ABOUT VERY PERFECT NUMBERS

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Abstract.
In this short paper we prove that the square of an odd prime number cannot be a very perfect number.

Introduction.

A natural number n is called *very perfect* if $\sigma(\sigma(n)) = 2n$ (see [1]), where $\sigma(x)$ means the sum of all positive divisors of the natural number x.

We now prove the following result:

Theorem. The square of an odd prime number cannot be a very perfect number.

Proof: Let's consider $n = p^2$, where p is an odd prime number, then

$$\sigma(n) = 1 + p + p^2, \quad \sigma(\sigma(n)) = \sigma(1 + p + p^2) = 2p^2.$$

We decompose $\sigma(n)$ in canonical form, from where $1 + p + p^2 = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$. Because $p(p+1)+1$ is odd, in the canonical decompose there must be only odd numbers.

$$\sigma(\sigma(n)) = (1 + p_1 + ... + p_1^{\alpha})...(1 + p_k + ... + p_k^{\alpha_k}) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} ... \frac{p_k^{\alpha_k+1} - 1}{p_k - 1} = 2p^2$$

Because

$$\frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} > 2, ..., \frac{p_k^{\alpha_k+1} - 1}{p_k - 1} > 2$$

one obtains that $2p^2$ cannot be decomposed in more than two factors, then each one is > 2, therefore $k \leq 2$.

Case 1. For $k = 1$ we find $\sigma(n) = 1 + p + p^2 = p_1^{\alpha_1}$, from where one obtains

$$p_1^{\alpha_1+1} = p_1\left(1+p+p^2\right) \text{ and}$$

$$\sigma(\sigma(n)) = \frac{p_1^{\alpha_1+1}-1}{p_1-1} = 2p^2,\ p_1\left(1+p+p^2\right)-1 = 2p^2\left(p_1-1\right),$$

from where

$$p_1 - 1 = p(pp_1 - 2p - p_1).$$

The right side is divisible by p, thus p_1-1 is a p multiple. Because $p_1>2$ it results that

$$p_1 \geq p-1 \text{ and } p_1^2 \geq \left(p+1\right)^2 > p^2+p+1 = p_1^{\alpha_1},$$

thus $\alpha_1 = 1$ and

$$\sigma(n) = p^2+p+1 = p_1,\ \sigma(\sigma(n)) = \sigma(p_1) = 1+p_1.$$

If n is very perfect then $1+p_1 = 2p^2$ or $p^2+p+2 = 2p^2$. The solutions of the equation are $p=-1$, and $p=2$ which is a contradiction.

Case 2. For $k=2$ we have $\sigma(n) = p^2+p+1 = p_1^{\alpha_1}p_2^{\alpha_2}$.

$$\sigma\left(\sigma(n)\right) = \left(1+p_1+...+p_1^{\alpha}\right)\left(1+p_2+...+p_2^{\alpha_2}\right) = \frac{p_1^{\alpha_1+1}-1}{p_1-1}\cdot\frac{p_2^{\alpha_2+1}-1}{p_2-1} = 2p^2.$$

Because

$$\frac{p_1^{\alpha_1+1}-1}{p_1-1} > 2 \text{ and } \frac{p_2^{\alpha_2+1}-1}{p_2-1} > 2,$$

it results

$$\frac{p_1^{\alpha_1+1}}{p_1-1} = p \text{ and } \frac{p_2^{\alpha_2+1}-1}{p_2-1} = 2p$$

(or inverse), thus

$$p_1^{\alpha_1+1}-1 = p\left(p_1-1\right),\ p_2^{\alpha_2+1}-1 = 2p(p_2-1),$$

then

$$p_1^{\alpha_1+1}p_2^{\alpha_2+1} - p_1^{\alpha_1+1} - p_2^{\alpha_2+1} + 1 = 2p^2(p_1-1)(p_2-1),$$

thus

$$\sigma(n) = p^2+p+1 = p_1^{\alpha_1+1}p_2^{\alpha_2+1}$$

and

$$p_1p_2\left(p^2+p+1\right) = 2p^2\left(p_1-1\right)\left(p_2-1\right) + p_1^{\alpha_1+1} + p_2^{\alpha_2+1} - 1$$

or

$$p_1p_2p\left(p+1\right) + p_1p_2 - 1 = 2p^2\left(p_1-1\right)\left(p_2-1\right) + \left(p_1^{\alpha_1+1}-1\right) + \left(p_2^{\alpha_2+1}-1\right) =$$
$$= 2p^2(p_1-1)(p_2-1) + p(p_1-1) + 2p(p_2-1)$$

accordingly p divides p_1p_2-1, thus $p_1p_2 > p+1$ and

$$p_1^2p_2^2 \geq (p+1)^2 > p^2+p+1 = p_1^{\alpha_1}p_2^{\alpha_2}.$$

Hence:

Π_1) If $\alpha_1 = 1$ and $n = 2p^2$, then

$$\sigma(n) = p^2 + p + 1 = p_1 p_2^{\alpha_2} \text{ and } \frac{p_1^2 - 1}{p_1 - 1} = p \text{, and } \frac{p_2^{\alpha_2 + 1} - 1}{p_2 - 1} = 2p,$$

thus $p_1 + 1 = p$ which is a contradiction.

Π_2) If $\alpha_2 = 1$ and $n = 2p^2$,

then

$$\sigma(n) = p^2 + p + 1 = p_1^{\alpha_1} p_2 \text{ and } \frac{p_1^{\alpha_1 + 1} - 1}{p_1 - 1} = p \text{, and}$$

$$\frac{p_2^2 - 1}{p_2 - 1} = 2p,$$

thus

$$p_2 + 1 = 2p, \ p_2 = 2p - 1$$

and

$$\sigma(n) = p^2 + p + 1 = p_1^{\alpha_1}(2p + 1),$$

from where

$$4\sigma(n) = (2p - 1)(2p + 3) + 7 = 4p_1^{\alpha_1}(2p - 1),$$

accordingly 7 is divisible by $2p - 1$ and thus p is divisible by 4 which is a contradiction.

Reference:

[1] Suryanarayama – Elemente der Mathematik – 1969.

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