Back and Forth Factorials

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Let n>k≥1 be two integers. Then the Smarandacheial is defined as:

$$!n!_k = \prod_{\substack{0<|n-k\cdot i|\leq n \\ i\in N}} (n-k\cdot i)$$

For examples:

1) In the case k=1:

$$!n!_1 \stackrel{conv}{\equiv} !n! = \prod_{\substack{0<|n-i|\leq n \\ i=0,1,2,\ldots}} (n-i) = n(n-1)(n-2)\ldots(2)(1)(-1)(-2)\ldots(-n+2)(-n+1)(-n) = (-1)^n(n!)^2.$$

Thus !5! = 5(5-1)(5-2)(5-3)(5-4)(5-6)(5-7)(5-8)(5-9)(5-10)=5·4·3·2·1·(-1)·(-2)·(-3)·(-4)·(-5) = -14400.

**The sequence is: 4, -36, 576, -14400, 518400, -25401600, 1625702400, -131681894400, 13168189440000,
-1593350922240000, 229442532802560000, -38775788043632640000, 7600054456551997440000, -1710012252724199424000000, … .**

2) In the case k=2:
a) If n is odd, then

$$!n!_2 = \prod_{\substack{0<|n-2i|\leq n \\ i=0,1,2,\ldots}} (n-2i) = n(n-2)(n-4)\ldots(3)(1)(-1)(-3)\ldots(-n+4)(-n+2)(-n) = (-1)^{(n+1)/2}(n!!)^2.$$

a) If n is even, then

$$!n!_2 = \prod_{\substack{0<|n-2i|\leq n \\ i=0,1,2,\ldots}} (n-2i) = n(n-2)(n-4)\ldots(4)(2)(-2)(-4)\ldots(-n+4)(-n+2)(-n) = (-1)^{n/2}(n!!)^2.$$

Thus: $!3!_2$ = 3(3-2)(3-4)(3-6) = 9 and $!4!_2$ = 4(4-2)(4-6)(4-8) = 64.

The sequence is: 9, 64, -225, -2304, 11025, 147456, -893025, -14745600, 108056025, 2123366400, … .

3) In the case k=3:

$$!n!_3 = \prod_{\substack{0<|n-3i|\leq n \\ i=0,1,2,\ldots}} (n-3i) = n(n-3)(n-6)\ldots .$$

Thus $!7!_3$ = 7(7-3)(7-6)(7-9)(7-12) = 7(4)(1)(-2)(-5) = 280.

The sequence is: -8, 40, 324, 280, -2240, -26244, -22400, 246400, 3779136, 3203200, -44844800, … .

4) In the case k=4:

$$!n!_4 = \prod_{\substack{0<|n-4i|\leq n \\ i=0,\,1,\,2,\,\dots}} (n-4i) = n(n-4)(n-8)\dots\,.$$

Thus $!9!_4 = 9(9-4)(9-8)(9-12)(9-16) = 9(5)(1)(-3)(-7) = 945$.

The sequence is: -15, 144, 105, 1024, 945, -14400, -10395, -147456, -135135, 2822400, 2027025, … .

5) In the case k=5:

$$!n!_5 = \prod_{\substack{0<|n-5i|\leq n \\ i=0,\,1,\,2,\,\dots}} (n-5i) = n(n-5)(n-10)\dots\,.$$

Thus $!11!_5 = 11(11-5)(11-10)(11-15)(11-20) = 11(6)(1)(-4)(-9) = 2376$.

The sequence is: -24, -42, 336, 216, 2500, 2376, 4032, -52416, -33264, -562500, -532224, -891072, 16039296, … .

More general:
Let $n>k\geq 1$ be two integers and $m\geq 1$ another integer. Then the generalized Smarandacheial is defined as:

$$!n!m_k = \prod_{\substack{0<|n-k\cdot i|[m \\ i\ c N}} (n-k\cdot i)$$

For examples:
$!7!3_2 = 7(7-2)(7-4)(7-6)(7-8)(7-10) = 7(5)(3)(1)(-1)(-3) = 315$.
$!7!9_2 = 7(7-2)(7-4)(7-6)(7-8)(7-10)(7-12)(7-14)(7-16) = 7(5)(3)(1)(-1)(-3)(-5)(-7)(-9)$
$= -99225$.

References:

J. Dezert, editor, "Smarandacheials", Mathematics Magazine, Aurora, Canada, No. 4/2004; http://www.mathematicsmagazine.com/corresp/J_Dezert/JDezert.htm, and www.gallup.unm.edu/~smarandache/Smarandacheials.htm.
F. Smarandache, "Back and Forth Factorials", Arizona State Univ., Special Collections, 1972.

[These Back and Forth Factorials have been called Smarandacheials.]