

ON SOLVING DIOPHANTINE QUADRATIC EQUATIONS WITH TWO UNKNOWN IN \mathbf{Z} AND \mathbf{N}

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Abstract: In this short note we study the existence and number of solutions in the set of integers (\mathbf{Z}) and in the set of natural numbers (\mathbf{N}) of Diophantine equations of second degree with two unknowns of the general form $ax^2 - by^2 = c$.

Property 1: The equation $x^2 - y^2 = c$ admits integer solutions if and only if c belongs to $4\mathbf{Z}$ or is odd.

Proof: The equation $(x - y)(x + y) = c$ admits solutions in \mathbf{Z} iff there exist c_1 and c_2 in \mathbf{Z} such that $x - y = c_1$, $x + y = c_2$, and $c_1c_2 = c$. Therefore $x = \frac{c_1 + c_2}{2}$ and $y = \frac{c_2 - c_1}{2}$.

But x and y are integers if and only if $c_1 + c_2 \in 2\mathbf{Z}$, i.e.:

1) or c_1 and c_2 are odd, then c is odd (and reciprocally).

2) or c_1 and c_2 are even, then $c \in 4\mathbf{Z}$.

Reciprocally, if $c \in 4\mathbf{Z}$, then we can decompose up c into two even factors c_1 and c_2 , such that $c_1c_2 = c$.

Remark 1:

Property 1 is true also for solving in \mathbf{N} , because we can suppose $c \geq 0$ (in the contrary case, we can multiply the equation by (-1)), and we can suppose $c_2 \geq c_1 \geq 0$, from which $x \geq 0$ and $y \geq 0$.

Property 2: The equation $x^2 - dy^2 = c^2$ (where d is not a perfect square) admits an infinity of solutions in \mathbf{N} .

Proof: Let's consider $x = ck_1$, $k_1 \in \mathbf{N}$ and $y = ck_2$, $k_2 \in \mathbf{N}$, $c \in \mathbf{N}$. It results that $k_1^2 - dk_2^2 = 1$, which we can recognize as being the Pell-Fermat's equation, which admits an infinity of solutions in \mathbf{N} , (u_n, v_n) . Therefore $x_n = cu_n$, $y_n = cv_n$ constitute an infinity of natural solutions for our equation.

Property 3: The equation $ax^2 - by^2 = c$, $c \neq 0$ where $ab = k^2$, ($k \in \mathbf{Z}$), admits a finite number of natural solutions.

Proof: We can consider a , b , c as positive numbers, otherwise, we can multiply the equation by (-1) and we can rename the variables.

Let us multiply the equation by a , then we will have:

$$z^2 - t^2 = d \text{ with } z = ax \in \mathbf{N}, t = ky \in \mathbf{N} \text{ and } d = ac > 0. \quad (1)$$

We will solve it as in property 1, which gives z and t .

But in (1) there is a finite number of natural solutions, because there is a finite number of integer divisors for a number in \mathbf{N}^* . Because the pairs (z, t) are in a limited number, it results that the pairs $(z/a, t/k)$ also are in a limited number, and the same for the pairs (x, y) .

Property 4: If $ax^2 - by^2 = c$, where $ab \neq k^2$ ($k \in \mathbf{Z}$) admits a particular nontrivial solution in \mathbf{N} , then it admits an infinity of solutions in \mathbf{N} .

Proof: Let's consider:

$$(2) \quad \begin{cases} x_n = x_0 u_n + b y_0 v_n \\ y_n = y_0 u_n + a x_0 v_n \end{cases} \quad (n \in \mathbf{N})$$

where (x_0, y_0) is the particular natural solution for the initial equation, and $(u_n, v_n)_{n \in \mathbf{N}}$ is the general natural solution for the equation $u^2 - abv^2 = 1$, called the solution Pell, which admits an infinity of solutions.

$$\text{Then } ax_n^2 - by_n^2 = (ax_0^2 - by_0^2)(u_n^2 - abv_n^2) = c.$$

Therefore (2) verifies the initial equation.

[1984]