Dedicated to Prof.Feng Tian on Occasion of his 70th Birthday

COMBINATORIAL WORLD

----*Applications of Voltage Assignament to Principal Fiber Bundles*

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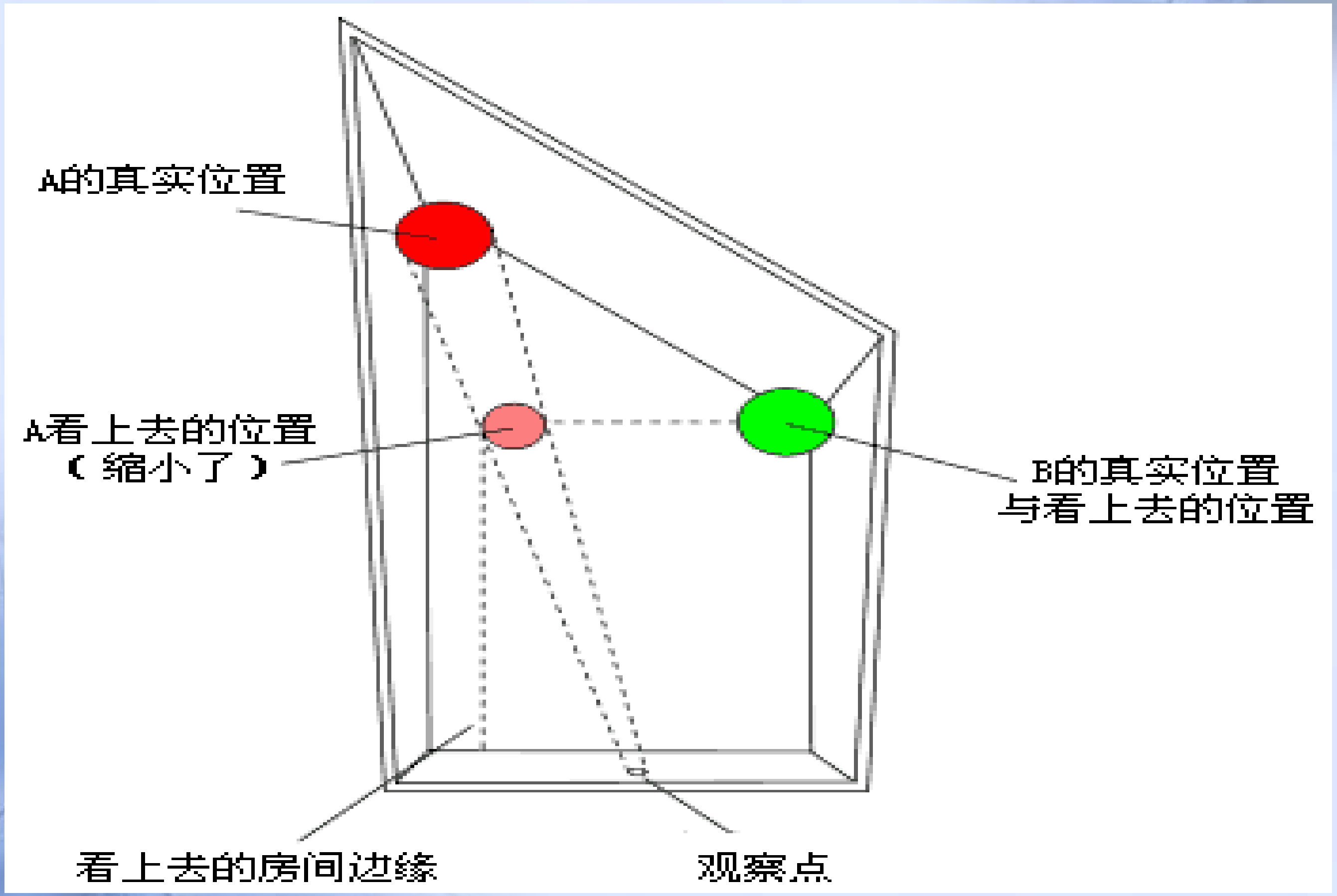
Contents

1. Why Is It Combinatorial?

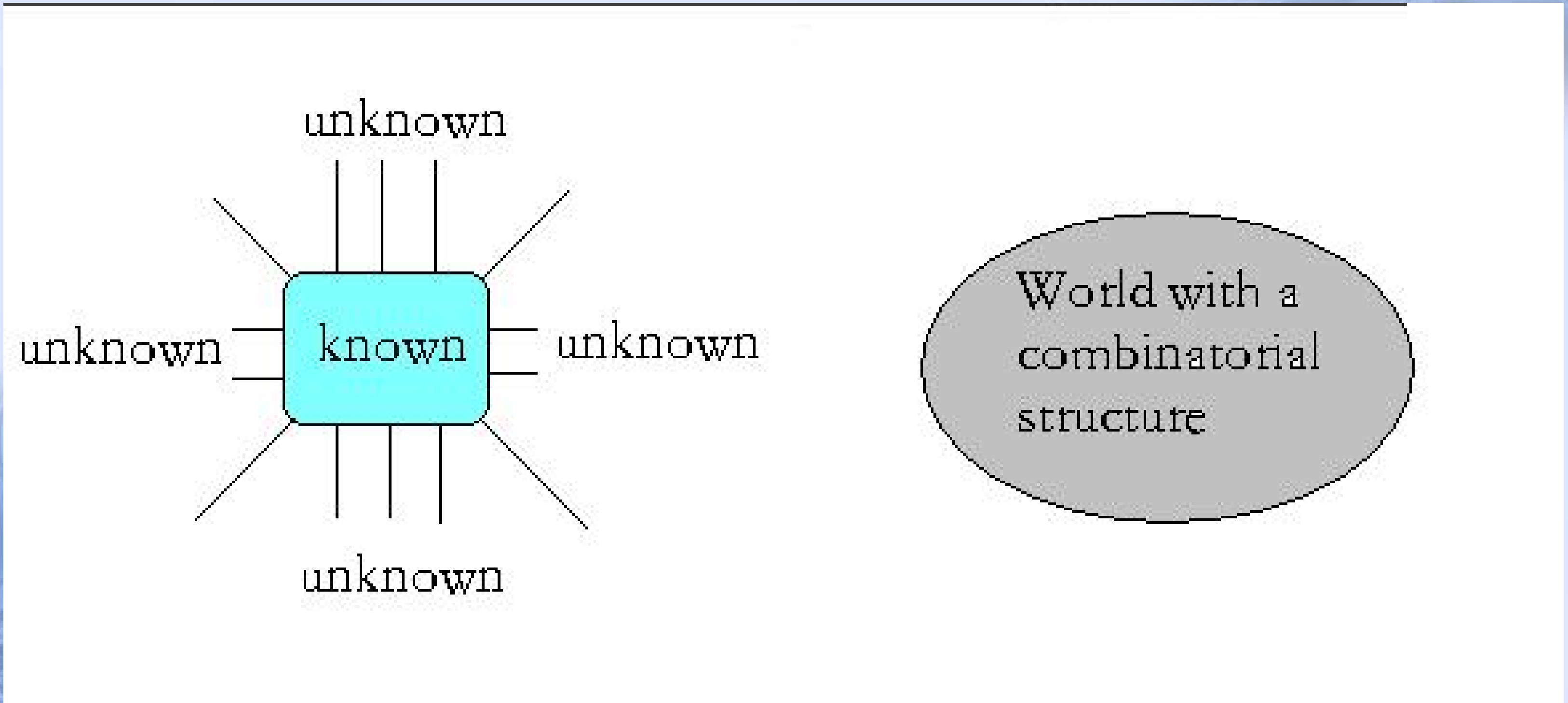
- Ames Room—It isn't all right of our visual sense.



● **Blind men with an elephant**

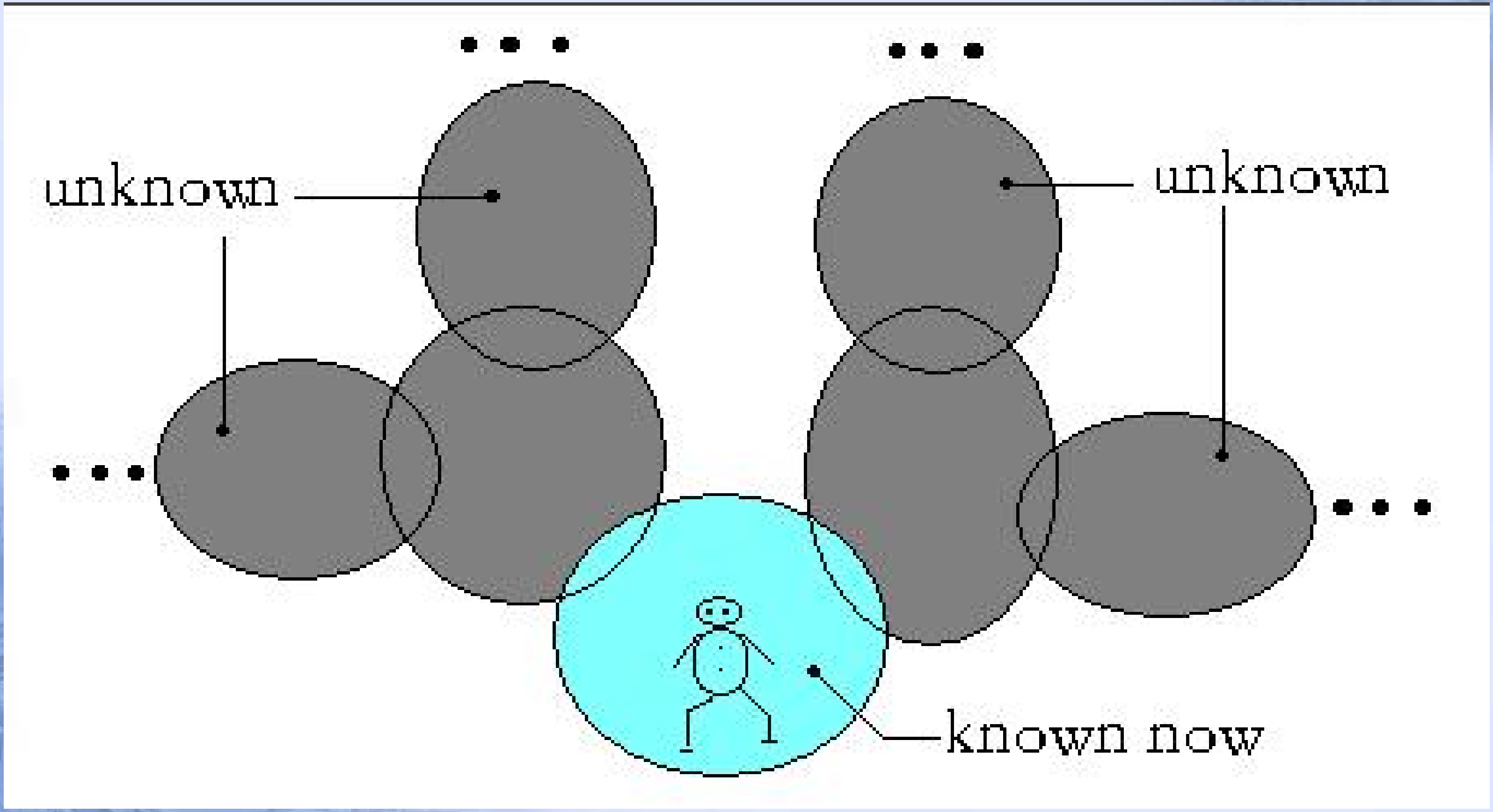
The man touched its leg, tail, trunk, ear, belly or tusk claims that the elephant is like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, respectively. All of you are right! A wise man said.

- ***What is the structure of the world?***



It is out order? No! in order! Any thing has itself reason for existence.

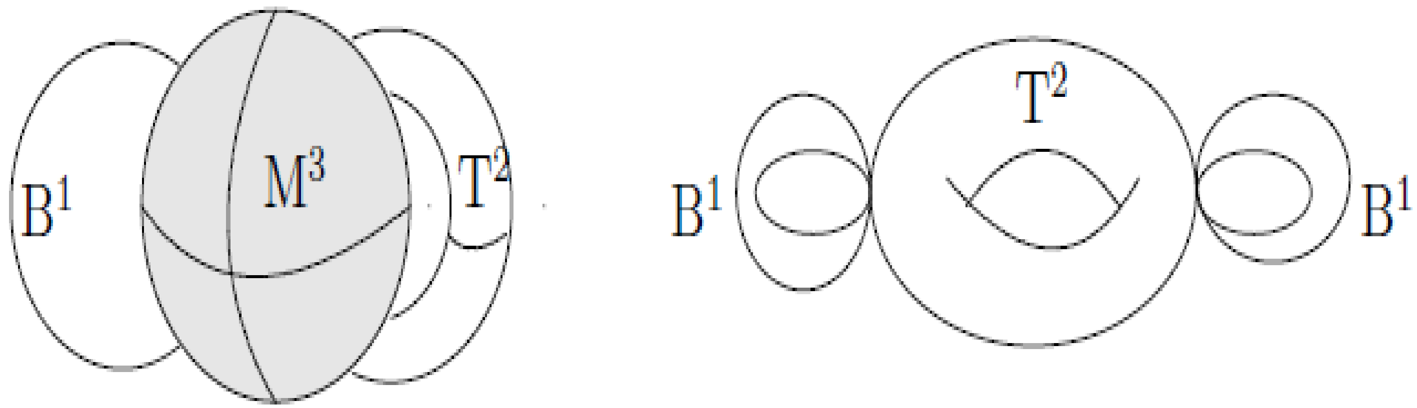
- **A depiction of the world by combinatoricians**



How to characterize it by mathematics? Manifold!

2. What is a Combinatorial Manifold?

Loosely speaking, a combinatorial manifold is a combination of finite manifolds, such as those shown in the next figure.

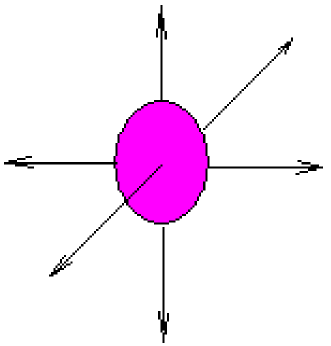


(a) (b)

2.1 Euclidean Fan-Space

A *combinatorial fan-space* $\widetilde{\mathrm{R}}(n_1, \cdots, n_m)$ is the combinatorial Euclidean space $\mathscr{E}_{K_m}(n_1, \cdots, n_m)$ of $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ such that for any integers i, j, $1 \leq i \neq j \leq m$,

$$\mathbf{R}^{n_i} \bigcap \mathbf{R}^{n_j} = \bigcap_{k=1}^{m} \mathbf{R}^{n_k}.$$



For $\forall p \in \widetilde{\mathrm{R}}(n_1, \cdots, n_m)$ we can present it by an $m \times n_m$ coordinate matrix $[\overline{x}]$ following with $x_{il} = \frac{x_l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$.

$$[\overline{x}] = \left[\begin{array}{cccccccc} x_{11} & \cdots & x_{1\widehat{m}} & x_{1(\widehat{m})+1)} & \cdots & x_{1n_1} & \cdots & 0 \\ x_{21} & \cdots & x_{2\widehat{m}} & x_{2(\widehat{m}+1)} & \cdots & x_{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ x_{m1} & \cdots & x_{m\widehat{m}} & x_{m(\widehat{m}+1)} & \cdots & \cdots & x_{mn_m-1} & x_{mn_m} \end{array}\right]$$

2.2 Topological Combinatorial Manifold

Definition 2.1 For a given integer sequence $n_1, n_2, \cdots, n_m$, $m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_m$, a *combinatorial manifold* $\widetilde{M}$ is a Hausdorff space such that for any point $p \in \widetilde{M}$, there is a local chart (U_p, φ_p) of p, i.e., an open neighborhood U_p of p in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \rightarrow \widetilde{\mathbf{R}}(n_1(p), n_2(p), \cdots, n_{s(p)}(p))$, a combinatorial fan-space with

$$\{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} \subseteq \{n_1, n_2, \cdots, n_m\},$$

$$\bigcup_{p \in \widetilde{M}} \{n_1(p), n_2(p), \cdots, n_{s(p)}(p)\} = \{n_1, n_2, \cdots, n_m\},$$

and $$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, \cdots, n_m))\}$$

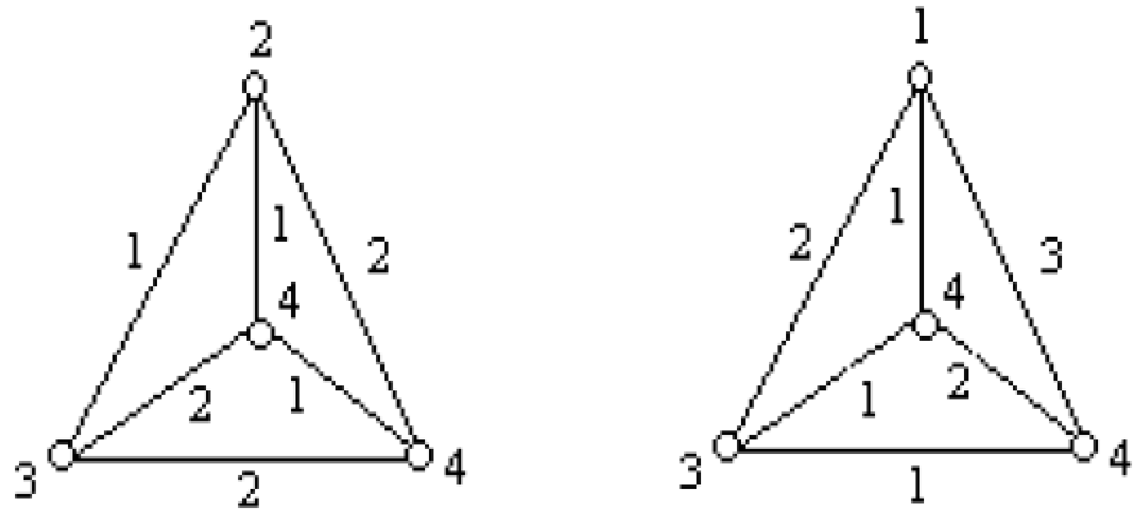
an atlas on $\widetilde{M}(n_1, n_2, \cdots, n_m)$. The maximum value of $s(p)$ and the dimension $\widehat{s}(p) = \dim(\bigcap_{i=1}^{s(p)} \mathbf{R}^{n_i(p)})$ are called the dimension and the intersectional dimension of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ at the point p and is *finite* if it is combined by finite manifolds with an underlying combinatorial structure G without one manifold contained in the union of others.

2.3 Vertex-Edge Labeled Graphs

A *vertex-edge labeled graph* $G([1,k],[1,l])$ is a connected graph $G=(V,E)$ with two mappings

$$\tau_1 : V \rightarrow \{1,2,\cdots,k\}, \qquad \tau_2 : E \rightarrow \{1,2,\cdots,l\}$$

for integers k and l. For example, two vertex-edge labeled graphs with an underlying graph K_4 are shown in the next figure.



$\mathcal{H}(n_1, n_2, \cdots, n_m)$ — all finitely combinatorial manifolds $\widetilde{M}(n_1, n_2, \cdots, n_m)$

$\mathcal{G}[0, n_m]$ ——— all vertex-edge labeled graphs $G([0, n_m], [0, n_m])$ with

(1) Each induced subgraph by vertices labeled with 1 in G is a union of complete graphs and vertices labeled with 0 can only be adjacent to vertices labeled with 1.

(2) For each edge $e = (u, v) \in E(G)$, $\tau_2(e) \leq \min\{\tau_1(u), \tau_1(v)\}$.

Theorem 2.1 *Let $1 \leq n_1 < n_2 < \cdots < n_m, m \geq 1$ be a given integer sequence. Then every finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1, n_2, \cdots, n_m)$ defines a vertex-edge labeled graph $G([0, n_m], [0, n_m]) \in \mathcal{G}[0, n_m]$. Conversely, every vertex-edge labeled graph $G([0, n_m], [0, n_m]) \in \mathcal{G}[0, n_m]$ defines a finitely combinatorial manifold $\widetilde{M} \in \mathcal{H}(n_1, n_2, \cdots, n_m)$ with a $1-1$ mapping $\theta : G([0, n_m], [0, n_m]) \to \widetilde{M}$ such that $\theta(u)$ is a $\theta(u)$-manifold in $\widetilde{M}$, $\tau_1(u) = \dim\theta(u)$ and $\tau_2(v, w) = \dim(\theta(v) \bigcap \theta(w))$ for $\forall u \in V(G([0, n_m], [0, n_m]))$ and $\forall (v, w) \in E(G([0, n_m], [0, n_m]))$.*

2.4 Fundamental d-Group

Definition 2.2 *For two points p, q in a finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$, if there is a sequence $B_1, B_2, \cdots, B_s$ of d-dimensional open balls with two conditions following hold.*

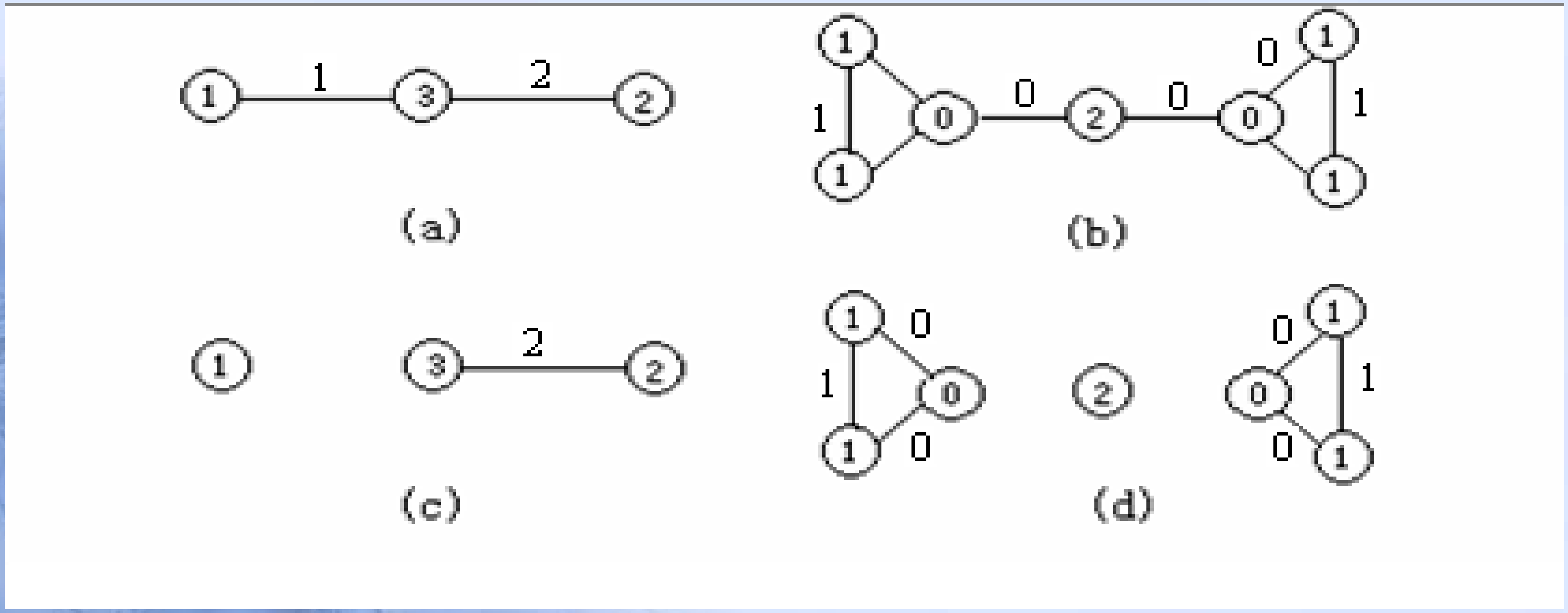
(1) $B_i \subset \widetilde{M}(n_1, n_2, \cdots, n_m)$ *for any integer* $i, 1 \leq i \leq s$ *and* $p \in B_1$, $q \in B_s$;

(2) The dimensional number $\dim(B_i \bigcap B_{i+1}) \geq d$ *for* $\forall i, 1 \leq i \leq s-1$.

Then points p, q are called d-dimensional connected in $\widetilde{M}(n_1, n_2, \cdots, n_m)$ and the sequence $B_1, B_2, \cdots, B_e$ a d-dimensional path connecting p and q, denoted by $P^d(p, q)$.

If each pair p, q of points in the finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is d-dimensional connected, then $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is called d-pathwise connected and say its connectivity$\geq d$.

Choose a graph with vertex set being manifolds labeled by its dimension and two manifold adjacent with a label of the dimension of the intersection if there is a d-path in this combinatorial manifold. Such graph is denoted by G^d. d=1 in (a) and (b), d=2 in (c) and (d) in the next figure.



Definition 2.3 *Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a finitely combinatorial manifold of d-arcwise connectedness for an integer $d, 1 \leq d \leq n_1$ and $\forall x_0 \in \widetilde{M}(n_1, n_2, \cdots, n_m)$, a fundamental d-group at the point x_0, denoted by $\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x_0)$ is defined to be a group generated by all homotopic classes of closed d-pathes based at x_0.*

A combinatorial Euclidean space $\mathscr{E}_G(\overbrace{d, d, \cdots, d}^{m})$ of $\mathbf{R}^d$ underlying a combinatorial structure $G, |G| = m$ is called a *d-dimensional graph*, denoted by $\widetilde{M}^d[G]$ if

(1) $\widetilde{M}^d[G] \setminus V(\widetilde{M}^d[G])$ is a disjoint union of a finite number of open subsets $e_1, e_2, \cdots, e_m$, each of which is homeomorphic to an open ball B^d;

(2) the boundary $\overline{e}_i - e_i$ of e_i consists of one or two vertices B^d, and each pair $(\overline{e}_i, e_i)$ is homeomorphic to the pair $(\overline{B}^d, S^{d-1})$,

Theorem 2.2 $\pi^d(\widetilde{M}^d[G], x_0) \cong \pi_1(G, x_0)$, $x_0 \in G$.

Theorem 2.3 *Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a d-connected finitely combinatorial manifold for an integer d, $1 \leq d \leq n_1$. If $\forall (M_1, M_2) \in E(G^L[\widetilde{M}(n_1, n_2, \cdots, n_m)])$, $M_1 \cap M_2$ is simply connected, then*

(1) *for $\forall x_0 \in G^d$, $M \in V(G^L[\widetilde{M}(n_1, n_2, \cdots, n_m)])$ and $x_{0M} \in M$,*

$$\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x_0) \cong \Big(\bigoplus_{M \in V(G^d)} \pi^d(M, x_{M0})\Big) \bigoplus \pi(G^d, x_0),$$

where $G^d = G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)]$ in which each edge (M_1, M_2) passing through a given point $x_{M_1M_2} \in M_1 \cap M_2$, $\pi^d(M, x_{M0}), \pi(G^d, x_0)$ denote the fundamental d-groups of a manifold M and the graph G^d, respectively and

(2) *for $\forall x, y \in \widetilde{M}(n_1, n_2, \cdots, n_m)$,*

$$\pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), x) \cong \pi^d(\widetilde{M}(n_1, n_2, \cdots, n_m), y).$$

2.5 Homology Group

For a subspace A of a topological space S and an inclusion mapping $i: A \hookrightarrow S$, it is readily verified that the induced homomorphism $i_\sharp : C_p(A) \to C_p(S)$ is a monomorphism. Let $C_p(S, A)$ denote the quotient group $C_p(S)/C_p(A)$.

$$Z_p(S, A) = \mathrm{Ker}\partial_p = \{ \ u \in C_p(S, A) \mid \partial_p(u) = 0 \ \},$$

$$B_p(S, A) = \mathrm{Im}\partial_{p+1} = \partial_{p+1}(C_{p+1}(S, A)).$$

The *pth relative homology group* $H_p(S, A)$ is defined to be

$$H_p(S, A) = Z_p(S, A)/B_p(S, A).$$

Theorem 2.4 *Let $\widetilde{M}^d(G)$ be a d-dimensional graph with $E(\widetilde{M}^d(G)) = \{e_1, e_2, \cdots, e_m\}$. Then the inclusion $(e_l, \dot{e}_l) \hookrightarrow (\widetilde{M}^d(G), V(\widetilde{M}^d(G)))$ induces a monomorphism $H_p(e_l, \dot{e}_l) \to H_p(\widetilde{M}^d(G), V(\widetilde{M}^d(G)))$ for $l = 1, 2 \cdots, m$ and $H_p(\widetilde{M}^d(G), V(\widetilde{M}^d(G)))$ is a direct sum of the image subgroups, which follows that*

$$H_p(\widetilde{M}^d(G), V(\widetilde{M}^d(G))) \cong \begin{cases} \underbrace{\mathbf{Z} \oplus \cdots \mathbf{Z}}_{m}, & if \ p = d, \\ 0, & if \ p \neq d. \end{cases}$$

3. What is a Differentiable Combinatorial Manifold?

3.1 Definition

Definition 3.1 *For a given integer sequence $1 \leq n_1 < n_2 < \cdots < n_m$, a combinatorial C^h-differential manifold $(\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$ is a finitely combinatorial manifold $\widetilde{M}(n_1, n_2, \cdots, n_m)$, $\widetilde{M}(n_1, n_2, \cdots, n_m) = \bigcup_{i \in I} U_i$, endowed with a atlas $\widetilde{\mathcal{A}} = \{(U_\alpha; \varphi_\alpha) | \alpha \in I\}$ on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ for an integer $h, h \geq 1$ with conditions following hold.*

(1) *$\{U_\alpha; \alpha \in I\}$ is an open covering of $\widetilde{M}(n_1, n_2, \cdots, n_m)$.*

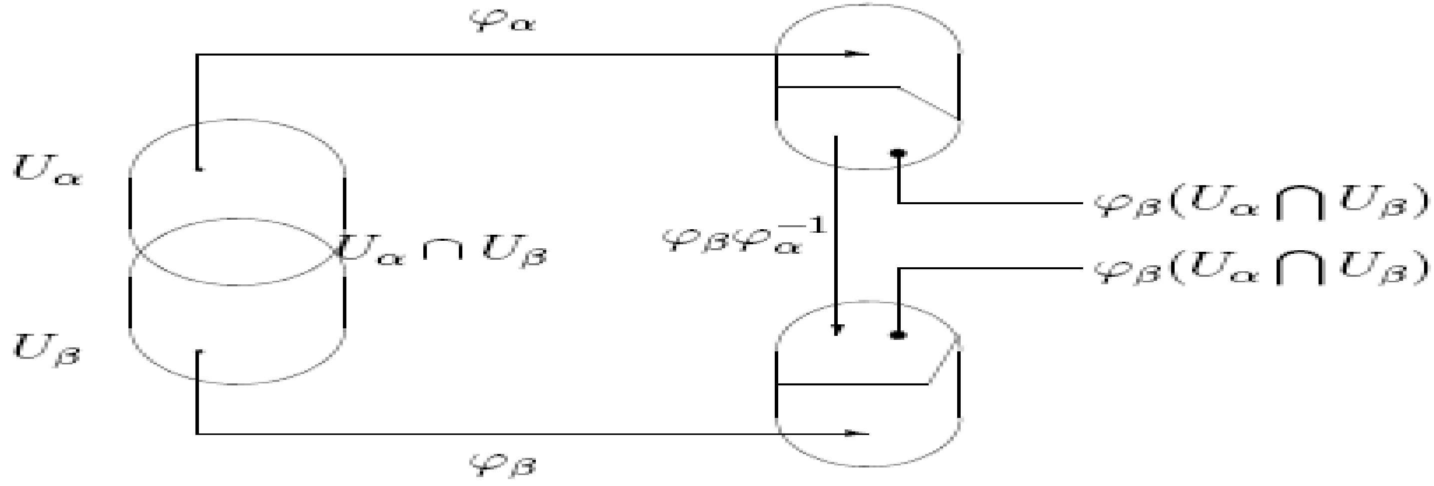
(2) *For $\forall \alpha, \beta \in I$, local charts $(U_\alpha; \varphi_\alpha)$ and $(U_\beta; \varphi_\beta)$ are equivalent, i.e., $U_\alpha \bigcap U_\beta = \emptyset$ or $U_\alpha \bigcap U_\beta \neq \emptyset$ but the overlap maps*

$$\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \bigcap U_\beta) \to \varphi_\beta(U_\beta) \text{ and } \varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \bigcap U_\beta) \to \varphi_\alpha(U_\alpha)$$

are C^h-mappings.

(3) *$\widetilde{\mathcal{A}}$ is maximal, i.e., if $(U; \varphi)$ is a local chart of $\widetilde{M}(n_1, n_2, \cdots, n_m)$ equivalent with one of local charts in $\widetilde{\mathcal{A}}$, then $(U; \varphi) \in \widetilde{\mathcal{A}}$.*

Explains for condition (2)



Extence Theorem *Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a finitely combinatorial manifold and $d, 1 \leq d \leq n_1$ an integer. If $\forall M \in V(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)])$ is C^h-differential and $\forall (M_1, M_2) \in E(G^d[\widetilde{M}(n_1, n_2, \cdots, n_m)])$ there exist atlas*

$$\mathcal{A}_1 = \{(V_x; \varphi_x) | \forall x \in M_1\} \quad \mathcal{A}_2 = \{(W_y; \psi_y) | \forall y \in M_2\}$$

such that $\varphi_x|_{V_x \bigcap W_y} = \psi_y|_{V_x \bigcap W_y}$ for $\forall x \in M_1, y \in M_2$, then there is a differential structures

$$\widetilde{\mathcal{A}} = \{(U_p; [\varpi_p]) | \forall p \in \widetilde{M}(n_1, n_2, \cdots, n_m)\}$$

such that $(\widetilde{M}(n_1, n_2, \cdots, n_m); \widetilde{\mathcal{A}})$ is a combinatorial C^h-differential manifold.

3.2 Local Properties of Combinatorial Manifolds

Denote by $\mathscr{X}_p$ all these C^∞-functions at a point $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$.

Definition 3.2 *Let $(\widetilde{M}(n_1, n_2, \cdots, n_m), \widetilde{\mathcal{A}})$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. A tangent vector $\overline{v}$ at p is a mapping $\overline{v} : \mathscr{X}_p \to \mathbf{R}$ with conditions following hold.*

(1) $\forall g, h \in \mathscr{X}_p, \forall \lambda \in \mathbf{R},\ \overline{v}(h + \lambda h) = \overline{v}(g) + \lambda \overline{v}(h)$;

(2) $\forall g, h \in \mathscr{X}_p, \overline{v}(gh) = \overline{v}(g)h(p) + g(p)\overline{v}(h)$.

Theorem 3.2 *For any point $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$ with a local chart $(U_p; [\varphi_p])$, the dimension of $T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ is*

$$\dim T_p\widetilde{M}(n_1, n_2, \cdots, n_m) = \widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p))$$

with a basis matrix $[\frac{\partial}{\partial \overline{x}}]_{s(p) \times n_{s(p)}} =$

$$\begin{bmatrix} \frac{1}{s(p)}\frac{\partial}{\partial x^{11}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{1\widehat{s}(p)}} & \frac{\partial}{\partial x^{1(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{1n_1}} & \cdots & 0 \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{21}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{2\widehat{s}(p)}} & \frac{\partial}{\partial x^{2(\widehat{s}(p)+1)}} & \cdots & \frac{\partial}{\partial x^{2n_2}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)1}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)\widehat{s}(p)}} & \frac{\partial}{\partial x^{s(p)(\widehat{s}(p)+1)}} & \cdots & \cdots & \frac{\partial}{\partial x^{s(p)(n_{s(p)}-1)}} & \frac{\partial}{\partial x^{s(p)n_{s(p)}}} \end{bmatrix}$$

where $x^{il} = x^{jl}$ for $1 \leq i, j \leq s(p), 1 \leq l \leq \widehat{s}(p)$.

3.3 Tensor Field

Definition 3.3 *Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. A tensor of type (r, s) at the point p on $\widetilde{M}(n_1, n_2, \cdots, n_m)$ is an $(r+s)$-multilinear function τ,*

$$\tau : \underbrace{T_p^*\widetilde{M} \times \cdots \times T_p^*\widetilde{M}}_{r} \times \underbrace{T_p\widetilde{M} \times \cdots \times T_p\widetilde{M}}_{s} \to \mathbf{R},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$.

Theorem 3.3 *Let $\widetilde{M}(n_1, n_2, \cdots, n_m)$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, \cdots, n_m)$. Then*

$$T_s^r(p, \widetilde{M}) = \underbrace{T_p\widetilde{M} \otimes \cdots \otimes T_p\widetilde{M}}_{r} \otimes \underbrace{T_p^*\widetilde{M} \otimes \cdots \otimes T_p^*\widetilde{M}}_{s},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, \cdots, n_m)$ and $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, \cdots, n_m)$, particularly,

$$\dim T_s^r(p, \widetilde{M}) = (\widehat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \widehat{s}(p)))^{r+s}.$$

3.4 Curvature Tensor

Definition 3.4 *Let $\widetilde{M}$ be a smoothly combinatorial manifold. A connection on tensors of $\widetilde{M}$ is a mapping $\widetilde{D} : \mathscr{X}(\widetilde{M}) \times T_s^r\widetilde{M} \to T_s^r\widetilde{M}$ with $\widetilde{D}_X\tau = \widetilde{D}(X,\tau)$ such that for $\forall X, Y \in \mathscr{X}\widetilde{M}$, $\tau, \pi \in T_s^r(\widetilde{M})$, $\lambda \in \mathbf{R}$ and $f \in C^\infty(\widetilde{M})$,*

(1) $\widetilde{D}_{X+fY}\tau = \widetilde{D}_X\tau + f\widetilde{D}_Y\tau$; and $\widetilde{D}_X(\tau + \lambda\pi) = \widetilde{D}_X\tau + \lambda\widetilde{D}_X\pi$;

(2) $\widetilde{D}_X(\tau \otimes \pi) = \widetilde{D}_X\tau \otimes \pi + \sigma \otimes \widetilde{D}_X\pi$;

(3) *for any contraction C on $T_s^r(\widetilde{M})$,* $\widetilde{D}_X(C(\tau)) = C(\widetilde{D}_X\tau)$.

A *combinatorial connection space* is a 2-tuple $(\widetilde{M}, \widetilde{D})$ consisting of a smoothly combinatorial manifold $\widetilde{M}$ with a connection $\widetilde{D}$ on its tensors.

For $\forall X, Y \in \mathscr{X}(\widetilde{M})$, a combinatorial curvature operator

$$\widetilde{\mathcal{R}}(X,Y) : \mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$$

is defined by

$$\widetilde{\mathcal{R}}(X,Y)Z = \widetilde{D}_X\widetilde{D}_YZ - \widetilde{D}_Y\widetilde{D}_XZ - \widetilde{D}_{[X,Y]}Z$$

for $\forall Z \in \mathscr{X}(\widetilde{M})$.

Definition 3.5 *Let $\widetilde{M}$ be a smoothly combinatorial manifold and $g \in A^2(\widetilde{M}) = \bigcup_{p\in\widetilde{M}} T_2^0(p,\widetilde{M})$. If g is symmetrical and positive, then $\widetilde{M}$ is called a combinatorial Riemannian manifold, denoted by $(\widetilde{M}, g)$. In this case, if there is a connection $\widetilde{D}$ on $(\widetilde{M}, g)$ with equality following hold*

$$Z(g(X,Y)) = g(\widetilde{D}_Z, Y) + g(X, \widetilde{D}_Z Y)$$

then $\widetilde{M}$ is called a combinatorial Riemannian geometry, denoted by $(\widetilde{M}, g, \widetilde{D})$.

In this case, $\widetilde{R} = \widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} dx^{\sigma\varsigma} \otimes dx^{\eta\theta} \otimes dx^{\mu\nu} \otimes dx^{\kappa\lambda}$ *with*

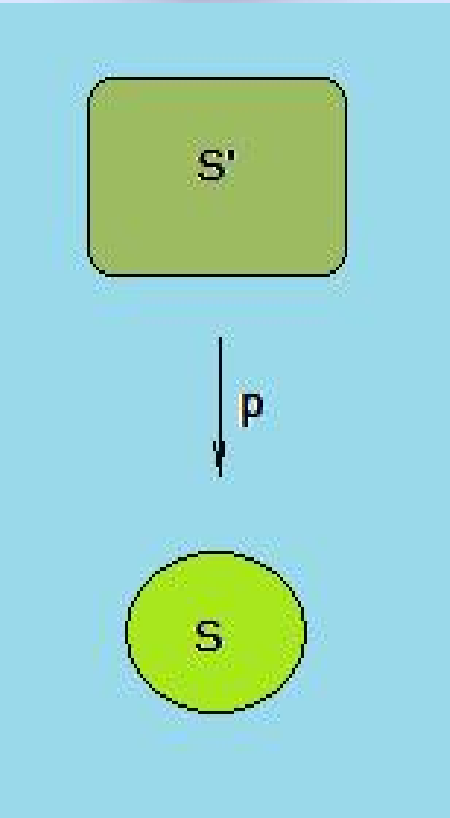
$$\begin{aligned}\widetilde{R}_{(\sigma\varsigma)(\eta\theta)(\mu\nu)(\kappa\lambda)} &= \frac{1}{2}\left(\frac{\partial^2 g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}\partial x^{\eta\theta}} + \frac{\partial^2 g_{(\kappa\lambda)(\eta\theta)}}{\partial x^{\mu\nu\nu}\partial x^{\sigma\varsigma}} - \frac{\partial^2 g_{(\mu\nu)(\eta\theta)}}{\partial x^{\kappa\lambda}\partial x^{\sigma\varsigma}} - \frac{\partial^2 g_{(\kappa\lambda)(\sigma\varsigma)}}{\partial x^{\mu\nu}\partial x^{\eta\theta}}\right) \\ &+ \Gamma^{\vartheta\iota}_{(\mu\nu)(\sigma\varsigma)}\Gamma^{\xi o}_{(\kappa\lambda)(\eta\theta)} g_{(\xi o)(\vartheta\iota)} - \Gamma^{\xi o}_{(\mu\nu)(\eta\theta)}\Gamma_{(\kappa\lambda)(\sigma\varsigma)^{\vartheta\iota}} g_{(\xi o)(\vartheta\iota)},\end{aligned}$$

where $g_{(\mu\nu)(\kappa\lambda)} = g(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\kappa\lambda}})$.

4. What is a Principal Fiber Bundle?

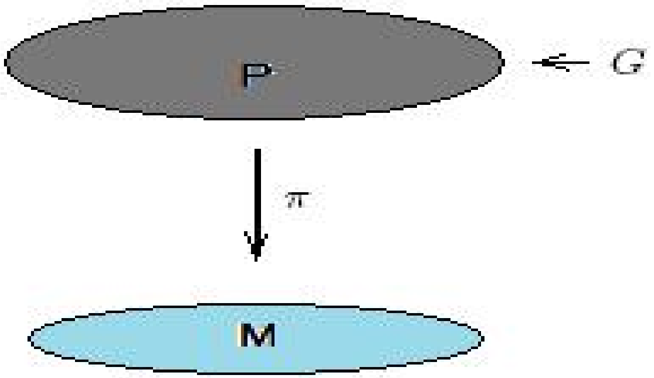
4.1 Covering Space

A covering space S' of S consisting of a space S' with a continuous mapping $p : S' \rightarrow S$ such that each point $x \in S$ has an arcwise connected neighborhood U_x and each arcwise connected component of $p^{-1}(U_x)$ is mapped topologically onto U_x by p. An opened neighborhoods U_x that satisfies the condition just stated is called an elementary neighborhood and p is often called a projection from S' to S.



4.2 Principal Fiber Bundle

A principal fiber bundle (PFB) consists of a manifold P, a projection $\pi: P \rightarrow M$, a base manifold M, and a Lie group G, which is a manifold with group operation $G \times G \rightarrow$ given by $(g, h) \rightarrow g \circ h$ being C^{∞} map, denoted by (P, M, π, G) such that (1), (2) and (3) following hold.



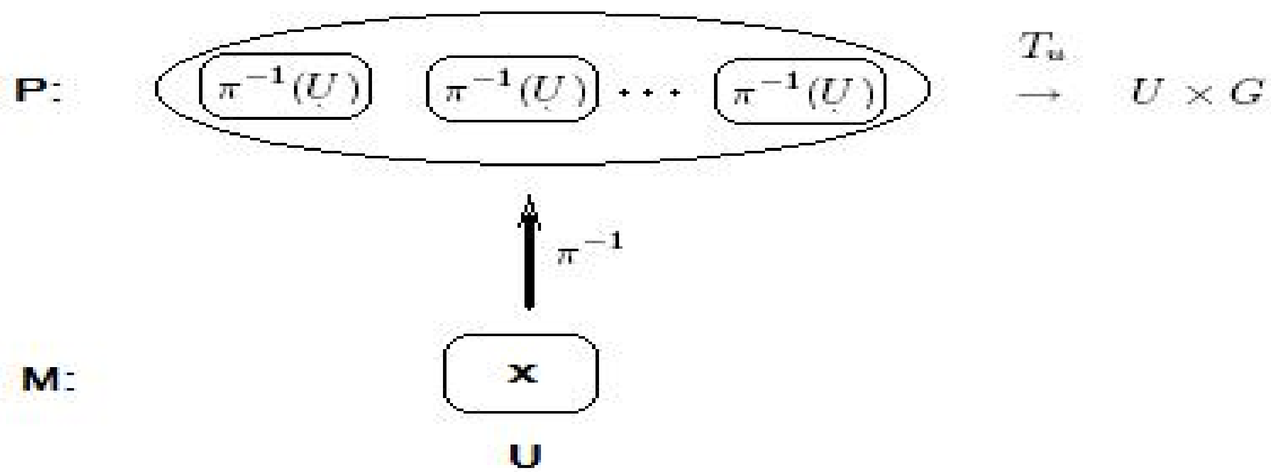
(1) There is a right freely action of G on P, i.e., for $\forall g \in G$, there is a diffeomorphism $R_g: P \rightarrow P$ with $R_g(p) = p\, g$ for $\forall p \in P$ such that

$$p\,(g_1 g_2) = (p\, g_1) g_2 \text{ for } \forall p \in P, \ \forall g_1, g_2 \in G \text{ and } p\, e = p$$

for some $p \in P$, $e \in G$ if and only if e is the identity element of G.

(2) *The map* $\pi: P \rightarrow M$ *is regular onto with* $\pi^{-1}(\pi(p)) = \{pg | g \in G\}$.

(3) *For* $\forall x \in M$ *there is an open set* U *with* $x \in U$ *and a diffeomorphism* $T_u: \pi^{-1}(U) \rightarrow U \times G$ *of the form* $T_u(p) = (\pi(p), s_u(p))$, *where* $s_u: \pi^{-1}(U) \rightarrow G$ *has the property* $s_u(pg) = s_u(p)g$ *for* $\forall g \in G, p \in \pi^{-1}(U)$.



Lie Group: A Lie group (G,·) is a smooth manifold M such that (a , b) → a·b^{-1} is C$^{\infty}$-differentiable for any a, b in G.

5. A Question

For a family of k principal fiber bundles

$$P_1(M_1,G_1), P_2(M_2,G_2),\ldots, P_k(M_k, G_k)$$

over manifolds $M^1, M^2, \cdots, M^l$, how can we construct principal fiber bundles on a smoothly combinatorial manifold consisting of $M^1, M^2, \cdots, M^l$ underlying a connected graph G?

6. **Voltage Graph with Its Lifting**

6.1 **Voltage Assignment**

Let G be a connected graph and $(\mathcal{G}, \circ)$ a group. For each edge e∈E(G), e = uv, an orientation on e is an orientation on e from u to v, denoted by e = (u, v) , called plus orientation and its minus orientation, from v to u, denoted by e^{-1} = (v, u). For a given graph G with plus and minus orientation on its edges, a voltage assignment on G is a mapping α from the plus-edges of G into a group $(\mathcal{G}, \circ)$ satisfying $\alpha(e^{-1}) = \alpha^{-1}(e)$, e∈E(G). These elements α(e), e∈E(G) are called voltages, and (G, α) a voltage graph over the group $(\mathcal{G}, \circ)$.

6.2 Lifting of Voltage Graph

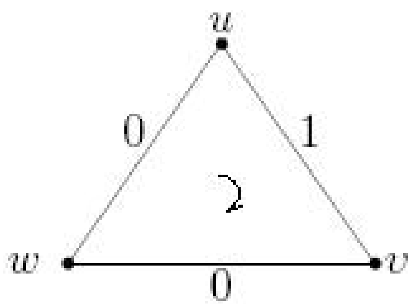
For a voltage graph (G, α), its lifting $G^\alpha = (V(G^\alpha), E(G^\alpha); I(G^\alpha))$ is defined by

$$V(G^\alpha) = V(G) \times \Gamma, \ (u, a) \in V(G) \times \Gamma \text{ abbreviated to } u_a;$$

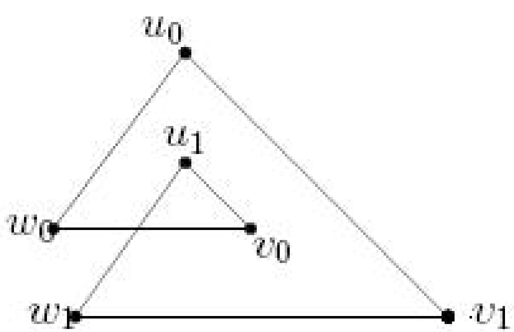
$$E(G^\alpha) = \{(u_a, v_{a \circ b}) | e^+ = (u, v) \in E(G), \alpha(e^+) = b\} \quad \text{and}$$

$$I(G^\alpha) = \{(u_a, v_{a \circ b}) | I(e) = (u_a, v_{a \circ b}) \ if \ e = (u_a, v_{a \circ b}) \in E(G^\alpha)\}.$$

For example, let $G = K_3$ and $\Gamma = Z_2$.



(G, α)



G^α

6.3 Voltage Vertex-Edge Labeled Graph with Its Lifting

Let G^L be a connected vertex-edge labeled graph with $\theta_L : V(G) \cup E(G) \to L$ of a label set and Γ a finite group. A *voltage labeled graph* on a vertex-edge labeled graph G^L is a 2-tuple $(G^L; \alpha)$ with a voltage assignments $\alpha : E(G^L) \to \Gamma$ such that

$$\alpha(u,v) = \alpha^{-1}(v,u), \quad \forall (u,v) \in E(G^L).$$

Similar to voltage graphs such as those shown in Example 3.1.3, the importance of voltage labeled graphs lies in their *labeled lifting* G^{L_α} defined by

$$V(G^{L_\alpha}) = V(G^L) \times \Gamma, \ \ (u,g) \in V(G^L) \times \Gamma \text{ abbreviated to } u_g;$$

$$E(G^L_\alpha) = \{ \ (u_g, v_{g\circ h}) \mid \text{for } \ \forall (u,v) \in E(G^L) \text{ with } \alpha(u,v) = h \ \}$$

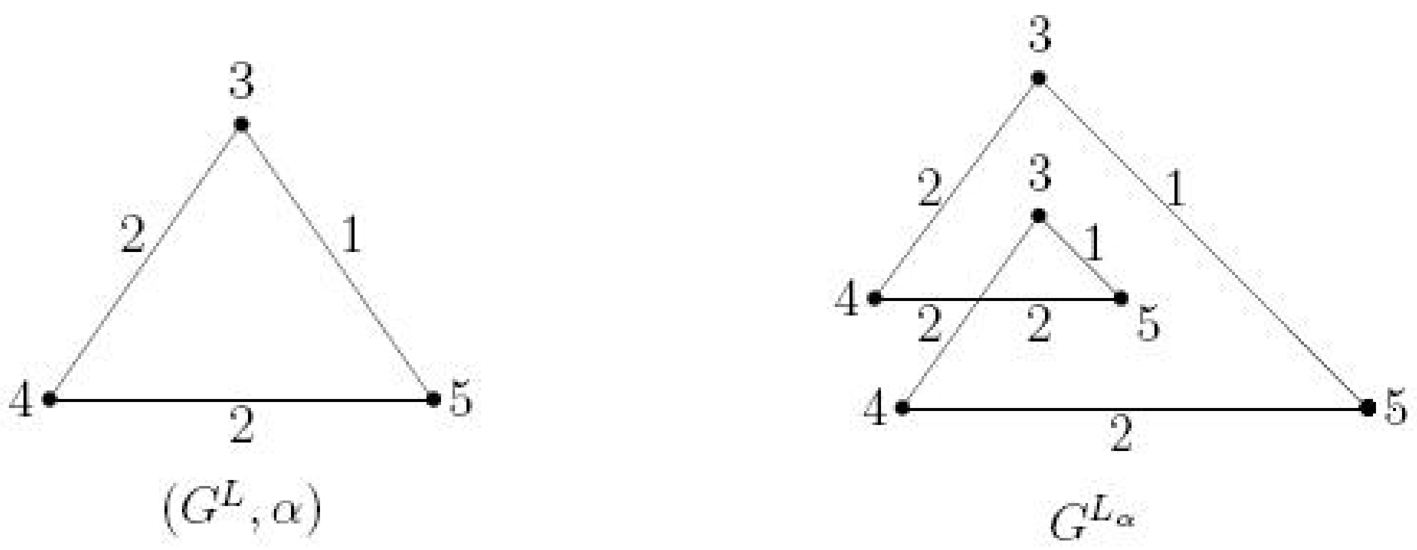
with labels $\Theta_L : G^{L_\alpha} \to L$ following:

$$\Theta_L(u_g) = \theta_L(u), \quad \text{and} \quad \Theta_L(u_g, v_{g\circ h}) = \theta_L(u,v)$$

for $u, v \in V(G^L)$, $(u,v) \in E(G^L)$ with $\alpha(u,v) = h$ and $g, h \in \Gamma$.

Example:

Let $G^L = C_3^L$ and $\Gamma = Z_2$.



6.4 Lifting of Automorphism of Graph

A mapping $g : G^L \to G^L$ is *acting on a labeled graph* G^L with a labeling $\theta_L : G^L \to L$ if $g\theta_L(x) = \theta_L g(x)$ for $\forall x \in V(G^L) \cup E(G^L)$, and a group Γ is acting on a labeled graph G^L if each $g \in \Gamma$ is acting on G^L.

Let A be a group of automorphisms of G^L. A voltage labeled graph (G^L, α) is called *locally* A*-invariant* at a vertex $u \in V(G^L)$ if for $\forall f \in A$ and $W \in \pi_1(G^L, u)$, we have

$$\alpha(W) = identity \Rightarrow \alpha(f(W)) = identity$$

and *locally* f*-invariant* for an automorphism $f \in \mathrm{Aut}G^L$ if it is locally invariant with respect to the group $\langle f \rangle$ in $\mathrm{Aut}G^L$.

Theorem 6.1 *Let* (G^L, α) *be a voltage labeled graph with* $\alpha : E(G^L) \to \Gamma$ *and* $f \in \mathrm{Aut}G^L$*. Then* f *lifts to an automorphism of* G^{L_α} *if and only if* (G^L, α) *is locally* f*-invariant.*

7. Combinatorial Fiber Bundle

7.1 Definition

Definition 7.1 *A combinatorial fiber bundle is a 4-tuple* $(\widetilde{M}^*, \widetilde{M}, p, G)$ *consisting of a covering combinatorial manifold* $\widetilde{M}^*$, *a group* G, *a combinatorial manifold* $\widetilde{M}$ *and a projection mapping* $p : \widetilde{M}^* \to \widetilde{M}$ *with properties following:*

(*i*) G *acts freely on* $\widetilde{M}^*$ *to the right.*

(*ii*) *the mapping* $p : \widetilde{M}^* \to \widetilde{M}$ *is onto, and for* $\forall x \in \widetilde{M}$, $p^{-1}(p(x)) = \mathrm{fib}_x = \{x_g | \forall g \in \Gamma\}$ *and* $l_x : \mathrm{fib}_x \to \Gamma$ *is a bijection.*

(*iii*) *for* $\forall x \in M$ *with its a open neighborhood* U_x, *there is an open set* $\widetilde{U}_x$ *and a mapping* $T_x : p^{-1}(U_x) \to \widetilde{U}_x \times \Gamma$ *of the form* $T_x(y) = (p(y), s_x(y))$, *where* $s_x : p^{-1}(U_x) \to \Gamma$ *has the property that* $s_x(yg) = s_x(y)g$ *for* $\forall g \in G$ *and* $y \in p^{-1}(U_x)$.

7.2 Theorem

Theorem 7.1 *Let $\widetilde{M}$ be a finite combinatorial manifold and $(G^L([\widetilde{M}]),\alpha)$ a voltage labeled graph with $\alpha : E(G^L([\widetilde{M}]) \to \Gamma$. Then $(\widetilde{M}^*, \widetilde{M}, p^*, \Gamma)$ is a combinatorial fiber bundle, where $\widetilde{M}^*$ is the combinatorial manifold correspondent to the lifting $G^{L_\alpha}([\widetilde{M}]$, $p^* : \widetilde{M}^* \to \widetilde{M}$ a natural projection determined by $p^* = h_s \circ \varsigma_M^{-1} p \varsigma_M$ with $h_s : M \to M$ a self-homeomorphism of $\widetilde{M}$ and $\varsigma_M : x \to M$ a mapping defined by $\varsigma_M(x) = M$ for $\forall x \in M$.*

Can we introduce differential structure on combinatorial Principal fiber bundles? The answer is YES!

8. Principal Fiber Bundle(PFB)

8.1 Lie Multi-Group

A *Lie multi-group* $\mathscr{L}_G$ is a smoothly combinatorial manifold $\widetilde{M}$ endowed with a multi-group $(\widetilde{\mathscr{A}}(\mathscr{L}_G); \mathscr{O}(\mathscr{L}_G))$, where $\widetilde{\mathscr{A}}(\mathscr{L}_G) = \bigcup_{i=1}^{m} \mathscr{H}_i$ and $\mathscr{O}(\mathscr{L}_G) = \bigcup_{i=1}^{m} \{\circ_i\}$ such that

(i) $(\mathscr{H}_i; \circ_i)$ is a group for each integer i, $1 \leq i \leq m$;

(ii) $G^L[\widetilde{M}] = G$;

(iii) the mapping $(a, b) \to a \circ_i b^{-1}$ is C^∞-differentiable for any integer i, $1 \leq i \leq m$ and $\forall a, b \in \mathscr{H}_i$.

8.2 Principal Fiber Bundle (PFB)

Let $\widetilde{P}$, $\widetilde{M}$ be a differentiably combinatorial manifolds and $\mathscr{L}_G$ a Lie multi-group $(\widetilde{\mathscr{A}}(\mathscr{L}_G); \mathscr{O}(\mathscr{L}_G))$ with

$$\widetilde{P} = \bigcup_{i=1}^{m} P_i, \ \widetilde{M} = \bigcup_{i=1}^{s} M_i, \ \widetilde{\mathscr{A}}(\mathscr{L}_G) = \bigcup_{i=1}^{m} \mathscr{H}_{\circ_i}, \mathscr{O}(\mathscr{L}_G) = \bigcup_{i=1}^{m} \{\circ_i\}.$$

A *differentiable principal fiber bundle over* $\widetilde{M}$ *with group* $\mathscr{L}_G$ consists of a differentiably combinatorial manifold $\widetilde{P}$, an action of $\mathscr{L}_G$ on $\widetilde{P}$ satisfying following conditions PFB1-PFB3:

PFB1. *For any integer* i, $1 \leq i \leq m$, $\mathscr{H}_{\circ_i}$ *acts differentiably on* P_i *to the right without fixed point, i.e.,*

$$(x, g) \in P_i \times \mathscr{H}_{\circ_i} \to x \circ_i g \in P_i \text{ and } x \circ_i g = x \text{ implies that } g = 1_{\circ_i};$$

PFB2. *For any integer i, $1 \leq i \leq m$, $M_{\circ_i}$ is the quotient space of a covering manifold $P \in \Pi^{-1}(M_{\circ_i})$ by the equivalence relation R induced by $\mathscr{H}_{\circ_i}$:*

$$R_i = \{(x, y) \in P_{\circ_i} \times P_{\circ_i} | \exists g \in \mathscr{H}_{\circ_i} \Rightarrow x \circ_i g = y\},$$

written by $M_{\circ_i} = P_{\circ_i}/\mathscr{H}_{\circ_i}$, i.e., an orbit space of $P_{\circ_i}$ under the action of $\mathscr{H}_{\circ_i}$. These is a canonical projection $\Pi : \widetilde{P} \to \widetilde{M}$ such that $\Pi_i = \Pi|_{P_{\circ_i}} : P_{\circ_i} \to M_{\circ_i}$ is differentiable and each fiber $\Pi_i^{-1}(x) = \{p \circ_i g | g \in \mathscr{H}_{\circ_i}, \Pi_i(p) = x\}$ is a closed

PFB3. *For any integer i, $1 \leq i \leq m$, $P \in \Pi^{-1}(M_{\circ_i})$ is locally trivial over $M_{\circ_i}$, i.e., any $x \in M_{\circ_i}$ has a neighborhood U_x and a diffeomorphism $T : \Pi^{-1}(U_x) \to U_x \times \mathscr{L}_G$ with*

$$T|_{\Pi_i^{-1}(U_x)} = T_i^x : \Pi_i^{-1}(U_x) \to U_x \times \mathscr{H}_{\circ_i};\ x \to\ T_i^x(x) = (\Pi_i(x), \epsilon(x)),$$

called a local trivialization (abbreviated to LT) such that $\epsilon(x \circ_i g) = \epsilon(x) \circ_i g$ for $\forall g \in \mathscr{H}_{\circ_i}$, $\epsilon(x) \in \mathscr{H}_{\circ_i}$.

8.3 Construction by Voltage Assignment

For a family of principal fiber bundles over manifolds $M_1, M_2, \cdots, M_l$, *such as those shown in Fig.* **8.1**,

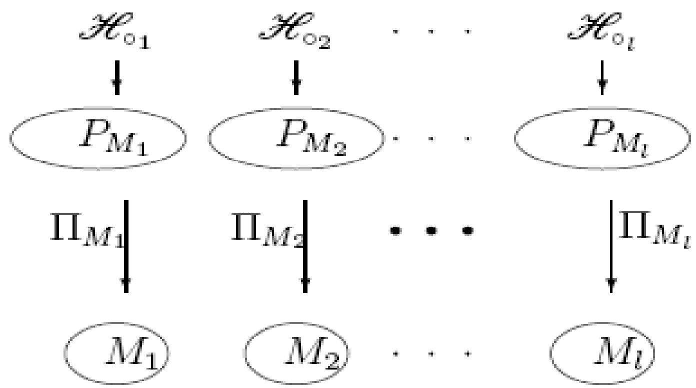


Fig. 8.1

where $\mathscr{H}_{\circ_i}$ *is a Lie group acting on* P_{M_i} *for* $1 \leq i \leq l$ *satisfying conditions PFB1-PFB3, let* $\widetilde{M}$ *be a differentiably combinatorial manifold consisting of* M_i, $1 \leq i \leq l$ *and* $(G^L[\widetilde{M}], \alpha)$ *a voltage graph with a voltage assignment* $\alpha : G^L[\widetilde{M}] \rightarrow \mathfrak{G}$ *over a finite group* $\mathfrak{G}$, *which naturally induced a projection* $\pi : G^L[\widetilde{P}] \rightarrow G^L[\widetilde{M}]$. *For* $\forall M \in V(G^L[\widetilde{M}])$, *if* $\pi(P_M) = M$, *place* P_M *on each lifting vertex* M^{L_α} *in the fiber* $\pi^{-1}(M)$ *of* $G^{L_\alpha}[\widetilde{M}]$, *such as those shown in Fig.* **8.2**.

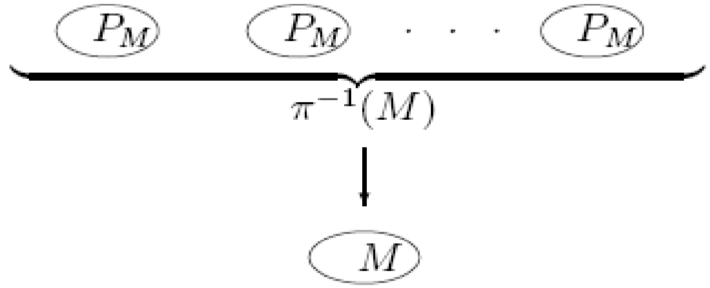
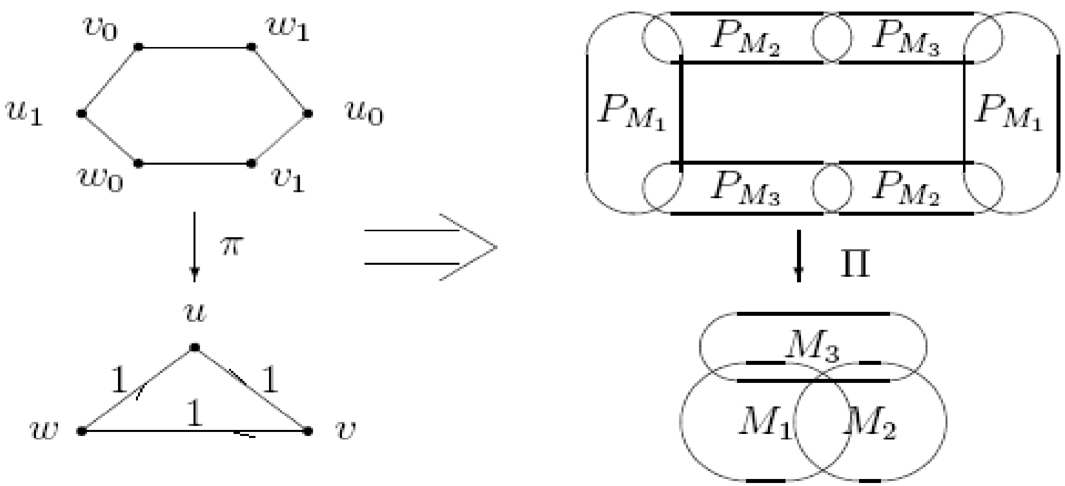


Fig. 8.2

Let $\Pi = \pi\Pi_M\pi^{-1}$ *for* $\forall M \in V(G^L[\widetilde{M}])$. *Then* $\widetilde{P} = \bigcup\limits_{M\in V(G^L[\widetilde{M}])} P_M$ *is a smoothly combinatorial manifold and* $\mathscr{L}_G = \bigcup\limits_{M\in V(G^L[\widetilde{M}])} \mathscr{H}_M$ *a Lie multi-group by definition. Such a constructed combinatorial fiber bundle is denoted by* $\widetilde{P}^{L_\alpha}(\widetilde{M}, \mathscr{L}_G)$.



8.4 Results

Theorem 8.1 *A combinatorial fiber bundle* $\widetilde{P}^{\alpha}(\widetilde{M},\mathscr{L}_G)$ *is a principal fiber bundle if and only if for* $\forall(M',M'')\in E(G^L[\widetilde{M}])$ *and* $(P_{M'},P_{M''})=(M',M'')^{L_\alpha}\in E(G^L[\widetilde{P}])$, $\Pi_{M'}|_{P_{M'}\cap P_{M''}}=\Pi_{M''}|_{P_{M'}\cap P_{M''}}$.

Theorem 8.2 *Let* $\widetilde{P}^{\alpha}(\widetilde{M},\mathscr{L}_G)$ *be a principal fiber bundle. Then*

$$\mathrm{Aut}\widetilde{P}^{\alpha}(\widetilde{M},\mathscr{L}_G)\geq\langle\mathfrak{L}\rangle\ ,$$

where $\mathfrak{L}=\{\ \widehat{h}\omega_i\mid\widehat{h}:P_{M_i}\to P_{M_i}$ *is* $1_{P_{M_i}}$ *determined by* $h((M_i)_g)=(M_i)_{g\circ_i h}$ *for* $h\in\mathfrak{G}$ *and* $g_i\in\mathrm{Aut}P_{M_i}(M_i,\mathscr{H}_{\circ_i}),\ 1\leq i\leq l\}$.

A principal fiber bundle $\widetilde{P}(\widetilde{M},\mathscr{L}_G)$ is called to be *normal* if for $\forall u,v\in\widetilde{P}$, there exists an $\omega\in\mathrm{Aut}\widetilde{P}(\widetilde{M},\mathscr{L}_G)$ such that $\omega(u)=v$. We get the necessary and sufficient conditions of normally principal fiber bundles $\widetilde{P}^{\alpha}(\widetilde{M},\mathscr{L}_G)$ following.

Theorem 8.3 $\widetilde{P}^{\alpha}(\widetilde{M},\mathscr{L}_G)$ *is normal if and only if* $P_{M_i}(M_i,\mathscr{H}_{\circ_i})$ *is normal,* $(\mathscr{H}_{\circ_i};\circ_i)=(\mathscr{H};\circ)$ *for* $1\leq i\leq l$ *and* $G^{L_\alpha}[\widetilde{M}]$ *is transitive by diffeomorphic automorphisms in* $\mathrm{Aut}G^{L_\alpha}[\widetilde{M}]$.

9. Connection on PFB

A *local connection* on a principal fiber bundle $\widetilde{P}^\alpha(\widetilde{M}, \mathscr{L}_G)$ is a linear mapping ${}^i\Gamma_u : T_x(\widetilde{M}) \to T_u(\widetilde{P})$ for an integer i, $1 \le i \le l$ and $u \in \Pi_i^{-1}(x) = {}^iF_x$, $x \in M_i$, enjoys the following properties:

(*i*) $(d\Pi_i){}^i\Gamma_u$ = identity mapping on $T_x(\widetilde{M})$;

(*ii*) ${}^i\Gamma_{{}^iR_g \circ_i u} = d\,{}^iR_g \circ_i {}^i\Gamma_u$, where ${}^iR_g$ denotes the right translation on P_{M_i};

(*iii*) the mapping $u \to {}^i\Gamma_u$ is C^∞.

Similarly, a *global connection* on a principal fiber bundle $\widetilde{P}^\alpha(\widetilde{M}, \mathscr{L}_G)$ is a linear mapping $\Gamma_u : T_x(\widetilde{M}) \to T_u(\widetilde{P})$ for a $u \in \Pi^{-1}(x) = F_x$, $x \in \widetilde{M}$ with conditions following hold:

(*i*) $(d\Pi)\Gamma_u$ = identity mapping on $T_x(\widetilde{M})$;

(*ii*) $\Gamma_{R_g \circ u} = dR_g \circ \Gamma_u$ for $\forall g \in \mathscr{L}_G$ and $\forall \circ \in \mathscr{O}(\mathscr{L}_G)$, where R_g denotes the right translation on $\widetilde{P}$;

(*iii*) the mapping $u \to \Gamma_u$ is C^∞.

Theorem 9.1 *There are always exist global connections on a normally principal fiber bundle* $\widetilde{P}^\alpha(\widetilde{M}, \mathscr{L}_G)$.

Theorem 9.2 (E.Cartan) *Let* ${}^{i}\omega$, $1 \leq i \leq l$ *and* ω *be local or global connection forms on a principal fiber bundle* $\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G)$. *Then*

$$(d\,{}^{i}\omega)(X,Y) = -[\,{}^{i}\omega(X),{}^{i}\omega(Y)] + {}^{i}\Omega(X,Y)$$

and

$$d\omega(X,Y) = -[\omega(X),\omega(Y)] + \Omega(X,Y)$$

for vector fields $X, Y \in \mathscr{X}(P_{M_i})$ *or* $\mathscr{X}(\widetilde{P})$.

Theorem 9.3 (Bianchi) *Let* ${}^{i}\omega$, $1 \leq i \leq l$ *and* ω *be local or global connection forms on a principal fiber bundle* $\widetilde{P}^{\alpha}(\widetilde{M}, \mathscr{L}_G)$. *Then*

$$(d\,{}^{i}\Omega)h = 0, \quad \textit{and} \quad (d\Omega)h = 0.$$

10. Applications to Gauge Field

A *gauge field* is such a mathematical model with local or global symmetries under a group, a finite-dimensional Lie group in most cases action on its gauge basis at an individual point in space and time, together with a set of techniques for making physical predictions consistent with the symmetries of the model, which is a generalization of Einstein's principle of covariance to that of internal field.

Gauge Invariant Principle *A gauge field equation, particularly, the Lagrange density of a gauge field is invariant under gauge transformations on this field.*

Combinatorial Gauge Field. A *globally* or *locally combinatorial gauge field* is a combinatorial field $\widetilde{M}$ under a gauge transformation $\tau_{\widetilde{M}} : \widetilde{M} \to \widetilde{M}$ independent or dependent on the field variable $\overline{x}$.

If a combinatorial gauge field $\widetilde{M}$ is consisting of gauge fields $M_1, M_2, \cdots, M_m$, we can easily find that $\widetilde{M}$ *is a globally combinatorial gauge field only if each gauge field is global.*

Whence, we can find infinite combinatorial gauge fields by application of principal fiber bundle.

Background:

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2.(2005)Formally presented **CC Conjecture**:

A mathematical science can be reconstructed from or made by combinatorialization.

3. (2006) **Smarandache Multi-Space Theory**, Hexis, Phoenix, American. (Reviewer: An algebraic geometry book)

4.(**2006)Selected Papers on Mathematical Combinatorics**, World Academic Union.

5.(2007)Sponsored journal: ***International J. Mathematical Combinatorics***, USA.

6.(**2009)Combinatorial Geometry with Application to Field Theory**, USA.