. One plots the lines $A_1 M_3, A_2 M_4$ and $A_3 M_5$ which intersect in M .

Let us have $K = \frac{\overline{M_3 A_3}}{\overline{M_3 A_4}} \cdot \frac{\overline{M_4 A_4}}{\overline{M_4 A_5}} \cdot \frac{\overline{M_5 A_5}}{\overline{M_5 A_1}}$

Then one plots the line $A_4 M_1$ such that it does not pass through M and such that it forms the ratio:

(2) $\frac{\overline{M_1 A_1}}{\overline{M_1 A_2}} = 1/K$ or $2/K$. (One chooses one of these values, for which

$A_4 M_1$ does not pass through M).

At the end one traces $A_5 M_2$ which forms the ratio $\frac{\overline{M_2 A_2}}{\overline{M_2 A_3}} = -1$ or $-\frac{1}{2}$ in function of (2). Therefore the product:

$\prod_{i=1}^5 \frac{\overline{M_i A_i}}{\overline{M_i A_{p(i)}}}$ without which the respective lines are concurrent.

Consequence 1.2: Under the conditions of the theorem, if for all i and $j, j \notin \{i, p^{-1}(i)\}$, one notes $M_{ij} = A_i M \cap A_j A_{p(j)}$ and $M_{ij} \notin \{A_j, A_{p(j)}\}$ then one has:

$$\prod_{i,j=1}^n \frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_{p(j)}}} = (-1)^n.$$

$$j \notin \{i, p^{-1}(i)\}$$

In effect one has $s = 1$, $t = n - 2$, and therefore $2s + t = n$.

Consequence 1.3: For $n = 3$, it comes $s = 1$ and $t = 1$, therefore one obtains (as a particular case) the theorem of Ceva.

An Application of the Generalizations of Ceva's Theorem is presented below.

Theorem 2: Let us consider a polygon $A_1 A_2 \dots A_n$ inserted in a circle. Let s and t be two non zero natural numbers such that $2s + t = n$. By each vertex A_i passes a line d_i which intersects the lines $A_{i+s} A_{i+s+1}, \dots, A_{i+s+t-1} A_{i+s+t}$ at the points $M_{i,i+s}, \dots, M_{i,i+s+t-1}$ respectively and the circle at the point M'_i . Then one has:

$$\prod_{i=1}^n \prod_{j=i+s}^{i+s+t-1} \frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_{j+1}}} = \prod_{i=1}^n \frac{\overline{M'_i A_{i+s}}}{\overline{M'_i A_{i+s+t}}}.$$

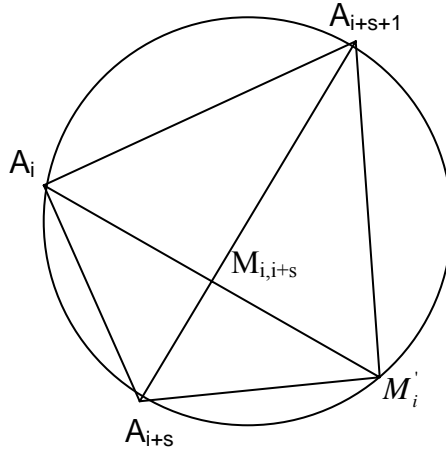
Proof:

Let i be fixed.

1) The case where the point $M_{i,i+s}$ is inside the circle.

There are the triangles $A_i M_{i,i+s} A_{i+s}$ and $M'_i M_{i,i+s} A_{i+s+1}$ similar, since the angles $M_{i,i+s} A_i A_{i+s}$ and $M_{i,i+s} A_{i+s+1} M'_i$ on one side, and $A_i M_{i,i+s} A_{i+s}$ and $A_{i+s+1} M_{i,i+s} M'_i$ are equal. It results from it that:

$$(1) \quad \frac{\overline{M_{i,i+s} A_i}}{\overline{M_{i,i+s} A_{i+s+1}}} = \frac{\overline{A_i A_{i+s}}}{\overline{M'_i A_{i+s+1}}}$$

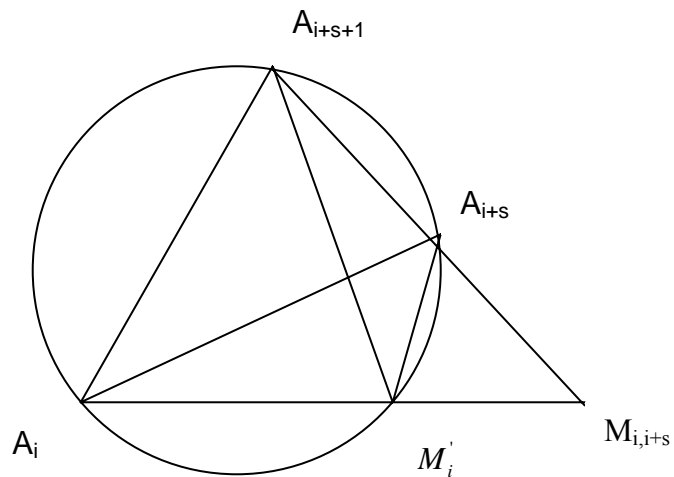


In a similar manner, one shows that the triangles $M_{i,i+s}A_iA_{i+s+1}$ and $M_{i,i+s}A_{i+s}M'_i$ are similar, from which:

$$(2) \quad \frac{\overline{M_{i,i+s}A_i}}{\overline{M_{i,i+s}A_{i+s}}} = \frac{\overline{A_iA_{i+s+1}}}{\overline{M'_iA_{i+s}}}. \text{ Dividing (1) by (2) we obtain:}$$

$$(3) \quad \frac{\overline{M_{i,i+s}A_{i+s}}}{\overline{M_{i,i+s}A_{i+s+1}}} = \frac{\overline{M'_iA_{i+s}}}{\overline{M'_iA_{i+s+1}}} \cdot \frac{\overline{A_iA_{i+s}}}{\overline{A_iA_{i+s+1}}}.$$

2) The case where $M_{i,i+s}$ is exterior to the circle is similar to the first, because the triangles (notations as in 1) are similar also in this new case. There are the same interpretations and the same ratios; therefore one has also the relation (3).



Let us calculate the product:

$$\begin{aligned}
\prod_{j=i+s}^{i+s+t-1} \frac{\overline{M_{ij}A_j}}{\overline{M_{ij}A_{j+1}}} &= \prod_{j=i+s}^{i+s+t-1} \left(\frac{\overline{M'_i A_j}}{\overline{M'_i A_{j+1}}} \cdot \frac{\overline{A_i A_j}}{\overline{A_i A_{j+1}}} \right) = \\
&= \frac{\overline{M'_i A_{i+s}}}{\overline{M'_i A_{i+s+1}}} \cdot \frac{\overline{M'_i A_{i+s+1}}}{\overline{M'_i A_{i+s+2}}} \cdots \frac{\overline{M'_i A_{i+s+t-1}}}{\overline{M'_i A_{i+s+t}}} \cdot \\
&\quad \cdot \frac{\overline{A_i A_{i+s}}}{\overline{A_i A_{i+s+1}}} \cdot \frac{\overline{A_i A_{i+s+1}}}{\overline{A_i A_{i+s+2}}} \cdots \frac{\overline{A_i A_{i+s+t-1}}}{\overline{A_i A_{i+s+t}}} = \frac{\overline{M'_i A_{i+s}}}{\overline{M'_i A_{i+s+t}}} \cdot \frac{\overline{A_i A_{i+s}}}{\overline{A_i A_{i+s+t}}}
\end{aligned}$$

Therefore the initial product is equal to:

$$\prod_{i=1}^n \left(\frac{\overline{M'_i A_{i+s}}}{\overline{M'_i A_{i+s+t}}} \cdot \frac{\overline{A_i A_{i+s}}}{\overline{A_i A_{i+s+t}}} \right) = \prod_{i=1}^n \frac{\overline{M'_i A_{i+s}}}{\overline{M'_i A_{i+s+t}}}$$

since:

$$\begin{aligned}
\prod_{i=1}^n \frac{\overline{A_i A_{i+s}}}{\overline{A_i A_{i+s+t}}} &= \frac{\overline{A_1 A_{1+s}}}{\overline{A_1 A_{1+s+t}}} \cdot \frac{\overline{A_2 A_{2+s}}}{\overline{A_2 A_{2+s+t}}} \cdots \frac{\overline{A_s A_{2s}}}{\overline{A_{s+1} A_1}} \cdot \\
&\quad \cdot \frac{\overline{A_{s+2} A_{2s+2}}}{\overline{A_{s+2} A_2}} \cdots \frac{\overline{A_{s+t} A_n}}{\overline{A_{s+t} A_t}} \cdot \frac{\overline{A_{s+t+1} A_1}}{\overline{A_{s+t+1} A_{t+1}}} \cdot \frac{\overline{A_{s+t+2} A_2}}{\overline{A_{s+t+2} A_{t+2}}} \cdots \frac{\overline{A_n A_s}}{\overline{A_n A_{s+t}}} = 1
\end{aligned}$$

(by taking into account the fact that $2s + t = n$).

Consequence 2.1: If there is a polygon $A_1 A_2, \dots, A_{2s-1}$ inscribed in a circle, and from each vertex A_i one traces a line d_i which intersects the opposite side $A_{i+s-1} A_{i+s}$ in M_i and the circle in M'_i then:

$$\prod_{i=1}^n \frac{\overline{M'_i A_{i+s-1}}}{\overline{M'_i A_{i+s}}} = \prod_{i=1}^n \frac{\overline{M'_i A_{i+s-1}}}{\overline{M'_i A_{i+s}}}$$

In fact for $t = 1$, one has n odd and $s = \frac{n+1}{2}$.

If one makes $s = 1$ in this consequence, one finds the mathematical note from [1], pages 35-37.

Application: If in the theorem, the lines d_i are concurrent, one obtains:

$$\prod_{i=1}^n \frac{\overline{M'_i A_{i+s}}}{\overline{M'_i A_{i+s+t}}} = (-1)^n \text{ (For this, see [2]).}$$

References:

- [1] Dan Barbilian - Ion Barbu – “Pagini inedite”, Editura Albatros, Bucharest, 1981 (Ediție îngrijită de Gerda Barbilian, V. Protopopescu, Viorel Gh. Vodă).
- [2] Florentin Smarandache – “Généralisation du théorème de Ceva”.