Two Triangles with the Same Orthocenter and a Vectorial Proof of Stevanovic's Theorem

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Abstract. In this article we'll emphasize on two triangles and provide a vectorial proof of the fact that these triangles have the same orthocenter. This proof will, further allow us to develop a vectorial proof of the Stevanovic's theorem relative to the orthocenter of the Fuhrmann's triangle.

Lemma 1

Let ABC an acute angle triangle, H its orthocenter, and A', B', C' the symmetrical points of H in rapport to the sides BC, CA, AB.

We denote by X, Y, Z the symmetrical points of A, B, C in rapport to $B'C', C'A', A'B'$ The orthocenter of the triangle XYZ is H.

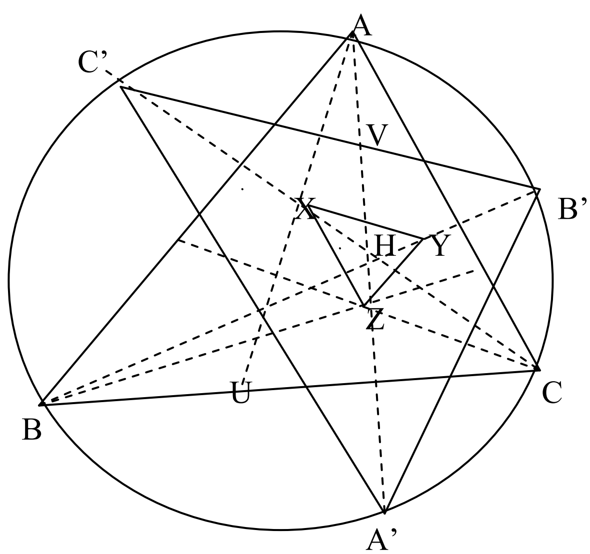

Fig. 1

Proof

We will prove that $XH \perp YZ$, by showing that $\overrightarrow{XH} \cdot \overrightarrow{YZ} = 0$.
We have (see Fig.1)

$$\overrightarrow{VH} = \overrightarrow{AH} - \overrightarrow{AX}$$
$$\overrightarrow{BC} = \overrightarrow{BY} + \overrightarrow{YZ} + \overrightarrow{ZC},$$

from here

$$\overrightarrow{YZ} = \overrightarrow{BC} - \overrightarrow{BY} - \overrightarrow{ZC}$$

Because Y is the symmetric of B in rapport to $A'C'$ and Z is the symmetric of C in rapport to $A'B'$, the parallelogram's rule gives us that:

$$\overrightarrow{BY} = \overrightarrow{BC'} + \overrightarrow{BA'}$$
$$\overrightarrow{CZ} = \overrightarrow{CB'} + \overrightarrow{CA'}.$$

Therefore

$$\overrightarrow{YZ} = \overrightarrow{BC} - \left(\overrightarrow{BC'} + \overrightarrow{BA'}\right) + \overrightarrow{B'C} + \overrightarrow{A'C}$$

But

$$\overrightarrow{BC'} = \overrightarrow{BH} + \overrightarrow{HC'}$$
$$\overrightarrow{BA'} = \overrightarrow{BH} + \overrightarrow{HA'}$$
$$\overrightarrow{CB'} = \overrightarrow{CH} + \overrightarrow{HB'}$$
$$\overrightarrow{CA'} = \overrightarrow{CH} + \overrightarrow{HA'}$$

By substituting these relations in the $\overrightarrow{YZ}$, we find:

$$\overrightarrow{YZ} = \overrightarrow{BC} + \overrightarrow{C'B'}$$

We compute

$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = \left(\overrightarrow{AH} - \overrightarrow{AX}\right) \cdot \left(\overrightarrow{BC} + \overrightarrow{C'B'}\right) = \overrightarrow{AX} \cdot \overrightarrow{BC} + \overrightarrow{AH} \cdot \overrightarrow{C'B'} - \overrightarrow{AX} \cdot \overrightarrow{BC} - \overrightarrow{AX} \cdot \overrightarrow{C'B'}$$

Because

$$AH \perp BC$$

we have

$$\overrightarrow{AH} \cdot \overrightarrow{BC} = 0,$$

also

$$AX \perp B'C'$$

and therefore

$$\overrightarrow{AX} \cdot \overrightarrow{B'C'} = 0.$$

We need to prove also that

$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = \overrightarrow{AH} \cdot \overrightarrow{C'B'} - \overrightarrow{AX} \cdot \overrightarrow{BC}$$

We note:

$$\{U\} = AX \cap BC \text{ and } \{V\} = AH \cap B'C'$$
$$\overrightarrow{AX} \cdot \overrightarrow{BC} = AX \cdot BC \cdot cox\sphericalangle\left(AX, BC\right) = AX \cdot BC \cdot cox\left(\sphericalangle AUC\right)$$
$$\overrightarrow{AH} \cdot \overrightarrow{C'B'} = AH \cdot C'B' \cdot cox\sphericalangle\left(AH, C'B'\right) = AH \cdot C'A' \cdot cox\left(\sphericalangle AVC'\right)$$

We observe that

$\sphericalangle AUC \equiv \sphericalangle AVC'$ (angles with the sides respectively perpendicular).

The point B' is the symmetric of H in rapport to AC, consequently

$$\sphericalangle HAC \equiv \sphericalangle CAB',$$

also the point C' is the symmetric of the point H in rapport to AB, and therefore

$$\sphericalangle HAB \equiv \sphericalangle BAC'.$$

From these last two relations we find that

$$\sphericalangle B'AC' = 2\sphericalangle A.$$

The sinus theorem applied in the triangles $AB'C'$ and ABC gives:

$$B'C' = 2R \cdot \sin 2A$$
$$BC = 2R \sin A$$

We'll show that

$$AX \cdot BC = AH \cdot C'B',$$

and from here

$$AX \cdot 2R\sin A = AH \cdot 2R \cdot \sin 2A$$

which is equivalent to

$$AX = 2AH\cos A$$

We noticed that

$$\sphericalangle B'AC' = 2A,$$

Because

$$AX \perp B'C',$$

it results that

$$\sphericalangle TAB \equiv \sphericalangle A,$$

we noted $\{T\} = AX \cap B'C'$.

On the other side

$$AC' = AH, \ AT = \frac{1}{2}AY,$$

and

$$AT = AC'\cos A = AH\cos A,$$

therefore

$$\overrightarrow{XH} \cdot \overrightarrow{YZ} = 0.$$

Similarly, we prove that

$$YH \perp XZ,$$

and therefore H is the orthocenter of triangle XYZ.

Lemma 2

Let ABC a triangle inscribed in a circle, I the intersection of its bisector lines, and A', B', C' the intersections of the circumscribed circle with the bisectors AI, BI, CI respectively.

The orthocenter of the triangle $A'B'C'$ is I.

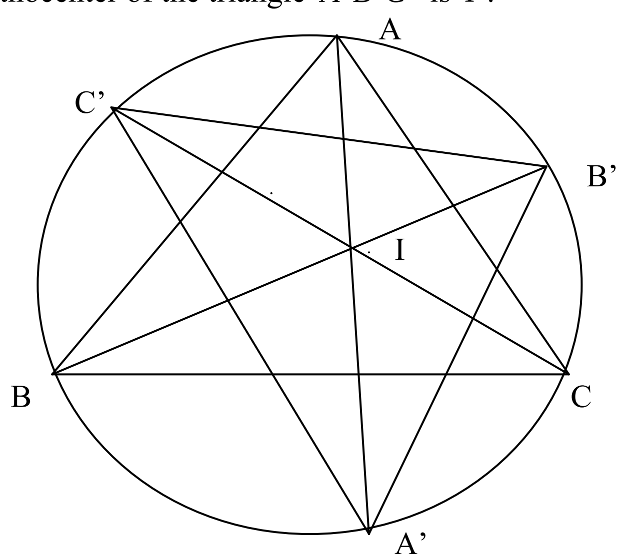

Fig. 2

Proof

We'll prove that $A'I \perp B'C'$.

Let

$$\alpha = m\left(\overset{\frown}{A'C}\right) = m\left(\overset{\frown}{A'B}\right),$$

$$\beta = m\left(\overset{\frown}{B'C}\right) = m\left(\overset{\frown}{B'A}\right)$$

$$\gamma = m\left(\overset{\frown}{C'A}\right) = m\left(\overset{\frown}{C'B}\right)$$

Then

$$m\sphericalangle\left(A'IC'\right) = \frac{1}{2}(\alpha+\beta+\gamma)$$

Because

$$2(\alpha+\beta+\gamma) = 360^\circ$$

it results

$$m\sphericalangle\left(A'IC'\right) = 90^\circ,$$

therefore

$$A'I \perp B'C'.$$

Similarly, we prove that

$$B'I \perp A'C',$$

and consequently the orthocenter of the triangle $A'B'C'$ is I, the center of the circumscribed circle of the triangle ABC.

Definition

Let ABC a triangle inscribed in a circle with the center in O and A', B', C' the middle of the arcs $\overset{\frown}{BC}, \overset{\frown}{CA}, \overset{\frown}{AB}$ respectively. The triangle XYZ formed by the symmetric of the points A', B', C' respectively in rapport to BC, CA, AB is called the Fuhrmann triangle of the triangle ABC.

Note

In 2002 the mathematician Milorad Stevanovic proved the following theorem:

Theorem (M. Stevanovic)

In an acute angle triangle the orthocenter of the Fuhrmann's triangle coincides with the center of the circle inscribed in the given triangle.

Proof

We note $A'B'C'$ the given triangle and let A, B, C respectively the middle of the arcs $\overset{\frown}{B'C'}, \overset{\frown}{C'A'}, \overset{\frown}{A'B'}$ (see Fig. 1). The lines AA', BB', CC' being bisectors in the triangle $A'B'C'$ are concurrent in the center of the circle inscribed in this triangle, which will note H, and which, in conformity with Lemma 2 is the orthocenter of the triangle ABC. Let XYZ the Fuhrmann triangle of the triangle $A'B'C'$, in conformity with Lemma 1, the orthocenter of XYZ coincides with H the orthocenter of ABC, therefore with the center of the inscribed circle in the given triangle $A'B'C'$.