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Neutrosophic filters in BE-algebras

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Abstract

In this paper, we introduce the notion of (implicative) neutrosophic filters in BE-algebras. The relation between implicative neutrosophic filters and neutrosophic filters is investigated and we show that in self distributive BE-algebras these notions are equivalent.

Keywords: BE-algebra, neutrosophic set, (implicative) neutrosophic filter.

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1 Introduction

Neutrosophic set theory was introduced by Smarandache in 1998 ([10]). Neutrosophic sets are a new mathematical tool for dealing with uncertainties which are free from many difficulties that have troubled the usual theoretical approaches. Research works on neutrosophic set theory for many applications such as information fussion, probability theory, control theory, decision making, measurement theory, etc. Kandasamy and Smarandache introduced the concept of neutrosophic algebraic structures ([3, 4, 5]). Since then many researchers worked in this area and lots of literatures had been produced about the theory of neutrosophic set. In the neutrosophic set one can have elements which have paraconsistent information (sum of components > 1), others incomplete information (sum of components < 1), others consistent information (in the case when the sum of components =1) and others interval-valued components (with no restriction on their superior or inferior sums).

H.S. Kim and Y.H. Kim introduced the notion of a BE-algebra as a generalization of a dual BCK-algebra ([6]). B.L. Meng give a procedure which generated a filter by a subset in a transitive BE-algebra ([7]). A. Walendziak introduced the notion of a normal filter in BE-algebras and showed that there is a bijection between congruence relations and filters in commutative BE-algebras ([11]). A. Borumand Saeid and et al. defined some types of filters in BE-algebras and showed the relationship between them ([1]). A. Rezaei and et al. discussed on the relationship between BE-algebras and Hilbert algebras ([9]). Recently, A. Rezaei and et al. introduced the notion of hesitant fuzzy (implicative) filters and get some results on BE-algebras ([8]).

In this paper, we introduce the notion of (implicative) neutrosophic filters and study it in details. In fact, we show that in self distributive BE-algebras concepts of implicative neutrosophic filter and neutrosophic filter are equivalent.

2 Preliminaries

In this section, we cite the fundamental definitions that will be used in the sequel:

Definition 2.1. [6] By a BE-algebra we shall mean an algebra $\mathfrak{X} = (X; *, 1)$ of type $(2, 0)$ satisfying the following axioms:

(BE1) $x * x = 1$,

(BE2) $x * 1 = 1$,

(BE3) $1 * x = x$,

(BE4) $x * (y * z) = y * (x * z)$, for all $x, y, z \in X$.

From now on, $\mathfrak{X}$ is a BE-algebra, unless otherwise is stated. We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$. A BE-algebra $\mathfrak{X}$ is said to be self distributive if $x * (y * z) = (x * y) * (x * z)$, for all $x, y, z \in X$. A BE-algebra $\mathfrak{X}$ is said to be commutative if satisfies:

$$(x * y) * y = (y * x) * x, \text{ for all } x, y \in X.$$

Proposition 2.1. [11] If $\mathfrak{X}$ is a commutative BE-algebra, then for all $x, y \in X$,

$$x * y = 1 \text{ and } y * x = 1 \text{ imply } x = y.$$

We note that "$\leq$" is reflexive by (BE1). If $\mathfrak{X}$ is self distributive then relation "$\leq$" is a transitive ordered set on X, because if $x \leq y$ and $y \leq z$, then

$$x * z = 1 * (x * z) = (x * y) * (x * z) = x * (y * z) = x * 1 = 1.$$

Hence $x \leq z$. If $\mathfrak{X}$ is commutative then by Proposition 2.1, relation "$\leq$" is anti-symmetric. Hence if $\mathfrak{X}$ is a commutative self distributive BE-algebra, then relation "$\leq$" is a partial ordered set on $\mathfrak{X}$.

Proposition 2.2. [6] In a BE-algebra $\mathfrak{X}$, the following hold:

(i) $x * (y * x) = 1$,

(ii) $y * ((y * x) * x) = 1$, for all $x, y \in X$.

A subset F of X is called a filter of $\mathfrak{X}$ if it satisfies: (F1) $1 \in F$, (F2) $x \in F$ and $x * y \in F$ imply $y \in F$. Define

$$A(x, y) = \{z \in X : x * (y * z) = 1\},$$

which is called an upper set of x and y. It is easy to see that $1, x, y \in A(x, y)$, for any $x, y \in X$. Every upper set $A(x, y)$ need not be a filter of $\mathfrak{X}$ in general.

Definition 2.2. [1] A non-empty subset F of X is called an implicative filter if satisfies the following conditions:

(IF1) $1 \in F$,

(IF2) $x * (y * z) \in F$ and $x * y \in F$ imply that $x * z \in F$, for all $x, y, z \in X$.

If we replace x of the condition (IF2) by the element 1, then it can be easily observed that every implicative filter is a filter. However, every filter is not an implicative filter as shown in the following example.

Example 2.1. Let $X = \{1, a, b\}$ be a BE-algebra with the following table:

$*$	1	a	b
1	1	a	b
a	1	1	a
b	1	a	1

Then $F = \{1, a\}$ is a filter of X, but it is not an implicative filter, since
$1 * (a * b) = 1 * a = a \in F$ and $1 * a = a \in F$ but $1 * b = b \notin F$.

Definition 2.3. [10] Let X be a set. A neutrosophic subset A of X is a triple (T_A, I_A, F_A) where $T_A : X \to [0,1]$ is the membership function, $I_A : X \to [0,1]$ is the indeterminacy function and $F_A : X \to [0,1]$ is the nonmembership function. Here for each $x \in X$, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are all standard real numbers in $[0,1]$.

We note that $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$, for all $x \in X$. The set of neutrosophic subset of X is denoted by NS(X).

Definition 2.4. [10] Let A and B be two neutrosophic sets on X. Define $A \leq B$ if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$, for all $x \in X$.

Definition 2.5. Let $\mathfrak{X}_1 = (X_1; *, 1)$ and $\mathfrak{X}_2 = (X_2; \circ, 1')$ be two BE-algebras. Then a mapping $f : X_1 \to X_2$ is called a homomorphism if, for all $x_1, x_2 \in X_1$ $f(x_1 * x_2) = f(x_1) \circ f(x_2)$. It is clear that if $f : X_1 \to X_2$ is a homomorphism, then $f(1) = 1'$.

3 Neutrosophic Filters

Definition 3.1. A neutrosophic set A of $\mathfrak{X}$ is called a *neutrosophic filter if satisfies the following conditions:*

(NF1) $T_A(x) \leq T_A(1)$, $I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$,

(NF2) $\min\{T_A(x * y), T_A(x)\} \leq T_A(y)$, $\min\{I_A(x * y), I_A(x)\} \geq I_A(y)$ and $\min\{F_A(x * y), F_A(x)\} \geq F_A(y)$, *for all* $x, y \in X$.

The set of neutrosophic filter of $\mathfrak{X}$ is denoted by NF($\mathfrak{X}$).

Example 3.1. In Example 2.1, put $T_A(1) = 0.9$, $T_A(a) = T_A(b) = 0.5$,
$I_A(1) = 0.2$, $I_A(a) = I_A(b) = 0.35$ and $F_A(1) = 0.1$, $F_A(a) = F_A(b) = 0$.
Then $A = (T_A, I_A, F_A)$ is a neutrosophic filter.

Proposition 3.1. *Let* $A \in \mathrm{NF}(\mathfrak{X})$. *Then*

(i) if $x \leq y$, *then* $T_A(x) \leq T_A(y)$, $I_A(x) \geq I_A(y)$ and $F_A(x) \geq F_A(y)$,

(ii) $T_A(x) \leq T_A(y * x)$, $I_A(x) \geq I_A(y * x)$ and $F_A(x) \geq F_A(y * x)$,

(iii) $\min\{T_A(x), T_A(y)\} \leq T_A(x * y)$, $\min\{I_A(x), I_A(y)\} \geq I_A(x * y)$ and $\min\{F_A(x), F_A(y)\} \geq F_A(x * y)$,

(iv) $T_A(x) \leq T_A((x*y)*y)$, $I_A(x) \geq I_A((x*y)*y)$ and $F_A(x) \geq F_A((x*y)*y)$,

(v) $\min\{T_A(x), T_A(y)\} \leq T_A((x * (y * z)) * z)$,
$\min\{I_A(x), I_A(y)\} \geq I_A((x * (y * z)) * z)$ and
$\min\{F_A(x), F_A(y)\} \geq F_A((x * (y * z)) * z)$,

(vi) *if* $\min\{T_A(y), T_A((x * y) * z)\} \leq T_A(z * x)$, *then* T_A *is order reversing and* I_A, F_A *are order* (i.e. if $x \leq y$, then $T_A(y) \leq T_A(x)$, $I_A(y) \geq I_A(x)$ and $F_A(y) \geq F_A(x)$)

(vii) *if* $z \in A(x, y)$, *then* $\min\{T_A(x), T_A(y)\} \leq T_A(z)$,
$\min\{I_A(x), I_A(y)\} \geq I_A(z)$ and $\min\{F_A(x), F_A(y)\} \geq F_A(z)$

(viii) *if* $\prod_{i=1}^{n} a_i * x = 1$, *then* $\bigwedge_{i=1}^{n} T_A(a_i) \leq T_A(x)$, $\bigwedge_{i=1}^{n} I_A(a_i) \geq I_A(x)$ and $\bigwedge_{i=1}^{n} F_A(a_i) \geq F_A(x)$ *where* $\prod_{i=1}^{n} a_i * x = a_n * (a_{n-1} * (\ldots (a_1 * x) \ldots))$.

Proof. (i). Let $x \leq y$. Then $x * y = 1$ and so

$$T_A(x) = \min\{T_A(x), T_A(1)\} = \min\{T_A(x), T_A(x * y)\} \leq T_A(y),$$

$$I_A(x) = \min\{I_A(x), I_A(1)\} = \min\{I_A(x), I_A(x * y)\} \geq I_A(y),$$

$$F_A(x) = \min\{F_A(x), F_A(1)\} = \min\{F_A(x), F_A(x * y)\} \geq F_A(y).$$

(ii). Since $x \leq y * x$, by using (i) the proof is clear.
(iii). By using (ii) we have

$$\min\{T_A(x), T_A(y)\} \leq T_A(y) \leq T_A(x * y),$$

$$\min\{I_A(x), I_A(y)\} \geq I_A(y) \geq I_A(x * y),$$

$$\min\{F_A(x), F_A(y)\} \geq F_A(y) \geq F_A(x * y).$$

(iv). It follows from Definition 3.1,

$$\begin{aligned} T_A(x) &= \min\{T_A(x), T_A(1)\} \\ &= \min\{T_A(x), T_A((x * y) * (x * y))\} \\ &= \min\{T_A(x), T_A(x * ((x * y) * y))\} \\ &\leq T_A((x * y) * y). \end{aligned}$$

Also, we have

$$\begin{aligned} I_A(x) &= \min\{I_A(x), I_A(1)\} \\ &= \min\{I_A(x), I_A((x*y)*(x*y))\} \\ &= \min\{I_A(x), I_A(x*((x*y)*y))\} \\ &\geq I_A((x*y)*y) \end{aligned}$$

and

$$\begin{aligned} F_A(x) &= \min\{F_A(x), F_A(1)\} \\ &= \min\{F_A(x), F_A((x*y)*(x*y))\} \\ &= \min\{F_A(x), F_A(x*((x*y)*y))\} \\ &\geq F_A((x*y)*y). \end{aligned}$$

(v). From (iv) we have

$$\begin{aligned} \min\{T_A(x), T_A(y)\} &\leq \min\{T_A(x), T_A((y*(x*z))*(x*z))\} \\ &= \min\{T_A(x), T_A((x*(y*z))*(x*z))\} \\ &= \min\{T_A(x), T_A(x*(x*(y*z))*z))\} \\ &\leq T_A((x*(y*z))*z)), \end{aligned}$$

$$\begin{aligned} \min\{I_A(x), I_A(y)\} &\geq \min\{I_A(x), I_A((y*(x*z))*(x*z))\} \\ &= \min\{I_A(x), I_A((x*(y*z))*(x*z))\} \\ &= \min\{I_A(x), I_A(x*(x*(y*z))*z))\} \\ &\geq I_A((x*(y*z))*z)) \end{aligned}$$

and

$$\begin{aligned} \min\{F_A(x), F_A(y)\} &\geq \min\{F_A(x), F_A((y*(x*z))*(x*z))\} \\ &= \min\{F_A(x), F_A((x*(y*z))*(x*z))\} \\ &= \min\{F_A(x), F_A(x*(x*(y*z))*z))\} \\ &\geq F_A((x*(y*z))*z)). \end{aligned}$$

(vi). Let $x \leq y$, that is, $x*y = 1$.

$T_A(y) = \min\{T_A(y), T_A(1*1)\} = \min\{T_A(y), T_A((x*y)*1)\} \leq T_A(1*x) = T_A(x)$,

$I_A(y) = \min\{I_A(y), I_A(1*1)\} = \min\{I_A(y), I_A((x*y)*1)\} \geq I_A(1*x) = I_A(x)$,

$$F_A(y) = \min\{F_A(y), F_A(1*1)\} = \min\{F_A(y), F_A((x*y)*1)\} \geq F_A(1*x) = F_A(x).$$

(vii). Let $z \in A(x,y)$. Then $x*(y*z) = 1$. Hence

$$\begin{aligned}\min\{T_A(x), T_A(y)\} &= \min\{T_A(x), T_A(y), T_A(1)\}\\ &= \min\{T_A(x), T_A(y), T_A(x*(y*z))\}\\ &\leq \min\{T_A(y), T_A(y*z)\}\\ &\leq T_A(z).\end{aligned}$$

Also, we have

$$\begin{aligned}\min\{I_A(x), I_A(y)\} &= \min\{I_A(x), I_A(y), I_A(1)\}\\ &= \min\{I_A(x), I_A(y), I_A(x*(y*z))\}\\ &\geq \min\{I_A(y), I_A(y*z)\}\\ &\geq I_A(z),\end{aligned}$$

and

$$\begin{aligned}\min\{F_A(x), F_A(y)\} &= \min\{F_A(x), F_A(y), F_A(1)\}\\ &= \min\{F_A(x), F_A(y), F_A(x*(y*z))\}\\ &\geq \min\{F_A(y), F_A(y*z)\}\\ &\geq F_A(z).\end{aligned}$$

(viii). The proof is by induction on n. By (vii) it is true for $n = 1, 2$. Assume that it satisfies for $n = k$, that is,

$\prod\limits_{i=1}^{k} a_i * x = 1 \Rightarrow \bigwedge\limits_{i=1}^{k} T_A(a_i) \leq T_A(x), \bigwedge\limits_{i=1}^{k} I_A(a_i) \geq I_A(x)$ and $\bigwedge\limits_{i=1}^{k} F_A(a_i) \geq F_A(x)$ for all $a_1, \ldots, a_k, x \in X$.

Suppose that $\prod\limits_{i=1}^{k+1} a_i * x = 1$, for all $a_1, \ldots, a_k, a_{k+1}, x \in X$. Then

$\bigwedge\limits_{i=2}^{k+1} T_A(a_i) \leq T_A(a_1 * x)$, $\bigwedge\limits_{i=2}^{k+1} I_A(a_i) \geq I_A(a_1 * x)$, and $\bigwedge\limits_{i=2}^{k+1} F_A(a_i) \geq F_A(a_1 * x)$.

Since A is a neutrosophic filter of $\mathfrak{X}$, we have

$$\bigwedge_{i=1}^{k+1} T_A(a_i) = \min\{(\bigwedge_{i=2}^{k+1} T_A(a_i)), T_A(a_1)\} \leq \min\{T_A(a_1 * x), T_A(a_1)\} \leq T_A(x),$$

$$\bigwedge_{i=1}^{k+1} I_A(a_i) = \min\{(\bigwedge_{i=2}^{k+1} I_A(a_i)), I_A(a_1)\} \geq \min\{I_A(a_1 * x), I_A(a_1)\} \geq I_A(x)$$

and

$$\bigwedge_{i=1}^{k+1} F_A(a_i) = \min\{(\bigwedge_{i=2}^{k+1} F_A(a_i)), F_A(a_1)\} \geq \min\{F_A(a_1 * x), F_A(a_1)\} \geq F_A(x).$$

□

Theorem 3.1. *If $\{A_i\}_{i \in I}$ is a family of neutrosophic filters in $\mathfrak{X}$, then $\bigcap_{i \in I} A_i$ is too.*

Theorem 3.2. *Let $A \in \mathrm{NF}(\mathfrak{X})$. Then the sets*

(i) $X_{T_A} = \{x \in X : T_A(x) = T_A(1)\}$,

(ii) $X_{I_A} = \{x \in X : I_A(x) = I_A(1)\}$,

(iii) $X_{F_A} = \{x \in X : F_A(x) = F_A(1)\}$,

are filters of $\mathfrak{X}$.

Proof. (i). Obviously, $1 \in X_{h_A}$. Let $x, x * y \in X_{T_A}$. Then $T_A(x) = T_A(x * y) = T_A(1)$. Now, by (NF1) and (NF2), we have

$$T_A(1) = \min\{T_A(x), T_A(x * y)\} \leq T_A(y) \leq T_A(1).$$

Hence $T_A(y) = T_A(1)$. Therefore, $y \in X_{T_A}$.
The proofs of (ii) and (iii) are similar to (i).□

Definition 3.2. A neutrosophic set A of $\mathfrak{X}$ is called an implicative neutrosophic filter of $\mathfrak{X}$ if satisfies the following conditions:

(INF1) $T_A(1) \geq T_A(x)$,

(INF2) $T_A(x * z) \geq \min\{T_A(x * (y * z)), T_A(x * y)\}$,
$I_A(x * z) \leq \min\{I_A(x * (y * z)), I_A(x * y)\}$ and
$F_A(x * z) \leq \min\{F_A(x * (y * z)), F_A(x * y)\}$, for all $x, y, z \in X$.

The set of implicative neutrosophic filter of $\mathfrak{X}$ is denoted by INF($\mathfrak{X}$). If we replace x of the condition (INF2) by the element 1, then it can be easily observed that every implicative neutrosophic filter is a neutrosophic filter. However, every neutrosophic filter is not an implicative neutrosophic filter as shown in the following example.

Example 3.2. Let $X = \{1, a, b, c, d\}$ be a BE-algebra with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then $\mathfrak{X} = (X; *, 1)$ is a BE-algebra. Define a neutrosophic set A on $\mathfrak{X}$ as follows:

$$T_A(x) = \begin{cases} 0.85 & \text{if } x = 1, a \\ 0.12 & otherwise \end{cases}$$

and $I_A(x) = F_A(x) = 0.5$, for all $x \in X$.

Then clearly $A = (T_A, I_A, F_A)$ is a neutrosophic filter of $\mathfrak{X}$, but it is not an implicative neutrosophic filter of $\mathfrak{X}$, since

$$T_A(b * c) \ngeq \min\{T_A(b * (d * c)), T_A(b * d)\}.$$

Theorem 3.3. *Let $\mathfrak{X}$ be a self distributive BE-algebra. Then every neutrosophic filter is an implicative neutrosophic filter.*

Proof. Let $A \in \mathrm{NF}(\mathfrak{X})$ and $x \in X$. Obvious that $T_A(x) \leq T_A(1)$, $I_A(x) \geq I_A(1)$ and $F_A(x) \geq F_A(1)$. By self distributivity and (NF2), we have

$$\min\{T_A(x*(y*z)), T_A(x*y)\} = \min\{T_A((x*y)*(x*z)), T_A(x*y)\} \leq T_A(x*z),$$

$$\min\{I_A(x*(y*z)), I_A(x*y)\} = \min\{I_A((x*y)*(x*z)), I_A(x*y)\} \geq I_A(x*z)$$

and

$$\min\{F_A(x*(y*z)), F_A(x*y)\} = \min\{F_A((x*y)*(x*z)), F_A(x*y)\} \geq F_A(x*z).$$

Therefore $A \in \mathrm{INF}(\mathfrak{X})$.□

Let $t \in [0, 1]$. For a neutrosophic filter A of $\mathfrak{X}$, t-level subset which denoted by $U(A; t)$ is defined as follows:

$$U(A; t) := \{x \in A : t \leq T_A(x), I_A(x) \leq t \text{ and } F_A(x) \leq t\}$$

and strong t-level subset which denoted by $U(A; t)_>$ as

$$U(A; t)_> := \{x \in A : t < T_A(x), I_A(x) < t \text{ and } F_A(x) < t\}.$$

Theorem 3.4. *Let* $A \in \mathrm{NS}(\mathfrak{X})$. *The following are equivalent:*

(i) $A \in \mathrm{NF}(\mathfrak{X})$,

(ii) $(\forall t \in [0, 1])$ $U(A; t) \neq \emptyset$ *imply* $U(A; t)$ *is a filter of* $\mathfrak{X}$.

Proof. (i)⇒(ii). Let $x, y \in X$ be such that $x, x * y \in U(A; t)$, for any $t \in [0, 1]$. Then $t \leq T_A(x)$ and $t \leq T_A(x*y)$. Hence $t \leq \min\{T_A(x), T_A(x*y)\} \leq T_A(y)$. Also, $I_A(x) \leq t$ and $I_A(x * y) \leq t$ and so $t \geq \min\{I_A(x), I_A(x * y)\} \geq I_A(y)$. By a similar argument we have $t \geq \min\{F_A(x), F_A(x * y)\} \geq F_A(y)$. Therefore, $y \in U(A; t)$.

(ii)⇒(i). Let $U(A; t)$ be a filter of $\mathfrak{X}$, for any $t \in [0, 1]$ with $U(A; t) \neq \emptyset$. Put $T_A(x) = I_A(x) = F_A(x) = t$, for any $x \in X$. Then $x \in U(A; t)$. Since $U(A; t)$ is a filter of $\mathfrak{X}$, we have $1 \in U(A; t)$ and so $T_A(x) = t \leq T_A(1)$. Now, for any $x, y \in X$, let $T_A(x * y) = I_A(x * y) = F_A(x * y) = t_{x*y}$ and $T_A(x) = I_A(x) = F_A(x) = t_x$. Put $t = \min\{t_{x*y}, t_x\}$. Then $x, x * y \in U(A; t)$, so $y \in U(A; t)$. Hence $t \leq T_A(y)$, $t \geq I_A(y)$, $t \geq F_A(y)$ and so

$$\min\{T_A(x * y), T_A(x)\} = \min\{t_{x*y}, t_x\} = t \leq T_A(y),$$

$$\min\{I_A(x * y), I_A(x)\} = \min\{t_{x*y}, t_x\} = t \geq I_A(y),$$

and

$$\min\{F_A(x * y), F_A(x)\} = \min\{t_{x*y}, t_x\} = t \geq F_A(y).$$

Therefore, $A \in \mathrm{NF}(\mathfrak{X})$.□

Theorem 3.5. *Let* $A \in \mathrm{NF}(\mathfrak{X})$. *Then we have*

$$(\forall a, b \in X)\ (\forall t \in [0, 1])\ (a, b \in U(A; t) \ \Rightarrow\ A(a, b) \subseteq U(A; t)).$$

Proof. Assume that $A \in \mathrm{NF}(\mathfrak{X})$. Let $a, b \in X$ be such that $a, b \in U(A;t)$. Then $t \leq T_A(a)$ and $t \leq T_A(b)$. Let $c \in A(a,b)$. Hence $a * (b * c) = 1$. Now, by Proposition 3.1(v) and (BE3), we have

$$t \leq \min\{T_A(a), T_A(b)\} \leq T_A((a * (b * c) * c)) = T_A(1 * c) = T_A(c),$$

$$t \geq \min\{I_A(a), I_A(b)\} \geq I_A((a * (b * c) * c)) = I_A(1 * c) = I_A(c)$$

and

$$t \geq \min\{F_A(a), F_A(b)\} \geq F_A((a * (b * c) * c)) = F_A(1 * c) = F_A(c).$$

Then $c \in U(A;t)$. Therefore, $A(a,b) \subseteq U(A;t)$).□

Corolary 3.1. *Let* $A \in \mathrm{NF}(\mathfrak{X})$. *Then*

$$(\forall t \in [0,1])\ (U(A;t) \neq \emptyset \ \Rightarrow\ U(A;t) = \bigcup_{a,b \in U(A;t)} A(a,b)).$$

Proof. It is sufficient prove that $U(A;t) \subseteq \bigcup_{a,b \in U(A;t)} A(a,b)$. For this, assume that $x \in U(A;t)$. Since $x * (1 * x) = 1$, we have $x \in A(x,1)$. Hence

$$U(A;t) \subseteq A(x,1) \subseteq \bigcup_{x \in U(A;t)} A(x,1) \subseteq \bigcup_{x,y \in U(A;t)} A(x,y).$$

□

Theorem 3.6. *Let* $\mathfrak{X}$ *be a self distributive BE-algebra and* $A \in \mathrm{NF}(\mathfrak{X})$. *Then the following conditions are equivalent:*

(i) $A \in \mathrm{INF}(\mathfrak{X})$,

(ii) $T_A(y * (y * x)) \leq T_A(y * x)$, $I_A(y * (y * x)) \geq I_A(y * x)$ *and* $F_A(y * (y * x)) \geq F_A(y * x)$,

(iii) $\min\{T_A((z * (y * (y * x))), T_A(z)\} \leq T_A(y * x)$, $\min\{I_A((z * (y * (y * x))), I_A(z)\} \geq I_A(y * x)$ *and* $\min\{F_A((z * (y * (y * x))), F_A(z)\} \geq F_A(y * x)$.

Proof. (i)⇒(ii). Let $A \in \mathrm{NF}(\mathfrak{X})$. By (INF1) and (BE1) we have

$$\begin{aligned} T_A(y*(y*x)) &= \min\{T_A(y*(y*x)), T_A(1)\} \\ &= \min\{T_A(y*(y*x)), T_A(y*y)\} \\ &\leq T_A(y*x), \end{aligned}$$

$$\begin{aligned} I_A(y*(y*x)) &= \min\{I_A(y*(y*x)), I_A(1)\} \\ &= \min\{I_A(y*(y*x)), I_A(y*y)\} \\ &\geq I_A(y*x) \end{aligned}$$

and

$$\begin{aligned} F_A(y*(y*x)) &= \min\{F_A(y*(y*x)), F_A(1)\} \\ &= \min\{F_A(y*(y*x)), F_A(y*y)\} \\ &\geq F_A(y*x). \end{aligned}$$

(ii)⇒(iii). Let A be a neutrosophic filter of $\mathfrak{X}$ satisfying the condition (ii). By using (NF2) and (ii) we have

$$\begin{aligned} \min\{T_A(z*(y*(y*x))), T_A(z)\} &\leq T_A(y*(y*x)) \\ &\leq T_A(y*x), \end{aligned}$$

$$\begin{aligned} \min\{I_A(z*(y*(y*x))), I_A(z)\} &\geq I_A(y*(y*x)) \\ &\geq I_A(y*x) \end{aligned}$$

and

$$\begin{aligned} \min\{F_A(z*(y*(y*x))), F_A(z)\} &\geq F_A(y*(y*x)) \\ &\geq F_A(y*x). \end{aligned}$$

(iii)⇒(i). Since

$$x*(y*z) = y*(x*z) \leq (x*y)*(x*(x*z)),$$

we have $T_A(x*(y*z)) \leq T_A((x*y)*(x*(x*z)))$,
$I_A(x*(y*z)) \geq I_A((x*y)*(x*(x*z)))$ and
$F_A(x*(y*z)) \geq F_A((x*y)*(x*(x*z)))$, by Proposition 3.1(i). Thus

$$\begin{aligned} \min\{T_A(x*(y*z)), T_A(x*y)\} &\leq \min\{T_A((x*y)*(x*(x*z))), T_A(x*y)\} \\ &\leq T_A(x*z). \end{aligned}$$

$$\begin{array}{rcl}\min\{I_A(x*(y*z)), I_A(x*y)\} & \geq & \min\{I_A((x*y)*(x*(x*z))), I_A(x*y)\} \\ & \geq & I_A(x*z)\end{array}$$

and

$$\begin{array}{rcl}\min\{F_A(x*(y*z)), F_A(x*y)\} & \geq & \min\{F_A((x*y)*(x*(x*z))), F_A(x* \\ y)\} & \geq & F_A(x*z).\end{array}$$

Therefore, $A \in \mathrm{INF}(\mathfrak{X})$. Let $f : X \to Y$ be a homomorphism of BE-algebras

and $A \in \mathrm{NS}(\mathfrak{X})$.
Define tree maps $T_{A^f} : X \to [0,1]$ such that $T_{A^f}(x) = T_A(f(x))$,
$I_{A^f} : X \to [0,1]$ such that $I_{A^f}(x) = I_A(f(x))$ and $F_{A^f} : X \to [0,1]$ such that $F_{A^f}(x) = F_A(f(x))$, for all $x \in X$. Then T_{A^f}, I_{A^f} and F_{A^f} are well-define and $A^f = (T_{A^f}, I_{A^f}, F_{A^f}) \in \mathrm{NS}(\mathfrak{X})$.□

Theorem 3.7. *Let $f : X \to Y$ be an onto homomorphism of BE-algebras and $A \in \mathrm{NS}(\mathfrak{Y})$. Then $A \in \mathrm{NF}(\mathfrak{Y})$ (resp. $A \in \mathrm{INF}(\mathfrak{Y})$) if and only if $A^f \in \mathrm{NF}(\mathfrak{X})$ (resp. $A^f \in \mathrm{INF}(\mathfrak{X})$).*

Proof. Assume that $A \in \mathrm{NF}(\mathfrak{Y})$. For any $x \in X$, we have

$$T_{A^f}(x) = T_A(f(x)) \leq T_A(1_Y) = T_A(f(1_X)) = T_{A^f}(1_X),$$

$$I_{A^f}(x) = I_A(f(x)) \geq I_A(1_Y) = I_A(f(1_X)) = I_{A^f}(1_X)$$

and

$$F_{A^f}(x) = F_A(f(x)) \geq F_A(1_Y) = F_A(f(1_X)) = F_{A^f}(1_X).$$

Hence (NF1) is valid. Now, let $x, y \in X$. By (NF1) we have

$$\begin{array}{rcl}\min\{T_{A^f}(x*y), T_{A^f}(x)\} & = & \min\{T_A(f(x*y)), T_A(f(x))\} \\ & = & \min\{T_A(f(x)*f(y)), T_A(f(x))\} \\ & \leq & T_A(f(y)) \\ & = & T_{A^f}(y)\end{array}$$

Also,

$$\begin{array}{rcl}\min\{I_{A^f}(x*y), I_{A^f}(x)\} & = & \min\{I_A(f(x*y)), I_A(f(x))\} \\ & = & \min\{I_A(f(x)*f(y)), I_A(f(x))\} \\ & \geq & I_A(f(y)) \\ & = & I_{A^f}(y).\end{array}$$

By a similar argument we have $\min\{F_{A^f}(x*y), F_{A^f}(x)\} \geq F_{A^f}(y)$. Therefore, $A^f \in \mathrm{NF}(\mathfrak{X})$.

Conversely, Assume that $A^f \in \mathrm{NF}(\mathfrak{X})$. Let $y \in Y$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. Then

$$T_A(y) = T_A(f(x)) = T_{A^f}(x) \leq T_{A^f}(1_X) = T_A(f(1_X)) = T_A(1_Y),$$

$$I_A(y) = I_A(f(x)) = I_{A^f}(x) \geq I_{A^f}(1_X) = I_A(f(1_X)) = I_A(1_Y)$$

and

$$F_A(y) = F_A(f(x)) = F_{A^f}(x) \geq F_{A^f}(1_X) = F_A(f(1_X)) = F_A(1_Y),$$

Now, let $x, y \in Y$. Then there exists $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Hence we have

$$\begin{array}{rcl}\min\{T_A(x*y), T_A(x)\} & = & \min\{T_A(f(a)*f(b)), T_A(f(a))\} \\ & = & \min\{T_A(f(a*b)), T_A(f(a))\} \\ & = & \min\{T_{A^f}(a*b), T_{A^f}(a)\} \\ & \leq & T_{A^f}(b) \\ & = & T_A(f(b)) \\ & = & T_A(y).\end{array}$$

Also, we have

$$\begin{array}{rcl}\min\{I_A(x*y), I_A(x)\} & = & \min\{I_A(f(a)*f(b)), I_A(f(a))\} \\ & = & \min\{I_A(f(a*b)), I_A(f(a))\} \\ & = & \min\{I_{A^f}(a*b), I_{A^f}(a)\} \\ & \geq & I_{A^f}(b) \\ & = & I_A(f(b)) \\ & = & I_A(y).\end{array}$$

By a similar argument we have $\min\{F_A(x*y), F_A(x)\} \geq F_A(y)$.
Therefore, $A \in \mathrm{NF}(\mathfrak{Y})$.□

4 Conclusion

F. Smarandache as an extension of intuitionistic fuzzy logic introduced the concept of neutrosophic logic and then several researchers have studied of some neutrosophic algebraic structures. In this paper, we applied the theory of neutrosophic sets to BE-algebras and introduced the notions of (implicative) neutrosophic filters and many related properties are investigated.

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