Radical Axis of Lemoine Circles

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In this article, we emphasize the radical axis of the Lemoine Circles. For the start, let us remind:

Theorem 1.

The parallels taken through the simmedian center K of a triangle to the sides of the triangle determine on them six concyclic points (The First Lemoine Circle).

Theorem 2.

The antiparallels taken through the simmedian center of a triangle to the sides of a triangle determine on them six concyclic points (The Second Lemoine Circle).

Remark 1.

If ABC is a scalene triangle and K is its simmedian center, then L, the center of the First Lemoine Circle, is the middle of the segment $[OK]$, where O is the center of the circumscribed circle, and the center of the Second Lemoine Circle is K. It follows that the radical axis of Lemoine circles is perpendicular on the line of the centers LK, therefore on the line OK.

Proposition 1.

The radical axis of Lemoine Circles is perpendicular on the line OK raised in the simmedian center K.

Proof.

Let A_1A_2 be the antiparallel to BC taken through K, then KA_1 is the radius R_{L_2} of the Second Lemoine Circle; we have:

$$R_{L_2} = \frac{abc}{a^2+b^2+c^2}.$$

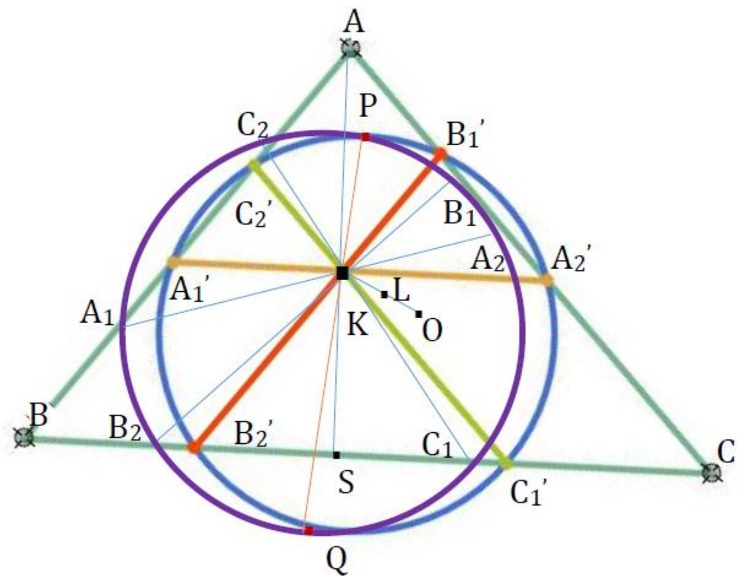

Figura 1

Let $A_1'A_2'$ be the Lemoine parallel taken to BC; we evaluate the power of K towards the First Lemoine Circle. We have:

$$\overrightarrow{KA_1'} \cdot \overrightarrow{KA_2'} = LK^2 - R_{L_1}^2. \quad (1)$$

Let S be the simmedian leg from A; it follows that:

$$\frac{KA_1'}{BS} = \frac{AK}{AS} - \frac{KA_2'}{SC}.$$

We obtain:

$$KA_1' = BS \cdot \frac{AK}{AS} \text{ and } KA_2' = SC \cdot \frac{AK}{AS},$$

but $\frac{BS}{SC} = \frac{c^2}{b^2}$ and $\frac{AK}{AS} = \frac{b^2+c^2}{a^2+b^2+c^2}$.

Therefore:

$$\overrightarrow{KA_1'} \cdot \overrightarrow{KA_2'} = -BS \cdot SC \cdot \left(\frac{AK}{AS}\right)^2 = \frac{-a^2b^2c^2}{(b^2+c^2)^2} \cdot \frac{(b^2+c^2)^2}{(a^2+b^2+c^2)^2} = -R_{L_2}^2 . \qquad (2)$$

We draw the perpendicular in K on the line LK and denote by P and Q its intersection to the First Lemoine Circle; we have $\overrightarrow{KP} \cdot \overrightarrow{KQ} = -R_{L_2}^2$; by the other hand, $KP = KQ$ (PQ is a chord which is perpendicular to the diameter passing through K).

It follows that $KP = KQ = R_{L_2}$, so P and Q are situated on the Second Lemoine Circle.

Because PQ is a chord which is common to the Lemoine Circles, it follows that PQ is the radical axis.

Comment 1.

After equalizing relations (1) and (2) or by the Pythagorean theorem in the triangle PKL, we can calculate R_{L_1}. It is known that:

$$OK^2 = R^2 - \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2},$$

and since $LK = \frac{1}{2} OK$, we find that:

$$R_{L_1}^2 = \frac{1}{4} \cdot \left[R^2 + \frac{a^2b^2c^2}{(a^2+b^2+c^2)^2}\right].$$

Remark 2.

The proven *Proposition* regarding the radical axis of the Lemoine Circles is a particular case of the following *Proposition*, which we leave it to the reader to prove.

Proposition 2.

If $\mathcal{C}(O_1, R_1)$ şi $\mathcal{C}(O_2, R_2)$ are two circles such as the power of center O_1 towards $\mathcal{C}(O_2, R_2)$ is $-R_1^2$, then the radical axis of the circles is the perpendicular in O_1 on the line of centers O_1O_2.

Bibliography.

[1] F. Smarandache, Ion Patrascu: *The Geometry of Homological Triangles*, Education Publisher, Ohio, USA, 2012.

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