Several Similarity Measures of Neutrosophic Sets

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Abstract- ***Smarandache (1995) defined the notion of neutrosophic sets, which is a generalization of Zadeh's fuzzy set and Atanassov's intuitionistic fuzzy set. In this paper, we first develop some similarity measures of neutrosophic sets. We will present a method to calculate the distance between neutrosophic sets (NS) on the basis of the Hausdorff distance. Then we will use this distance to generate a new similarity measure to calculate the degree of similarity between NS. Finally we will prove some properties of the proposed similarity measures.***

Keywords- **Neutrosophic Set, Matching Function, Hausdorff Distance, Similarity Measure.**

I-Introduction

Smarandache introduced a concept of neutrosophic set which has been a mathematical tool for handling problems involving imprecise, indeterminacy, and inconsistent data [1, 2].The concept of similarity is fundamentally important in almost every scientific field. Many methods have been proposed for measuring the degree of similarity between fuzzy sets (Chen, [11]; Chen et al., [12]; Hyung, Song, & Lee, [14]; Pappis&Karacapilidis, [10]; Wang, [13]...). But these methods are unsuitable for dealing with the similarity measures of neutrosophic set (NS). Few researchers have dealt with similarity measures for neutrosophic set ([3, 4]). Recently, Jun [3] discussed similarity measures on interval neutrosophic set (which an instance of NS) based on Hamming distance and Euclidean distance and showed how these measures may be used in decision making problems. Furthermore, A.A.Salama [4] defined the correlation coefficient, on the domain of neutrosophic sets, which is another kind of similarity measurement. In this paper we first extend the Hausdorff distance to neutrosophic set which plays an important role in practical application, especially in many visual tasks, computer assisted surgery and so on. After that a new series of similarity measures has been proposed for neutrosophic set using different approaches.

Similarity measures have extensive application in several areas such as pattern recognition, image processing, region extraction, psychology [5], handwriting recognition [6], decision making [7], coding theory etc.

This paper is organized as follows: Section2 briefly reviews the definition of Hausdorff distance and the neutrosophic set. Section 3 presents the new extended Hausdorff distance between neutrosophic sets. Section 4 provides the new series of similarity measure between neutrosophic sets, some of its properties are discussed. In section 5 a comparative study was done. Finally the section 6 outlines some conclusions.

II-Preliminaries

In this section we briefly review some definitions and examples which will be used in rest of the paper.

Definition 2.1: Hausdorff Distance

The Hausdorff distance (Nadler, 1978) is the maximum distance of a set to the nearest point in the other set. More formal description is given by the following

Given two finite sets $A = \{a_1, ..., a_p\}$ and $B = \{b_1, ..., b_q\}$, the Hausdorff distance H (A, B) is defined as:

$$H(A, B) = \max\{h(A, B), h(B, A)\}$$

where

$$H(A, B) = \max_{a\in A} \min_{b\in B} d(a, b)$$

a and b are elements of sets A and B respectively; d (a, b) is any metric between these elements.

The two distances h (A, B) and h (B, A) are called directed Hausdorff distances.

The function h (A, B) (the directed Hausdorff distance from A to B) ranks each element of A based on its distance to the nearest element of B, and then the largest ranked such element (the most mismatched element of A) specifies the value of the distance. Intuitively, if h(A, B) = c, then each element of A must be within distance c of some element of B, and there also is some element of A that is exactly distance c from the nearest element of B (the most mismatched element). In general h (A, B) and h (B, A) can attain very different values (the directed distances are not symmetric).

Let us consider the real space R, for any two intervals A= $[a_1,a_2]$ and B= $[b_1,b_2]$, the Hausdorff distance H(A,B) is given by

$$H(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$$

Definition 2.2 (see [2]). Let U be an universe of discourse then the neutrosophic set A is an object having the form A = {< x: $T_{A(x)}, I_{A(x)}, F_{A(x)}$ >,x ∈ U}, where the functions T, I, F : U→$]^-0,1^+[$ define respectively the degree of membership (or Truth) , the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element x ∈ U to the set A with the condition.

$$^-0 \le T_{A(x)} + I_{A(x)} + F_{A(x)} \le 3^+.$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$. So instead of $]^-0,1^+[$ we need to take the interval [0,1] for technical applications, because $]^-0,1^+[$will be difficult to apply in the real applications such as in scientific and engineering problems.

Definition 2.3 (see [2]). A neutrosophic set A is contained in another neutrosophic set B i.e. A ⊆ B if ∀x ∈ U, $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, $F_A(x) \ge F_B(x)$.

Definition 2.4 (see [2]). The complement of a neutrosophic set A is denoted by A^c and is defined as $T_{A^c(x)} = T_{A(x)}$, $I_{A^c(x)} = I_{A(x)}$, and $F_{A^c(x)} = F_{A(x)}$ for every x in X.

A complete study of the operations and application of neutrosophic set can be found in [1] [2].

In this paper we are concerned with neutrosophic sets whose T_A, I_A and F_A values are single points in [0, 1] instead of subintervals/subsets in [0, 1].

III. Extended Hausdorff Distance Between Two Neutrosophic Sets

Based on the Hausdorff metric, Eulalia Szmidt and Janusz Kacprzyk defined a new distance between intuitionistic fuzzy sets and/or interval-valued fuzzy sets in[8], taking into account three parameter representation (membership, non-membership values, and the hesitation margins) of A-IFSs which fulfill the properties of the Hausdorff distances. Their definition is defined by:

$$H_3(A,B) = \frac{1}{n}\sum_{i=1}^{n} max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, |\pi_A(x) - \pi_B(x)|\}$$

where A = {< x, $\mu_A(x)$, $\nu_A(x)$, $\pi_A(x)$ >} and B = {< x, $\mu_B(x)$, $\nu_B(x)$, $\pi_B(x)$>}.

The terms and symbols used in [8] are changed so that they are consistent with those in this section.

In this paper we are interested in extending the Hausdorff distance formulation in constructing a new distance for neutrosophic set due to its simplicity in the calculation.

Let X={$x_1,x_2, \ldots, x_n$} be a discrete finite set. Consider a neutrosophic set A in X, where $T_{A(xi)}$, $I_{A(xi)}$, $F_{A(xi)}$ ∈ [0, 1], for every x_i ∈ X, represent its membership, indeterminacy, and non-membership values respectively denoted by A = {< x, $T_{A(xi)}$, $I_{A(xi)}$, $F_{A(xi)}$ >}.

Then we propose a new distance between A ∈ NS and B ∈ NS defined by

$$d_H(A,B) = \frac{1}{n}\sum_{i=1}^{n} \max\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_B(x_i)|, |F_A(x_i) - F_B(x_i)|\}$$

Where $d_H(A,B) =$ H (A, B) denote the extended Hausdorff distance between two neutrosophic sets A and B.

Let A, B and C be three neutrosophic sets for all x_i ∈ X we have:

$d_H(A,B)$ = H (A, B) = max $\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_{B(x_i)}|, |F_A(x_i) - F_B(x_i)|\}$

The same between A and C are written as:

For all x_i ∈ X

H (A, C) = max $\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}$

and between B and C is written as:

For all $x_i \in X$

H (B , C) = max $\{|T_B(x_i) - T_C(x_i)|, |I_B(x_i) - I_C(x_i)|, |F_B(x_i) - F_C(x_i)|\}$

Proposition 3.1:

The above defined distance $d_H(A,B)$ between NS A and B satisfies the following properties (D1-D4):

(D1) $d_H(A,B) \geq 0$.

(D2) $d_H(A,B) = 0$ if and only if A = B; for all A, B ∈ NS.

(D3) $d_H(A,B) = d_H(B,A)$.

(D4) If A⊆B⊆C, C is an NS in X, then

$$d_H(A,C) \geq d_H(A,B)$$

And

$$d_H(A,C) \geq d_H(B,C)$$

Remark: Let A, B ∈ NS, A⊆ B if and only if , for all x_i in X

$$T_A(x_i) \leq T_B(x_i), I_A(x_i) \geq I_B(x_i), F_A(x_i) \geq F_B(x_i)$$

It is easy to see that the defined measure $d_H(A,B)$ satisfies the above properties (D1)-(D3). Therefore, we only prove (D4).

Proof of (D4) for the extended Hausdorff distance between two neutrosophic sets. Since

A⊆ B ⊆ C implies , for all x_i in X $T_A(x_i) \leq T_B(x_i) \leq T_C(x_i), I_A(x_i) \geq I_B(x_i) \geq I_C(x_i), F_A(x_i) \geq F_B(x_i) \geq F_C(x_i)$

We prove that $d_H(A,B) \leq d_H(A,C)$

α - If $|T_A(x_i) - T_C(x_i)| \geq |I_A(x_i) - I_C(x_i)| \geq |F_A(x_i) - F_C(x_i)|$

Then

H (A, C) = $|T_A(x_i) - T_C(x_i)|$ but we have

(i) For all x_i in X, $|I_A(x_i) - I_B(x_i)| \leq |I_A(x_i) - I_C(x_i)| \leq |T_A(x_i) - T_C(x_i)|$

And , for all x_i in X $|F_A(x_i) - F_B(x_i)| \leq |F_A(x_i) - F_C(x_i)| \leq |T_A(x_i) - T_C(x_i)|$

(ii) For all x_i in X, $|I_B(x_i) - I_C(x_i)| \leq |I_A(x_i) - I_C(x_i)| \leq |T_A(x_i) - T_C(x_i)|$

And ,for all x_i in X $|F_B(x_i) - F_C(x_i)| \leq |F_A(x_i) - F_C(x_i)| \leq |T_A(x_i) - T_C(x_i)|$

On the other hand we have, for all x_i in X

(iii) $|T_A(x_i) - T_B(x_i)| \leq |T_A(x_i) - T_C(x_i)|$
and $|T_B(x_i) - T_C(x_i)| \leq |T_A(x_i) - T_C(x_i)|$

Combining (i), (ii), and (iii) we obtain

Therefore, for all x_i in X

$$\frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_B(x_i)|, |F_A(x_i) - F_B(x_i)|\} \leq \frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}$$

And

$$\frac{1}{n}\sum_1^n \max\{|T_B(x_i) - T_C(x_i)|, |I_B(x_i) - I_C(x_i)|, |F_B(x_i) - F_C(x_i)|\} \leq \frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}$$

That is

$$d_H(A,B) \leq d_H(A,C) \text{ and } d_H(B,C) \leq d_H(A,C).$$

β - If $|\mathbf{T_A(x_i)} - \mathbf{T_C(x_i)}| \leq |\mathbf{F_A(x_i)} - \mathbf{F_C(x_i)}| \leq |\mathbf{I_A(x_i)} - \mathbf{I_C(x_i)}|$

Then

H (A, C) = $|I_A(x_i) - I_C(x_i)|$ but we have for all x_i in X

(a) $|T_A(x_i) - T_B(x_i)| \leq |T_A(x_i) - T_C(x_i)| \leq |I_A(x_i) - I_C(x_i)|$

And $|F_A(x_i) - F_B(x_i)| \leq |F_A(x_i) - F_C(x_i)| \leq |I_A(x_i) - I_C(x_i)|$

(b) $|T_B(x_i) - T_C(x_i)| \leq |T_A(x_i) - T_C(x_i)| \leq |I_A(x_i) - I_C(x_i)|$

And $|F_B(x_i) - F_C(x_i)| \leq |F_A(x_i) - F_C(x_i)| \leq |I_A(x_i) - I_C(x_i)|$

On the other hand we have for all $x_i \in X$:

(c) $|I_A(x_i) - I_B(x_i)| \leq |I_A(x_i) - I_C(x_i)|$ and $|I_B(x_i) - I_C(x_i)| \leq |I_A(x_i) - I_C(x_i)|$

Combining (a) and (c) we obtain:

Therefore, for all x_i in X

$$\frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_B(x_i)|, |F_A(x_i) - F_B(x_i)|\} \leq \frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}$$

And

$$\frac{1}{n}\sum_1^n \max\{|T_B(x_i) - T_C(x_i)|, |I_B(x_i) - I_C(x_i)|, |F_B(x_i) - F_C(x_i)|\} \leq \frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}$$

That is

$d_H(A,B) \leq d_H(A,C)$ and $d_H(B,C) \leq d_H(A,C)$

$\boldsymbol{\gamma}$ - If $|T_A(x_i) - T_C(x_i)| \leq |I_A(x_i) - I_C(x_i)| \leq |F_A(x_i) - F_C(x_i)|$

Then

H (A, C) = $|F_A(x_i) - F_C(x_i)|$ but we have for all x_i in X

(a) $|T_A(x_i) - T_B(x_i)| \leq |T_A(x_i) - T_C(x_i)| \leq |F_A(x_i) - F_C(x_i)|$

and $|I_A(x_i) - I_B(x_i)| \leq |I_A(x_i) - I_C(x_i)| \leq |F_A(x_i) - F_C(x_i)|$

(b) for all x_i in X $|T_B(x_i) - T_C(x_i)| \leq |T_A(x_i) - T_C(x_i)| \leq |F_A(x_i) - F_C(x_i)|$

and for all x_i in X $|I_B(x_i) - I_C(x_i)| \leq |I_A(x_i) - I_C(x_i)| \leq |F_A(x_i) - F_C(x_i)|$

On the other hand we have for all x_i in X

(c) $|F_A(x_i) - F_B(x_i)| \leq |F_A(x_i) - F_C(x_i)|$ and

$|F_B(x_i) - F_C(x_i)| \leq |F_A(x_i) - F_C(x_i)|$

Combining (a), (b), and (c) we obtain

Therefore, for all x_i in X

$$\frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_B(x_i)|, |F_A(x_i) - F_B(x_i)|\} \leq \frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}.$$

And

$$\frac{1}{n}\sum_1^n \max\{|T_B(x_i) - T_C(x_i)|, |I_B(x_i) - I_C(x_i)|, |F_B(x_i) - F_C(x_i)|\} \leq \frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}$$

That is

$d_H(A,B) \leq d_H(A,C)$ and $d_H(B,C) \leq d_H(A,C)$.

From α, β , and γ, we can obtain the property (D4).

3.2 Weighted Extended Hausdorff Distance Between Two Neutrosophic Sets.

In many situations the weight of the element $x_i \in X$ should be taken into account. Usually the elements have different importance. We need to consider the weight of the element so that we have the following weighted distance between NS. Assume that the weight of $x_i \in X$ is w_i where $X=\{x_1, x_2,.., x_n\}$, $w_i \in [0,1]$, $i=\{1,2,3,.., n\}$ and $\sum_1^n w_i = 1$. Then the weighted extended Hausdorff distance between NS A and B is defined as:

$$d_{Hw}(A,B) = \sum_1^n w_i \, d_H(A(x_i), B(x_i))$$

It is easy to check that $d_{Hw}(A,B)$ satisfies the four properties D1-D4 defined above.

IV. SOME NEW SIMILARITY MEASURES FOR NEUTROSOPHIC SETS

The distance measure between two NS is used in finding the similarity between neutrosophic sets.

We found in the literature different similarity measures, and we extend them to neutrosophic sets (NS), several of them defined below:

Liu [9] also gave an axiom definition for the similarity measure of fuzzy sets, which also can be expressed for neutrosophic sets (NS) as follow:

Definition 4.1*: *Axioms of a Similarity Measure

A mapping $S:NS(X)\times NS(X)\rightarrow[0,1]$, NS(X) denotes the set of all NS in $X=\{x_1,x_2,\ldots,x_n\}$, S(A, B) is said to be the degree of similarity between A∈ NS and B ∈ NS, if S(A,B) satisfies the properties of conditions (P1-P4):

(**P1**) S (A, B) = S (B, A).
(**P2**) S(A,B) = (1,0,0) = $\underline{1}$ .If A = B for all A,B ∈ NS.
(**P3**) $S_T(A,B) \geq 0$, $S_I(A,B) \geq 0$, $S_F(A,B) \geq 0$.
(**P4**) If A B C for all A, B, C ∈ NS, then S (A, B) ≥S (A, C) and S (B, C) ≥ S (A, C).

Numerical Example:

Let $A \leq B \leq C$. with $T_A \leq T_B \leq T_C$ and $I_A \geq I_B \geq I_C$ and $F_A \geq F_B \geq F_C$ for each $x_i \in$ NS.

For example:
A= { x_1 (0.2, 0.5, 0.6); x_2 (0.2, 0.4, 0.4) }
B= { x_1 (0.2, 0.4, 0.4); x_2 (0.4, 0.2, 0.3) }
C= { x_1 (0.3, 0.3, 0.4); x_2 (0.5, 0.0, 0.3) }
In the following we define a new similarity measure of neutrosophic set and discuss its properties.

4.2 Similarity Measures Based on the Set –Theoretic Approach.

In this section we extend the similarity measure for intuitionistic and fuzzy set defined by Hung and Yung [16] to neutrosophic set which is based on set-theoretic approach as follow.

Definition 4.2: Let A,B be two neutrosophic sets in $X=\{x_1,x_2,.., x_n\}$, if $A = \{< x, T_A(x_i), I_A(x_i), F_A(x_i) >\}$ and $B= \{< x, T_B(x_i), I_B(x_i), F_B(x_i) >\}$ are neutrosophic values of X in A and B respectively, then the similarity measure between the neutrosophic sets A and B can be evaluated by the function

For all x_i in X

$$S_T(A,B) = \left(\sum_1^N \left[\frac{min(T_A(x_i),T_B(x_i))}{Max(T_A(x_i),T_B(x_i))}\right]\right)/n$$

$$S_I(A,B) = 1 - \left(\sum_1^N \left[\frac{min(I_A(x_i),I_B(x_i))}{Max(I_A(x_i),I_B(x_i))}\right]\right)/n$$

$$S_F(A,B) = 1 - \left(\sum_1^N \left[\frac{min(F_A(x_i),F_B(x_i))}{Max(F_A(x_i),F_B(x_i))}\right]\right)/n$$

and $S(A,B) = (S_T(A,B), S_I(A,B), S_F(A,B))$ **eq. (1)**

where

$S_T(A,B)$ denote the degree of similarity (where we take only the T's).
$S_I(A,B)$ denote the degree of indeterminate similarity (where we take only the I's).
$S_F(A,B)$ denote degree of nonsimilarity (where we take only the F's).
Min denotes the minimum between each element of A and B.
Max denotes the minimum between each element of A and B.

Proof of (P4) for the eq. (1).

Since A⊆B⊆C implies, for all x_i in X

$T_A(x_i) \leq T_B(x_i) \leq T_C(x_i), I_A(x_i) \geq I_B(x_i) \geq I_C(x_i), F_A(x_i) \geq F_B(x_i) \geq F_C(x_i)$

Then, for all x_i in X

$$\frac{min(T_A(x_i), T_B(x_i))}{Max(T_A(x_i), T_B(x_i))} = \frac{T_A(x_i)}{T_B(x_i)}$$
$$\frac{min(T_A(x_i), T_C(x_i))}{Max(T_A(x_i), T_C(x_i))} = \frac{T_A(x_i)}{T_C(x_i)}$$
$$\frac{min(T_B(x_i), T_C(x_i))}{Max(T_B(x_i), T_C(x_i))} = \frac{T_B(x_i)}{T_C(x_i)}$$

Therefore, for all x_i in X

$$\frac{T_A(x_i)}{T_C(x_i)} = \frac{T_B(x_i)}{T_C(x_i)} + \frac{T_A(x_i) - T_B(x_i)}{T_C(x_i)} \leq \frac{T_B(x_i)}{T_C(x_i)} \quad (1)$$

(since $T_A(x_i) \leq T_B(x_i)$)

Furthermore, for all x_i in X

$$\frac{\boldsymbol{min(T_A(x_i), T_B(x_i))}}{\boldsymbol{Max(T_A(x_i), T_B(x_i))}} \geq \frac{\boldsymbol{min(T_A(x_i), T_C(x_i))}}{\boldsymbol{Max(T_A(x_i), T_C(x_i))}} \quad (2)$$

Or

$$\frac{\boldsymbol{T_A(x_i)}}{\boldsymbol{T_B(x_i)}} \geq \frac{\boldsymbol{T_A(x_i)}}{\boldsymbol{T_C(x_i)}} \text{ or } T_B(x_i) \leq T_C(x_i)$$

(since $T_C(x_i) \geq T_B(x_i)$)

Inequality (2) implies that, for all x_i in X

$$\frac{\boldsymbol{T_A(x_i)}}{\boldsymbol{T_C(x_i)}} \leq \frac{\boldsymbol{T_A(x_i)}}{\boldsymbol{T_B(x_i)}} \quad (3)$$

From the inequalities (1) and (3), the property (P4) for $S_T(A,B) \geq S_T(A,C)$ is proven.

In a similar way we can prove that $S_I(A,B)$ and $S_F(A,B)$.

We will to prove that $S_I(A,C) \geq S_I(A,B)$. For all $x_i \in$ X we have:

$S_I(A,C) = 1 - \frac{min(I_A(x_i), I_C(x_i))}{Max(I_A(x_i), I_C(x_i))} = 1 - \frac{I_C(x_i)}{I_A(x_i)} \geq 1 - \frac{I_B(x_i)}{I_A(x_i)}$

Since $I_C(x_i) \leq I_B(x_i)$

Similarly we prove $S_F(A,C) \geq S_F(A,B)$ for all x_i in X

$\boldsymbol{S_F(A,C) = 1 - \frac{min(F_A(x_i), F_C(x_i))}{Max(F_A(x_i), F_C(x_i))} = 1 - \frac{F_C(x_i)}{F_A(x_i)} \geq 1 - \frac{F_B(x_i)}{F_A(x_i)}}$

Since $F_C(x_i) \leq F_B(x_i)$

Then $S(A,C) \leq S(A,B)$ where $S(A,C) = (S_T(A,C), S_I(A,C), S_F(A,C))$ and
$S(A,B) = (S_T(A,B), S_I(A,B), S_F(A,B))$.

In a similar way we can prove that S (B, C) ≥ S (A, C). If A⊆B⊆C therefore S (A, B) satisfies (P4) of definition 4.1.
By applying eq. (1), the degree of similarity between the neutrosophic sets (A, B), (A, C) and (B, C) are:

S(A, B) =($S_T(A,B)$, $S_I(A,B)$, $S_F(A,B)$)= (0.75, 0.35, 0.30)
S (A, C) = ($S_T(A,C)$, $S_I(A,C)$, $S_F(A,C)$)= (0.53, 0.7, 0.30)
S (B, C) = $S_T(B,C)$, $S_I(B,C)$, $S_F(B,C)$)= (0.73, 0.63, 0)
Then eq. (1) satisfies property P4: S(A, C) ≤ S(A, B) and S(A, C) ≤ S(B, C).

Usually, the weight of the element $x_i \in$ X should be taken into account, then we present the following weighted similarity between NS. Assume that the weight of $x_i \in X=\{1,2,\ldots,n\}$ is w_i (i=1,2,…, n) when $w_i \in [0,1]$, $\sum_1^n w_i = 1$.

Denote $S_w^T(A,B) = (\sum_1^N w_i \left[\frac{min(T_A(x_i), T_B(x_i))}{Max(T_A(x_i), T_B(x_i))}\right])/n$

$$S_w^I(A,B) = 1 - (\sum_1^N w_i \left[\frac{min(I_A(x_i), I_B(x_i))}{Max(I_A(x_i), I_B(x_i))}\right])/n$$

$$S_w^F(A,B) = 1 - (\sum_1^N w_i \left[\frac{min(F_A(x_i), F_B(x_i))}{Max(F_A(x_i), F_B(x_i))}\right])/n$$

and $S_w(A,B) = (S_w^T((A,B), S_w^I((A,B), S_w^F((A,B))$

It is easy to check that $S_w(A,B)$ satisfies the four properties P1-P4 defined above.

4.3 Similarity Measure Based on the Type1 Geometric Distance Model

In the following, we express the definition of similarity measure between fuzzy sets based on the model of geometric distance proposed by Pappis and Karacapilidis in [10] to similarity of neutrosophic set.

Definition 4.3 : Let A,B be two neutrosophic sets in X={x_1, x_2,..., x_n}, if A = {< x, $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i)$ >} and B= {< x, $T_B(x_i)$, $I_B(x_i)$, $F_B(x_i)$ >} are neutrosophic values of X in A and B respectively, then the similarity measure between the neutrosophic sets A and B can be evaluated by the function

For all x_i in X

$$L_T(A,B) = 1 - \frac{\sum_1^n |T_A(x_i) - T_B(x_i)|}{\sum_1^n (T_A(x_i) + T_B(x_i))}$$
$$L_I(A,B) = \frac{\sum_1^n |I_A(x_i) - I_B(x_i)|}{\sum_1^n (I_A(x_i) + I_B(x_i))}$$
$$L_F(A,B) = \frac{\sum_1^n |F_A(x_i) - F_B(x_i)|}{\sum_1^n (F_A(x_i) + F_B(x_i))} \quad \text{and}$$
$$L(A,B) = (L_T(A,B),\ L_I(A,B),\ L_F(A,B)) \quad \text{eq. (2)}$$

We will prove this similarity measure satisfies the properties 1-4 as above. The property (P1) for the similarity measure eq. (2) is obtained directly from the definition 4.1.

Proof: obviously, eq. (2) satisfies P1-P3-P4 of definition 4.1. In the following L (A, B) will be proved to satisfy (P2) and (P4).
Proof of (P2) for the eq. 2

For all x_i in X

First of all, $L_T(A,B) = 1 \leftrightarrow \frac{\sum_1^n |T_A(x_i) - T_B(x_i)|}{\sum_1^n (T_A(x_i) + T_B(x_i))} = 0$

$$\leftrightarrow |T_A(x_i) - T_B(x_i)| = 0$$
$$\leftrightarrow T_A(x_i) = T_B(x_i)$$

$L_I(A,B) = 0 \leftrightarrow \frac{\sum_1^n |I_A(x_i) - I_B(x_i)|}{\sum_1^n (I_A(x_i) + I_B(x_i))} = 0$

$\leftrightarrow |I_A(x_i) - I_B(x_i)| = 0 \ \leftrightarrow I_A(x_i) = I_B(x_i)$

$L_F(A,B) = 0 \leftrightarrow \frac{\sum_1^n |F_A(x_i) - F_B(x_i)|}{\sum_1^n (F_A(x_i) + F_B(x_i))} = 0$

$\leftrightarrow |F_A(x_i) - F_B(x_i)| = 0 \leftrightarrow F_A(x_i) = F_B(x_i)$

Then $\boldsymbol{L}(A,B) = (L_T(A,B),\ L_I(A,B),\ L_F(A,B))$ = (1, 0, 0) if A=B for all A, B ∈ NS.

Proof of P3 for the eq .(2) is obvious.

By applying eq.2 the degree of similarity between the neutrosophic sets (A, B), (A, C) and (B, C) are:

L (A, B) =$(L_T(A,B), L_I(A,B), L_F(A,B))$= (0.8, 0.2, 0.17).
L (A, C) = $(L_T(A,C), L_I(A,C), L_F(A,C))$= (0.67, 0.5, 0.17).
L (B, C) = $(L_T(B,C), L_I(B,C), L_F(B,C))$= (0.85, 0.33, 0).

The result indicates that the degree of similarity between neutrosophic sets A and B ∈ [0, 1]. Then Eq.(2)satisfies property P4**:** L(A, C) ≤ L(A, B) and L(A, C) ≤ L(B, C).

4.4 Similarity Measure Based on the Type 2 Geometric Distance model

In this section we extend the similarity measure proposed by Yang and Hang [16] to neutrosophic set as follow:

***Definition 4.4*:** Let A, B be two neutrosophic set in X={x_1,x_2,.., x_n}, if A = {< x, $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i)$ >} and B= {<x, $T_B(x_i)$, $I_B(x_i)$, $F_B(x_i)$ >} are neutrosophic values of X in A and B respectively, then the similarity measure between the neutrosophic set A and B can be evaluated by the function:

For all x_i in X

M_T (A, B)= $\frac{1}{n}\sum_1^n (1 - \frac{|T_A(x_i) - T_B(x_i)|}{2})$.
M_I (A, B)= $\frac{1}{n}\sum_1^n (\frac{|I_A(x_i) - I_B(x_i)|}{2})$.
M_F (A, B)= $\frac{1}{n}\sum_1^n (\frac{|F_A(x_i) - F_B(x_i)|}{2})$.
And $M_{T,I,F} = (M_T(A,B), M_I(A,B), M_F(A,B))$ for all i={x_1,x_2 ,.., x_n} eq. (3)

The proofs of the properties P1-P2-P3 in definition 4.1 (Axioms of a Similarity Measure) of the similarity measure in definition 4.4 are obvious.

Proof of (P4) for the eq. (3).

Since for all x_i in X

$T_A(x_i) \leq T_B(x_i) \leq T_C(x_i), I_A(x_i) \geq I_B(x_i) \geq I_C(x_i), F_A(x_i) \geq F_B(x_i) \geq F_C(x_i)$ Then for all x_i in X

$$1-\frac{|T_C(x_i)-T_A(x_i)|}{2}=1-\frac{(T_C(x_i)-T_A(x_i))}{2}$$

$$=1-\left(\frac{(T_C(x_i)-T_B(x_i))}{2}+\frac{(T_B(x_i)-T_A(x_i))}{2}\right)$$

$$\leq 1-\left(\frac{(T_C(x_i)-T_B(x_i))}{2}\right)$$

$$=1-\frac{|T_C(x_i)-T_B(x_i)|}{2}$$

Then M_T(A, C) $\leq$ M_T(B, C).
Similarly, M_T(A, C) $\leq$ M_T(A, B) can be proved easily.
For M_I(A, C) $\geq$ M_I(B, C) and M_F(A, C) $\geq$ M_F(B, C) the proof is easy.
Then by the definition 4.4, (P4) for definition 4.1, is satisfied as well.
By applying eq. (3), the degree of similarity between the neutrosophic sets (A, B), (A, C) and (B, C) are:

M(A,B)=(M_T (A,B), M_I (A,B), M_F (A,B))=(0.95 , 0.075 , 0.075)
M(A,C)= (M_T (A,C), M_I (A,C), M_F (A,C))=(0.9, 0.15 , 0.075)
M(B,C)= (M_T (B,C), M_I (B,C), M_F (B,C))=(0.9, 0.075 , 0)

Then eq. (3) satisfies property P4**:**
M (A, C) $\leq$ M (A, B) and M (A, C) $\leq$ M (B, C).

Another way of calculating similarity (degree) of neutrosophic sets is based on their distance. There are more approaches on how the relation between the two notions in form of a function can be expressed. Two of them are presented below (in section 4.5 and 4.6).

4.5 Similarity Measure Based on the Type3 Geometric Distance Model.

In the following we extended the similarity measure proposed by Koczy in [15] to neutrosophic set (NS).

Definition 4.5: Let A, B be two neutrosophic sets in X={x_1,x_2,.., x_n}, if A = {< x, $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i)$ >} and B= {< x, $T_B(x_i)$, $I_B(x_i)$, $F_B(x_i)$ >} are neutrosophic values of x in A and B respectively, then the similarity measure between the neutrosophic sets A and B can be evaluated by the function

$\mathbf{H_T(A,B)}=\frac{\mathbf{1}}{\mathbf{1+d_\infty^T(A,B)}}$ denotes the degree of similarity.

$\mathbf{H_I(A,B)}=\mathbf{1}-\frac{\mathbf{1}}{\mathbf{1+d_\infty^I(A,B)}}$ denotes the degree of indeterminate similarity.

$\mathbf{H_F(A,B)}=\mathbf{1}-\frac{\mathbf{1}}{\mathbf{1+d_\infty^F(A,B)}}$ denotes degree of non-similarity

where $d_\infty^T(A,B)$, $d_\infty^I(A,B)$, and $d_\infty^F(A,B)$ are the distance measure of two neutrosophic sets **A** and B.

For all x_i in X

$d_\infty^T(A,B)=\max\{|T_A(x_i)-T_B(x_i)|\}$.

$d_\infty^I(A,B)=\max\{|I_A(x_i)-I_B(x_i)|\}$.

$d_\infty^F(A,B)=\max\{|F_A(x_i)-F_B(x_i)|\}$.

and H (A, B) = (H_T(A, B), H_I(A, B), H_F(A, B)). Eq. (4)

By applying the Eq. (4) in numerical example we obtain:

$d_\infty(A,B)$= (0.2, 0.2, 0.2), then H (A, B) = (0.83, 0.17, 0.17).

$d_\infty(A,C)$= (0.3, 0.4, 0.1), then H (A, C) = (0.76, 0.29, 0.17).

$d_\infty(B,C)$= (0.1, 0.2, 0), then H (B, C) = (0.90, 0.17, 0).

It can be verified that H (A, B) also has the properties (P1)-(P4).

4.6 Similarity Measure Based on Extended Hausdorff Distance

It is well known that similarity measures can be generated from distance measures. Therefore, we may use the proposed distance measure based on extended Hausdorff distance to define similarity measures. Based on the relationship of similarity measures and distance measures, we can define a new similarity measure between NS A and B as follows:

$$N(A,B) = 1 - d_H(A,B) \quad \text{eq. (5)}$$

where $d_H(A,B)$ represent the extended Hausdorff distance between neutrosophic sets (NS) A and B.

According to the above distance properties (D1-D4).It is easy to check that the similarity measure eq. (5) satisfies the four properties of axiom similarity defined in 4.1

By applying the eq. (5) in numerical example we obtain:

$N(A,B)$ =0.8
$N(A,C)$ =0.7
$N(B,C)$ =0.85
Then eq. (5) satisfies property P4**:**
N(A, C) ≤ N(A, B) and N(A, C) ≤ N(B, C)

Remark: It is clear that the larger the value of N(A, B), the more the similarity between NS A and B.
Next we define similarity measure between NS A and B using a matching function.

4.7 Similarity Measure of two Neutrosophic Sets Based on Matching Function.

Chen [11] and Chen et al. [12] introduced a matching function to calculate the degree of similarity between fuzzy sets. In the following, we extend the matching function extend to deal with the similarity measure of NS.

Definition 4.7 Let F and E be two neutrosophic sets over U. Then the similarity between them, denoted by K (F, G) or $K_{F,G}$ has been defined based on the matching function as:

For all x_i in X

$$K(F,G) = K_{F,G} = \frac{\sum_1^n (T_F(x_i)\cdot T_G(x_i) + I_F(x_i)\cdot I_G(x_i) + F_F(x_i)\cdot F_G(x_i))}{\max\left(\sum_1^n ((T_F(x_i))^2 + (I_F(x_i))^2 + (F_F(x_i))^2), \sum_1^n ((T_G(x_i))^2 + (I_G(x_i))^2 + (F_G(x_i))^2)\right)}$$

Eq. (6)

Considering the weight $w_j \in [0, 1]$ of each element $x_i \in X$, we get the weighting similarity measure between NS as:

For all x_i in X

$$K_w(F,G) = \frac{\sum_1^n w_i (T_F(x_i)\cdot T_G(x_i) + I_F(x_i)\cdot I_G(x_i) + F_F(x_i)\cdot F_G(x_i))}{\max\left(\sum_1^n w_i ((T_F(x_i))^2 + (I_F(x_i))^2 + (F_F(x_i))^2), \sum_1^n w_i ((T_G(x_i))^2 + (I_G(x_i))^2 + (F_G(x_i))^2)\right)}$$

Eq. (7)

If each element $x_i \in X$ has the same importance, then Eq.(7) is reduced to eq. (6). The larger the value of $K(F,G)$ the more the similarity between F and G. Here $K(F,G)$ has all the properties described as listed in the definition 4.1.

By applying the eq. (6) in numerical example we obtain:

$K(A,B) = 0.75$, $K(A,C) = 0.66$, and $K(B,C) = 0.92$

Then Eq. (6) satisfies property P4: K(A, C) ≤ K(A, B) and K(A, C) ≤ K(B, C)

V. Comparison of various similarity measures

In this section, we make a comparison among similarity measures proposed in the paper. Table I show the comparison of various similarity measures between two neutrosophic sets respectively.

Table I . Example results obtained from the similarity measures between neutrosophic sets A , B and C.

	A, B	A, C	B, C
Eq. (1)	(0.75, 0.35, 0.3)	(0.53, 0.7, 0.3)	(0.73, 0.63, 0)
Eq. (2)	(0.8, 0.2,0.17)	(0.67, 0.5, 0.17)	(0.85, 0.33, 0)
Eq. (3)	(0.95, 0.075, 0.075)	(0.9, 0.15, 0.075)	(0.9, 0.075, 0)
Eq. (4)	(0.83, 0.17, 0.17)	(0.76, 0.29, 0.17)	(0.9, 0.17, 0)
Eq. (5)	0.8	0.7	0.85
Eq. (6)	0.75	0.66	0.92

Each similarity measure expression has its own measuring, they all evaluate the similarities in neutrosophic sets, and they can meet all or most of the properties of similarity measure.

In definition 4.1, that is P1-P4. It seem from the table above that from the results of similarity measures between neutrosophic sets can be classified in two type of similarity measures: the first type which we called “crisp similarity measure” is illustrated by similarity measures (N and K) and the second type called “neutrosophic similarity measures” illustrated by similarity measures (S, L, M and H). The computation of measure **H , N** and **S** are much simpler than that of **L, M** and **K**

CONCLUSIONS

In this paper we have presented a new distance called "extended Hausdorff distance for neutrosophic sets" or "neutrosophic Hausdorff distance", then we defined a new series of similarity measures to calculate the similarity between neutrosophic sets. It’s hoped that our findings will help enhancing this study on neutrosophic set for researchers.

ACKNOWLEDGMENT

The authors are thankful to the anonymous referee for his valuable and constructive remarks that helped to improve the clarity and the completeness of this paper.

REFERENCES

[1] Smarandache, F. (1998). A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic. Rehoboth: American Research Press

[2] F. Smarandache, Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, Inter. J.Pure Appl. Math. 24 (2005) 287–297.

[3] Jun Ye .Similarity measures between interval neutrosophic sets and their multicriteria decision-making method, SOCO-D-11-00309,soft computing, pp 1-15

[4] A.A .Salama and S.A. AL- Blowi(2012) ; Correlation of Neutrosophic Data ,International Refereed Journal of Engineering and Science (IRJES) ISSN (Online) 2319-183X, (Print) 2319-1821 Volume 1, Issue 2 (October 2012), PP.39-43.

[5] R.M. Nosofsky, “Choice, Similarity, and the Context Theory of Classification”, Jr. of Exp. Psychology: Learning, Memory, and Cognition, Vol. 10 (1984),pp. 104-114

[6] W. Y. Leng and S.M. Shamsuddin, “Writer Identification for chinese handwriting”, Int. J. Advance. Soft Comput. Appl., Vol.2, No.2, (2010), pp.142-173

[7] J. Williams, N. Steele: Difference, distance and similarity as a basis for fuzzy decision support based on prototypical decision classes, Fuzzy Sets and Systems 131 (2002) 35-46.

[8] Eulalia Szmidt and Janusz Kacprzyk ,A note on the Hausdorff distance between Atanassov’s intuitionistic fuzzy sets, NIFS Vol. 15 (2009), No. 1, 1-12.

[9] Liu Xue chang, Entropy, distance measure and similarity measure of Fuzzy sets and their relations, Fuzzy Sets and Systems 52 (1992) 305-318.

[10] C.P. Pappis, N.I. Karacapilidis, A comparative assessment of measures of similarity of fuzzy values. Fuzzy Sets and Systems, 56:171174, 1993.

[11] Chen, S. M. (1988). A new approach to handling fuzzy decision making problems. IEEE Transactions on Systems, Man,and Cybernetics-18, 1012–1016

[12] Chen,S.M.,Yeh,S.M.,&Hsiao,P.H.(1995).A comparison of similarity measures of fuzzy values.Fuzzy Sets and Systems, 72,79–89.

[13] Wang, W. J. (1997). New similarity measures on fuzzy sets and elements. Fuzzy Sets and Systems, 85,305–309.

[14] Hyung, L. K., Song, Y. S., & Lee, K. M. (1994). Similarity measure between fuzzy sets and between elements. Fuzzy Sets and Systems, 62, 291–293

[15] Kóczy T. László, Tikk Domonkos: Fuzzy rendszerek, Typotex, 2000.

[16] Hung W.L and Yang M.S. (2008) On similarity measures between intuitionistic fuzzy sets. International Journal of Intelligent Systems 23 (3): 364–383

Published in *Neutrosophic Sets and Systems*, 54-62, Vol. 1, 2013.