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Presentation of DSmT

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Abstract: *This chapter presents a general overview and foundations of the DSmT, i.e. the recent theory of plausible and paradoxical reasoning developed by the authors, specially for the static or dynamic fusion of information arising from several independent but potentially highly conflicting, uncertain and imprecise sources of evidence. We introduce and justify here the basis of the DSmT framework with respect to the Dempster-Shafer Theory (DST), a mathematical theory of evidence developed in 1976 by Glenn Shafer. We present the DSm combination rules and provide some simple illustrative examples and comparisons with other main rules of combination available in the literature for the combination of information for simple fusion problems. Detailed presentations on recent advances and applications of DSmT are presented in the next chapters of this book.*

1.1 Introduction

The Dezert-Smarandache Theory (DSmT) of plausible and paradoxical reasoning proposed by the authors in recent years [9, 10, 36] can be considered as an extension of the classical Dempster-Shafer theory (DST) [33] but includes fundamental differences with the DST. DSmT allows to formally combine any types of independent sources of information represented in term of belief functions, but is mainly focused on the fusion of uncertain, highly conflicting and imprecise sources of evidence. DSmT is able to solve complex static or dynamic fusion problems beyond the limits of the DST framework, specially when conflicts between sources become large and when the refinement of the frame of the problem under consideration, denoted Θ, becomes inaccessible because of the vague, relative and imprecise nature of elements of Θ [10].

The foundation of DSmT is based on the definition of the Dedekind's lattice D^Θ also called *hyper-power set* of the frame Θ in the sequel. In the DSmT framework, Θ is first considered as only a set $\{\theta_1, \ldots, \theta_n\}$ of n exhaustive elements (closed world assumption) without introducing other constraint (exclusivity or non-existential constraints). This corresponds to the *free DSm model* on which is based the *classic DSm rule of combination*. The exhaustivity (closed world) assumption is not fundamental actually, because one can always close any open world theoretically, say Θ_{Open} by including into it an extra element/hypothesis θ_0 (although not precisely identified) corresponding to all missing hypotheses of Θ_{Open} to work with the new closed frame $\Theta = \{\theta_0\} \cup \Theta_{\text{Open}} = \{\theta_0, \theta_1, \ldots, \theta_n\}$. This idea has been already proposed and defended by Yager, Dubois & Prade and Testemale in [45, 13, 30] and differs from the Transferable Belief Model (TBM) of Smets [42]. The proper use of the free DSm model for the fusion depends on the intrinsic nature of elements/concepts θ_i involved in the problem under consideration and becomes naturally justified when dealing with vague/continuous elements which cannot be precisely defined and separated (e.g. the relative concepts of smallness/tallness, pleasure/pain, hot/cold, colors (because of the continuous spectrum of the light), etc) so that no refinement of Θ in a new larger set Θ_{ref} of exclusive refined hypotheses is possible. In such case, we just call Θ the *frame* of the problem.

When a complete refinement (or maybe sometimes an only partial refinement) of Θ is possible and thus allows us to work on Θ_{ref}, then we call Θ_{ref} the *frame of discernment* (resp. *frame of partial discernment*) of the problem because some elements of Θ_{ref} are truly exclusive and thus they become (resp. partially) discernable. The refined frame of discernment assuming exclusivity of all elements $\theta_i \in \Theta$ corresponds to the *Shafer's model* on which is based the DST and can be obtained from the free DSm model by introducing into it all exclusivity constraints. All fusion problems dealing with truly exclusive concepts must obviously be based on such model since it describes adequately the real and intrinsic nature of hypotheses. Actually, any constrained model (including Shafer's model) corresponds to what we called an *hybrid DSm model*. DSmT provides a generalized hybrid DSm rule of combination for working with any kind of hybrid models including exclusivity and non-existential constraints as well and it is not only limited to the most constrained one, i.e. Shafer's model (see chapter 4 for a detailed presentation and examples on the hybrid DSm rule). Before going further into this DSmT presentation it is necessary to briefly present the foundations of the DST [33] for pointing out the important differences between these two theories for managing the combination of evidence.

1.2 Short introduction to the DST

In this section, we present a short introduction to the Dempster-Shafer theory. A complete presentation of the Mathematical Theory of Evidence proposed by Glenn Shafer can be found in his milestone book in [33]. Advances on DST can be found in [34, 48] and [49].

1.2.1 Shafer's model and belief functions

Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be the *frame of discernment* of the fusion problem under consideration having n *exhaustive* and *exclusive elementary* hypotheses θ_i. This corresponds to Shafer's model of the problem. Such a model assumes that an ultimate refinement of the problem is possible (exists and is achievable) so that θ_i are well precisely defined/identified in such a way that we are sure that they are exclusive and exhaustive (closed-world assumption).

The set of all subsets of Θ is called the *power set* of Θ and is denoted 2^Θ. Its cardinality is $2^{|\Theta|}$. Since 2^Θ is closed under unions, intersections, and complements, it defines a *Boolean algebra*.

By example, if $\Theta = \{\theta_1, \theta_2, \theta_3\}$ then $2^\Theta = \{\emptyset, \theta_1, \theta_2, \theta_3, \theta_1 \cup \theta_2, \theta_1 \cup \theta_3, \theta_2 \cup \theta_3, \theta_1 \cup \theta_2 \cup \theta_3\}$.

In Shafer's model, a *basic belief assignment* (bba) $m(.) : 2^\Theta \to [0, 1]$ associated to a given body of evidence $\mathcal{B}$ (also called corpus of evidence) is defined by [33]

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{A \in 2^\Theta} m(A) = 1 \tag{1.1}$$

Glenn Shafer defines the belief (credibility) and plausibility functions of $A \subseteq \Theta$ as

$$\text{Bel}(A) = \sum_{B \in 2^\Theta, B \subseteq A} m(B) \tag{1.2}$$

$$\text{Pl}(A) = \sum_{B \in 2^\Theta, B \cap A \neq \emptyset} m(B) = 1 - \text{Bel}(\bar{A}) \tag{1.3}$$

where $\bar{A}$ denotes the complement of the proposition A in Θ.

The belief functions $m(.)$, Bel(.) and Pl(.) are in one-to-one correspondence [33]. The set of elements $A \in 2^\Theta$ having a positive basic belief assignment is called the *core/kernel* of the source of evidence under consideration and is denoted $\mathcal{K}(m)$.

1.2.2 Dempster's rule of combination

Let $\text{Bel}_1(.)$ and $\text{Bel}_2(.)$ be two belief functions provided by two independent (and a priori equally reliable) sources/bodies of evidence $\mathcal{B}_1$ and $\mathcal{B}_2$ over the same frame of discernment Θ and their corresponding bba $m_1(.)$ and $m_2(.)$. Then the combined global belief function denoted $\text{Bel}(.) = \text{Bel}_1(.) \oplus \text{Bel}_2(.)$ is obtained by combining the bba $m_1(.)$ and $m_2(.)$ through the following Dempster rule of combination [33] $m(.) = [m_1 \oplus m_2](.)$ where

$$\begin{cases} m(\emptyset) = 0 \\ m(A) = \dfrac{\displaystyle\sum_{\substack{X,Y \in 2^\Theta \\ X \cap Y = A}} m_1(X) m_2(Y)}{1 - \displaystyle\sum_{\substack{X,Y \in 2^\Theta \\ X \cap Y = \emptyset}} m_1(X) m_2(Y)} \qquad \forall (A \neq \emptyset) \in 2^\Theta \end{cases} \tag{1.4}$$

$m(.)$ is a proper basic belief assignment if and only if the denominator in equation (1.4) is non-zero. The *degree of conflict* between the sources $\mathcal{B}_1$ and $\mathcal{B}_2$ is defined by

$$k_{12} \triangleq \sum_{\substack{X,Y \in 2^\Theta \\ X \cap Y = \emptyset}} m_1(X) m_2(Y) \tag{1.5}$$

The effect of the normalizing factor $1 - k_{12}$ in (1.4) consists in eliminating the conflicting pieces of information between the two sources to combine, consistently with the intersection operator. When $k_{12} = 1$, the combined bba $m(.)$ does not exist and the bodies of evidences $\mathcal{B}_1$ and $\mathcal{B}_2$ are said to be in *full contradiction*. Such a case arises when there exists $A \subset \Theta$ such that $\text{Bel}_1(A) = 1$ and $\text{Bel}_2(\bar{A}) = 1$. The core of the bba $m(.)$ equals the intersection of the cores of m_1 and m_2, i.e $\mathcal{K}(m) = \mathcal{K}(m_1) \cap \mathcal{K}(m_2)$. Up to the normalization factor $1 - k_{12}$, Dempster's rule is formally nothing but a random set intersection under stochastic assumption and it corresponds to the conjunctive consensus [13]. Dempster's rule of combination can be directly extended for the combination of N independent and equally reliable sources of evidence and its major interest comes essentially from its commutativity and associativity properties [33]. A recent discussion on Dempster's and Bayesian rules of combination can be found in [5].

1.2.3 Alternatives to Dempster's rule of combination

The DST is attractive for the Information Fusion community because it gives a nice mathematical model for the representation of uncertainty and it includes Bayesian theory as a special case [33] (p. 4). Although very appealing, the DST presents some weaknesses and limitations [27] already reported by Zadeh [50, 51, 52, 53] and Dubois & Prade in the eighties [12] and reinforced by Voorbraak in [43] because of the lack of complete theoretical justification of Dempster's rule of combination, but mainly because of our low confidence to trust the result of Dempster's rule of combination when the conflict becomes important between sources (i.e. $k_{12} \nearrow 1$). Indeed, there exists an infinite class of cases where Dempster's rule of combination can assign certainty to a minority opinion (other infinite classes of counter-examples are discussed in chapter 5) or where the "ignorance" interval disappears forever whenever a single piece of evidence commits all its belief to a proposition and its negation [29]. Moreover, elements of sets with larger cardinality can gain a disproportionate share of belief [43]. These drawbacks have fed intensive debates and research works for the last twenty years:

- either to interpret (and justify as best as possible) the use of Dempster's rule by several approaches and to circumvent numerical problems with it when conflict becomes high. These approaches are mainly based on the extension of the domain of the probability functions from the propositional logic domain to the modal propositional logic domain [31, 32, 28] or on the hint model [22] and probabilistic argumentation systems [14, 15, 1, 2, 16, 17, 18, 19, 20]. Discussions on these interpretations of DST can be found in [38, 40, 42], and also in chapter 12 of this book which analyzes and compares Bayesian reasoning, Dempster-Shafer's reasoning and DSm reasoning on a very simple but interesting example drawn from [28].
- or to propose new alternative rules. DSmT fits in this category since it extends the foundations of DST and also provides a new combination rules as it will be shown in next sections.

Several interesting and valuable alternative rules have thus been proposed in literature to circumvent the limitations of Dempster's rule of combination. The major common alternatives are listed in this section and most of the current available combination rules have been recently unified in a nice general framework by Lefèvre, Colot and Vanoorenberghe in [25]. Their important contribution, although strongly criticized by Haenni in [19] but properly justified by Lefevre et al. in [26], shows clearly that an infinite number of possible rules of combinations can be built from Shafer's model depending on the choice for transfer of the conflicting mass (i.e. k_{12}). A justification of Dempster's rule of combination has been proposed afterwards in the nineties by the axiomatic of Philippe Smets [37, 24, 41, 42] based on his Transferable Belief Model (TBM) related to anterior works of Cheng and Kashyap in [6], a non-probabilistic interpretation of Dempster-Shafer theory (see [3, 4] for discussion).

Here is the list of the most common rules of combination[1] for two independent sources of evidence proposed in the literature in the DST framework as possible alternatives to Dempster's rule of combination to overcome its limitations. Unless explicitly specified, the sources are assumed to be equally reliable.

- **The disjunctive rule of combination** [11, 13, 39]: This commutative and associative rule proposed by Dubois & Prade in 1986 and denoted here by the index $\cup$ is examined in details in chapter 9. $m_{\cup}(.)$ is defined $\forall A \in 2^{\Theta}$ by

$$\begin{cases} m_{\cup}(\emptyset) = 0 \\ m_{\cup}(A) = \displaystyle\sum_{\substack{X,Y \in 2^{\Theta} \\ X \cup Y = A}} m_1(X) m_2(Y) \qquad \forall (A \neq \emptyset) \in 2^{\Theta} \end{cases} \tag{1.6}$$

[1]The MinC rule of combination is not included here since it is covered in details in chapter 10.

The core of the belief function given by $m_\cup$ equals the union of the cores of Bel_1 and Bel_2. This rule reflects the disjunctive consensus and is usually preferred when one knows that one of the source $\mathcal{B}_1$ or $\mathcal{B}_2$ is mistaken but without knowing which one among $\mathcal{B}_1$ and $\mathcal{B}_2$.

- **Murphy's rule of combination** [27]: This commutative (but not associative) trade-off rule, denoted here with index M, drawn from [46, 13] is a special case of convex combination of bba m_1 and m_2 and consists actually in a simple arithmetic average of belief functions associated with m_1 and m_2. $\text{Bel}_M(.)$ is then given $\forall A \in 2^\Theta$ by:

$$\text{Bel}_M(A) = \frac{1}{2}[\text{Bel}_1(A) + \text{Bel}_2(A)] \tag{1.7}$$

- **Smets' rule of combination** [41, 42]: This commutative and associative rule corresponds actually to the non-normalized version of Dempster's rule of combination. It allows positive mass on the null/empty set $\emptyset$. This eliminates the division by $1 - k_{12}$ involved in Dempster's rule (1.4). Smets' rule of combination of two independent (equally reliable) sources of evidence (denoted here by index S) is given by:

$$\begin{cases} m_S(\emptyset) \equiv k_{12} = \displaystyle\sum_{\substack{X,Y \in 2^\Theta \\ X \cap Y = \emptyset}} m_1(X) m_2(Y) \\ m_S(A) = \displaystyle\sum_{\substack{X,Y \in 2^\Theta \\ X \cap Y = A}} m_1(X) m_2(Y) \qquad \forall (A \neq \emptyset) \in 2^\Theta \end{cases} \tag{1.8}$$

- **Yager's rule of combination** [45, 46, 47]: Yager admits that in case of conflict the result is not reliable, so that k_{12} plays the role of an absolute discounting term added to the weight of ignorance. The commutative (but not associative) Yager rule, denoted here by index Y is given[2] by:

$$\begin{cases} m_Y(\emptyset) = 0 \\ m_Y(A) = \displaystyle\sum_{\substack{X,Y \in 2^\Theta \\ X \cap Y = A}} m_1(X) m_2(Y) \qquad \forall A \in 2^\Theta, A \neq \emptyset, A \neq \Theta \\ m_Y(\Theta) = m_1(\Theta) m_2(\Theta) + \displaystyle\sum_{\substack{X,Y \in 2^\Theta \\ X \cap Y = \emptyset}} m_1(X) m_2(Y) \quad \text{when} \quad A = \Theta \end{cases} \tag{1.9}$$

- **Dubois & Prade's rule of combination** [13]: We admit that the two sources are reliable when they are not in conflict, but one of them is right when a conflict occurs. Then if one observes a value in set X while the other observes this value in a set Y, the truth lies in $X \cap Y$ as long $X \cap Y \neq \emptyset$. If $X \cap Y = \emptyset$, then the truth lies in $X \cup Y$ [13]. According to this principle, the commutative (but

[2] Θ represents here the full ignorance $\theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ on the frame of discernment according the notation used in [33].

not associative) Dubois & Prade hybrid rule of combination, denoted here by index DP, which is a reasonable trade-off between precision and reliability, is defined[3] by:

$$\begin{cases} m_{DP}(\emptyset) = 0 \\ m_{DP}(A) = \sum\limits_{\substack{X,Y\in 2^{\Theta} \\ X\cap Y=A \\ X\cap Y\neq\emptyset}} m_1(X)m_2(Y) + \sum\limits_{\substack{X,Y\in 2^{\Theta} \\ X\cup Y=A \\ X\cap Y=\emptyset}} m_1(X)m_2(Y) \qquad \forall A\in 2^{\Theta}, A\neq\emptyset \end{cases} \tag{1.10}$$

1.2.3.1 The unified formulation for rules of combinations involving conjunctive consensus

We present here the unified framework recently proposed by Lefèvre, Colot and Vanoorenberghe in [25] to embed all the existing (and potentially forthcoming) combination rules involving conjunctive consensus in the same general mechanism of construction. Here is the principle of their general formulation based on two steps.

- **Step 1**: Computation of the total conflicting mass based on the conjunctive consensus

$$k_{12} \triangleq \sum_{\substack{X,Y\in 2^{\Theta} \\ X\cap Y=\emptyset}} m_1(X)m_2(Y) \tag{1.11}$$

- **Step 2**: This step consists in the reallocation (convex combination) of the conflicting masses on $(A\neq\emptyset)\subseteq\Theta$ with some given coefficients $w_m(A)\in[0,1]$ such that $\sum_{A\subseteq\Theta} w_m(A)=1$ according to

$$\begin{cases} m(\emptyset) = w_m(\emptyset)k_{12} \\ m(A) = [\sum\limits_{\substack{X,Y\in 2^{\Theta} \\ X\cap Y=A}} m_1(X)m_2(Y)] + w_m(A)k_{12} \qquad \forall (A\neq\emptyset)\in 2^{\Theta} \end{cases} \tag{1.12}$$

The particular choice of the set of coefficients $w_m(.)$ provides a particular rule of combination. Actually this nice and important general formulation shows there exists an infinite number of possible rules of combination. Some rules are then justified or criticized with respect to the other ones mainly on their ability to, or not to, preserve the associativity and commutativity properties of the combination. It can be easily shown in [25] that such general procedure provides all existing rules involving conjunctive consensus developed in the literature based on Shafer's model. As examples:

- **Dempster's rule of combination** (1.4) can be obtained from (1.12) by choosing $\forall A\neq\emptyset$

$$w_m(\emptyset) = 0 \qquad \text{and} \qquad w_m(A) = \frac{1}{1-k_{12}} \sum_{\substack{X,Y\in 2^{\Theta} \\ X\cap Y=A}} m_1(X)m_2(Y) \tag{1.13}$$

[3]taking into account the the correction of the typo error in formula (56) given in [13], page 257.

- **Yager's rule of combination** (1.9) is obtained by choosing

$$w_m(\Theta) = 1 \qquad \text{and} \qquad w_m(A \neq \Theta) = 0 \tag{1.14}$$

- **Smets' rule of combination** (1.8) is obtained by choosing

$$w_m(\emptyset) = 1 \qquad \text{and} \qquad w_m(A \neq \emptyset) = 0 \tag{1.15}$$

- **Dubois and Prade's rule of combination** (1.10) is obtained by choosing

$$\forall A \subseteq \mathcal{P}, \qquad w_m(A) = \frac{1}{1-k_{12}} \sum_{\substack{A_1,A_2 \mid A_1 \cup A_2 = A \\ A_1 \cap A_2 = \emptyset}} m^\star \tag{1.16}$$

where $m^\star \triangleq m_1(A_1)m_2(A_2)$ corresponds to the partial conflicting mass which is assigned to $A_1 \cup A_2$. $\mathcal{P}$ is the set of all subsets of 2^Θ on which the conflicting mass is distributed. $\mathcal{P}$ is defined by [25]

$$\mathcal{P} \triangleq \{A \in 2^\Theta \mid \exists A_1 \in \mathcal{K}(m_1), \exists A_2 \in \mathcal{K}(m_2), A_1 \cup A_2 = A \text{ and } A_1 \cap A_2 = \emptyset\} \tag{1.17}$$

The computation of the weighting factors $w_m(A)$ of Dubois and Prade's rule of combination does not depend only on propositions they are associated with, but also on belief mass functions which have cause the partial conflicts. Thus the belief mass functions leading to the conflict allow to compute that part of conflicting mass which must be assigned to the subsets of $\mathcal{P}$ [25]. Yager's rule coincides with the Dubois and Prade's rule of combination when $\mathcal{P} = \{\Theta\}$.

1.2.4 The discounting of sources of evidence

Most of the rules of combination proposed in the literature are based on the assumption of the same reliability of sources of evidence. When the sources are known not being equally reliable and the reliability of each source is perfectly known (or at least has been properly estimated when it's possible [42, 25]), then is it natural and reasonable to discount each unreliable source proportionally to its corresponding reliability factor according to method proposed by Shafer in [33], chapter 11. Two methods are usually used for discounting the sources:

- **Classical discounting method** [33, 13, 42, 25]:

  Assume that the reliability/confidence[4] factor $\alpha \in [0,1]$ of a source is known, then the discounting of the bba $m(.)$ provided by the unreliable source is done to obtain a new (discounted) bba $m'(.)$ as follows:

$$\begin{cases} m'(A) = \alpha \cdot m(A), \qquad \forall A \in 2^\Theta, A \neq \Theta \\ m'(\Theta) = (1-\alpha) + \alpha \cdot m(\Theta) \end{cases} \tag{1.18}$$

[4]We prefer to use here the terminology *confidence* rather than *reliability* since the notion of reliability is closely related to the repetition of experiments with random outputs which may not be always possible in the context of some information fusion applications (see example 1.6 given by Shafer on the life on Sirius in [33], p.23)

$\alpha = 1$ means the total confidence in the source while $\alpha = 0$ means a complete calling in question of the reliability of the source.

- **Discounting by convex combination of sources** [13]: This method of discounting is based on the convex combination of sources by their relative reliabilities, assumed to be known. Let consider two independent unreliable sources of evidence with reliability factors α_1 and α_2 with $\alpha_1, \alpha_2 \in [0, 1]$, then the result of the combination of the discounted sources will be given $\forall A \in 2^\Theta$ by

$$\text{Bel}(A) = \frac{\alpha_1}{\alpha_1 + \alpha_2}\text{Bel}_1(A) + \frac{\alpha_2}{\alpha_1 + \alpha_2}\text{Bel}_2(A) \tag{1.19}$$

When the sources are highly conflicting and they have been sufficiently discounted, Shafer has shown in [33], p. 253, that the combination of a large number n of equally reliable sources using Dempster's rule on equally discounted belief functions, becomes similar to the convex combination of the n sources with equal reliability factors $\alpha_i = 1/n$. A detailed presentation of discounting methods can be found in [13].

It is important to note that such discounting methods must not be chosen as an ad-hoc tool to adjust the result of the fusion (once obtained) in case of troubles if a counter-intuitive or bad result arises, but only *beforehand* when one has prior information on the quality of sources. In the sequel of the book we will assume that sources under consideration are a priori equally reliable/trustable, unless specified explicitly. Although being very important for practical issues, the case of the fusion of known unreliable sources of information is not considered in this book because it depends on the own choice of the discounting method adopted by the system designer (this is also highly related with the application under consideration and the types of the sources to be combined). Fundamentally the problem of combination of unreliable sources of evidence is the same as working with new sets of basic belief assignments and thus has little interest in the framework of this book.

1.3 Foundations of the DSmT

1.3.1 Notion of free and hybrid DSm models

The development of the DSmT arises from the necessity to overcome the inherent limitations of the DST which are closely related with the acceptance of Shafer's model (the frame of *discernment* Θ defined as a finite set of *exhaustive* and *exclusive* hypotheses θ_i, $i = 1, \dots, n$), the third middle excluded principle (i.e. the existence of the complement for any elements/propositions belonging to the power set of Θ), and the acceptance of Dempter's rule of combination (involving normalization) as the framework for the combination of independent sources of evidence. We argue that these three fundamental conditions of the DST can be removed and another new mathematical approach for combination of evidence is possible.

The basis of the DSmT is the refutation of the principle of the third excluded middle and Shafer's model, since for a wide class of fusion problems the intrinsic nature of hypotheses can be only vague and imprecise in such a way that precise refinement is just impossible to obtain in reality so that the exclusive elements θ_i cannot be properly identified and precisely separated. Many problems involving fuzzy continuous and relative concepts described in natural language and having no absolute interpretation like tallness/smallness, pleasure/pain, cold/hot, Sorites paradoxes, etc, enter in this category. DSmT starts with the notion of *free DSm model*, denoted $\mathcal{M}^f(\Theta)$, and considers Θ only as a frame of exhaustive elements θ_i, $i = 1, \ldots, n$ which can potentially overlap. This model is *free* because no other assumption is done on the hypotheses, but the weak exhaustivity constraint which can always been satisfied according the closure principle explained in the introduction of this chapter. No other constraint is involved in the free DSm model. When the free DSm model holds, the classic commutative and associative DSm rule of combination (corresponding to the conjunctive consensus defined on the free Dedekind's lattice - see next subsection) is performed.

Depending on the intrinsic nature of the elements of the fusion problem under consideration, it can however happen that the free model does not fit the reality because some subsets of Θ can contain elements known to be truly exclusive but also truly non existing at all at a given time (specially when working on dynamic fusion problem where the frame Θ varies with time with the revision of the knowledge available). These integrity constraints are then explicitly and formally introduced into the free DSm model $\mathcal{M}^f(\Theta)$ in order to adapt it properly to fit as close as possible with the reality and permit to construct a *hybrid DSm model* $\mathcal{M}(\Theta)$ on which the combination will be efficiently performed. Shafer's model, denoted $\mathcal{M}^0(\Theta)$, corresponds to a very specific hybrid DSm model including all possible exclusivity constraints. The DST has been developed for working only with $\mathcal{M}^0(\Theta)$ while the DSmT has been developed for working with any kind of hybrid model (including Shafer's model and the free DSm model), to manage as efficiently and precisely as possible imprecise, uncertain and potentially high conflicting sources of evidence while keeping in mind the possible dynamicity of the information fusion problematic. The foundations of the DSmT are therefore totally different from those of all existing approaches managing uncertainties, imprecisions and conflicts. DSmT provides a new interesting way to attack the information fusion problematic with a general framework in order to cover a wide variety of problems. A detailed presentation of hybrid DSm models and hybrid DSm rule of combination is given in chapter 4.

DSmT refutes also the idea that sources of evidence provide their beliefs with the same absolute interpretation of elements of the same frame Θ and the conflict between sources arises not only because of the possible unreliabilty of sources, but also because of possible different and relative interpretation of Θ, e.g. what is considered as good for somebody can be considered as bad for somebody else. There is some unavoidable subjectivity in the belief assignments provided by the sources of evidence, otherwise it would mean that all bodies of evidence have a same objective and universal interpretation (or measure) of the phenomena under consideration, which unfortunately rarely occurs in reality, but when bba are based on some *objective probabilities* transformations. But in this last case, probability theory can handle properly and efficiently the information, and the DST, as well as the DSmT, becomes useless. If we now get out of the probabilistic background argumentation for the construction of bba, we claim that in most of cases, the sources of evidence provide their beliefs about elements of the frame of the fusion problem only based on their own limited knowledge and experience without reference to the (inaccessible) absolute truth of the space of possibilities.

The DSmT includes the possibility to deal with evidences arising from different sources of information which do not have access to the absolute and same interpretation of the elements of Θ under consideration. The DSmT, although not based on probabilistic argumentation can be interpreted as an extension of Bayesian theory and Dempster-Shafer theory in the following sense. Let $\Theta = \{\theta_1, \theta_2\}$ be the simplest frame made of only two hypotheses, then

- the probability theory deals, under the assumptions on exclusivity and exhaustivity of hypotheses, with basic probability assignments (bpa) $m(.) \in [0, 1]$ such that

$$m(\theta_1) + m(\theta_2) = 1$$

- the DST deals, under the assumptions on exclusivity and exhaustivity of hypotheses, with bba $m(.) \in [0, 1]$ such that

$$m(\theta_1) + m(\theta_2) + m(\theta_1 \cup \theta_2) = 1$$

- the DSmT theory deals, under only assumption on exhaustivity of hypotheses (i.e. the free DSm model), with the generalized bba $m(.) \in [0, 1]$ such that

$$m(\theta_1) + m(\theta_2) + m(\theta_1 \cup \theta_2) + m(\theta_1 \cap \theta_2) = 1$$

1.3.2 Notion of hyper-power set D^Θ

One of the cornerstones of the DSmT is the notion of hyper-power set (see chapters 2 and 3 for examples and a detailed presentation). Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a finite set (called frame) of n exhaustive elements[5]. The Dedekind's lattice, also called in the DSmT framework *hyper-power set* D^Θ is defined as the set of all composite propositions built from elements of Θ with $\cup$ and $\cap$ operators[6] such that:

[5] We do not assume here that elements θ_i are necessary exclusive. There is no restriction on θ_i but the exhaustivity.
[6] Θ generates D^Θ under operators $\cup$ and $\cap$

1. $\emptyset, \theta_1, \dots, \theta_n \in D^\Theta$.

2. If $A, B \in D^\Theta$, then $A \cap B \in D^\Theta$ and $A \cup B \in D^\Theta$.

3. No other elements belong to D^Θ, except those obtained by using rules 1 or 2.

The dual (obtained by switching $\cup$ and $\cap$ in expressions) of D^Θ is itself. There are elements in D^Θ which are self-dual (dual to themselves), for example α_8 for the case when $n = 3$ in the example below. The cardinality of D^Θ is majored by 2^{2^n} when the cardinality of Θ equals n, i.e. $|\Theta| = n$. The generation of hyper-power set D^Θ is closely related with the famous Dedekind problem [8, 7] on enumerating the set of isotone Boolean functions. The generation of the hyper-power set is presented in chapter 2. Since for any given finite set Θ, $|D^\Theta| \geq |2^\Theta|$ we call D^Θ the *hyper-power set* of Θ.

Example of the first hyper-power sets D^Θ

- For the degenerate case ($n = 0$) where $\Theta = \{\}$, one has $D^\Theta = \{\alpha_0 \triangleq \emptyset\}$ and $|D^\Theta| = 1$.

- When $\Theta = \{\theta_1\}$, one has $D^\Theta = \{\alpha_0 \triangleq \emptyset, \alpha_1 \triangleq \theta_1\}$ and $|D^\Theta| = 2$.

- When $\Theta = \{\theta_1, \theta_2\}$, one has $D^\Theta = \{\alpha_0, \alpha_1, \dots, \alpha_4\}$ and $|D^\Theta| = 5$ with $\alpha_0 \triangleq \emptyset$, $\alpha_1 \triangleq \theta_1 \cap \theta_2$, $\alpha_2 \triangleq \theta_1$, $\alpha_3 \triangleq \theta_2$ and $\alpha_4 \triangleq \theta_1 \cup \theta_2$.

- When $\Theta = \{\theta_1, \theta_2, \theta_3\}$, one has $D^\Theta = \{\alpha_0, \alpha_1, \dots, \alpha_{18}\}$ and $|D^\Theta| = 19$ with

$$
\begin{array}{ll}
\alpha_0 \triangleq \emptyset & \\
\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 & \alpha_{10} \triangleq \theta_2 \\
\alpha_2 \triangleq \theta_1 \cap \theta_2 & \alpha_{11} \triangleq \theta_3 \\
\alpha_3 \triangleq \theta_1 \cap \theta_3 & \alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3 \\
\alpha_4 \triangleq \theta_2 \cap \theta_3 & \alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2 \\
\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3 & \alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1 \\
\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2 & \alpha_{15} \triangleq \theta_1 \cup \theta_2 \\
\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1 & \alpha_{16} \triangleq \theta_1 \cup \theta_3 \\
\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) & \alpha_{17} \triangleq \theta_2 \cup \theta_3 \\
\alpha_9 \triangleq \theta_1 & \alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3
\end{array}
$$

Note that the complement $\bar{A}$ of any proposition A (except for $\emptyset$ and for the total ignorance $I_t \triangleq \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$), is not involved within DSmT because of the refutation of the third excluded middle. In other words, $\forall A \in D^\Theta$ with $A \neq \emptyset$ or $A \neq I_t$, $\bar{A} \notin D^\Theta$. Thus $(D^\Theta, \cap, \cup)$ does not define a Boolean algebra. The cardinality of hyper-power set D^Θ for $n \geq 1$ follows the sequence of Dedekind's numbers [35], i.e. 1,2,5,19,167,7580,7828353,... (see next chapter for details).

Elements θ_i, $i = 1, \ldots, n$ of Θ constitute the finite set of hypotheses/concepts characterizing the fusion problem under consideration. D^Θ constitutes what we call the *free DSm model* $\mathcal{M}^f(\Theta)$ and allows to work with fuzzy concepts which depict a continuous and relative intrinsic nature. Such kinds of concepts cannot be precisely refined in an absolute interpretation because of the unapproachable universal truth.

However for some particular fusion problems involving discrete concepts, elements θ_i are truly exclusive. In such case, all the exclusivity constraints on θ_i, $i = 1, \ldots, n$ have to be included in the previous model to characterize properly the true nature of the fusion problem and to fit it with the reality. By doing this, the hyper-power set D^Θ reduces naturally to the classical power set 2^Θ and this constitutes the most restricted hybrid DSm model, denoted $\mathcal{M}^0(\Theta)$, coinciding with Shafer's model. As an exemple, let's consider the 2D problem where $\Theta = \{\theta_1, \theta_2\}$ with $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$ and assume now that θ_1 and θ_2 are truly exclusive (i.e. Shafer's model $\mathcal{M}^0$ holds), then because $\theta_1 \cap \theta_2 \overset{\mathcal{M}^0}{=} \emptyset$, one gets $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2 \overset{\mathcal{M}^0}{=} \emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\} = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\} \equiv 2^\Theta$.

Between the class of fusion problems corresponding to the free DSm model $\mathcal{M}^f(\Theta)$ and the class of fusion problems corresponding to Shafer's model $\mathcal{M}^0(\Theta)$, there exists another wide class of hybrid fusion problems involving in Θ both fuzzy continuous concepts and discrete hypotheses. In such (hybrid) class, some exclusivity constraints and possibly some non-existential constraints (especially when working on dynamic[7] fusion) have to be taken into account. Each hybrid fusion problem of this class will then be characterized by a proper hybrid DSm model $\mathcal{M}(\Theta)$ with $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$ and $\mathcal{M}(\Theta) \neq \mathcal{M}^0(\Theta)$, see examples presented in chapter 4.

1.3.3 Generalized belief functions

From a general frame Θ, we define a map $m(.) : D^\Theta \to [0, 1]$ associated to a given body of evidence $\mathcal{B}$ as

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{A \in D^\Theta} m(A) = 1 \tag{1.20}$$

The quantity $m(A)$ is called the *generalized basic belief assignment/mass* (gbba) of A.

The *generalized belief and plausibility functions* are defined in almost the same manner as within the DST, i.e.

$$\text{Bel}(A) = \sum_{\substack{B \subseteq A \\ B \in D^\Theta}} m(B) \tag{1.21}$$

$$\text{Pl}(A) = \sum_{\substack{B \cap A \neq \emptyset \\ B \in D^\Theta}} m(B) \tag{1.22}$$

[7] i.e. when the frame Θ is changing with time.

These definitions are compatible with the definitions of classical belief functions in the DST framework when D^Θ reduces to 2^Θ for fusion problems where Shafer's model $\mathcal{M}^0(\Theta)$ holds. We still have $\forall A \in D^\Theta$, $\text{Bel}(A) \leq \text{Pl}(A)$. Note that when working with the free DSm model $\mathcal{M}^f(\Theta)$, one has always $\text{Pl}(A) = 1$ $\forall A \neq \emptyset \in D^\Theta$ which is normal.

1.3.4 The classic DSm rule of combination

When the free DSm model $\mathcal{M}^f(\Theta)$ holds for the fusion problem under consideration, the classic DSm rule of combination $m_{\mathcal{M}^f(\Theta)} \equiv m(.) \triangleq [m_1 \oplus m_2](.)$ of two independent sources of evidences $\mathcal{B}_1$ and $\mathcal{B}_2$ over the same frame Θ with belief functions $\text{Bel}_1(.)$ and $\text{Bel}_2(.)$ associated with gbba $m_1(.)$ and $m_2(.)$ corresponds to the conjunctive consensus of the sources. It is given by [9, 10]:

$$\forall C \in D^\Theta, \qquad m_{\mathcal{M}^f(\Theta)}(C) \equiv m(C) = \sum_{\substack{A,B \in D^\Theta \\ A \cap B = C}} m_1(A) m_2(B) \tag{1.23}$$

Since D^Θ is closed under $\cup$ and $\cap$ set operators, this new rule of combination guarantees that $m(.)$ is a proper generalized belief assignment, i.e. $m(.) : D^\Theta \to [0, 1]$. This rule of combination is commutative and associative and can always be used for the fusion of sources involving fuzzy concepts. This rule can be directly and easily extended for the combination of $k > 2$ independent sources of evidence (see the expression for $S_1(.)$ in the next section and chapter 4 for details).

This classic DSm rule of combination becomes very expensive in terms of computations and memory size due to the huge number of elements in D^Θ when the cardinality of Θ increases. This remark is however valid only if the cores (the set of focal elements of gbba) $\mathcal{K}_1(m_1)$ and $\mathcal{K}_2(m_2)$ coincide with D^Θ, i.e. when $m_1(A) > 0$ and $m_2(A) > 0$ for all $A \neq \emptyset \in D^\Theta$. Fortunately, it is important to note here that in most of the practical applications the sizes of $\mathcal{K}_1(m_1)$ and $\mathcal{K}_2(m_2)$ are much smaller than $|D^\Theta|$ because bodies of evidence generally allocate their basic belief assignments only over a subset of the hyper-power set. This makes things easier for the implementation of the classic DSm rule (1.23).

The DSm rule is actually very easy to implement. It suffices for each focal element of $\mathcal{K}_1(m_1)$ to multiply it with the focal elements of $\mathcal{K}_2(m_2)$ and then to pool all combinations which are equivalent under the algebra of sets according to figure 1.1.

The figure 1.1 represents the *DSm network architecture* of the DSm rule of combination. The first layer of the network consists in all gbba of focal elements $A_i, i = 1, \ldots, n$ of $m_1(.)$. The second layer of the network consists in all gbba of focal elements $B_j, j = 1, \ldots, k$ of $m_2(.)$. Each node of layer 2 is connected with each node of layer 1. The output layer (on the right) consists in the combined basic belief assignments of all possible intersections $A_i \cap B_j$, $i = 1, \ldots, n$ and $j = 1, \ldots, k$. The last step of the classic DSm rule (not included on the figure) consists in the compression of the output layer by regrouping (summing up) all the combined belief assignments corresponding to the same focal elements (by example if $X = A_2 \cap B_3 = A_4 \cap B_5$, then $m(X) = m(A_2 \cap B_3) + m(A_4 \cap B_5)$). If a third body of evidence provides a new gbba $m_3(.)$, the one combines it by connecting the output layer with the layer associated to $m_3(.)$, and so on. Because of commutativity and associativity properties of the classic DSm rule, the DSm network can be designed with any order of the layers.

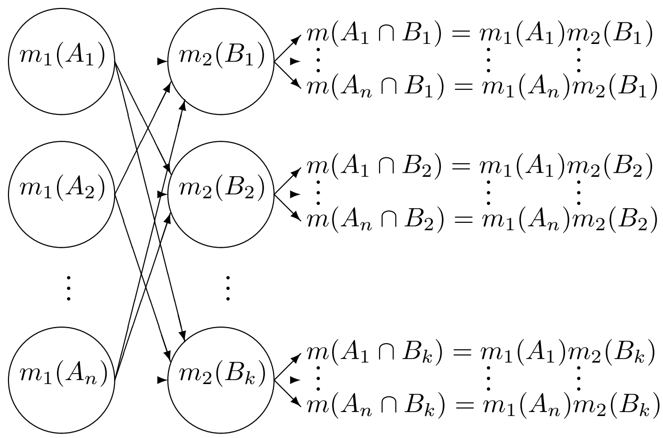

Figure 1.1: Representation of the classic DSm rule on $\mathcal{M}^f(\Theta)$

1.3.5 The hybrid DSm rule of combination

When the free DSm model $\mathcal{M}^f(\Theta)$ does not hold due to the true nature of the fusion problem under consideration which requires to take into account some known integrity constraints, one has to work with a proper hybrid DSm model $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$. In such case, the hybrid DSm rule of combination based on the chosen hybrid DSm model $\mathcal{M}(\Theta)$ for $k \geq 2$ independent sources of information is defined for all $A \in D^\Theta$ as (see chapter 4 for details):

$$m_{\mathcal{M}(\Theta)}(A) \triangleq \phi(A)\Big[S_1(A) + S_2(A) + S_3(A)\Big] \tag{1.24}$$

where $\phi(A)$ is the *characteristic non-emptiness function* of a set A, i.e. $\phi(A) = 1$ if $A \notin \boldsymbol{\emptyset}$ and $\phi(A) = 0$ otherwise, where $\boldsymbol{\emptyset} \triangleq \{\boldsymbol{\emptyset}_\mathcal{M}, \emptyset\}$. $\boldsymbol{\emptyset}_\mathcal{M}$ is the set of all elements of D^Θ which have been forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical/universal empty set. $S_1(A) \equiv m_{\mathcal{M}^f(\theta)}(A)$, $S_2(A)$, $S_3(A)$ are defined by

$$S_1(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ (X_1 \cap X_2 \cap \ldots \cap X_k) = A}} \prod_{i=1}^{k} m_i(X_i) \tag{1.25}$$

$$S_2(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k\in\boldsymbol{\emptyset} \\ [\mathcal{U}=A]\vee[(\mathcal{U}\in\boldsymbol{\emptyset})\wedge(A=I_t)]}} \prod_{i=1}^{k} m_i(X_i) \tag{1.26}$$

$$S_3(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k\in D^\Theta \\ (X_1\cup X_2\cup\ldots\cup X_k)=A \\ (X_1\cap X_2\cap\ldots\cap X_k)\in\boldsymbol{\emptyset}}} \prod_{i=1}^{k} m_i(X_i) \tag{1.27}$$

with $\mathcal{U} \triangleq u(X_1) \cup u(X_2) \cup \ldots \cup u(X_k)$ where $u(X)$ is the union of all singletons θ_i that compose X and $I_t \triangleq \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ is the total ignorance. $S_1(A)$ corresponds to the classic DSm rule of combination for k independent sources based on the free DSm model $\mathcal{M}^f(\Theta)$; $S_2(A)$ represents the mass of all relatively and absolutely empty sets which is transferred to the total or relative ignorances; $S_3(A)$ transfers the sum of relatively empty sets to the non-empty sets.

The hybrid DSm rule of combination generalizes the classic DSm rule of combination and is not equivalent to Dempter's rule. It works for any models (the free DSm model, Shafer's model or any other hybrid models) when manipulating *precise* generalized (or eventually classical) basic belief functions. An extension of this rule for the combination of *imprecise* generalized (or eventually classical) basic belief functions is presented in chapter 6 and is not reported in this presentation of DSmT.

1.3.6 On the refinement of the frames

Let's bring here a clarification on the notion of refinement and its consequences with respect to DSmT and DST. The refinement of a set of overlapping hypotheses $\Theta = \{\theta_i, i = 1, \ldots, n\}$ consists in getting a new finer set of hypotheses θ'_i, $i = 1, \ldots, n'$, $n' > n\}$ such that we are sure that θ'_i are truly exclusive and $\cup_{i=1}^{n}\theta_i \equiv \cup_{i=1}^{n'}\theta'_i$, i.e. $\Theta = \{\theta'_i, i = 1, \ldots, n' > n\}$. The DST starts with the notion of frame of discernment (finite set of exhaustive and exclusive hypotheses). The DST assumes therefore that a refinement exists to describe the fusion problem and is achievable while DSmT does not make such assumption at its starting. The assumption of existence of a refinement process appears to us as a very strong assumption which reduces drastically the domain of applicability of the DST because the frames for most of problems described in terms of natural language manipulating vague/continuous/relative concepts cannot be formally refined at all. Such an assumption is not fundamental and is relaxed in DSmT.

As a very simple but illustrative example, let's consider Θ defined as $\Theta = \{\theta_1 = \text{Small}, \theta_2 = \text{Tall}\}$. The notions of smallness (θ_1) and tallness (θ_2) cannot be interpreted in an absolute manner actually since these notions are only defined with respect to some reference points chosen arbitrarily. Two independent sources of evidence (human "experts" here) can provide a different interpretation of θ_1 and θ_2 just because they usually do not share the same reference point. θ_1 and θ_2 represent actually fuzzy concepts carrying only a relative meaning. Moreover, these concepts are linked together by a continuous path.

Let's examine now a numerical example. Consider again the frame $\Theta = \{\theta_1 \triangleq \text{Small}, \theta_2 \triangleq \text{Tall}\}$ on the size of person with two independent witnesses providing belief masses

$$m_1(\theta_1) = 0.4 \qquad m_1(\theta_2) = 0.5 \qquad m_1(\theta_1 \cup \theta_2) = 0.1$$

$$m_2(\theta_1) = 0.6 \qquad m_2(\theta_2) = 0.2 \qquad m_2(\theta_1 \cup \theta_2) = 0.2$$

If we admit that θ_1 and θ_2 cannot be precisely refined according to the previous justification, then the result of the classic DSm rule (denoted by index $DSmc$ here) of combination yields:

$$m_{DSmc}(\emptyset) = 0 \quad m_{DSmc}(\theta_1) = 0.38 \quad m_{DSmc}(\theta_2) = 0.22 \quad m_{DSmc}(\theta_1 \cup \theta_2) = 0.02 \quad m_{DSmc}(\theta_1 \cap \theta_2) = 0.38$$

Starting now with the same information, i.e. $m_1(.)$ and $m_2(.)$, we volontary assume that a refinement is possible (even if it does not make sense actually here) in order to compare the previous result with the result one would obtain with Dempster's rule of combination. So, let's assume the existence of an hypothetical refined frame of discernment $\Theta_{ref} \triangleq \{\theta'_1 = \text{Small'}, \theta'_2 \triangleq \text{Medium}, \theta'_3 = \text{Tall'}\}$ where θ'_1, θ'_2 and θ'_3 correspond to some virtual exclusive hypotheses such that $\theta_1 = \theta'_1 \cup \theta'_2$, $\theta_2 = \theta'_2 \cup \theta'_3$ and $\theta_1 \cap \theta_2 = \theta'_2$ and where Small' and Tall' correspond respectively to a *finer notion* of smallness and tallness than in original frame Θ. Because, we don't change the information we have available (that's all we have), the initial bba $m_1(.)$ and $m_2(.)$ expressed now on the virtual refined power set $2^{\Theta_{ref}}$ are given by

$$m'_1(\theta'_1 \cup \theta'_2) = 0.4 \qquad m'_1(\theta'_2 \cup \theta'_3) = 0.5 \qquad m'_1(\theta'_1 \cup \theta'_2 \cup \theta'_3) = 0.1$$

$$m'_2(\theta'_1 \cup \theta'_2) = 0.6 \qquad m'_2(\theta'_2 \cup \theta'_3) = 0.2 \qquad m'_2(\theta'_1 \cup \theta'_2 \cup \theta'_3) = 0.2$$

Because Θ_{ref} is a refined frame, DST works and Dempster's rule applies. Because there is no positive masses for conflicting terms $\theta'_1 \cap \theta'_2$, $\theta'_1 \cap \theta'_3$, $\theta'_2 \cap \theta'_3$ or $\theta'_1 \cap \theta'_2 \cap \theta'_3$, the degree of conflict reduces to $k_{12} = 0$ and the normalization factor involved in Dempster's rule is 1 in this refined example. One gets formally, where index DS denotes here Dempster's rule, the following result:

$$\begin{aligned}
m_{DS}(\emptyset) &= 0 \\
m_{DS}(\theta'_2) &= m'_1(\theta'_1 \cup \theta'_2)m'_2(\theta'_2 \cup \theta'_3) + m'_2(\theta'_1 \cup \theta'_2)m'_1(\theta'_2 \cup \theta'_3) = 0.2 \cdot 0.4 + 0.5 \cdot 0.6 = 0.38 \\
m_{DS}(\theta'_1 \cup \theta'_2) &= m'_1(\theta'_1 \cup \theta'_2)m'_2(\theta'_1 \cup \theta'_2) + m'_1(\theta'_1 \cup \theta'_2 \cup \theta'_3)m'_2(\theta'_1 \cup \theta'_2) + m'_2(\theta'_1 \cup \theta'_2 \cup \theta'_3)m'_1(\theta'_1 \cup \theta'_2) \\
&= 0.4 \cdot 0.6 + 0.1 \cdot 0.6 + 0.2 \cdot 0.4 = 0.38 \\
m_{DS}(\theta'_2 \cup \theta'_3) &= m'_1(\theta'_2 \cup \theta'_3)m'_2(\theta'_2 \cup \theta'_3) + m'_1(\theta'_1 \cup \theta'_2 \cup \theta'_3)m'_2(\theta'_2 \cup \theta'_3) + m'_2(\theta'_1 \cup \theta'_2 \cup \theta'_3)m'_1(\theta'_2 \cup \theta'_3) \\
&= 0.2 \cdot 0.5 + 0.1 \cdot 0.2 + 0.2 \cdot 0.5 = 0.22 \\
m_{DS}(\theta'_1 \cup \theta'_2 \cup \theta'_3) &= m'_1(\theta'_1 \cup \theta'_2 \cup \theta'_3)m'_2(\theta'_1 \cup \theta'_2 \cup \theta'_3) = 0.1 \cdot 0.2 = 0.02
\end{aligned}$$

But since $\theta'_2 = \theta_1 \cap \theta_2$, $\theta'_1 \cup \theta'_2 = \theta_1$, $\theta'_2 \cup \theta'_3 = \theta_2$ and $\theta'_1 \cup \theta'_2 \cup \theta'_3 = \theta_1 \cup \theta_2$, one sees that Dempster's rule reduces to the classic DSm rule of combination, which means that the refinement of the frame Θ does not help to get a more specific (better) result from the DST when the inputs of the problem remain the same. Actually, working on Θ_{ref} with DST does not bring a difference with DSmT, but just brings an useless complexity in derivations. Note that the hybrid DSm rule of combination can also be applied on Shafer's model associated with Θ_{ref}, but it naturally provides the same result as with the classic DSm rule in this case.

If the inputs of the problem are now *changed* by re-asking (assuming that such process is possible) the sources to provide their revised belief assignents directly on Θ_{ref}, with $m'_i(\theta'_1) > 0$, $m'_i(\theta'_2) > 0$ and $m'_i(\theta'_3) > 0$ $(i = 1, 2)$ rather than on Θ, then the hybrid DSm rule of combination will be applied instead of Dempster's rule when adopting the DSmT. The fusion results will then differ, which is normal since the hybrid DSm rule is not equivalent to Dempster's rule, except when the conflict is zero.

1.3.7 On the combination of sources over different frames

In some fusion problems, it can happen that sources provide their basic belief assignment over distinct frames (which can moreover sometimes partially overlap). As simple example, let's consider two equally reliable sources of evidence $\mathcal{B}_1$ and $\mathcal{B}_2$ providing their belief assignments repectively on distinct frames Θ_1 and Θ_2 defined as follows

$$\Theta_1 = \{P \triangleq \text{Plane}, H \triangleq \text{Helicopter}, M \triangleq \text{Missile}\}$$

$$\Theta_2 = \{S \triangleq \text{Slow motion}, F \triangleq \text{Fast motion}\}$$

In other words, $m_1(.)$ associated with $\mathcal{B}_1$ is defined either on D_1^Θ or 2_1^Θ (if Shafer's model is assumed to hold) while $m_2(.)$ associated with $\mathcal{B}_2$ is defined either on D_2^Θ or 2_2^Θ. The problem relates here to the combination of $m_1(.)$ with $m_2(.)$.

The basic solution of this problem consists in working on the global frame[8] $\Theta = \{\Theta_1, \Theta_2\}$ and in following the deconditionning method proposed by Smets in [39] based on the principle on the minimum of specificity to revise the basic belief assignments $m_1(.)$ and $m_2(.)$ on Θ. When additional information on compatibility links between elements of Θ_1 and Θ_2 is known, then the refined method proposed by Janez in [21] is preferred. Once the proper model $\mathcal{M}(\Theta)$ for Θ has been chosen to fit with the true nature of hypotheses and the revised bba $m_1^{rev}(.)$ and $m_2^{rev}(.)$ defined on D^Θ are obtained, the fusion of belief assignments is performed with the hybrid DSm rule of combination.

[8]with suppression of possible redundant elements when Θ_1 and Θ_2 overlap partially.

1.4 Comparison of different rules of combinations

1.4.1 First example

In this section, we compare the results provided by the most common rules of combinations on the following very simple numerical example where only 2 independent sources (a priori assumed equally reliable) are involved and providing their belief initially on the 3D frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$. It is assumed in this example that Shafer's model holds and thus the belief assignments $m_1(.)$ and $m_2(.)$ do not commit belief to internal conflicting information. $m_1(.)$ and $m_2(.)$ are chosen as follows:

$$m_1(\theta_1) = 0.1 \qquad m_1(\theta_2) = 0.4 \qquad m_1(\theta_3) = 0.2 \qquad m_1(\theta_1 \cup \theta_2) = 0.1$$

$$m_2(\theta_1) = 0.5 \qquad m_2(\theta_2) = 0.1 \qquad m_2(\theta_3) = 0.3 \qquad m_2(\theta_1 \cup \theta_2) = 0.1$$

These belief masses are usually represented in the form of a belief mass matrix $\mathbf{M}$ given by

$$\mathbf{M} = \begin{bmatrix} 0.1 & 0.4 & 0.2 & 0.3 \\ 0.5 & 0.1 & 0.3 & 0.1 \end{bmatrix} \tag{1.28}$$

where index i for the rows corresponds to the index of the source no. i and the indexes j for columns of $\mathbf{M}$ correspond to a given choice for enumerating the focal elements of all sources. In this particular example, index $j = 1$ corresponds to θ_1, $j = 2$ corresponds to θ_2, $j = 3$ corresponds to θ_3 and $j = 4$ corresponds to $\theta_1 \cup \theta_2$.

Now let's imagine that one finds out that θ_3 is actually truly empty because some extra and certain knowledge on θ_3 is received by the fusion center. As example, θ_1, θ_2 and θ_3 may correspond to three suspects (potential murders) in a police investigation, $m_1(.)$ and $m_2(.)$ corresponds to two reports of independent witnesses, but it turns out that finally θ_3 has provided a strong alibi to the criminal police investigator once arrested by the policemen. This situation corresponds to set up a hybrid model $\mathcal{M}$ with the constraint $\theta_3 \overset{\mathcal{M}}{\equiv} \emptyset$ (see chapter 4 for a detailed presentation on hybrid models).

Let's examine the result of the fusion in such situation obtained by the Smets', Yager's, Dubois & Prade's and hybrid DSm rules of combinations. First note that, based on the free DSm model, one would get by applying the classic DSm rule (denoted here by index $DSmc$) the following fusion result

$$m_{DSmc}(\theta_1) = 0.21 \qquad m_{DSmc}(\theta_2) = 0.11 \qquad m_{DSmc}(\theta_3) = 0.06 \qquad m_{DSmc}(\theta_1 \cup \theta_2) = 0.03$$

$$m_{DSmc}(\theta_1 \cap \theta_2) = 0.21 \qquad m_{DSmc}(\theta_1 \cap \theta_3) = 0.13 \qquad m_{DSmc}(\theta_2 \cap \theta_3) = 0.14$$

$$m_{DSmc}(\theta_3 \cap (\theta_1 \cup \theta_2)) = 0.11$$

But because of the exclusivity constraints (imposed here by the use of Shafer's model and by the non-existential constraint $\theta_3 \overset{\mathcal{M}}{\equiv} \emptyset$), the total conflicting mass is actually given by

$$k_{12} = 0.06 + 0.21 + 0.13 + 0.14 + 0.11 = 0.65 \qquad \text{(conflicting mass)}$$

- If one applies the **Disjunctive rule** (1.6), one gets:

$$\begin{aligned}
m_{\cup}(\emptyset) &= 0 \\
m_{\cup}(\theta_1) &= m_1(\theta_1)m_2(\theta_1) = 0.1 \cdot 0.5 = 0.05 \\
m_{\cup}(\theta_2) &= m_1(\theta_2)m_2(\theta_2) = 0.4 \cdot 0.1 = 0.04 \\
m_{\cup}(\theta_3) &= m_1(\theta_3)m_2(\theta_3) = 0.2 \cdot 0.3 = 0.06 \\
m_{\cup}(\theta_1 \cup \theta_2) &= [m_1(\theta_1 \cup \theta_2)m_2(\theta_1 \cup \theta_2)] + [m_1(\theta_1)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2)] \\
&\quad + [m_1(\theta_1)m_2(\theta_1 \cup \theta_2) + m_2(\theta_1)m_1(\theta_1 \cup \theta_2)] \\
&\quad + [m_1(\theta_2)m_2(\theta_1 \cup \theta_2) + m_2(\theta_2)m_1(\theta_1 \cup \theta_2)] \\
&= [0.3 \cdot 0.1] + [0.01 + 0.20] + [0.01 + 015] + [0.04 + 0.03] \\
&= 0.03 + 0.21 + 0.16 + 0.007 = 0.47 \\
m_{\cup}(\theta_1 \cup \theta_3) &= m_1(\theta_1)m_2(\theta_3) + m_2(\theta_1)m_1(\theta_3) = 0.03 + 0.10 = 0.13 \\
m_{\cup}(\theta_2 \cup \theta_3) &= m_1(\theta_2)m_2(\theta_3) + m_2(\theta_2)m_1(\theta_3) = 0.12 + 0.02 = 0.14 \\
m_{\cup}(\theta_1 \cup \theta_2 \cup \theta_2) &= m_1(\theta_3)m_2(\theta_1 \cup \theta_2) = 0.02 + 0.09 = 0.11
\end{aligned}$$

- If one applies the **hybrid DSm rule** (1.24) (denoted here by index $DSmh$) for 2 sources ($k = 2$), one gets:

$$\begin{aligned}
m_{DSmh}(\emptyset) &= 0 \\
m_{DSmh}(\theta_1) &= 0.21 + 0.13 = 0.34 \\
m_{DSmh}(\theta_2) &= 0.11 + 0.14 = 0.25 \\
m_{DSmh}(\theta_1 \cup \theta_2) &= 0.03 + [0.2 \cdot 0.1 + 0.3 \cdot 0.3] + [0.1 \cdot 0.1 + 0.5 \cdot 0.4] + [0.2 \cdot 0.3] = 0.41
\end{aligned}$$

- If one applies **Smets' rule** (1.8), one gets:

$$\begin{aligned}
m_S(\emptyset) = m(\emptyset) &= 0.65 \qquad \text{(conflicting mass)} \\
m_S(\theta_1) &= 0.21 \\
m_S(\theta_2) &= 0.11 \\
m_S(\theta_1 \cup \theta_2) &= 0.03
\end{aligned}$$

- If one applies **Yager's rule** (1.9), one gets:

$$m_Y(\emptyset) = 0$$
$$m_Y(\theta_1) = 0.21$$
$$m_Y(\theta_2) = 0.11$$
$$m_Y(\theta_1 \cup \theta_2) = 0.03 + k_{12} = 0.03 + 0.65 = 0.68$$

- If one applies **Dempster's rule** (1.4) (denoted here by index DS), one gets:

$$m_{DS}(\emptyset) = 0$$
$$m_{DS}(\theta_1) = 0.21/[1 - k_{12}] = 0.21/[1 - 0.65] = 0.21/0.35 = 0.600000$$
$$m_{DS}(\theta_2) = 0.11/[1 - k_{12}] = 0.11/[1 - 0.65] = 0.11/0.35 = 0.314286$$
$$m_{DS}(\theta_1 \cup \theta_2) = 0.03/[1 - k_{12}] = 0.03/[1 - 0.65] = 0.03/0.35 = 0.085714$$

- If one applies **Murphy's rule** (1.7), i.e average of masses, one gets:

$$m_M(\emptyset) = (0 + 0)/2 = 0$$
$$m_M(\theta_1) = (0.1 + 0.5)/2 = 0.30$$
$$m_M(\theta_2) = (0.4 + 0.1)/2 = 0.25$$
$$m_M(\theta_3) = (0.2 + 0.3)/2 = 0.25$$
$$m_M(\theta_1 \cup \theta_2) = (0.3 + 0.1)/2 = 0.20$$

But if one finds out with certainty that $\theta_3 = \emptyset$, where does $m_M(\theta_3) = 0.25$ go to? Either one accepts here that $m_M(\theta_3)$ goes to $m_M(\theta_1 \cup \theta_2)$ as in Yager's rule, or $m_M(\theta_3)$ goes to $m_M(\emptyset)$ as in Smets' rule. Catherine Murphy does not provide a solution for such a case in her paper [27].

- If one applies **Dubois & Prade's rule** (1.10), one gets because $\theta_3 \overset{\mathcal{M}}{\equiv} \emptyset$:

$$m_{DP}(\emptyset) = 0 \qquad \text{(by definition of Dubois \& Prade's rule)}$$
$$\begin{aligned} m_{DP}(\theta_1) &= [m_1(\theta_1)m_2(\theta_1) + m_1(\theta_1)m_2(\theta_1 \cup \theta_2) + m_2(\theta_1)m_1(\theta_1 \cup \theta_2)] \\ &\quad + [m_1(\theta_1)m_2(\theta_3) + m_2(\theta_1)m_1(\theta_3)] \\ &= [0.1 \cdot 0.5 + 0.1 \cdot 0.1 + 0.5 \cdot 0.3] + [0.1 \cdot 0.3 + 0.5 \cdot 0.2] = 0.21 + 0.13 = 0.34 \end{aligned}$$
$$m_{DP}(\theta_2) = [0.4 \cdot 0.1 + 0.4 \cdot 0.1 + 0.1 \cdot 0.3] + [0.4 \cdot 0.3 + 0.1 \cdot 0.2] = 0.11 + 0.14 = 0.25$$
$$\begin{aligned} m_{DP}(\theta_1 \cup \theta_2) &= [m_1(\theta_1 \cup \theta_2)m_2(\theta_1 \cup \theta_2)] + [m_1(\theta_1 \cup \theta_2)m_2(\theta_3) + m_2(\theta_1 \cup \theta_2)m_1(\theta_3)] \\ &\quad + [m_1(\theta_1)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2)] \\ &= [0.30.1] + [0.3 \cdot 0.3 + 0.1 \cdot 0.2] + [0.1 \cdot 0.1 + 0.5 \cdot 0.4] = [0.03] + [0.09 + 0.02] + [0.01 + 0.20] \\ &= 0.03 + 0.11 + 0.21 = 0.35 \end{aligned}$$

Now if one adds up the masses, one gets $0+0.34+0.25+0.35 = 0.94$ which is less than 1. Therefore Dubois & Prade's rule of combination does not work when a singleton, or an union of singletons, becomes empty (in a dynamic fusion problem). The products of such empty-element columns of the mass matrix $\mathbf{M}$ are lost; this problem is fixed in DSmT by the sum $S_2(.)$ in (1.24) which transfers these products to the total or partial ignorances.

In this particular example, using the hybrid DSm rule, one transfers the product of the empty-element θ_3 column, $m_1(\theta_3)m_2(\theta_3) = 0.2 \cdot 0.3 = 0.06$, to $m_{DSmh}(\theta_1 \cup \theta_2)$, which becomes equal to $0.35+0.06 = 0.41$.

In conclusion, DSmT is a natural extension of DST and Yager's, Smets' and Dubois & Prade's approaches. When there is no singleton nor union of singletons empty, DSmT is consistent with Dubois & Prade's approach, getting the same results (because the sum $S_2(.)$ is not used in this case in the hybrid DSm rule of combination). Otherwise, Dubois & Prade's rule of combination does not work (giving a sum of fusionned masses less than 1) for dynamic fusion problems involving non existential constraints. Murphy's rule does not work either in this case because the masses of empty sets are not transferred. If the conflict is k_{12} is total (i;e. $k_{12} = 1$, DST does not work at all (one gets 0/0 in Dempster's rule of combination), while Smets' rule gives $m_S(\emptyset) = 1$ which is upon to us for the reasons explained in this introduction and in chapter 5 not necessary justified. When the conflict is total, the DSm rule is consistent with Yager's and Dubois & Prade's rules.

The general hybrid DSm rule of combination works on any models for solving static and dynmaic fusion problems and is designed for all kinds of conflict: $0 \leq m(\text{conflict}) \leq 1$. When the conflict is converging towards zero, all rules (Dempster's, Yager's, Smets', Murphy's, Dubois & Prade's, DSmT) are converging towards the same result. This fact is important because it shows the connection among all of them. But if the conflict is converging towards 1, the results among these rules diverge more and more, getting the point when some rules do not work at all (Dempster's rule). Murphy's rule is the only one which is idempotent (being the average of masses). Dubois & Prade's rule does not work in the Smets' case (when $m(\emptyset) > 0$). For models with all intersections empty (Shafer's model) and conflict 1, Dempster's rule is not defined. See below example on $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ with all θ_i, $i = 1, 2, 3, 4$ exclusive:

$$m_1(\theta_1) = 0.1 \qquad m_1(\theta_2) = 0 \qquad m_1(\theta_3) = 0.7 \qquad m_1(\theta_4) = 0$$

$$m_2(\theta_1) = 0 \qquad m_2(\theta_2) = 0.6 \qquad m_2(\theta_3) = 0 \qquad m_2(\theta_4) = 0.4$$

Using Dempster's rule, one gets 0/0, undefined. Conflicting mass is 1.

Yager's rule provides in this case $m_Y(\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4) = 1$ which does not bring specific information, while Smets' rule gives $m(\emptyset) = 1$ which is also not very useful. Murphy's rule gives $m_M(\theta_1) = 0.15$, $m_M(\theta_2) = 0.30$, $m_M(\theta_3) = 0.35$ and $m_M(\theta_4) = 0.20$ which is very specific while the hybrid DSm rule provides $m_{DSmh}(\theta_1 \cup \theta_2) = 0.18$, $m_{DSmh}(\theta_1 \cup \theta_4) = 0.12$, $m_{DSmh}(\theta_2 \cup \theta_3) = 0.42$ and $m_{DSmh}(\theta_3 \cup \theta_4) = 0.28$ which is less specific than Murphy's result but characterizes adequately the internal conflict between sources after the combination and partial ignorances.

The disjunctive rule gives in this last example $m_\cup(\theta_1 \cup \theta_2) = m_1(\theta_1)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2) = 0.18$. Similarly, one gets $m_\cup(\theta_1 \cup \theta_4) = 0.12$, $m_\cup(\theta_2 \cup \theta_3) = 0.42$ and $m_\cup(\theta_3 \cup \theta_4) = 0.28$. This coincides with the hybrid DSm rule when all intersections are empty.

1.4.2 Second example

This example is an extension of Zadeh's example discussed in chapter 5. Let's consider two independent sources of evidences over the frame $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and assume that Shafer's model holds. The basic belief assignments are chosen as follows:

$$m_1(\theta_1) = 0.998 \qquad m_1(\theta_2) = 0 \qquad m_1(\theta_3) = 0.001 \qquad m_1(\theta_4) = 0.001$$

$$m_2(\theta_1) = 0 \qquad m_2(\theta_2) = 0.998 \qquad m_2(\theta_3) = 0 \qquad m_2(\theta_4) = 0.02$$

In this simple numerical example, Dempster's rule of combination gives the counter-intuitive result

$$m_{DS}(\theta_4) = \frac{0.001 \cdot 0.002}{0.998 \cdot 0.998 + 0.998 \cdot 0.002 + 0.998 \cdot 0.001 + 0.998 \cdot 0.001 + 0.001 \cdot 0.002} = \frac{0.000002}{0.000002} = 1$$

Yager's rule gives $m_Y(\theta_4) = 0.000002$ and $m_Y(\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4) = 0.999998$.

Smets' rule gives $m_S(\theta_4) = 0.000002$ and $m_S(\emptyset) = 0.999998$.

Murphy's rule gives $m_M(\theta_1) = 0.499$, $m_M(\theta_2) = 0.499$, $m_M(\theta_3) = 0.0005$ and $m_M(\theta_4) = 0.0015$.

Dubois & Prade's rule gives $m_{DP}(\theta_4) = 0.000002$, $m_{DP}(\theta_1 \cup \theta_2) = 0.996004$, $m_{DP}(\theta_1 \cup \theta_4) = 0.001996$, $m_{DP}(\theta_2 \cup \theta_3) = 0.000998$, $m_{DP}(\theta_2 \cup \theta_4) = 0.000998$ and $m_{DP}(\theta_3 \cup \theta_4) = 0.000002$. Dubois & Prade's rule works only in Shafer's model $\mathcal{M}^0(\Theta)$, i.e. when all intersections are empty. For other hybrid models, Dubois & Prade's rule of combination fails to provide a reliable and reasonable solution to the combination of sources (see next example).

The classic DSm rule of combination provides $m_{DSmc}(\theta_4) = 0.000002$, $m_{DSmc}(\theta_1 \cap \theta_2) = 0.996004$, $m_{DSmc}(\theta_1 \cap \theta_4) = 0.001996$, $m_{DSmc}(\theta_2 \cap \theta_3) = 0.000998$, $m_{DSmc}(\theta_2 \cap \theta_4) = 0.000998$ and $m_{DSmc}(\theta_3 \cap \theta_4) =$

0.000002. If one now applies the hybrid DSm rule since one assumes here that Shafer's model holds, one gets the same result as Dubois & Prade's. The disjunctive rule coincides with Dubois & Prade's rule and the hybrid DSm rule when all intersections are empty.

1.4.3 Third example

Here is an exemple for the Smets' case (i.e. TBM) when $m(\emptyset) > 0$ where Dubois & Prade's rule of combination does not work. Let's consider the following extended[9] belief assignments

$$m_1(\emptyset) = 0.2 \qquad m_1(\theta_1) = 0.4 \qquad m_1(\theta_2) = 0.4$$

$$m_2(\emptyset) = 0.3 \qquad m_2(\theta_1) = 0.6 \qquad m_2(\theta_2) = 0.1$$

In this specific case, the Dubois & Prade's rule of combination gives (assuming all intersections empty)

$$m_{DP}(\emptyset) = 0 \qquad \text{(by definition)}$$

$$m_{DP}(\theta_1) = m_1(\theta_1)m_2(\theta_1) + [m_1(\emptyset)m_2(\theta_1) + m_2(\emptyset)m_1(\theta_1)] = 0.24 + [0.12 + 0.12] = 0.48$$

$$m_{DP}(\theta_2) = m_1(\theta_2)m_2(\theta_2) + [m_1(\emptyset)m_2(\theta_2) + m_2(\emptyset)m_1(\theta_2)] = 0.04 + [0.02 + 0.12] = 0.18$$

$$m_{DP}(\theta_1 \cup \theta_2) = m_1(\theta_1)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2) = 0.04 + 0.24 = 0.28$$

The sum of masses is $0.48+0.18+0.28 = 0.94 < 1$. Where goes the mass $m_1(\emptyset)m_2(\emptyset) = 0.2 \cdot 0.3 = 0.06$? When using the hybrid DSm rule of combination, one gets $m_{DSmh}(\emptyset) = 0$, $m_{DSmh}(\theta_1) = 0.48$, $m_{DSmh}(\theta_2) = 0.18$ and

$$m_{DSmh}(\theta_1 \cup \theta_2) = [m_1(\theta_1)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2)] + [m_1(\emptyset)m_2(\emptyset)] = [0.28] + [0.2 \cdot 0.3] = 0.34$$

and the masses add up to 1.

The disjunctive rule gives in this example

$$m_\cup(\theta_1) = m_1(\theta_1)m_2(\theta_1) + [m_1(\emptyset)m_2(\theta_1) + m_2(\emptyset)m_1(\theta_1)] = 0.24 + [0.12 + 0.12] = 0.48$$

$$m_\cup(\theta_2) = m_1(\theta_2)m_2(\theta_2) + [m_1(\emptyset)m_2(\theta_2) + m_2(\emptyset)m_1(\theta_2)] = 0.04 + [0.02 + 0.12] = 0.18$$

$$m_\cup(\theta_1 \cup \theta_2) = m_1(\theta_1)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2) = 0.04 + 0.24 = 0.28$$

$$m_\cup(\emptyset) = m_1(\emptyset)m_2(\emptyset) = 0.06 > 0$$

One gets the same results for $m_\cup(\theta_1)$, $m_\cup(\theta_2)$ as with Dubois & Prade's rule and as with the hybrid DSm rule. The distinction is in the reallocation of the empty mass $m_1(\emptyset)m_2(\emptyset) = 0.06$ to $\theta_1 \cup \theta_2$ in the hybrid DSm rule, while in Dubois & Prade's and disjunctive rules it is not.

[9]We mean here non-normalized masses allowing weight of evidence on the empty set as in the TBM of Smets.

A major difference among the hybrid DSm rule and all other combination rules is that DSmT uses from the beginning a hyper-power set, which includes intersections, while other combination rules need to do a refinement in order to get intersections.

1.4.4 Fourth example

Here is another example where Dempster's rule does not work properly (this is different from Zadeh's example). Let's consider $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and assume that Shafer's model holds. The basic belief assignments are now chosen as follows:

$$m_1(\theta_1) = 0.99 \qquad m_1(\theta_2) = 0 \qquad m_1(\theta_3 \cup \theta_4) = 0.01$$

$$m_2(\theta_1) = 0 \qquad m_2(\theta_2) = 0.98 \qquad m_2(\theta_3 \cup \theta_4) = 0.02$$

Applying Dempster's rule, one gets $m_{DS}(\theta_1) = m_{DS}(\theta_2) = 0$ and

$$m_{DS}(\theta_3 \cup \theta_4) = \frac{0.01 \cdot 0.02}{1 - [0.99 \cdot 0.98 + 0.99 \cdot 0.02 + 0.98 \cdot 0.01]} = \frac{0.0002}{1 - 0.9998} = \frac{0.0002}{0.0002} = 1$$

which is abnormal.

The hybrid DSm rule gives $m_{DSmh}(\theta_1 \cup \theta_2) = 0.99 \cdot 0.98 = 0.9702$, $m_{DSmh}(\theta_1 \cup \theta_3 \cup \theta_4) = 0.0198$, $m_{DSmh}(\theta_2 \cup \theta_3 \cup \theta_4) = 0.0098$ and $m_{DSmh}(\theta_3 \cup \theta_4) = 0.0002$. In this case, Dubois & Prade's rule gives the same results as the hybrid DSm rule. The disjunctive rule provides a combined belief assignment $m_{\cup}(.)$ which is same as $m_{DSmh}(.)$ and $m_{DP}(.)$.

Yager's rule gives $m_Y(\theta_3 \cup \theta_4) = 0.0002$, $m_Y(\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4) = 0.9998$ and Smets' rule gives $m_S(\theta_3 \cup \theta_4) = 0.0002$, $m_S(\emptyset) = 0.9998$. Both Yager's and Smets' results are less specific than the result obtained with the hybrid DSm rule. There is a loss of information somehow when using Yager's or Smets' rules.

1.4.5 Fifth example

Suppose one extends Dubois & Prade's rule from the power set 2^Θ to the hyper-power set D^Θ. It can be shown that Dubois & Prade's rule does not work when (because $S_2(.)$ term is missing):

a) at least one singleton is empty and the element of its column are all non zero

b) at least an union of singletons is empty and elements of its column are all non zero

c) or at least an intersection is empty and the elements of its column are non zero

Here is an example with intersection (Dubois & Prade's rule extended to the hyper-power set). Let's consider two independent sources on $\Theta = \{\theta_1, \theta_2\}$ with

$$m_1(\theta_1) = 0.5 \qquad m_1(\theta_2) = 0.1 \qquad m_1(\theta_1 \cap \theta_2) = 0.4$$

$$m_2(\theta_1) = 0.1 \qquad m_2(\theta_2) = 0.6 \qquad m_2(\theta_1 \cap \theta_2) = 0.3$$

Then the extended Dubois & Prade rule on the hyper-power set gives $m_{DP}(\emptyset) = 0$, $m_{DP}(\theta_1) = 0.05$, $m_{DP}(\theta_2) = 0.06$, $m_{DP}(\theta_1 \cap \theta_2) = 0.04 \cdot 0.3 + 0.5 \cdot 0.6 + 0.5 \cdot 0.3 + 0.1 \cdot 0.4 + 0.1 \cdot 0.3 + 0.6 \cdot 0.4 = 0.89$.

Now suppose one finds out that $\theta_1 \cap \theta_2 = \emptyset$, then the revised masses become

$$\begin{aligned}
m'_{DP}(\emptyset) &= 0 \qquad \text{(by definition)}\\
m'_{DP}(\theta_1) &= 0.05 + [m_1(\theta_1)m_2(\theta_1 \cap \theta_2) + m_2(\theta_1)m_1(\theta_1 \cap \theta_2)] = 0.05 + [0.5 \cdot 0.3 + 0.1 \cdot 0.4] = 0.24\\
m'_{DP}(\theta_2) &= 0.06 + [m_1(\theta_2)m_2(\theta_1 \cap \theta_2) + m_2(\theta_2)m_1(\theta_1 \cap \theta_2)] = 0.06 + [0.1 \cdot 0.3 + 0.6 \cdot 0.4] = 0.33\\
m'_{DP}(\theta_1 \cup \theta_2) &= m_1(\theta_2)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2) = 0.5 \cdot 0.6 + 0.1 \cdot 0.1 = 0.31
\end{aligned}$$

The sum of the masses is $0.24 + 0.33 + 0.31 = 0.88 < 1$. The mass product $m_1(\theta_1 \cap \theta_2)m_2(\theta_1 \cap \theta_2) = 0.4 \cdot 0.3 = 0.12$ has been lost.

When applying the classic DSm rule in this case, one gets exactly the same results as Dubois & Prade, i.e. $m_{DSmc}(\emptyset) = 0$, $m_{DSmc}(\theta_1) = 0.05$, $m_{DSmc}(\theta_2) = 0.06$, $m_{DSmc}(\theta_1 \cap \theta_2) = 0.89$. Now if one takes into account the integrity constraint $\theta_1 \cap \theta_2 = \emptyset$ and using the hybrid DSm rule of combination, one gets

$$\begin{aligned}
m_{DSmh}(\emptyset) &= 0 \qquad \text{(by definition)}\\
m_{DSmh}(\theta_1) &= 0.05 + [m_1(\theta_1)m_2(\theta_1 \cap \theta_2) + m_2(\theta_1)m_1(\theta_1 \cap \theta_2)] = 0.05 + [0.5 \cdot 0.3 + 0.1 \cdot 0.4] = 0.24\\
m_{DSmh}(\theta_2) &= 0.06 + [m_1(\theta_2)m_2(\theta_1 \cap \theta_2) + m_2(\theta_2)m_1(\theta_1 \cap \theta_2)] = 0.06 + [0.1 \cdot 0.3 + 0.6 \cdot 0.4] = 0.33\\
m_{DSmh}(\theta_1 \cup \theta_2) &= [m_1(\theta_2)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2)] + \underbrace{[m_1(\theta_1 \cap \theta_2)m_2(\theta_1 \cap \theta_2)]}_{S_2 \text{ in hybrid DSm rule eq.}} = [0.31] + [0.12] = 0.43
\end{aligned}$$

Thus the sum of the masses obtained by the hybrid DSm rule of combination is $0.24 + 0.33 + 0.43 = 1$.

The disjunctive rule extended on the hyper-power set gives for this example

$$\begin{aligned}
m_{\cup}(\emptyset) &= 0\\
m_{\cup}(\theta_1) &= [m_1(\theta_1)m_2(\theta_1)] + [m_1(\theta_1)m_2(\theta_1 \cap \theta_2) + m_2(\theta_1)m_1(\theta_1 \cap \theta_2)] = 0.05 + [0.15 + 0.04] = 0.24\\
m_{\cup}(\theta_2) &= [m_1(\theta_2)m_2(\theta_2)] + [m_1(\theta_2)m_2(\theta_1 \cap \theta_2) + m_2(\theta_2)m_1(\theta_1 \cap \theta_2)] = 0.06 + [0.15 + 0.04] = 0.33\\
m_{\cup}(\theta_1 \cup \theta_2) &= [m_1(\theta_2)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2)] = 0.31\\
m_{\cup}(\theta_1 \cap \theta_2) &= m_1(\theta_1 \cap \theta_2)m_2(\theta_1 \cap \theta_2) = 0.4 \cdot 0.3 = 0.12
\end{aligned}$$

If now one finds out that $\theta_1 \cap \theta_2 = \emptyset$, then the revised masses $m'_{\cup}(.)$ become $m'_{\cup}(\theta_1) = m_{\cup}(\theta_1)$, $m'_{\cup}(\theta_2) = m_{\cup}(\theta_2)$, $m'_{\cup}(\theta_1 \cup \theta_2) = m_{\cup}(\theta_1 \cup \theta_2)$ but $m'_{\cup}(\emptyset) \equiv m_{\cup}(\theta_1 \cap \theta_2) = 0.12 > 0$.

1.5 Summary

DSmT has to be viewed as a general flexible *Bottom-Up* approach for managing uncertainty and conflicts for a wide class of static or dynamic fusion problems where the information to combine is modelled as a finite set of belief functions provided by different independent sources of evidence. The development of DSmT emerged from the fact that the conflict between the sources of evidence arises not only from the unreliability of sources themselves (which can be handled by classical discounting methods), but also from a different interpretation of the frame itself by the sources of evidence due to their limited knowlege and own (local) experience; not to mention the fact that elements of the frame cannot be truly refined at all in many problems involving only fuzzy and continuous concepts. Based on this matter of fact, DSmT proposes, according to the general block-scheme in Figure 1.2, a new appealing mathematical framework.

Here are the major steps for managing uncertain and conflicting information arising from independent sources of evidence in the DSmT framework, once expressed in terms of basic belief functions:

1. **Bottom Level**: The ground level of DSmT is to start from the free DSm model $\mathcal{M}^f(\Theta)$ associated with the frame Θ and the notion of hyper-power set (free Dedekind's lattice) D^Θ. At this level, DSmT provides a general commutative and associative rule of combination of evidences (the conjunctive consensus) to work on $\mathcal{M}^f(\Theta)$.

2. **Higher Level** (only used when necessary): Depending on the absolute true intrinsic nature (assumed to be known by the fusion center) of the elements of the frame Θ of the fusion problem under consideration (which defines a set of integrity constraints on $\mathcal{M}^f(\Theta)$ leading to a particular hybrid DSm model $\mathcal{M}(\Theta)$), DSmT automatically adapts the combination process to work on any hybrid DSm model with the general hybrid DSm rule of combination explaine in details in chapter 4. The taking into account of an integrity constraint consists just in forcing some elements of the Dedekind's lattice D^Θ to be empty, when they truly are, given the problem under consideration.

3. **Decision-Making**: Once the combination is obtained after step 1 (or step 2 when necessary), the Decision-making step follows. Although no real general consensus has emerged in literature over last 30 years to give a well-accepted solution for the decision-making problem in the DST framework, we follow here Smets' idea and his justifications to work at the pignistic level [42] rather than at the credal level when a final decision has to be taken from any combined belief mass $m(.)$. A generalized pignistic transformation is then proposed in chapter 7 based on DSmT.

Decision-making

▲

Hybrid DSm rule for hybrid model $\mathcal{M}(\Theta)$

$$\forall A \in D^\Theta, \quad m_{\mathcal{M}(\Theta)}(A) = \phi(A)\Big[m_{\mathcal{M}^f(\Theta)}(A) + S_2(A) + S_3(A)\Big]$$

▲

Introduction of integrity constraints into D^Θ
Hybrid model $\mathcal{M}(\Theta)$

▲

Classic DSm rule based on free model $\mathcal{M}^f(\Theta)$

$$\forall A \in D^\Theta, \quad m_{\mathcal{M}^f(\Theta)}(A) = \sum_{\substack{X_1,\ldots,X_k \in D^\Theta \\ (X_1 \cap \ldots \cap X_k) = A}} \prod_{i=1}^{k} m_i(X_i)$$

▲ ▲ ▲

$m_1(.) : D^\Theta \rightarrow [0,1]$
Source s_1

$m_k(.) : D^\Theta \rightarrow [0,1]$
Source s_k

Figure 1.2: Block Scheme of the principle for the DSm fusion

The introduction of a specific integrity constraint in step 2 is like *pushing an elevator button* for going a bit up in the complexity of the processing for managing uncertainty and conflict through the hybrid DSm rule of combination. If one needs to go to a higher level, then one can take into account several integrity constraints as well in the framework of DSmT. If we finally want to take into account *all possible exclusivity constraints only* (when we really know that all elements of the frame of the given problem are truly exclusive), then we go directly to the Top Level (i.e. Shafer's model which serves as foundation for Shafer's mathematical theory of evidence), but we still apply the hybrid DSm rule instead of Dempster's rule of combination. The DSmT approach for modelling the frame and combining information is more general than previous approaches which have been mainly[10] based on the Shafer model (which is a very specific and constrained DSm hybrid model) and works for static fusion problems.

The DSmT framework can easily handle *not only* exclusivity constraints, *but also* non existential constraints or mixed constraints as well which is very useful in some dynamic fusion problems as it is shown in chapter 4. Depending on the nature of the problem, we claim that it is unnecessary to try working at the Top Level (as DST does), when working directly at a lower level is sufficient to manage properly the information to combine using the hybrid DSm rule of combination.

[10]except the Transferable Belief Model of Smets [41] and the trade-off/averaging combination rules.

It is also important to reemphasize here that the general hybrid DSm rule of combination is definitely not equivalent to Dempster's rule of combination (and to all its alternatives involving conjunctive consensus based on the Top level and especially when working with dynamic problems) because DSmT allows to work at any level of modelling for managing uncertainty and conflicts, depending on the intrinsic nature of the problem. The hybrid DSm rule and Dempster's rule do not provide the same results even when working on Shafer's model as it has been shown in examples of the previous section and explained in details in forthcoming chapters 4 and 5.

DSmT differs from DST because it is based on the free Dedekind lattice. It works for any model (free DSm model and hybrid models - including Shafer's model as a special case) which fits adequately with the true nature of the fusion problem under consideration. This ability of DSmT allows to deal formally with any fusion problems expressed in terms of belief functions which can mix discrete concepts with vague/continuous/relative concepts. The DSmT deals with static and dynamic fusion problematics in the same theoretical way taking into account the integrity constraints into the model which are considered either as static or eventually changing with time when necessary. The general hybrid DSm rule of combination of independent sources of evidence works for all possible static or dynamic models and does not require a normalization step. It differs from Dempster's rule of combination and from all its concurrent alternatives. The hybrid DSm rule of combination has been moreover extended to work for the combination of imprecise admissible belief assignments as well. The approach proposed by the DSmT to attack the fusion problematic throughout this book is therefore totally new both by its foundations, its applicability and the solution provided.

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