# About Factorial Sums

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**Abstract.** In this paper, we present some new inequalities for factorial sum.

**Application 1**.We have the following inequality

$$\sum_{k=1}^{n} k! \leq \frac{2((n+1)!-1)}{n+1}$$

**Proof**. If $x_k,\ y_k > 0\ \ (k = 1, 2, ..., n)$, have the same monotonity, then

$$\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right) \leq \frac{1}{n}\sum_{k=1}^{n} x_k y_k \qquad (1)$$

the Chebishev's inequality.

If $x_k,\ y_k$ have different monotonity, then holds true the reverse inequality, we take $x_k = k,\ y_k = k!\ \ (k = 1, 2, ..., n)$ and use that $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$.

**Application 2**. We have the following inequality

$$\sum_{k=1}^{n} k! \leq \frac{3(n+1)(n+1)!}{n^2 + 3n + 5}$$

**Proof**. In (1) we take

$$x_k = k^2 + k + 1\,;$$
$$y_k = k!\ \ (k = 1, 2, ..., n)$$

and the identity

$$\sum_{k=1}^{n} (k^2 + k + 1)k! = (n+1)(n+1)!$$

**Application 3**. We have the following inequality

$$\sum_{k=1}^{n} \frac{1}{k!} \geq \frac{n^2(n+1)}{2((n+1)!-1)}$$

**Proof.** Using the Application 1, we take

$$\sum_{k=1}^{n}\frac{1}{k!}\geq\frac{n^2}{\sum_{k=1}^{n}k!}\geq\frac{n^2(n+1)}{2((n+1)!-1)}$$

**Application 4**. We have the following inequality

$$\sum_{k=1}^{n}\frac{1}{k!}\geq\frac{n^2(n^2+3n+5)}{3(n+1)(n+1)!}$$

**Proof.** Using the Application 2, we take

$$\sum_{k=1}^{n}\frac{1}{k!}\geq\frac{n^2}{\sum_{k=1}^{n}k!}\geq\frac{n^2(n^2+3n+5)}{3(n+1)(n+1)!}$$

**Application 5.** We have the following inequality:

$$\sum_{k=1}^{n}\frac{1}{k!}\geq 1+\frac{2}{n}\left(1-\frac{1}{n!}\right)$$

**Proof.** In (1) we take $x_k=k,\quad y_k=\frac{1}{(k+1)!},\quad (k=1,2,...,n)$ and we obtain

$$\frac{1}{n}\left(\sum_{k=1}^{n}k\right)\left(\sum_{k=1}^{n}\frac{1}{(k+1)!}\right)\geq\sum_{k=1}^{n}\frac{k}{(k+1)!}=1-\frac{1}{(n+1)!}$$

therefore

$$\left(\sum_{k=1}^{n}\frac{1}{(k+1)!}\right)\geq\frac{2}{n+1}\left(1-\frac{1}{(n+1)!}\right)$$

or

$$\sum_{k=2}^{n}\frac{1}{k!}\geq\frac{2}{n}\left(1-\frac{1}{n!}\right)$$

therefore

$$\left(\sum_{k=1}^{n}\frac{1}{k!}\right)\geq 1+\frac{2}{n}\left(1-\frac{1}{n!}\right)$$

**Application 6**. We have the following inequality:

$$\left(\sum_{k=1}^{n}\frac{1}{(k+2)^2k!}\right)\geq\frac{2}{n+5}\left(1-\frac{1}{(n+2)!}\right)$$

**Proof.** In (1) we take $x_k=k+2,\ y_k=\frac{1}{(k+2)^2k!},\ (k=1,2,...,n)$

therefore

$$\frac{1}{n}\left(\sum_{k=1}^{n}(k+2)\right)\sum_{k=1}^{n}\frac{1}{(k+2)^2k!}\geq\sum_{k=1}^{n}\frac{1}{(k+2)^2k!}=1-\frac{1}{(n+2)!}$$

therefore

$$\sum_{k=1}^{n}\frac{1}{(k+2)^2 k!}\geq\frac{2}{n+5}\left(1-\frac{1}{(n+2)!}\right)$$

**Application 7**.We have the following inequality:

$$\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)!}\geq\frac{6}{2n^2+9n+1}\left(\frac{1}{2}-\frac{1}{(n+1)(n+2)!}\right)$$

**Proof.** In (1) we take

$$x_k=k^2+2k+2,\ y_k=\frac{1}{k(k+1)(k+2)!},\ (k=1,2,...,n)$$

then

$$\frac{1}{n}\sum_{k=1}^{n}(k^2+2k+2)\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)!}\geq\sum_{k=1}^{n}\frac{k^2+2k+2}{k(k+1)(k+2)!}=$$

$$=\sum_{k=1}^{n}\frac{1}{k(k+1)!}-\frac{1}{(k+1)(k+2)!}=\frac{1}{2}-\frac{1}{(n+1)(n+2)!}$$

**Application 8.** We have the following inequality:

$$\sum_{k=1}^{n}\frac{1}{4k^4+1}\geq\frac{n}{2n^2+2n+1}$$

**Proof.** In (1) we take $x_k=4k,\ y_k=\frac{1}{4k^4+1},\ (k=1,2,...,n)$,

therefore

$$\frac{1}{n}\left(\sum_{k=1}^{n}4k\right)\left(\sum_{k=1}^{n}\frac{1}{4k^4+1}\right)\geq\sum_{k=1}^{n}\frac{4k}{4k^4+1}=\sum_{k=1}^{n}\left(\frac{1}{2k^2-2k+1}-\frac{1}{2k^2+2k+1}\right)=\frac{2n(n+1)}{2n^2+2n+1}$$

**Application 9**. We have the following inequality:

$$\sum_{k=1}^{n}\frac{1}{4k^4-1}\geq\frac{3n}{(2n+1)^2}$$

**Proof.** In (1) we take $x_k=k^2,\ y_k=\frac{1}{4k^2-1},\ (k=1,2,...,n)$ then

$$\frac{1}{n}\left(\sum_{k=1}^{n}k^2\right)\left(\sum_{k=1}^{n}\frac{1}{4k^2-1}\right)\geq\sum_{k=1}^{n}\frac{k^2}{4k^2-1}=\frac{n(n+1)}{2(2n+1)},\text{ etc.}$$

**Reference:**

[1] Octogon Mathematical Magazine (1993-2007)

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