Nedians and triangles with the same coefficient of deformation

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In [1] Dr. Florentin Smarandache generalized several properties of the nedians. Here, we will continue the series of these results and will establish certain connections with the triangles which have the same coefficient of deformation.

Definition 1

The line segments that have their origin in the triangle's vertex and divide the opposite side in n equal segments are called nedians.

We call the nedian AA_i being of order i ($i \in N^*$), in the triangle ABC, if A_i divides the side (BC) in the rapport $\frac{i}{n}$ ($\overrightarrow{BA_i} = \frac{i}{n} \cdot \overrightarrow{BC}$ or $\overrightarrow{CA_i} = \frac{i}{n} \cdot \overrightarrow{CB}$, $1 \le i \le n-1$)

Observation 1

The medians of a triangle are nedians of order 1, in the case when $n = 3$, these are called tertian.

We'll recall from [1] the following:

Proposition 1

Using the nedians of the same of a triangle, we can construct a triangle.

Proposition 2

The sum of the squares of the lengths of the nedians of order i of a triangle ABC is given by the following relation:

$$AA_i^2 + BB_i^2 + CC_i^2 = \frac{i^2 - in + n^2}{n^2}\left(a^2 + b^2 + c^2\right) \qquad (1)$$

We'll prove

Proposition3.

The sum of the squares of the lengths of the sides of the triangle $A_0B_0C_0$, determined by the intersection of the nedians of order i of the triangle ABC is given by the following relation:

$$A_0B_0^2 + B_0C_0^2 + C_0A_0^2 = \frac{(n-2i)^2}{i^2 - in + n^2}\left(a^2 + b^2 + c^2\right) \qquad (2)$$

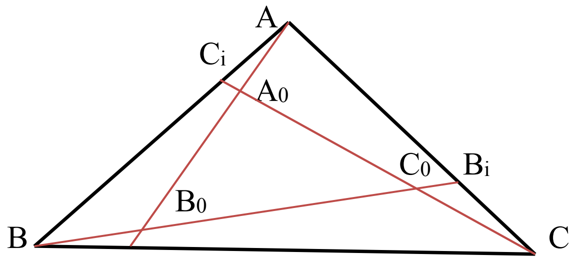


Fig. 1

Ai

We noted

$$\{A_0\} = CC_i \cap AA_i,\ \{B_0\} = AA_i \cap BB_i,\ \{C_0\} = BB_i \cap CC_i.$$

Proof

We'll apply the Menelaus 'theorem in the triangle AA_iC for the transversals $B - B_0 - B_i$, see Fig. 1.

$$\frac{BA_i}{BC} \cdot \frac{B_iC}{B_iA} \cdot \frac{B_0A}{B_0A_i} = 1 \qquad (3)$$

Because $BA_i = \frac{ia}{n},\ B_iC = \frac{ib}{n},\ B_iA = \frac{(n-i)b}{n}$, from (3) it results that:

$$B_0A = \frac{n(n-i)}{i^2 - in + n^2} AA_i \qquad (4)$$

The Menelaus 'theorem applied in the triangle AA_iB for the transversal $C - C_0 - C_i$ gives

$$\frac{CA_i}{CB} \cdot \frac{C_iB}{C_iA} \cdot \frac{A_0A}{A_0A_i} = 1 \qquad (5)$$

But $CA_i = \frac{(n-i)a}{n},\ C_iB = \frac{(n-i)c}{n},\ C_iA = \frac{ic}{n}$, which substituted in (5), gives

$$AA_0 = \frac{in}{i^2 - in + n^2} AA_i \qquad (6)$$

It is observed that $A_0B_0 = AB_0 - AA_0$ and using the relation (4) and (6) we find:

$$A_0B_0 = \frac{n(n-2i)}{i^2 - in + n^2} AA_i \qquad (7)$$

Similarly, we obtain:

$$B_0C_0 = \frac{n(n-2i)}{i^2 - in + n^2} BB_i \qquad (8)$$

$$C_0A_0 = \frac{n(n-2i)}{i^2 - in + n^2} CC_i \qquad (9)$$

Using the relations (7), (8) and (9), after a couple of computations we obtain the relation (2).

Observation 2.

The triangle formed by the nedians of order i as sides is similar with the triangle formed by the intersections of the nedians of order i.

Indeed, the relations (7), (8) and (9) show that the sides $A_0B_0,\ B_0C_0,\ C_0A_0$ are proportional with $AA_i,\ BB_i,\ CC_i$

The Russian mathematician V. V. Lebedev introduces in [2] the notion of coefficient of deformation of a triangle. To define this notion we need a couple of definitions and observations.

Definition 2

If ABC is a triangle and in its exterior on its sides are constructed the equilateral triangles BCA_1, CAB_1, ABC_1, then the equilateral triangle $O_1O_2O_3$ formed by the centers of the circumscribed circles to the equilateral triangles, described above, is called the exterior triangle of Napoleon.

If the equilateral triangles BCA_1, CAB_1, ABC_1 intersect in the interior of the triangle ABC then the equilateral triangle $O_1'O_2'O_3'$ formed by the centers of the circumscribed circles to these triangles is called the interior triangle of Napoleon.

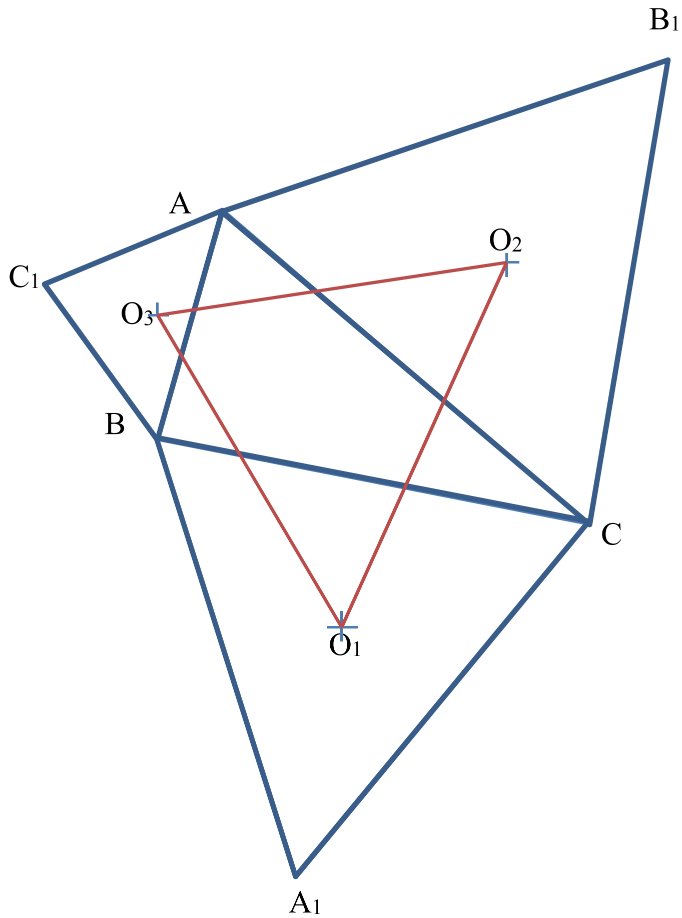
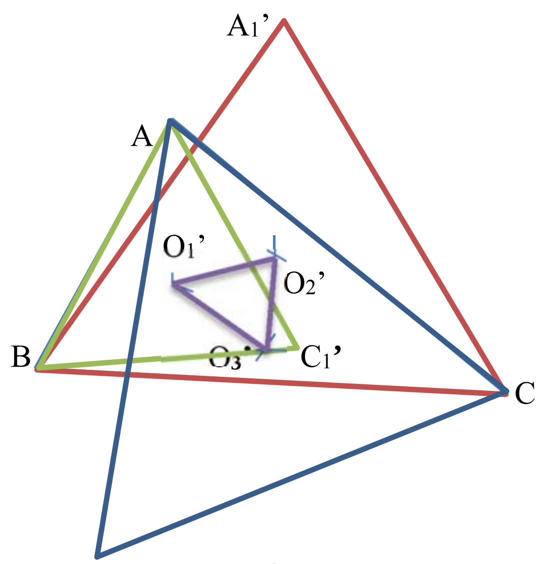


Fig. 2



B1’

Fig. 3

Observation 3

In figure 2 is represented the external triangle of Napoleon and in figure 3 is represented the interior triangle of Napoleon.

Definition 3

A coefficient of deformation of a triangle is the rapport between the side of the interior triangle of Napoleon and the side of the exterior triangle of Napoleon corresponding to the same triangle.

Observation 4

The coefficient of deformation of the triangle ABC is

$$k = \frac{O_1'O_2'}{O_1O_2}$$

Proposition 4

The coefficient of deformation k of triangle ABC ha the following formula:

$$k = \left(\frac{a^2 + b^2 + c^2 - 4s\sqrt{3}}{a^2 + b^2 + c^2 + 4s\sqrt{3}} \right)^{\frac{1}{2}} \tag{10}$$

where s is the aria of the triangle ABC.

Proof

We’ll apply the cosine theorem in the triangle $CO_1'O_2'$ (see Fig. 3), in which

$$CO_1' = \frac{a\sqrt{3}}{3},\ \ CO_2' = \frac{b\sqrt{3}}{3}\ \text{, and } m\left(\sphericalangle O_1CO_2'\right) = C - 60^\circ .$$

We have

$$O_1'O_2'^2 = \frac{3a^2}{9} + \frac{3b^2}{9} - 2\frac{ab}{3} cos\left(C - 60^\circ\right)$$

Because

$$cos\left(C - 60^\circ\right) = cos\,C \cdot cos\,60^\circ + sin\,60^\circ \cdot sin\,C = \frac{1}{2} cos\,C + \frac{\sqrt{2}}{2} sin\,C \text{ and}$$

$$cosC = \frac{b^2 + a^2 - c^2}{2ab}\text{, and}$$

$$ab\,sin\,C = 2s\,,$$

we obtain

$$O_1'O_2'^2 = \frac{a^2 + b^2 + c^2 - 4s\sqrt{3}}{6} \tag{11}$$

Similarly

$$O_1O_2^2 = \frac{a^2 + b^2 + c^2 + 4s\sqrt{3}}{6} \tag{12}$$

By dividing the relations (11) and (12) and resolving the square root we proved the proposition.

Observation 5

In an equilateral triangle the deformation coefficient is $k=0$. In general, for a triangle ABC, $0 \le k < 1$.

Observation 6

From (11) it results that in a triangle is true the following inequality:

$$a^2+b^2+c^2 \ge 4s\sqrt{3} \tag{13}$$

which is the inequality Weitzeböck.

Observation 7

In a triangle there following inequality – stronger than (13) – takes also place:

$$a^2+b^2+c^2 \ge 4s\sqrt{3}+(a-b)^2+(b-c)^2+(c-a)^2 \tag{14}$$

which is the inequality of Finsher - Hadwiger.

Observation 8

It can be proved that in a triangle the coefficient of deformation can be defined by the

$$k=\frac{AA_1'}{AA_1} \tag{15}$$

Definition 4

We define the Brocard point in triangle ABC the point Ω from the triangle plane, with the property:

$$\sphericalangle \Omega AB \equiv \sphericalangle \Omega BC \equiv \sphericalangle \Omega CA \tag{16}$$

The common measure of the angles from relation (16) is called the Brocard angle and is noted

$$\sphericalangle \Omega AB = \omega$$

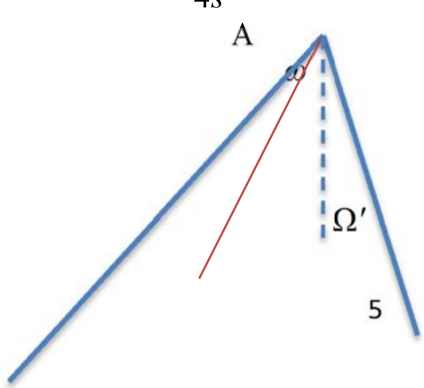
Observation 9

A triangle ABC has, in general, two points Brocard Ω and Ω' which are isogonal conjugated (see Fig. 4)

Proposition 5

In a triangle ABC takes place the following relation:

$$ctg\omega=\frac{a^2+b^2+c^2}{4s} \tag{17}$$



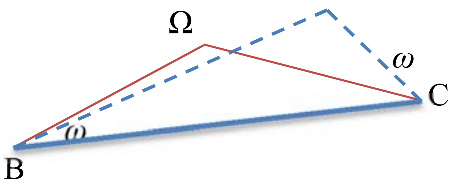


Fig. 4

Proof

We'll show, firstly, that in a non-rectangle triangle ABC is true the following relation:

$$ctg\omega = ctgA + ctgB + ctgC \tag{18}$$

Applying the sin theorem in triangle $A\Omega B$ and $A\Omega C$, we obtain

$$\frac{B\Omega}{\sin\omega} = \frac{c}{\sin B\Omega A} \text{ and } \frac{A\Omega}{\sin\omega} = \frac{b}{\sin A\Omega C}$$

Because $m(\sphericalangle B\Omega A) = 180° - m(\sphericalangle B)$ and $m(\sphericalangle A\Omega C) = 180° - m(\sphericalangle A)$ from the precedent relations we retain that

$$\frac{A\Omega}{B\Omega} = \frac{b}{c}\frac{\sin B}{\sin A} \tag{19}$$

On the other side also from the sin theorem in triangle $A\Omega B$, we obtain

$$\frac{A\Omega}{B\Omega} = \frac{\sin(B-\omega)}{\sin\omega} \tag{20}$$

Working out $\sin(B-\omega)$, taking into account that $\frac{b}{c} = \frac{\sin B}{\sin C}$ and that $\sin B = \sin(A+C)$, we obtain (18).

In a triangle ABC is true the relation $ctgA = \frac{a^2+b^2+c^2}{4s}$ (19) and the analogues.

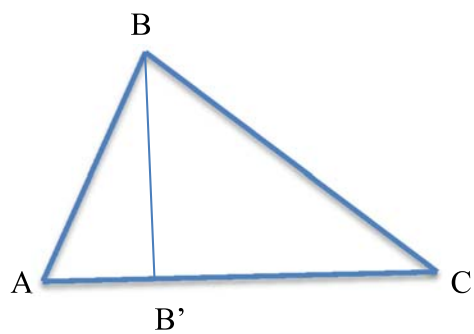


Fig. 5

Indeed, if $m(\sphericalangle A) < 90°$ and B' is the orthogonal projection of B on AC (see Fig. 5), then

$$ctgA = \frac{AB'}{BB'} = \frac{c \cdot \cos A}{BB'}$$

Because $BB' = \frac{2s}{b}$ it results that $ctgA = \frac{2bc\cos A}{4s}$

From the cosine theorem we get

$$2bc\cos A = b^2 + c^2 - a^2$$

Replacing in (18) the $ctgA,\ ctgB,\ ctgC$, we obtain (17)

Observation 10
The coefficient of deformation k of triangle ABC is given by

$$k = \left(\frac{ctg\omega - \sqrt{3}}{ctg\omega + \sqrt{3}} \right)^{\frac{1}{2}} \quad (21)$$

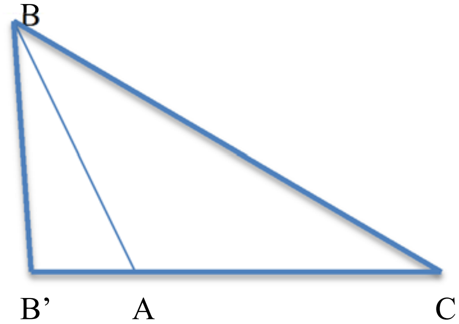


Fig. 6

Indeed, from (10) and (17), it results, without difficulties (21)

Proposition 6 (V.V. Lebedev)
The necessary and sufficient condition for two triangles to have the same coefficient of deformation is to have the same Brocard angle.
Proof
If the triangles ABC and $A_1B_1C_1$ have equal coefficients of deformation $k = k_1$ then from relation 21 it results

$$\frac{ctg\omega - \sqrt{3}}{ctg\omega + \sqrt{3}} = \frac{ctg\omega_1 - \sqrt{3}}{ctg\omega_1 + \sqrt{3}}$$

Which leads to $ctg\omega = ctg\omega_1$ with the consequence that $\omega = \omega_1$.
Reciprocal, if $\omega = \omega_1$, immediately results, using (21), that takes place $k = k_1$.

Proposition 7
Two triangles ABC and $A_1B_1C_1$ have the same coefficient of deformation if and only if

$$\frac{s_1}{s} = \frac{a_1^2 + b_1^2 + c_1^2}{a^2 + b^2 + c^2} \quad (22)$$

(s_1 being the aria of triangle $A_1B_1C_1$, with the sides a_1, b_1, c_1)
Proof
If ω, ω_1 are the Brocard angles of triangles ABC and $A_1B_1C_1$ then, taking into consideration (17) and Proposition 6, we'll obtain (22). Also from (22) taking into consideration of (17) and Proposition 6, we'll get $k = k_1$.

Proposition 8
Triangle $A_iB_iC_i$ formed by the legs of the nediands of order i of triangle ABC and triangle ABC have the same coefficient of deformation.

Proof

We'll use Proposition 7, applying the cosine theorem in triangle $A_iB_iC_i$, we'll obtain

$$B_iC_i^2 = AC_i^2 + AB_i^2 - 2AC_iAB_i \cos A$$

Because

$$AC_i = \frac{ic}{n},\ AB_i = \frac{(n-i)b}{n}$$

it results

$$B_iC_i^2 = \frac{i^2c^2}{n^2} + \frac{(n-i)^2 b^2}{n^2} - \frac{2i(n-i)bc \cos A}{n^2}$$

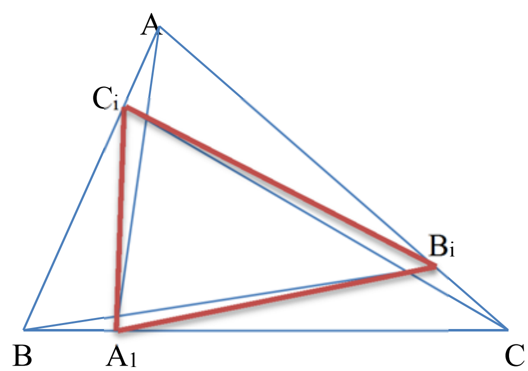


Fig. 7

The cosin theorem in the triangle ABC gives

$$2bc \cos A = b^2 + c^2 - a^2$$

which substituted above gives

$$B_iC_i^2 = \frac{i^2c^2 + (n-i)^2 b^2 + i(n-i)(a^2 - b^2 - c^2)}{n^2}$$

$$B_iC_i^2 = \frac{a^2(in - i^2) + b^2(n^2 - 3in + 2i^2) + c^2(2i^2 - in)}{n^2}$$

Similarly we'll compute $C_iA_i^2$ and $A_iB_i^2$

It results

$$\frac{A_iB_i^2 + B_iC_i^2 + C_iA_i^2}{a^2 + b^2 + c^2} = \frac{n^2 - 2in + 3i^2}{n^2} \tag{23}$$

If we note

$$s_i = Aria_{\triangle} A_iB_iC_i$$

We obtain

$$s_i = s - (Aria_{\triangle} AB_iC_i + Aria_{\triangle} BA_iC_i + Aria_{\triangle} CA_iB_i) \tag{24}$$

But

$$Aria_{\triangle} AB_iC_i = \frac{1}{2} AC_i \cdot AB_i \sin A$$

$$Aria_{\triangle}AB_iC_i = \frac{1}{2}\frac{i(n-i)b\cdot c}{n^2}\sin A = \frac{i(n-i)\cdot s}{n^2}$$

Similarly, we find that

$$Aria_{\triangle}BA_iC_i = Aria_{\triangle}CA_iB_i = \frac{i(n-i)\cdot s}{n^2}$$

Revisiting (23) we get that

$$s_i = \frac{sn^2 - 3in + 3i^2}{n^2}$$

therefore,

$$\frac{s_i}{s} = \frac{n^2 - 3in + 3i^2}{n^2} \quad (25)$$

The relations (23), (25) and Proposition 7 will imply the conclusion.

Proposition 9

The triangle formed by the medians of a given triangle, as sides, and the given triangle have the same coefficient of deformation.

Proof

The medians are nedians of order I. Using (1), it results

$$AA_i^2 + BB_i^2 + CC_i^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right) \quad (26)$$

The proposition will be proved if we'll show that the rapport between the aria of the formed triangle with the medians of the given triangle and the aria of the given triangle is $\frac{3}{4}$.

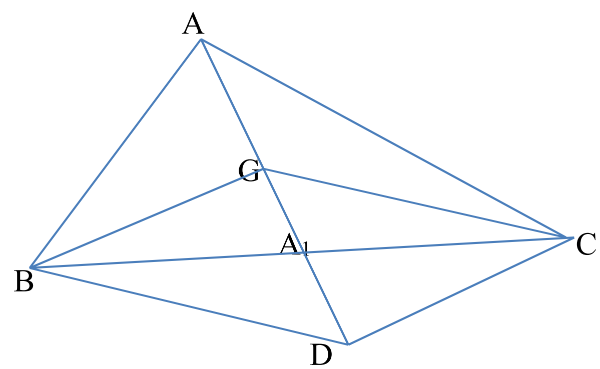


Fig. 9

If in triangle ABC we prolong the median AA_1 such that $A_1D = GA_1$ (G being the center of gravity of the triangle ABC), then the quadrilateral $BGCD$ is a parallelogram (see Fig. 9). Therefore $CD = BG$. It is known that $BG = \frac{2}{3}BB_1$, $CG = \frac{2}{3}CC_1$ and from construction we have that $GD = \frac{2}{3}AA_1$. Triangle GDC has the sides equal to $\frac{2}{3}$ from the length of the medians of the triangle ABC. Because the median of a triangle divides the triangle in two equivalent triangles and the gravity center of the triangle forms with the vertexes of the triangle three equivalent triangle, it results that $Aria_{\triangle}GDC = \frac{1}{3}s$. On the other side the rapport of the arias of two similar triangles is equal with the squared of their similarity rapport, therefore, if we note s_1 the aria of the triangle formed by the medians, we have $\frac{Aria_{\triangle}GDC}{s_1} = \left(\frac{2}{3}\right)^2$.

We find that $\frac{s_1}{s} = \frac{3}{4}$, which proves the proposition.

Proposition 10

The triangle formed by the intersections of the tertianes of a given triangle and the given triangle have the same coefficient of deformation.

Proof

If $A_0B_0C_0$ is the triangle formed by the intersections of the tertianes, from relation (2) we'll find

$$\frac{A_0B_0^2 + B_0C_0^2 + C_0A_0^2}{a^2 + b^2 + c^2} = \frac{1}{7}$$

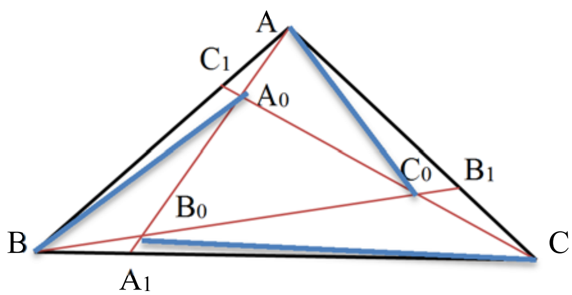


Fig 10

We note s_0 the aria of triangle $A_0B_0C_0$, we'll prove that $\frac{s_0}{s} = \frac{1}{7}$.

From the formulae (6) and (7), it is observed that $A_0 = A_0B_0$ and $CC_0 = C_0A_0$.

Using the median's theorem in a triangle to determine that in that triangle two triangle are equivalent, we have that:

$$Aria_{\triangle}AA_0C_0 = Aria_{\triangle}AC_0C = Aria_{\triangle}A_0B_0C_0 =$$
$$= Aria_{\triangle}CB_0C_0 = Aria_{\triangle}CBB_0 = Aria_{\triangle}BB_0A_0 = Aria_{\triangle}ABA_0$$

Because the sum of the aria of these triangles is s, it results that $s_0 = \frac{1}{7}s$, which shows what we had to prove.

Proposition 11

We made the observation that the triangle $A_0B_0C_0$ and the triangle formed by the tertianes AA_1, BB_1, CC_1 as sides are similar. Two similar triangles have the same Brocard angle, therefore the same coefficient of deformation. Taking into account Proposition 10, we obtain the proof of the statement

Observation 11

From the precedent observations it results that being given a triangle, the triangles formed by the tertianes intersections with the triangle as sides, the intersections of the tertianes of the triangle have the same coefficient of deformation.

References

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